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Internal constraints in the theories of immiscible mixtures for soils

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Abstract

In this paper we formulate balance principles for an immiscible mixture of continua with microstructure in the broadest sense to include, *e.g.*, diffusion and adsorption phenomena, strain gradient effects and chemical reactions, and introduce an additional balance of micromomentum for each constituent to describe the microstructural effects. Next, we describe a method for taking into account the general internal constraints in entropy inequality, based on an 'extended' principle of thermodynamic determinism, and obtain a set of 'pure' constitutive equations. Finally we consider some examples of interest for thermo-mechanics of soils, such as granular materials dispersed in a fluid or the flow of fluids in a porous solid. Particular solutions are obtained in linear approximations for mixtures of packed granular materials in rarefied air, and solids with nano-pores filled by a gas.

Key words: Immiscible mixtures, complex materials, multi-scale representations, internal constraints, soil mechanics, balance laws, constitutive equations, numerical solutions.

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1 Introduction

A general formulation of balance principles for a mixture of continua with microstructure in the widest sense was presented in [41] to study diffusion, adsorption and broader chemical-physical phenomena. The theory generalizes the multiphase mixtures presented in [76] where each constituent has a simple geometric structure characterized only by a scalar kinematic parameter, its volume fraction. In fact, by considering kinematical parameters on a differentiable manifold, we unify proposals (such as those in the essay [11] or in [62] dealing with granular and porous media, or Cosserat and micromorphic continua. An example of mixtures for micromorphic materials has been studied in [90], and applied to micropolar media, although, there, the authors consider for each constituent of the mixture an additional balance equation, parallel, in a certain sense, to the mass constituent balance: that of microinertia moment. Instead, such equation has been shown to be a simple consequence of the definition of the tensor field of microstructural inertia (see, *e.g.*, §21 of [11], or equations (4.12) and (34) of [17] and [43], respectively).

Moreover, in [41] a new expression of the integral balances of moment of momentum appears evident in the theory, in which the presence of the various microstructures is taken into account, while the mass density fields can no longer be regarded as determined by the deformation fields because chemical reactions are present, thus the constitutive assumptions must allow for a dependence on a larger number of variables (see, also, [14] for reacting mixtures of polar bodies).

The present work concerns the use of internal constraints [18] in order to study the essential features of some classical models of soils capable of describing the effects of immiscibility and variable volume fractions, besides those associated to microstructural interactions. In the so-called immiscible mixtures the components do not form a mixture on the molecular level, as it is the case for fluids, but remain separated to the sub-structural level of observations: material systems such as rocks, solid filters, granular and porous media, biological tissues, clays, etc. belong to this class (see, for example, § 5A.4 of [10], otherwise [79], or further the Ch. 13 of [95]). Furthermore, due to the complexity of the interactions between components as well as among macro- and microstructure, it is assumed that the entropy flux is not equal to the heat flux divided by the temperature, as suggested in [68] and [51].

As special cases of our proposal we find the description of a theory of fluid suspensions, an incompressible mixture of a concentrated granular material immersed in a fluid, and a mechanical theory of poroelasticity, in which the constitutive equations for the solid elastic skeleton take into account for the micro-strain gradient effects due to the presence of nanopores.

Detailed numerical studies are also provided to describe the effects of microstructures, such as the influence of micro-rotations in a quasi-linear dilatant granular material with rotating grains, or the incidence of micro-vibrations in a linear thermoelastic solid with big pores.

2 Kinematics and microstructures

In this section we discuss the kinematics of motion and the equations of balance for a mixture of n continuous bodies $\mathcal{B}_i$, $i = 1, 2, \ldots, n$, each endowed with its own microstructure, and, in our developments and notations, we mainly follow Truesdell (Lecture 5 of [86]).

We assume that all of them are able to occupy regions of the three-dimensional Euclidean space $\mathcal{E}$, at a certain time τ in an interval $[\tau_0, \tau_1]$ during which the motion is observed, and indeed contemporaneously: for which every place x in the body is simultaneously occupied by a material particle x_i of each constituent at time τ. If X_i is the place taken by a particle of $\mathcal{B}_i$ in some reference placement, the motion of $\mathcal{B}_i$ is the smooth mapping

$$x = x_i(X_i, \tau) \tag{1}$$

of $\mathcal{B}_i$ onto a time-sequence of placements in space: each such motion has its own kinematics.

We shall use a subscript to indicate a constituent and a prime to denote the material time derivative following the motion of that constituent; therefore, v_i and a_i are the peculiar velocity and acceleration of constituent i, respectively:

$$v_i := \frac{\partial x_i}{\partial \tau}(X_i, \tau) = x_i'(x, \tau), \quad a_i := \frac{\partial^2 x_i}{\partial \tau^2}(X_i, \tau) = x_i''(x, \tau). \tag{2}$$

The i^{th} peculiar velocity gradient L_i and deformation gradient F_i of $X_i \in \mathcal{B}_i$ are

$$L_i = \operatorname{grad} v_i(x, \tau) \quad \text{and} \quad F_i = \frac{\partial x_i}{\partial X_i}(X_i, \tau), \tag{3}$$

respectively. Because of the assumptions made about the smoothness of x_i, it is

$$\iota_i := \det F_i > 0, \tag{4}$$

thus F_i^{-1} exists and, by the chain rule, it is easy to show that

$$L_i = \left(F_i' F_i^{-1}\right)(x, \tau) = D_i + W_i, \tag{5}$$

where the standard decomposition of the velocity gradients are used with $D_i := \frac{1}{2}(L_i + L_i^T)$ and $W_i := \frac{1}{2}(L_i - L_i^T)$ the i^{th} peculiar rate of deformation and spin tensor, respectively.

Each body $\mathcal{B}_i$ has its own bulk mass and consequently its mass density per unit volume ρ_i in the placement x_i at time τ, then the density and the velocity of the mixture are defined by

$$\rho := \sum \rho_i \quad \text{and} \quad v := \sum \xi_i v_i, \quad \text{with} \quad \xi_i = \frac{\rho_i}{\rho}, \tag{6}$$

respectively, where ξ_i is the concentration of the i^{th} constituent; here and henceforth, $\sum$ stands for summation from $i = 1$ to $i = n$. Moreover, if we introduce the diffusion velocity of the i^{th} constituent in the mixture

$$u_i = v_i - v, \tag{7}$$

the following property holds:

$$\sum \xi_i \, u_i = \sum \xi_i \, v_i - \left(\sum \xi_i\right) \, v = 0. \tag{8}$$

The hypothesis that the constituents $\mathcal{B}_i$ of the mixture have a Lagrangian microstructure (in the sense of Capriz [11]) means that each material element of a single body reveals a microscopic geometric order at a closer look; then it is there assigned a measure $\nu_i(x)$ of the peculiar microstructure, read on a smooth manifold $\mathcal{M}_i$ of finite dimension m_i: *e.g.*, the interval $[0, \bar{\nu})$ of real number, with $\bar{\nu} < 1$, for the volume fractions of fluids in an immiscible mixture [76], the projective plane in the theory of liquid crystals [29] or the space of definite positive symmetric tensor in the theory of solids with large pores [35]. For now, we do not fix the tensor rank of each order parameter ν_i.

Let us consider now two observers differing by a rotation of characteristic vector $q(\tau)$, with corresponding proper orthogonal tensor

$$Q(\tau) = \exp[-\varepsilon \, q(\tau)] \left[:= I - \varepsilon \, q + \frac{1}{2}(\varepsilon \, q)(\varepsilon \, q) - \ldots\right], \tag{9}$$

where exp is the basis of natural logarithms, ε is Ricci's three-dimensional alternating tensor and I is the identity tensor. They read two different values ν_i and $(\nu_i)_q$ of the i^{th} order parameters connected by the following relation (see §3 of [11] or §6 of [19]):

$$(\nu_i)_q = \nu_i + \mathcal{A}_i q + o(|q|), \tag{10}$$

where $\mathcal{A}_i(\nu_i)$ is the infinitesimal generator of the local action of the proper orthogonal group SO(3) over $\mathcal{M}_i$ defined by:

$$\mathcal{A}_i(\nu_i) := \left. \frac{d(\nu_i)_q}{dq} \right|_{q=0} ; \tag{11}$$

$\mathcal{A}_i$ is a linear operator mapping vectors of $\Re^3$ into elements of the tangent space $T_{\nu_i}\mathcal{M}_i$ to $\mathcal{M}_i$ at ν_i and, in its matrix representation, has three columns and a number of rows equal to the dimension m_i of $\mathcal{M}_i$. For the examples above, the volume fraction of a fluid is invariant for changes of observer, so $\mathcal{A}_i$ vanishes; in the theory of uniaxial liquid crystals, the order parameter is a unit vector d marking the alignment of rod-like molecules, therefore $\mathcal{A}_i$ coincides with (εd); finally, in the theory of solids with large pores, the order parameter is a symmetric tensor U, which changes as a 2^{nd} order tensor, thus $\mathcal{A}_i$ has the following components $(\mathcal{A}_i)_{\alpha\beta\iota} = U_{\alpha\gamma}\varepsilon_{\gamma\beta\iota} - \varepsilon_{\alpha\gamma\iota}U_{\gamma\beta}$ (see, also, §3 of [11] and §3.1 of [64] for more general changes in observers, currently not of interest in this context). The convention that repeated greek indices are summed is adopted throughout.

Now we suppose that, for each body $\mathcal{B}_i$, exists a non-negative kinetic energy $\kappa_i(\nu_i, \omega_i)$, associated with each time-rate of change of the i^{th} microstructure

$$\omega_i := \nu_i' = \frac{\partial \nu_i}{\partial \tau} + (\operatorname{grad} \nu_i)\, v_i = \dot{\nu}_i + (\operatorname{grad} \nu_i)\, u_i, \tag{12}$$

that is the material time derivative of ν_i with respect to the peculiar velocity v_i, and which will be called, briefly, the i^{th} microspeed; this kinetic energy κ_i is such that $\kappa_i(\nu_i, 0) = 0$ and $\frac{\partial^2 \kappa_i}{\partial \omega_i^2} \neq 0$. Here, $\dot{(\cdot)}$ $\left(:= \frac{\partial(\cdot)}{\partial\tau} + [\operatorname{grad}(\cdot)]\, v\right)$ is the material time derivative with respect to the mixture velocity v given by $(6)_2$.

Moreover, we can define the density of kinetic co-energy $\chi_i(\nu_i, \omega_i)$ related to κ_i by the Legendre transform:

$$\kappa_i = \frac{\partial \chi_i}{\partial \omega_i} \cdot \omega_i - \chi_i, \tag{13}$$

that is, χ_i is a solution of the system of linear partial differential equation of the first order (13) or, introducing coordinates ν_i^α in a local chart of the manifold $\mathcal{M}_i$,

$$\kappa_i(\nu_i^\alpha; (\omega_i)^\gamma) = \frac{\partial \chi_i}{\partial (\omega_i)^\beta} \cdot (\omega_i)^\beta - \chi_i, \tag{14}$$

with the usual convention for the sum over repeated indices. Of course, if χ_i were homogeneous of second degree in ω_i, then it would coincide with κ_i. On the contrary, even if κ_i were homogeneous of second degree in ω_i, χ_i need not coincide with κ_i (see [13]). In the absence of κ_i, the measure ν_i is rather termed internal (state) variable and ruled by a first order evolution equation instead of a balance equation (see, *e.g.*, equation (5A.4.11) of [10]).

3 Laws of balance for constituents

For a region $\mathcal{V}$ of space, we may consider the actions on the part of the constituent body $\mathcal{B}_i$ presently occupying $\mathcal{V}$ and calculate the rates of growth per unit volume of mass α_i^+, linear momentum m_i^+, micromomentum ϕ_i^+, rotational momentum z_i^+, energy ϵ_i^+ and entropy η_i^+ within it. It is meaningful to note that these quantities derive from interactions between constituents and should therefore be recognized as terms of interchange, thus the rules that underlie our formulation of the constituent balance laws are the first two metaphysical principles of Truesdell [86]: 1) all properties of the whole mixture must be mathematical consequences of properties of the constituents; 2) so as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.

Each constituent $\mathcal{B}_i$ undergoes actions of three kinds: 1) the contact actions, represented by the stress T_i, the microstress $\mathcal{S}_i$, the heating and entropy fluxes q_i and p_i, respectively; 2) the internal microactions per unit volume ζ_i; 3) the prescribed actions at a distance, depicted by the densities per unit mass of body force b_i, microforce δ_i, heating λ_i and entropy supply ϖ_i. Therefore, the integral equations of balance for the constituent i of the mixture, proposed in [41], are assumed in the general form, with the growth terms included in the source term:

$$
\begin{aligned}
&\int \alpha_i^+ \, dv \equiv \left(\int \rho_i \, dv \right)', \\
&\int m_i^+ \, dv \equiv \left(\int \rho_i v_i \, dv \right)' - \int \rho_i b_i \, dv - \oint T_i n \, da, \\
&\int \phi_i^+ \, dv \equiv \left(\int \rho_i \frac{\partial \chi_i}{\partial \omega_i} \, dv \right)' - \int \left[\rho_i \left(\frac{\partial \chi_i}{\partial \nu_i} + \delta_i \right) - \zeta_i \right] dv - \oint \mathcal{S}_i n \, da, \\
&\int \left(z_i^+ + r \times m_i^+ + \mathcal{A}_i^T \phi_i^+ \right) dv \equiv \left[\int \rho_i \left(r \times v_i + \mathcal{A}_i^T \frac{\partial \chi_i}{\partial \omega_i} \right) dv \right]' - \\
&\qquad - \int \rho_i \left(r \times b_i + \mathcal{A}_i^T \delta_i \right) dv - \oint \left[r \times T_i n + \mathcal{A}_i^T (\mathcal{S}_i n) \right] da, \\
&\int \epsilon_i^+ \, dv \equiv \left[\int \rho_i \left(\epsilon_i + \frac{1}{2} v_i^2 + \kappa_i \right) dv \right]' \\
&\qquad - \int \rho_i \left(\lambda_i + b_i \cdot v_i + \delta_i \cdot \omega_i \right) dv + \oint \left(q_i - T_i^T v_i - \mathcal{S}_i^T \omega_i \right) \cdot n \, da, \\
&\int \eta_i^+ \, dv \equiv \left(\int \rho_i \eta_i dv \right)' - \int \rho_i \varpi_i dv + \oint p_i \cdot n \, da.
\end{aligned}
\tag{15}
$$

In the equations (15), $\int$ denotes integration over the volume $\mathcal{V}$ and dv the element of volume; $\oint$ denotes integration over its boundary $\partial\mathcal{V}$ and da the element of surface, while n is the outward unit vector normal to the boundary $\partial\mathcal{V}$; the position vector field r is given by $(x - x_0)$ with x_0 a fixed point in $\mathcal{E}$; the transpose of the $(m_i + 1)^{\text{th}}$ order tensors $\mathcal{A}_i$ (or $\mathcal{S}_i$) has the following components $(\mathcal{A}_i^T)_{\alpha \dots \beta \iota} = (\mathcal{A}_i)_{\iota \alpha \dots \beta}$; ε_i and η_i are the i^{th} peculiar internal energy and entropy, respectively.

The formulation of the balance of rotational momentum in the form $(15)_4$ appears to be a novelty in the theories of general continua with microstructure, even if the deduction of its local form will be in agreement with that inferred, *e.g.*, from a theorem of kinetic energy in [11] and [40], or obtained with invariance requirements with respect to classes of changes in observers [63]. Moreover, if we consider polar continua of Cosserat brothers or oriented materials of Toupin, the order parameters ν_i and the operator $\mathcal{A}_i$ can be recognized as a proper orthogonal tensor R and the 3^{rd}-order tensor of components: $(\mathcal{A}_i)_{\alpha\beta\iota} = \varepsilon_{\alpha\gamma\iota} R_{\gamma\beta}$, respectively; therefore, for a single polar body, the balance $(15)_4$ in local form reduces to the Cosserats' law for balance of moment of momentum (205.10) of [89], or otherwise to the Toupin's spin momentum balance (98.26) of [88] (when the same spin momentum vanishes), that is

$$\operatorname{skw} T_i = \rho_i l + \operatorname{div} M, \tag{16}$$

where the skew part of a tensor C is defined as $\operatorname{skw} C := 2^{-1}(C - C^T)$, while the assigned couple field l and the couple stress tensor M are now

$$l_{\alpha\beta} := R_{[\alpha\delta}(\delta_i)_{\delta\beta]} \quad \text{and} \quad M_{\alpha\beta\iota} = R_{[\alpha\delta}(\mathcal{S}_i)_{\delta\beta]\iota} \tag{17}$$

respectively: here the square brackets specify the skew-symmetric part with respect to indicated indices.

Finally, there are appropriate additional terms in the energy equation $(15)_5$ corresponding to the work done by the respective field terms in the balance of micromomentum.

For suitably smooth regions and fields, it is possible to apply the divergence theorem and obtain the local statements of balance equations (15) for each constituent $\mathcal{B}_i$:

$$\begin{aligned}
\alpha_i^+ &= \rho_i' + \rho_i \operatorname{div} v_i, && (18)\\
m_i^+ - \alpha_i^+ v_i &= \rho_i v_i' - \rho_i b_i - \operatorname{div} T_i, && (19)\\
\phi_i^+ - \alpha_i^+ \frac{\partial \chi_i}{\partial \omega_i} &= \rho_i \left(\frac{\partial \chi_i}{\partial \omega_i}\right)' - \rho_i \left(\frac{\partial \chi_i}{\partial \nu_i} + \delta_i\right) - \operatorname{div} \mathcal{S}_i + \zeta_i, && (20)\\
z_i^+ &= \varepsilon\, T_i - \mathcal{A}_i^T \zeta_i - (\operatorname{grad} \mathcal{A}_i^T)\mathcal{S}_i, && (21)\\
\epsilon_i^+ - m_i^+ \cdot v_i - \phi_i^+ \cdot \omega_i - \alpha_i^+ \left(\epsilon_i - 2^{-1} v_i^2 - \kappa_i\right) &= && (22)\\
&= \rho_i \epsilon_i' - \rho_i \lambda_i + \operatorname{div} q_i - T_i \cdot L_i - \zeta_i \cdot \omega_i - \mathcal{S}_i \cdot \operatorname{grad} \omega_i, &&\\
\eta_i^+ &= \alpha_i^+ \eta_i + \rho_i \eta_i' - \rho_i \varpi_i + \operatorname{div} p_i. && (23)
\end{aligned}$$

We wish to observe that, to get equation (21), we used balances (18)-(20) and the invariance of χ_i under the galilean group, *i.e.*, $\left(\mathcal{A}_i^T\right)' \frac{\partial \chi_i}{\partial \omega_i} = -\mathcal{A}_i^T \frac{\partial \chi_i}{\partial \nu_i}$. Further, in equation (23), we followed Müller [68] by assuming that, in general, the entropy flux p_i is not equal to the heat flux q_i over the peculiar temperature θ_i: in fact the components of the mixture are complex bodies as it is the case, for example, for granular mixtures (see [92,44]). In this regard, the multiphase theory formulated in [76], and based on the continua of "Goodman & Cowin" type [51], ignores this assumption despite it being present in [51] itself: it could be neglected only if the related body has particular material symmetries [68].

4 Powers and objectivity

The kinetic energy $\mathcal{T}_i$ of the i^{th} constituent $\mathcal{B}_i$ occupying $\mathcal{V}$ in the present configuration is defined by

$$\mathcal{T}_i \equiv \int \rho_i \left(\frac{1}{2} v_i^2 + \kappa_i \right) dv, \tag{24}$$

while the mechanical power $\mathcal{P}_i$ developed on the i^{th} constituent occupying $\mathcal{V}$ is the rate of working of all forces acting on $\mathcal{B}_i$ by the exterior bodies:

$$\mathcal{P}_i = \mathcal{P}_i^{pro} + \mathcal{P}_i^{exc}, \tag{25}$$

where the decomposition of the power $\mathcal{P}_i$ into a proper part $\mathcal{P}_i^{pro}$, the one that can be attributed to $\mathcal{B}_i$ as if it was isolated, and an exchange part $\mathcal{P}_i^{pro}$, which accounts for the power of the direct actions exerted on $\mathcal{B}_i$ by other components, is motivated by the second metaphysical principle for mixtures, for which each constituent can be isolated from the rest provided that the interactions with the others are accounted for. Precisely, they are

$$\begin{aligned} \mathcal{P}_i^{pro} &:= \int \rho_i \left(v_i \cdot b_i + \omega_i \cdot \delta_i \right) dv + \oint \left[v_i \cdot (T_i\, n) + \omega_i \cdot (\mathcal{S}_i\, n) \right] da \quad \text{and} \\ \mathcal{P}_i^{exc} &:= \int \left[v_i \cdot \left(m_i^+ - \alpha_i^+ v_i \right) + \omega_i \cdot \left(\phi_i^+ - \alpha_i^+ \frac{\partial \chi_i}{\partial \omega_i} \right) + \frac{1}{2} \left(\operatorname{curl} v_i \right) \cdot z_i^+ \right] dv, \end{aligned} \tag{26}$$

where the curl of a vector u is defined as: $\operatorname{curl} u := -\varepsilon(\operatorname{grad} u)$. The macro- and micro-actions appearing in the expressions above have contact and bulk nature. In definition $(26)_1$, contact and bulk macro- and micro-actions include all mechanical actions, as introduced on the right hand-side of balances (15). Instead, the second category of bulk actions, which appears in $(26)_2$ for $\mathcal{P}_i^{exc}$, consists of macro- and micro-interactions expressed on the i^{th} constituent by all the other components: they are defined through the rates of growth and their expressions become evident on the left hand-side of balances (19-21) (see, also, [63,60]).

We are now able to obtain the expression for the net working $\mathcal{W}_i$ of the i^{th} constituent according to a classical theorem of kinetic energy first proved by

Stokes for simple bodies (see Lecture 2 of [86]). It is given by

$$\mathcal{W}_i \equiv \mathcal{P}_i - \dot{\mathcal{T}}_i \tag{27}$$

and can be inferred in a standard way from the definition (25) of the power $\mathcal{P}_i$, the divergence theorem and the balance equations (19) and (20) of macro- and micro-momentum, respectively; namely, taking the appropriate scalar product of both sides of (19) by v_i, operating similarly with ω_i on (20), integrating both sides over the region $\mathcal{V}$ by parts where possible, taking account for the balance of mass (18) and the Legendre transform (13) and summing finally term by term.

The result (27) asserts that the working $\mathcal{W}_i$ is the power of the forces exerted upon $\mathcal{B}_i$ by the exterior of $\mathcal{B}_i$, minus the rate of increase in kinetic energy of $\mathcal{B}_i$, in an inertial frame. Therefore, we are led to the following formula:

$$\mathcal{W}_i = \int \left(T_i \cdot L_i + \frac{1}{2}\, z_i^+ \cdot \operatorname{curl} v_i + \zeta_i \cdot \omega_i + \mathcal{S}_i \cdot \operatorname{grad} \omega_i \right) dv, \tag{28}$$

where the scalar under the sign of integral is the so-called net working per unit volume w_i of the body with microstructure $\mathcal{B}_i$ in the mixture; w_i is often called the stress power [86,18].

In the following we shall give a suitable definition of a continuum with microstructure subject to perfect internal kinematical constraints, for which the expression of the stress power w_i plays an essential rôle, thus we furnish here a version of w_i which clearly shows its independence from the observer by means of the use of the corotational time derivative $\breve{\nu}_i$, due to the spin tensor W_i, which is an objective measure of the microspeed ω_i (see, *e.g.*, equation (36.13) of [88] or §2.8.2.4 of [55]); precisely, if we introduce the spin vector

$$r_i := -\frac{1}{2} \varepsilon \left(F_i' F_i^{-1} \right) = \frac{1}{2} \operatorname{curl} v_i, \tag{29}$$

we have the subsequent expression for $\breve{\nu}_i$:

$$\breve{\nu}_i = \omega_i - \mathcal{A}_i\, r_i \tag{30}$$

(see comments in Remark 2 of §6 in [11]), while the standard decomposition $(5)_2$ of the velocity gradient L_i assumes here the following form: $L_i = D_i - \varepsilon r_i$.

At the end, we can obtain the requested objective version of the stress power w_i by the use of the balance of moment of momentum (21):

$$\begin{aligned} w_i = & \, T_i \cdot L_i + z_i^+ \cdot r_i + \zeta_i \cdot \omega_i + \mathcal{S}_i \cdot \operatorname{grad} \omega_i = \\ & = \operatorname{sym} \, T_i \cdot D_i - \operatorname{skw} \, T_i \cdot (\varepsilon r_i) + z_i^+ \cdot r_i + \zeta_i \cdot \omega_i + \mathcal{S}_i \cdot \operatorname{grad} \omega_i = \\ & = \operatorname{sym} \, T_i \cdot D_i - \left[\mathcal{A}_i^T \zeta_i + \left(\operatorname{grad} \mathcal{A}_i^T \right) \mathcal{S}_i \right] \cdot r_i + \zeta_i \cdot \omega_i + \mathcal{S}_i \cdot \operatorname{grad} \omega_i = \\ & = \operatorname{sym} \, T_i \cdot D_i + \zeta_i \cdot \breve{\nu}_i + \mathcal{S}_i \cdot \operatorname{grad} \breve{\nu}_i + \left(\mathcal{A}_i^T \mathcal{S}_i \right) \cdot \operatorname{grad} r_i, \end{aligned} \tag{31}$$

where, in the last row, D_i, $\breve{\nu}_i$, $\operatorname{grad} \breve{\nu}_i$ and $\operatorname{grad} r_i$ are all frame indifferent.

These few formulas provide us with a specific mechanical framework upon which a thermodynamic structure can be raised for constrained immiscible mixtures.

Moreover, by introducing the expression $(31)_1$ for the stress power w_i in the balance of energy (22), we obtain

$$\begin{aligned} \epsilon_i^+ - z_i^+ \cdot r_i - m_i^+ \cdot v_i - \phi_i^+ \cdot \omega_i - \alpha_i^+ \left(\epsilon_i - 2^{-1} v_i^2 - \kappa_i \right) = & \\ = \rho_i \epsilon_i' - \rho_i \lambda_i + \operatorname{div} q_i - w_i. & \end{aligned} \tag{32}$$

Before going further, we record here an expression for η_i^+ obtained by introducing the vector of extra entropy flux k_i, which means the difference between entropy flux p_i and heat flux q_i over the peculiar temperature θ_i, and assuming that the entropy supply ϖ_i from the external world is equal to the energy supply λ_i divided by the same temperature (see [68,51]):

$$k_i = p_i - \frac{q_i}{\theta_i}, \quad \varpi_i = \frac{\lambda_i}{\theta_i}, \tag{33}$$

with, in general, $k_i \neq 0$ for bodies with microstructure, as we specified in previous section; thus, it is

$$\eta_i^+ = \alpha_i^+ \eta_i + \rho_i \eta_i' - \rho_i {\theta_i}^{-1} \lambda_i + \operatorname{div} k_i + \operatorname{div} ({\theta_i}^{-1} q_i). \tag{34}$$

After, we can reduce it by elimination of the body heating λ_i through (32) and by use of the concept of Helmholtz's free energy per unit mass

$$\psi_i := \epsilon_i - \theta_i \eta_i, \tag{35}$$

of the chain rule and of the balances of mass (18) in order to recast the entropy equation (23) in the following form:

$$\begin{aligned} \eta_i^+ = & \operatorname{div} k_i + {\theta_i}^{-1} \Big[w_i - \rho_i \left(\psi_i' + \eta_i \theta_i' \right) - {\theta_i}^{-1} q_i \cdot \operatorname{grad} \theta_i + \\ & + \epsilon_i^+ - z_i^+ \cdot r_i - m_i^+ \cdot v_i - \phi_i^+ \cdot \omega_i - \alpha_i^+ \left(\psi_i - 2^{-1} v_i^2 - \kappa_i \right) \Big] . \end{aligned} \tag{36}$$

5 Balance laws for the mixture

In our mixture, we assume that physical transfers, and eventual chemical reactions, are exchanges rather than true processes of creation or destruction, thus we allow mass, linear momentum, rotational momentum, and energy of any constituents to change form, but do not allow the total mixture to produce these quantities, *i.e.*, they are conserved for the whole mixture and, from balances (15), give rise to the following four axioms of balance for mixtures (that satisfy the third metaphysical principle of Truesdell: the whole mixture behaves as a single body):

$$\sum \alpha_i^+ = 0, \quad \sum m_i^+ = 0, \quad \sum z_i^+ = 0, \quad \sum \epsilon_i^+ = 0. \tag{37}$$

In addition, following Lecture 5 of [86], we do not restrict η_i^+ except for the requirement that the total growth of entropy for the mixture remain non-negative, *i.e.*, our axiom of dissipation is

$$\sum \eta_i^+ \geq 0 \tag{38}$$

(see, also, [91]). We can give a more expressive form to the entropy inequality (38) by using the general equation of transport (5.16) of [86], which relates the time derivative $\dot{\eta}$ of $\eta = \sum \xi_i \eta_i$ following the mean motion of the mixture to the time derivative η_i' of η_i, following the motion of the body $\mathcal{B}_i$, that is:

$$\dot{\eta} = \sum \xi_i \eta_i' + \frac{1}{\rho} \left[\sum \alpha_i^+ \eta_i - \operatorname{div} \left(\sum \rho_i \eta_i u_i \right) \right]. \tag{39}$$

Therefore, the summation of (23) over all constituents i gives the so-called second law of thermodynamics for the mixture in the form

$$\mathcal{D} = \rho \dot{\eta} - \rho \mu + \operatorname{div} p \geq 0, \tag{40}$$

where $\mathcal{D} = \sum \eta_i^+$ is the dissipation, while

$$\mu := \sum \xi_i \varpi_i \quad \text{and} \quad p := \sum (p_i + \rho_i \eta_i u_i); \tag{41}$$

we wish to observe that not only the net entropy flux, but also diffusion may give rise to mixture entropy flux p: in fact we see that, even if p_I $(:= \sum p_i) = 0$, the resultant rate of entropy increase $\operatorname{div} p$ will not generally vanish if diffusion is occurring.

A reduced version of the dissipation inequality for microstructured mixtures (40) can be obtained by using in (38) the alternative expression (36) for η_i^+, where the heat flux λ_i does not occur due to the constitutive expression (33):

$$
\begin{aligned}
-\mathcal{D} = \sum \Big\{ {\theta_i}^{-1} \Big[& \rho_i \left(\psi_i' + \eta_i \theta_i'\right) - w_i + {\theta_i}^{-1} q_i \cdot \operatorname{grad}\theta_i - \epsilon_i^+ + z_i^+ \cdot r_i + \\
& + \phi_i^+ \cdot \omega_i + m_i^+ \cdot v_i + \alpha_i^+ \left(\psi_i - 2^{-1} v_i^2 - \kappa_i\right)\Big] - \operatorname{div} k_i \Big\} \leq 0. \qquad (42)
\end{aligned}
$$

The balance principles (18)-(22) of mass, linear momentum, micro-momentum, rotational momentum and energy make it possible to remove from (42) any one member of each of the five lists $\alpha_i^+, m_i^+, \phi_i^+, z_i^+$, and ϵ_i^+ if we wish to.

A final axiom for the mixtures was added in [74,76] to take into account the rôle of the particular microstructure treated therein, the volume fractions of constituents. But here the growth of micromomentum ϕ_i^+ take values on the cotangent space $T^*_{\nu_i}\mathcal{M}_i$ to $\mathcal{M}_i$ at ν_i that is, obviously, different for each constituent i, therefore, their sum is meaningless in our framework. Instead, in [41], we have proposed more generally that the micromomentum growths must assure the consistency of the axiom of dissipation with the constitutive equations, as we shall discuss in the following (see, also, [39,40,61]), that is,

$$\sum \theta_i^{-1} \left(\alpha_i^+ \kappa_i - \phi_i^+ \cdot \omega_i\right) = 0. \qquad (43)$$

Alternatively, it appears very interesting a recent proposal in [66] that suggests to modify the third metaphysical principle of Truesdell, which states: "The motion of the mixture is governed by the same equations of a single body". Axioms (37) assures the validity of the third principle, when all constituents are described as simple bodies, but they do not suffice for mixtures of complex bodies, when the constituent representations belong to different modeling classes: what kind of single body should be considered? Therefore, the extended proposal [66] is cogent, because it affirms that "the interactions between any pair of constituents appear only at the common level of description while the whole mixture behaves as a body admitting the richest description among those of the constituents".

Thus, at microstructural level, only for kinematic measures ν_i read on 'similar' manifolds $\mathcal{M}_i$, of same dimension m_i, we can assign possible exchanges of micromomentum and connections like formulas (37): this is the case of volume fractions in [76] where the whole mixture behaves as a single "Goodman & Cowin" granular materials (see [51]) for which all micromomentum growths are scalar quantities; moreover, the atypical immiscible mixture in [39], that describes the behaviour of soils and where constituents are a porous solid with ellipsoidal microstructures filled by a compressible fluid, the scalar fluid gain sums to zero only with the trace of the porous solid gain, that is a symmetric tensor with the deviatoric part null.

At the end, in general, for our mixtures of continua with microstructure, we shall have a single body with the richest microstructure necessary to completely describe its macro- and micro-motions, as we can see in [94,41,48] and, also, in the examples below, where the additional axiom of balance for micromomentum growths for mixtures is proposed on a case-by-case basis.

6 Internal constraints in microstructured mixtures

In the previous sections we tacitly supposed that the triple of variables x_i, ν_i, θ_i, for the constituent i, could take arbitrary values for each element of the body $\mathcal{B}_i$; however, there exists a wide class of microstructured mixtures for which, when an element of $\mathcal{B}_i$ has reached a certain state, the complete placement x_i, the microstate ν_i, the temperature θ_i are somehow restricted: as examples, we may think of uniaxial liquid crystals, usually modeled as perfect incompressible fluids with a unit vectorial microstructure, or of Cosserat's continua, where their tensorial microstructure is constrained to be a proper orthogonal tensor, or, finally, of a macroscopically rigid conductor where the introduction of a differentiable internal constraint on the microstructure ν_i, only depending by temperature variations, implies finite-speed heat conduction [65]. Therefore, if the choice is limited, we will say that the body is subject to an internal constraint, and so is the mixture itself. Accordingly we need to define here the class of immiscible microstructured mixtures with internal constraints and adopt an effective extended principle of thermodynamic determinism in order to analyze the consequences of their presence and to give a full thermodynamical description for a broad family of such peculiar immiscible mixtures.

The thermo-mechanical theory of internal constraints in media with microstructure, such as those we conceive of here, is a not trivial case of the abstract thermodynamical theory of constrained materials developed in [53], as generalized in [18]. In particular, to our knowledge, this is the first time that the constraints in a microstructured mixture theory are treated with the formalism of [18], in fact, usually, the theory of Lagrange multipliers is used. Hence, the body $\mathcal{B}_i$ is said to be internally constrained if the allowed velocity, microspeed and temperature rate distributions v_i, ω_i and θ_i', respectively, are such that not all values of the objective factors D_i in the space of symmetric tensors, $\breve{\nu}_i$ in the tangent space $T_{\nu_i}\mathcal{M}_i$, $\operatorname{grad}\breve{\nu}_i$ in the space of linear operators from the vectorial space of translations into $T_{\nu_i}\mathcal{M}_i$, $\operatorname{grad} r_i$, $\operatorname{grad}\theta_i$ and $\operatorname{grad}\theta_i'$ in the space of linear operators are accessible.

The *Extended principle of thermodynamic determinism* for mixtures of materials with microstructure subject to contraints asserts that each quantity, which in absence of the constraint is ruled by a constitutive prescription, as T_i, ζ_i, $\mathcal{S}_i, \ldots$, is now the direct sum of two components, one active and the other reactive:

$$T_i = T_i^a + T_i^r,\ \zeta_i = \zeta_i^a + \zeta_i^r,\ \mathcal{S}_i = \mathcal{S}_i^a + \mathcal{S}_i^r,\ \ldots \tag{44}$$

where only active components T_i^a, ζ_i^a, $\mathcal{S}_i^a, \ldots$ have to be specified, through suitable constitutive relations, by the independent thermo-kinetic variables. As for the reactive terms T_i^r, ζ_i^r, $\mathcal{S}_i^r, \ldots$, in general they remain undetermined unless some information on the physical mechanism which causes the constraint is given (see, also, §27 of [11] or, for the purely mechanical case, §IV.7 of [87] and §9 of [19]). Following [52], we do not include reaction terms ψ_i^r for the free energies in (44): in fact, if we enclose such quantities, the conclusion would be that they are constant in every process.

However, as we anticipated before, we shall consider the class of mixtures with perfect constraints, *i.e.*, internally frictionless, for which, in this thermomechanical context, reactive parts do not produce entropy (see, also, [71]): that is the contribution of reactions to the left hand part of (42) is identically zero for every process allowed by the constraints:

$$\begin{aligned}\sum \Big\{ \theta_i^{-1} \Big[\rho_i\, \eta_i^r\, \theta_i' - w_i^r + \theta_i^{-1} q_i^r \cdot \operatorname{grad} \theta_i - \epsilon_i^{+r} + z_i^{+r} \cdot r_i + \\ + \phi_i^{+r} \cdot \omega_i + m_i^{r+} \cdot v_i + \alpha_i^{+r} \left(\psi_i - 2^{-1} v_i^2 - \kappa_i \right) \Big] - \operatorname{div} k_i^r \Big\} = 0, \end{aligned} \tag{45}$$

where, in agreement with relation $(31)_4$,

$$w_i^r := \operatorname{sym} T_i^r \cdot D_i + \zeta_i^r \cdot \breve{\nu}_i + \mathcal{S}_i^r \cdot \operatorname{grad} \breve{\nu}_i + \left(\mathcal{A}_i^T \mathcal{S}_i^r \right) \cdot \operatorname{grad} r_i. \tag{46}$$

7 Constitutive choices in the thermomechanics of soils

For each body with microstructure $\mathcal{B}_i$, a thermokinetic process is a triple of fields on $\mathcal{B}_i \times \Re$ with values on $\mathcal{E} \times \mathcal{M}_i \times \Re^+$, namely

$$x_i(X_i, \tau), \quad \nu_i(x, \tau), \quad \theta_i(x, \tau); \tag{47}$$

an associated caloro-dynamic process involves not only the classical fields $T_i, \epsilon_i, q_i, \eta_i, p_i, \psi_i$, but also, besides, microstructural ones $\mathcal{S}_i, \zeta_i$, in addition to the growths

$$\alpha_i^+(x, \tau), \quad m_i^+(x, \tau), \quad \phi_i^+(x, \tau), \quad z_i^+(x, \tau), \quad \epsilon_i^+(x, \tau), \tag{48}$$

provided these be subject to the general balance axioms of the thermodynamics of microstructured mixtures, namely, (18)-(22), (36), (37) and (42), plus the (eventual) general axiom regarding micromomentum growths, as well as constraint conditions.

Therefore, to express the constitutive axioms, the prior fields have to satisfy the general principles governing constitutive equations, which are: 1. *Determinism*, 2. *Equipresence*, 3. *Local action*, 4. *Material frame-indifference*, 5.

Dissipation (see note 12, Lecture 5 of [86]). These principles are well known as those governing the constitutive relations for mixtures of simple bodies, and must be appropriately interpreted in our context: Determinism is specified in the previous section; Equipresence, "which forbids the theorists from choosing independent variables by caprice" (see Lecture 1 of [86]), imposes that the variables and analytic restrictions at the outset be the same for all constitutive functionals; Local action asserts that the response of components at any points X_i depends only on the thermokinetic process in the immediate neighbour of X_i; Material frame-indifference affirms that the behavior of constituents is independent of the observer; Dissipation requires that the response of constituents satisfy the reduced dissipation inequality (42) for every thermodynamics process.

We observe here that, to simplify the calculations and inferences from the entropy principle, in many mixture theories the Equipresence is sometimes replaced by the so-called principle of Phase Separation, which reduces the equipresence only to the growth functionals (48) (see, *e.g.*, [1,76,92]), but there are also valid plausibility arguments to reject that choice, as we do in our context of soil thermodynamics (see [91] for comments on this question).

The last feature of the constitutive relations concerns the material symmetries of each constituent, associated with its physical structure, therefore such symmetries must be appropriately reflected in the form of the constitutive equations of the whole mixture.

In the following sections we wish to present applications for immiscible mixtures of interest for the diffusion of pollutants, soil micromechanics, granular flows or poroelasticity, for which the individual constituents remain physically separate and thus the volume fractions influence the constitutive responses, in accordance with [10].

Therefore, it is necessary to distinguish between the bulk mass densities ρ_i and the true mass densities γ_i of the i^{th} constituent: the first one represents the mass of the i^{th} constituent per unit of mixture volume; the second one is its mass per unit of true volume, which is given as a fraction of the whole mixture by the introduction of the volume fraction β_i, a smooth scalar field representing the proportion of space occupied by the i^{th} component. For the hypotheses made, the bulk densities ρ_i are tied to the true densities γ_i by the following relation

$$\rho_i = \gamma_i \beta_i, \quad \text{with} \quad \beta_i \in]0,1[, \tag{49}$$

and the total volume fraction is

$$\beta_{tot} = \sum \beta_i; \tag{50}$$

the mixture is said saturated if $\beta_{tot} = 1$, unsaturated if $\beta_{tot} < 1$, *i.e.*, when there are void spaces in the mixture. Finally, we write again the mass balance

(18) inserting the decomposition (49)

$$\alpha_i^+ = \gamma_i'\beta_i + \gamma_i\beta_i' + \gamma_i\beta_i \operatorname{div} v_i. \tag{51}$$

Furthermore, we would like to conclude this section by introducing an additional important special case concerning temperatures: in particular, if the energy exchange between the constituents is so efficient that the mixture can be characterized by a single temperature

$$\theta_i = \theta, \quad \text{for } i = 1, \ldots, n, \tag{52}$$

then such an assumption is tantamount to restricting considerations to the mixture energy equation only, rather than to each energy equation separately [69]: in fact the terms in ϵ_i^+ drop out of (42) because of $(37)_4$, so the dissipation principle does not restrict the growths of energy and we need then work only with the mixture forms for reduced energy and entropy balances (see, also, Lecture 5 of [86] and [84,92]). Moreover, in this 'single temperature' model, constituent interactions are limited to those associated with mass, momentum, micromomentum and moment of momentum.

About the reduced dissipation inequality (42), inserting condition (52) we have that

$$\begin{aligned} -\theta\,\mathcal{D} = \sum \Big\{ \Big[\rho_i \left(\psi_i' + \eta_i \theta'^i \right) - w_i + \theta^{-1} q_i \cdot \operatorname{grad}\theta + z_i^+ \cdot r_i + \\ + \phi_i^+ \cdot \omega_i + m_i^+ \cdot v_i + \alpha_i^+ \left(\psi_i - 2^{-1} v_i^2 - \kappa_i \right) \Big] - \theta \operatorname{div} k_i \Big\} \leq 0, \end{aligned} \tag{53}$$

where $\theta'^i := \frac{\partial\theta}{\partial\tau} + (\operatorname{grad}\theta)\, v_i = \dot\theta + (\operatorname{grad}\theta)\, u_i$. Moreover, using the resultant free energy ψ defined as the concentration-weighted sum of the peculiar Helmholtz's free energies per unit mass (35)

$$\psi = \sum \xi_i \psi_i, \tag{54}$$

we obtain

$$\begin{aligned} -\theta\,\mathcal{D} = \rho \left(\dot\psi + \eta\, \dot\theta \right) - w + \theta^{-1} q \cdot \operatorname{grad}\theta - \theta \operatorname{div} k + \\ + \sum \left[z_i^+ \cdot r_i + \phi_i^+ \cdot \omega_i + m_i^+ \cdot v_i - \alpha_i^+ \left(2^{-1} v_i^2 + \kappa_i \right) \right] \leq 0, \end{aligned} \tag{55}$$

where

$$w := \sum w_i, \;\; q := q_I + \sum \rho_i \left(\psi_i + \theta\, \eta_i \right) u_i \text{ and } k := k_I - \theta^{-1} \sum \rho_i \psi_i u_i \tag{56}$$

with $q_I := \sum q_i$ and $k_I := \sum k_i$ the net heating and extra-entropy flux, respectively: as for $(41)_2$, also here the presence of diffusion can increase the additional heating and extra-entropy fluxes, even if q_I and k_I are null. From $(41)_2$ and $(56)_{2,3}$, we have also that

$$k = p - \theta^{-1} q. \tag{57}$$

In addition, we write the condition (45) of perfect constraints for reactions, when (52) holds,

$$\begin{aligned} &\rho\, \eta^r\, \dot{\theta} - w^r + \theta^{-1} q^r \cdot \operatorname{grad}\theta - \theta \operatorname{div} k^r + \\ &\qquad + \sum \left[z_i^{+r} \cdot r_i + \phi_i^{+r} \cdot \omega_i + m_i^{+r} \cdot v_i - \alpha_i^{+r} \left(2^{-1} v_i^2 + \kappa_i \right) \right] = 0. \end{aligned} \tag{58}$$

8 Applications

8.1 Two-phase suspensions

In this first application we specialize our theory to a saturated multiphase mixture of bodies with scalar microstructure, where all the order parameters ν_i represent the volume fraction β_i of the i^{th} constituent, as it is the case for granular materials of Goodman & Cowin type [51], or for fluid suspensions of Passman, Nunziato & Walsh [76], or for saturated solid-fluid mixtures of Wang & Hutter [91]. All these scalar measures β_i are not affected by rigid rotations, so that the infinitesimal generators $\mathcal{A}_i$ vanish, for each i, and $\breve{\nu}_i = \omega_i$ for relation (30).

We consider now a simple temperature model for a two-phase motion of spherical particles, of subscript $_2$, suspended in a fluid, of subscript $_1$, in circumstances where there are no chemical reactions: $\alpha_i^+ = 0$. The particles may be either solid or fluid and we make no restriction concerning diluteness, while we suppose that the saturation constraint $\beta_{tot} = 1$ applies, so that

$$\beta := \beta_2 = 1 - \beta_1 \quad \text{and} \quad \omega_i = (-1)^i \, \beta'^i, \tag{59}$$

with $\beta'^i := \frac{\partial \beta}{\partial \tau} + (\operatorname{grad}\beta) \cdot v_i$, for $i = 1, 2$. Moreover, the three axioms for the mixture balance $(37)_{2,3,4}$, plus the additional one for micro-momentum growth ϕ^+, are now, for $i = 1, 2$:

$$m^+ := (-1)^i \, m_i^+, \; z^+ := (-1)^i \, z_i^+, \; \epsilon^+ := (-1)^i \, \epsilon_i^+, \; \phi^+ := (-1)^i \, \phi_i^+. \tag{60}$$

The kinetic co-energies χ_i, $i = 1, 2$, are assumed to be quadratic forms in ω_i, as is customary for immiscible fluid, or fluid-like, mixtures (see, *e.g.*, [7,12,46,42]);

then χ_i coincides with the kinetic energy κ_i and it is

$$\kappa_i = \chi_i := 2^{-1}\mu_i(\beta_i)\,\omega_i^2 = 2^{-1}\mu_i(\beta_i)(\beta'^i)^2. \tag{61}$$

The kinetic energies express the inertia due to the local microvariations of the volume of inclusions, as well as that related to the admissible expansional motion of spherical boundaries (see, also, [34,43], other than references cited above, for explicit evaluations of non-negative coefficient $\mu_i(\beta_i)$).

With these hypotheses, the balance equations (51), (19), (20), (21), (32) reduce to the following ones for $i = 1, 2$:

$$\begin{aligned}
&\rho_i' + \rho_i \operatorname{div} v_i = 0, && (62)\\
&\rho_i v_i' - \operatorname{div} T_i - \rho_i b_i = (-1)^i\, m^+, && (63)\\
&(-1)^i \rho_i \left(\mu_i \beta''^i + \frac{1}{2}\frac{d\mu_i}{d\beta}(\beta'^i)^2\right) - \operatorname{div} \mathcal{S}_i + \zeta_i - \rho_i \delta_i = (-1)^i\, \phi^+, && (64)\\
&\varepsilon\, T_i = (-1)^i\, z^+, && (65)\\
&\rho\, \dot{\epsilon} - w + \operatorname{div} q - \rho\, \lambda = \\
&\qquad = z^+ \cdot (r_1 - r_2) - \phi^+(\beta'^1 + \beta'^2) + m^+ \cdot (u_1 - u_2). && (66)
\end{aligned}$$

with u_i defined in (7) and w and q given by $(56)_{1,2}$, respectively, while $\epsilon := \sum \xi_i \epsilon_i$ and $\lambda := \sum \xi_i \lambda_i$ are the concentration-weighted sums of the peculiar inner energies and heating supplies per unit mass, respectively; we have

$$\epsilon = \psi + \theta\eta. \tag{67}$$

The mixture energy balance (66) is obtained by summing peculiar energy equations (32) on $i = 1, 2$ and an expression, more pertinent to the third metaphysical principle of Truesdell, could be easily obtained by substituting values of m^+, z^+, ϕ^+ given by (63), (64) and (65) (see equation (2.18) of [86]).

We emphasize that $\phi^+, \delta_i, \operatorname{div} \mathcal{S}_i$ and ζ_i in equations (64) are all scalar fields and that the microstress vector $\mathcal{S}_i$ is normally related to the boundary microtractions, even if, in some cases, it could express weakly non-local internal effects; δ_i is interpreted as an externally controlled fluid pressure; ζ_i includes interactive forces between the gross and fine structures. Further, stress tensors T_i are not 'a priori' symmetric because, for (65) and the properties of the Ricci's alternating tensor ε, we have that

$$\text{skw } T_i = (-1)^i \, \frac{1}{2} \, \varepsilon\, z^+, \tag{68}$$

while only their sum is, inasmuch

$$\text{skw } T := \text{skw } T_1 + \text{skw } T_2 = 0. \tag{69}$$

We remark here that in the models based on 'void theories', as, *e.g.*, [75,76,92], unexpectedly only the first microinertia term are present in their balances of 'equilibrated' forces (*i.e.*, our micromomentum balances (64)): in fact those equations remain ambiguous because the complete Lagrangian derivative of the kinetic co-energy does not appear on the left hand side of the equations themselves.

For the presence of the perfect saturation constraint $(59)_1$, we impose the extended principle of thermodynamic determinism (44) and the condition (58) with (46), together with the identity $\beta'^{i} = \dot{\beta} + u_i \cdot \operatorname{grad}\beta$ deriving from (7), in order to obtain the following equation

$$\begin{aligned}&\rho\,\eta^r\,\dot{\theta} + \theta^{-1} q^r \cdot \operatorname{grad}\theta - \theta \operatorname{div} k^r - \operatorname{sym} T_1^r \cdot D_1 - \operatorname{sym} T_2^r \cdot D_2 + \\ &+(\zeta_1^r - \zeta_2^r + 2\phi^{+r})\dot{\beta} + \mathcal{S}_1^r \cdot \operatorname{grad}\beta'^1 - \mathcal{S}_2^r \cdot \operatorname{grad}\beta'^2 + z^{+r} \cdot (r_2 - r_1) + \\ &-[m^{+r} - (\zeta_1^r + \phi^{+r})\operatorname{grad}\beta] \cdot u_1 + [m^{+r} - (\zeta_2^r - \phi^{+r})\operatorname{grad}\beta] \cdot u_2 = 0. \quad (70)\end{aligned}$$

If we observe that the constraint $(59)_1$ leaves locally the choice of θ, $\dot{\theta}$, $\operatorname{grad}\theta$, D_1, D_2, $\dot{\beta}$, $\operatorname{grad}\beta'^1$, $\operatorname{grad}\beta'^2$, $(r_2 - r_1)$, u_1 and u_2 totally free, we deduce from (70) that the reactions are characterized by the following requirements:

$$\eta^r = 0,\ q^r = 0, \operatorname{div} k^r = 0,\ \operatorname{sym} T_i^r = 0,\ \mathcal{S}_i^r = 0,\ z^{+r} = 0, \quad (71)$$

$$\zeta_1^r + \phi^{+r} = \zeta_2^r - \phi^{+r} \quad \text{and} \quad m^{+r} = (\zeta_2^r - \phi^{+r}) \operatorname{grad}\beta. \quad (72)$$

Now we are able to obtain a set of *pure* balance equations which rules the thermo-mechanical evolution of our model of a saturated two-phase suspension; in fact, by splitting the stress tensors T_i into its symmetric and skew parts and by using the conditions $(71)_4$ together with the balances of moment of momentum (68) into the Cauchy balances (63), the following reaction-free expressions for it follows:

$$\rho_i(v_i' - b_i) - \operatorname{div}\left(\operatorname{sym} T_i^a + (-1)^i \frac{1}{2}\,\varepsilon\, z^{+a}\right) = (-1)^i (m^{+a} + \pi^{in} \operatorname{grad}\beta), \quad (73)$$

where

$$\pi^{in} := \zeta_2^r - \phi^{+r} \quad (74)$$

and the expressions (72) for $(\zeta_1^r + \phi^{+r})$ and m^{+r} were used; moreover, from the balance (64) with $i = 2$, it follows:

$$\pi^{in} = \rho_2 \left[\delta_2 - \mu_2 \beta''^2 - \frac{1}{2}\frac{d\mu_2}{d\beta}\left(\beta'^2\right)^2\right] + \operatorname{div}\mathcal{S}_2^a - \zeta_2^a + \phi^{+a} : \quad (75)$$

here, we physically interpret the coupled reactions π^{in} as the interface pressure between constituents, *i.e.*, the pressure that acts at the interface between the phases necessary to maintain the contact in order to satisfy the saturation constraint (see Appendix of [6]).

Moreover, if we sum (64), for $i = 1, 2$, use $(31)_4$ for w and (67) for ϵ_I, and substitute reactions (71) in (66), we obtain

$$\begin{aligned}
&\rho_1 \left[\mu_1 \beta''^1 + \frac{1}{2}\frac{d\mu_1}{d\beta}\left(\beta'^1\right)^2\right] + \rho_2 \left[\mu_2 \beta''^2 + \frac{1}{2}\frac{d\mu_2}{d\beta}\left(\beta'^2\right)^2\right] = \operatorname{div}\left(\mathcal{S}^a_2 - S^a_1\right) + \\
&\qquad + (\zeta^a_1 + \phi^{+a}) - (\zeta^a_2 - \phi^{+a}) + \rho_2\delta_2 - \rho_1\delta_1 \quad \text{and} \qquad (76) \\
&\rho\, \overline{(\psi^a + \theta\eta^a)}^{\boldsymbol{\cdot}} + \operatorname{div} q^a - \rho\,\lambda = \sum \left[\text{sym } T^a_i \cdot D_i + (-1)^i\, \mathcal{S}^a_i \cdot \operatorname{grad} \beta'^i\right] - \\
&-(\zeta^a_1 + \phi^{+a})\beta'^1 + (\zeta^a_2 - \phi^{+a})\beta'^2 + m^{+a} \cdot (u_1 - u_2) + z^{+a} \cdot (r_1 - r_2). \qquad (77)
\end{aligned}$$

We would like to observe that also the active components of the internal microactions ζ^a_i and the growth rates of the micromoment ϕ^{+a}, besides the reactive ones, are coupled in all the pure balances and so their constitutive laws will always be linked.

In conclusion, only the active parts of all fields, which will be the object of a constitutive prescription, appear in the Cauchy equations (73) (with π^{in} given by (75)), in the equation for micromomentum (76) and in the equation of evolution for the temperature of the whole mixture (77): these are the pure equations which rule the thermo-mechanical evolution of the body.

Once a motion is ensued from them, the corresponding reaction π^{in} to the constraint is obtained by the equation (75) (other than by (71) and (72)) within the intrinsic indeterminacy generated from the equation itself for ϕ^{+r} and ζ^r_2 (see, also, §205 and 227 of [89], or Remark 1, §3 of [15]).

About constitutive equations for immiscible mixtures, we assume that the overall response of the simple temperature model for a two-phase suspension depends only on the set $\mathcal{T}$ of the following thermokinetic variables

$$\mathcal{T} = \{\gamma_1, \gamma_2, \beta, \operatorname{grad}\gamma_1, \operatorname{grad}\gamma_2, \operatorname{grad}\beta, \theta, \operatorname{grad}\theta\}; \qquad (78)$$

by imposing the principle of equipresence, we postulate that the dependent constitutive quantities $\psi^a, \eta^a, q^a, k^a, T^a_i, \zeta^a_i, \mathcal{S}^a_i, m^{+a}, \phi^{+a}$ and z^{+a} are all twice continuously differentiable functions with respect to all constitutive fields and require the consistency with the reduced dissipation inequality (55), when the perfect constraint condition (70) applies, *i.e.*:

$$\begin{aligned}
&\rho\left(\dot{\psi}^a + \eta^a\,\dot{\theta}\right) + \theta^{-1} q^a \cdot \operatorname{grad}\theta - \theta \operatorname{div} k^a - \text{sym } T^a_1 \cdot D_1 - \\
&\quad -\text{sym } T^a_2 \cdot D_2 + (\zeta^a_1 + \phi^{+a})\,\beta'^1 + (\phi^{+a} - \zeta^a_2)\,\beta'^2 + \mathcal{S}^a_1 \cdot \operatorname{grad}\beta'^1 - \\
&\qquad -\mathcal{S}^a_2 \cdot \operatorname{grad}\beta'^2 + m^{+a} \cdot (u_2 - u_1) + z^{+a} \cdot (r_2 - r_1) \leq 0. \qquad (79)
\end{aligned}$$

Since the constitutive relations allow dependency on the true densities γ_i, the constituents have a fluid-like behavior and the frame-indifference implies that all response functions are isotropic functions of their variables [71]. Introducing the dependency of the active components on the set (78), using the mass conservations (51), with $\alpha_i^+ = 0$, the property (8) and the identities

$$
\begin{aligned}
&\dot{\gamma}_i = \gamma_i' - u_i \cdot \operatorname{grad}\gamma_i = -\gamma_i \beta_i^{-1}\beta_i' - \gamma_i \operatorname{div} u_i - u_i \cdot \operatorname{grad}\gamma_i, \\
&\dot{\beta} = \beta'^i - u_i \cdot \operatorname{grad}\beta,\ \dot{\overline{\operatorname{grad}\beta}} = \operatorname{grad}\beta'^i - L_i^T \operatorname{grad}\beta - (\operatorname{grad}^2\beta)u_i,
\end{aligned}
\tag{80}
$$

when the terms are appropriately ordered, produce the following new inequality:

$$
\begin{aligned}
&\rho\left(\eta^a + \frac{\partial\psi^a}{\partial\theta}\right)\dot{\theta} + \left(\frac{q^a}{\theta} - \theta\frac{\partial k^a}{\partial\theta}\right) \cdot \operatorname{grad}\theta + \rho\frac{\partial\psi^a}{\partial \operatorname{grad}\theta} \cdot \dot{\overline{\operatorname{grad}\theta}} - \\
&+ \sum\left\{\rho\frac{\partial\psi^a}{\partial\operatorname{grad}\gamma_i} \cdot \dot{\overline{\operatorname{grad}\gamma_i}} - \theta\left(\frac{\partial k^a}{\partial\gamma_i} \cdot \operatorname{grad}\gamma_i + \frac{\partial k^a}{\partial\operatorname{grad}\gamma_i} \cdot \operatorname{grad}^2\gamma_i\right) - \right. \\
&- \left.\left[\operatorname{sym}\, T_i^a + \rho\gamma_i\frac{\partial\psi^a}{\partial\gamma_i}I + \rho_i \operatorname{sym}\left(\operatorname{grad}\beta \otimes \frac{\partial\psi^a}{\partial\operatorname{grad}\beta}\right)\right] \cdot D_i\right\} - \\
&- \left(m^{+a} + \rho\frac{\partial\psi^a}{\partial\gamma_1}\operatorname{grad}\gamma_1 + \rho_1\frac{\partial\psi^a}{\partial\beta}\operatorname{grad}\beta\right) \cdot u_1 - \theta\frac{\partial k^a}{\partial\beta} \cdot \operatorname{grad}\beta + \\
&+ \left(m^{+a} - \rho\frac{\partial\psi^a}{\partial\gamma_2}\operatorname{grad}\gamma_2 - \rho_2\frac{\partial\psi^a}{\partial\beta}\operatorname{grad}\beta\right) \cdot u_2 - \theta\frac{\partial k^a}{\partial\operatorname{grad}\beta} \cdot \operatorname{grad}^2\beta + \\
&+ \left[z^{+a} + \rho\,\varepsilon\left(\operatorname{grad}\beta \otimes \frac{\partial\psi^a}{\partial\operatorname{grad}\beta}\right)\right] \cdot (r_2 - r_1) + \\
&+ \left(\mathcal{S}_1^a + \rho_1\frac{\partial\psi^a}{\partial\operatorname{grad}\beta}\right) \cdot \operatorname{grad}\beta'^1 - \left(\mathcal{S}_2^a - \rho_2\frac{\partial\psi^a}{\partial\operatorname{grad}\beta}\right) \cdot \operatorname{grad}\beta'^2 + \\
&+\beta'^1\left(\zeta_1^a + \phi^{+a} + \rho_1\frac{\partial\psi^a}{\partial\beta} + \frac{\rho\,\gamma_1}{1-\beta}\,\frac{\partial\psi^a}{\partial\gamma_1}\right) - \theta\frac{\partial k^a}{\partial\operatorname{grad}\theta} \cdot \operatorname{grad}^2\theta + \\
&-\beta'^2\left(\zeta_2^a - \phi^{+a} - \rho_2\frac{\partial\psi^a}{\partial\beta} + \frac{\rho\,\gamma_2}{\beta}\,\frac{\partial\psi^a}{\partial\gamma_2}\right) \leq 0.
\end{aligned}
\tag{81}
$$

Because the left-hand member is linear in the scalar fields $\dot{\theta}$ and β'^i, the vector fields $\operatorname{grad}\theta$, $\dot{\overline{\operatorname{grad}\theta}}$, $\dot{\overline{\operatorname{grad}\gamma_i}}$, u_i, $(r_2 - r_1)$ and $\operatorname{grad}\beta'^i$, and the symmetric tensor fields $\operatorname{grad}^2\theta$, $\operatorname{grad}^2\beta$ and D_i, the classical arguments of Coleman and Noll [20] assure us that the coefficients in the linear expressions must all vanish, and hence:

•) the active part of the Helmholtz free energy ψ^a is a function only of the array $\hat{\mathcal{T}} = \{\gamma_1, \gamma_2, \beta, \operatorname{grad}\beta, \theta\}$ and is a potential function for active parts of the entropy η^a, the symmetric part of stress tensors $\operatorname{sym}\, T_i^a$, the microstresses $\mathcal{S}_i^a$, the growths of linear and rotational momentum m^{+a} and z^{+a}, respectively, and the difference between the internal microactions ζ_i^a and the micromomentum ϕ^{+a}, in the sense that:

$$\eta^a = -\frac{\partial\psi^a}{\partial\theta}, \quad \mathcal{S}_1^a = -\gamma_1(1-\beta)\frac{\partial\psi^a}{\partial \mathrm{grad}\,\beta}, \quad \mathcal{S}_2^a = \gamma_2\beta\,\frac{\partial\psi^a}{\partial \mathrm{grad}\,\beta}, \tag{82}$$

$$\mathrm{sym}\; T_1^a = \frac{1-\beta}{\xi_1}\pi_1^t I - \mathrm{sym}\left[\gamma_1(1-\beta)\,\mathrm{grad}\,\beta\otimes\frac{\partial\psi^a}{\partial \mathrm{grad}\,\beta}\right],$$

$$\mathrm{sym}\; T_2^a = -\frac{\beta}{\xi_2}\pi_2^t I - \mathrm{sym}\left[\gamma_2\beta\,\mathrm{grad}\,\beta\otimes\frac{\partial\psi^a}{\partial \mathrm{grad}\,\beta}\right], \tag{83}$$

$$m^{+a} = \pi_2^c\mathrm{grad}\,\beta + \frac{\beta\pi_2^t}{\xi_2\gamma_2}\mathrm{grad}\,\gamma_2, \; z^{+a} = -\rho\,\varepsilon\left(\mathrm{grad}\,\beta\otimes\frac{\partial\psi^a}{\partial \mathrm{grad}\,\beta}\right), \tag{84}$$

$$\zeta_1^a + \phi^{+a} = -\pi_1^c - \xi_1^{-1}\pi_1^t, \quad \zeta_2^a - \phi^{+a} = \pi_2^c - \xi_2^{-1}\pi_2^t, \tag{85}$$

together with the compatibility condition on m^{+a}

$$(\pi_1^c + \pi_2^c)\,\mathrm{grad}\,\beta + \frac{(1-\beta)\pi_1^t}{\xi_1\gamma_1}\,\mathrm{grad}\,\gamma_1 + \frac{\beta\pi_2^t}{\xi_2\gamma_2}\,\mathrm{grad}\,\gamma_2 = 0, \tag{86}$$

where

$$\pi_i^c = \rho_i\frac{\partial\psi^a}{\partial\beta} \quad \text{and} \quad \pi_i^t = \gamma_i^2\frac{\partial\psi^a}{\partial\gamma_i}, \quad \text{for } 1 = 1, 2, \tag{87}$$

are the configuration and the thermodynamic pressures, respectively, while the last term of stress tensors in (83) are of Ericksen's type ($\mathrm{grad}\,\beta\otimes\frac{\partial\psi^a}{\partial \mathrm{grad}\,\beta}$) and justify the ability of granular suspension to sustain shear in equilibrium; in constitutive relations (82)-(86) we used the identities $(6)_{1,3}$, (7) and (49), for $i = 1, 2$.

•) Moreover, the active extra-entropy flux k^a must be such that

$$\frac{\partial k^a}{\partial \mathrm{grad}\,\gamma_i}\cdot\mathrm{grad}^2\gamma_i = 0, \frac{\partial k^a}{\partial \mathrm{grad}\,\theta}\cdot\mathrm{grad}^2\theta = 0, \frac{\partial k^a}{\partial \mathrm{grad}\,\beta}\cdot\mathrm{grad}^2\beta = 0, \tag{88}$$

which means that $\frac{\partial k^a}{\partial \mathrm{grad}\,\gamma_i}, \frac{\partial k^a}{\partial \mathrm{grad}\,\beta}$ and $\frac{\partial k^a}{\partial \mathrm{grad}\,\theta}$ are all skew-symmetric and, therefore, k^a must be collinear to $\mathrm{grad}\,\gamma_i$, $\mathrm{grad}\,\beta$ and $\mathrm{grad}\,\theta$ with the corresponding tensorial material coefficients being skew-symmetric; on the other hand, the isotropy of k^a requires that any such material tensors must be symmetric (see analogous computations for the constitutive part j_c of the mixture flux density j in [91]). To satisfy both conditions these tensors must vanish, making k^a independent of $\mathrm{grad}\,\gamma_i, \mathrm{grad}\,\beta$ and $\mathrm{grad}\,\theta$, and yielding its reduced form

$$k^a = \hat{k}^a(\gamma_1, \gamma_2, \beta, \theta), \tag{89}$$

but there is no isotropic vectorial function of only scalars, and thus the reduced form (89) necessarily implies $k^a = 0$ for thermo-elastic two-phase suspensions.

Accordingly, the reduced dissipation inequality reduces to the following one:

$$\theta^{-1} q^a \cdot \operatorname{grad} \theta \leq 0, \tag{90}$$

which apparently expresses the classical Fourier inequality for a single body, except that it is valid only for the active part of the heat flux q^a which, in addition, depends on the whole set $\mathcal{T}$.

•) Finally, as ψ^a is an isotropic function, its dependence on $\hat{\mathcal{T}}$ implies that

$$\psi^a(\hat{\mathcal{T}}) = \hat{\psi}^a(\gamma_1, \gamma_2, \beta, \operatorname{grad} \beta \cdot \operatorname{grad} \beta, \theta). \tag{91}$$

Substituting (91) into equations $(82)_{2,3}$ assert that the microstresses $\mathcal{S}_i$ have the following representation

$$\mathcal{S}_1^a = -\gamma_1 (1 - \beta)\, \varphi \operatorname{grad} \beta, \quad \mathcal{S}_2^a = \gamma_2 \beta\, \varphi \operatorname{grad} \beta, \tag{92}$$

where the so-called modulus of dilatancy φ is given by the following expression:

$$\varphi = \hat{\varphi}(\gamma_1, \gamma_2, \beta, \operatorname{grad} \beta \cdot \operatorname{grad} \beta, \theta) = 2 \frac{\partial \hat{\psi}^a}{\partial (\operatorname{grad} \beta \cdot \operatorname{grad} \beta)} \tag{93}$$

and thus, for the two-phase suspensions of the type considered here, the Cauchy stress tensor T_i is symmetric and the moment of momentum growth z^{+a} vanishes identically:

$$\begin{aligned} T_1^a &= (1 - \beta) \left(\xi_1^{-1} \pi_1^t\, I - \gamma_1\, \varphi \operatorname{grad} \beta \otimes \operatorname{grad} \beta \right), \\ T_2^a &= -\beta \left(\xi_2^{-1} \pi_2^t\, I + \gamma_2\, \varphi \operatorname{grad} \beta \otimes \operatorname{grad} \beta \right), \quad z^{+a} = 0. \end{aligned} \tag{94}$$

Now we are able to write the set of pure balance equations, which rules the thermo-elastic evolution of a two-phase suspension, inserting constitutive laws $(84)_1$, (85), (92), (93), (94), into balance equations (51) (with $\alpha^{+a} = 0$) and (73), for $i = 1, 2$, (76) and (77) to obtain:

$$\gamma_1'(1 - \beta) = \gamma_1 \left[\beta'^1 - (1 - \beta) \operatorname{div} v_1 \right], \ \ \gamma_2' \beta = -\gamma_2 \left(\beta'^2 + \beta \operatorname{div} v_2 \right), \tag{95}$$

$$\begin{aligned} \gamma_1 (1 - \beta) v_1' = \operatorname{div} &\left[(1 - \beta) \left(\xi_1^{-1} \pi_1^t\, I - \gamma_1\, \varphi \operatorname{grad} \beta \otimes \operatorname{grad} \beta \right) \right] + \\ &+ \gamma_1 (1 - \beta) b_1 - \pi_2^c \operatorname{grad} \beta - \frac{\beta \pi_2^t}{\xi_2 \gamma_2} \operatorname{grad} \gamma_2 - \pi^{in} \operatorname{grad} \beta, \end{aligned} \tag{96}$$

$$\begin{aligned} \gamma_2 \beta v_2' = -\operatorname{div} &\left[\beta \left(\xi_2^{-1} \pi_2^t\, I + \gamma_2\, \varphi \operatorname{grad} \beta \otimes \operatorname{grad} \beta \right) \right] + \gamma_2 \beta b_2 + \\ &+ \pi_2^c \operatorname{grad} \beta + \frac{\beta \pi_2^t}{\xi_2 \gamma_2} \operatorname{grad} \gamma_2 + \pi^{in} \operatorname{grad} \beta, \end{aligned} \tag{97}$$

$$\gamma_1(1-\beta)\left[\mu_1\beta''^1+\frac{1}{2}\frac{d\mu_1}{d\beta}\left(\beta'^1\right)^2\right]+\gamma_2\beta\left[\mu_2\beta''^2+\frac{1}{2}\frac{d\mu_2}{d\beta}\left(\beta'^2\right)^2\right]=$$

$$=\operatorname{div}\left(\rho\,\varphi\operatorname{grad}\beta\right)-\left(\pi_1^c+\pi_2^c\right)-\frac{\pi_1^t}{\xi_1}+\frac{\pi_2^t}{\xi_2}+\gamma_2\beta\delta_2-\gamma_1(1-\beta)\delta_1, \quad (98)$$

$$\rho\,\theta\frac{d}{d\tau}\left(\frac{\partial\psi^a}{\partial\theta}\right)=\operatorname{div}q^a-\rho\,\lambda \quad (99)$$

with

$$\pi^{in}=\gamma_2\beta\left[\delta_2-\mu_2\beta''^2-\frac{1}{2}\frac{d\mu_2}{d\beta}(\beta'^2)^2\right]+\operatorname{div}\left(\gamma_2\beta\,\varphi\operatorname{grad}\beta\right)+\left(\xi_2^{-1}\pi_2^t-\pi_2^c\right)$$

and where the evolution equation (77) for the temperature θ of the mixture is considerably simplified and, formally, is reduced to the classical one (99), the difference remains in the dependence on the set $\hat{\mathcal{T}}$.

Next we quote from [76] the mechanical balance equations (5C.7.5)-(5C.7.7), which should correspond to ours (95)-(98) when we suppose isothermal flows of a two-phase mixture of particles dispersed in an elastic fluid. In particular, identifying the different pressures as follows: $\pi=\pi^{in}$, $\beta_a=\pi_i^c$, $p_a=\pi_i^t$, putting their viscous coefficients μ_a, s_a, λ_a to zero, adding $(5C.7.7)_a$ to $(5C.7.6)_a$ for $a=1,2$, in order to delete the term $(\pi-p_a)$ containing the reaction π and after subtracting $(5C.7.7)_1$ to $(5C.7.6)_2$, the substance of the analysis in [76] would be easily recovered, still getting rid of some kinematic terms, such as $\rho_a k_a'(\varphi_a')^2/2$, in their definition (5C.4.9) of the dissipation function; moreover, the difference remains in the constitutive equations in which we applied the classical Principle of Equipresence of Truesdell, instead of that of the Phase Separation.

8.2 Concentrated suspensions with solid particles

For concentrated suspensions with density preserving constituents, *i.e.*, when true mass densities does not change, γ_i and $\operatorname{grad}\gamma_i$ are no more independent variables and then must be deleted from the list (78): the saturated two-phase immiscible mixture is subjected to other two constraints, the incompressibility of the suspending fluid and of the dispersed granular solid.

The balances of mass (62) are now

$$\beta'^1=(1-\beta)\operatorname{div}v_1, \quad \beta'^2=-\beta\operatorname{div}v_2, \quad (100)$$

for which, using the definition of peculiar time derivatives and summing (100), we obtain the condition of incompressibility for the whole mixture:

$$\operatorname{div}v=0. \quad (101)$$

Again, for the extended principle of thermodynamic determinism, the conditions of perfect constraints (58) gives now the following requirements for the reactive terms:

$$\eta^r = 0,\ q^r = 0, \operatorname{div} k^r = 0,\ \mathcal{S}_i^r = 0,\ m^{+r} = 0,\ z^{+r} = 0, \tag{102}$$
$$\operatorname{sym} T_1^r = (1-\beta)(\zeta_1^r + \phi^{+r})I \quad \text{and} \quad \operatorname{sym} T_2^r = \beta(\zeta_2^r - \phi^{+r})I, \tag{103}$$

while, using the constraints of saturation and incompressibility in the reduced dissipation inequality (55) and performing the same calculations made in the previous section, we obtain that the active part of the free energy ψ^a is always a potential function but, here, of the array $\tilde{\mathcal{T}} = \{\beta, \operatorname{grad}\beta, \theta\}$. The constitutive relations (82) still hold for η^a and $\mathcal{S}_i^a$, whereas now it is

$$\operatorname{sym} T_1^a = (1-\beta)\left[(\zeta_1^a + \phi^{+a} + \pi_1^c)I - \operatorname{sym}\left(\gamma_1 \operatorname{grad}\beta \otimes \frac{\partial \psi^a}{\partial \operatorname{grad}\beta}\right)\right],$$
$$\operatorname{sym} T_2^a = \beta\left[(\zeta_2^a - \phi^{+a} - \pi_2^c)I - \operatorname{sym}\left(\gamma_2 \operatorname{grad}\beta \otimes \frac{\partial \psi^a}{\partial \operatorname{grad}\beta}\right)\right], \tag{104}$$
$$m^{+a} = 0,\ z^{+a} = 0 \quad \text{and} \quad k^a = 0, \tag{105}$$

where π_i^c is defined in $(87)_1$ and the last two results are valid for the isotropy of ψ^a and k^a itself, as before.

According to reactions $(102)_6$ and (103), and constitutive laws (68), (104) and $(105)_2$, we can obtain the following expressions for Cauchy stress tensors:

$$\begin{aligned} T_1 &= \operatorname{sym} T_1^a + \operatorname{sym} T_1^r - 2^{-1}\varepsilon z^+ = \\ &= (1-\beta)\left[(\zeta_1 + \phi^+ + \pi_1^c)I - \gamma_1 \tilde{\varphi} \operatorname{grad}\beta \otimes \operatorname{grad}\beta\right], \end{aligned} \tag{106}$$
$$\begin{aligned} T_2 &= \operatorname{sym} T_2^a + \operatorname{sym} T_2^r + 2^{-1}\varepsilon z^+ = \\ &= \beta\left[(\zeta_2 - \phi^+ - \pi_2^c)I - \gamma_2 \tilde{\varphi} \operatorname{grad}\beta \otimes \operatorname{grad}\beta\right], \end{aligned} \tag{107}$$

where

$$\tilde{\varphi}(\beta, \operatorname{grad}\beta \cdot \operatorname{grad}\beta, \theta) = 2\frac{\partial \psi^a}{\partial(\operatorname{grad}\beta \cdot \operatorname{grad}\beta)}. \tag{108}$$

Now, using the reactions $(102)_{4,5}$, the constitutive laws $(82)_{2,3}$ and the mass balances (100) in the micromomentum balances (64), we get the following expressions

$$
\begin{aligned}
\zeta_1^r + \phi^{+r} &= -(\zeta_1^a + \phi^{+a}) - \gamma_1 \operatorname{div}\left[(1-\beta)\, \tilde{\varphi} \operatorname{grad} \beta\right] + \gamma_1 (1-\beta) \left\{ \delta_1 + \right. \\
&\left. + (1-\beta) \left[\mu_1 (\operatorname{div} v_1)^{\prime 1} + \left(\frac{1}{2}(1-\beta) \frac{d\mu_1}{d\beta} - \mu_1 \right) (\operatorname{div} v_1)^2 \right] \right\}, \qquad (109) \\
\zeta_2^r - \phi^{+r} &= -(\zeta_2^a - \phi^{+a}) + \gamma_2 \operatorname{div}\left(\beta\, \tilde{\varphi} \operatorname{grad} \beta\right) + \\
&+ \gamma_2 \beta \left\{ \beta \left[\mu_2 (\operatorname{div} v_2)^{\prime 2} - \left(\frac{1}{2} \beta \frac{d\mu_2}{d\beta} + \mu_2 \right) (\operatorname{div} v_2)^2 \right] + \delta_2 \right\}, \qquad (110)
\end{aligned}
$$

and, therefore, we are able to write the two pure balances of linear momentum, which governs the motion of a saturated two-phase granular suspension with incompressible components, replacing the Cauchy stresses (106) and (107) in the equations (63) and taking into account the expressions (109) and (110) now obtained:

$$
\begin{aligned}
\gamma_1 (1-\beta) v_1' &= \operatorname{grad}\left[(1-\beta)(\zeta_1 + \phi^+)\right] + \gamma_1 (1-\beta) b_1 + \\
&+ \operatorname{div}\left[(1-\beta)(\pi_1^c\, I - \gamma_1\, \tilde{\varphi} \operatorname{grad} \beta \otimes \operatorname{grad} \beta)\right], \qquad (111) \\
\gamma_2 \beta v_2' &= \operatorname{grad}\left[\beta(\zeta_2 - \phi^+)\right] + \gamma_2 \beta b_2 - \\
&- \operatorname{div}\left[\beta(\pi_2^c\, I + \gamma_2\, \tilde{\varphi} \operatorname{grad} \beta \otimes \operatorname{grad} \beta)\right]. \qquad (112)
\end{aligned}
$$

The pure evolution equation for the mixture temperature θ remains the same, *i.e.*, equation (99), as well as the Fourier inequality (90) with the active part of the heat flux q^a now depending on the set $\tilde{\mathcal{T}}$.

At the end, as soon as the motions of constituents are ensued from (111) and (112), the corresponding unknown reactions, that is, the saturation pressure ϕ^{+r} and the two pressures ζ_1^r and ζ_2^r due to the peculiar incompressibility of the components, are obtained by the equations (109), (110) and the corresponding one to (98)

$$
\begin{aligned}
\zeta_1^r + \zeta_2^r &= -(\pi_1^c + \pi_2^c) + \operatorname{div}\left\{\left[\gamma_2 \beta - \gamma_1 (1-\beta)\right] \tilde{\varphi} \operatorname{grad} \beta\right\} + \gamma_1 (1-\beta) \\
&\left\{ (1-\beta) \left[\mu_1 (\operatorname{div} v_1)^{\prime 1} + \left(\frac{1}{2}(1-\beta) \frac{d\mu_1}{d\beta} - \mu_1 \right) (\operatorname{div} v_1)^2 \right] + \delta_1 \right\} + \\
&+ \gamma_2 \beta \left\{ \beta \left[\mu_2 (\operatorname{div} v_2)^{\prime 2} - \left(\frac{1}{2} \beta \frac{d\mu_2}{d\beta} + \mu_2 \right) (\operatorname{div} v_2)^2 \right] + \delta_2 \right\}. \qquad (113)
\end{aligned}
$$

Moreover, if the constituents are closely packed, we infer that microstructural inertial effects are negligible, and thus set $\mu_i = 0$; in absence of body forces, equation (113) reduces to

$$
\zeta_1^r + \zeta_2^r = -(\pi_1^c + \pi_2^c) + \operatorname{div}\left\{\left[\gamma_2 \beta - \gamma_1 (1-\beta)\right] \tilde{\varphi} \operatorname{grad} \beta\right\}, \qquad (114)
$$

where the configuration pressures π_i^c and the dilatancy modulus $\tilde{\varphi}$ are functions of $\tilde{\mathcal{T}}$: therefore the total pressure in the solid and fluid phases results from the effects of intergranular contact forces, represented by π_i^c, and the local variation in the stress fields due to a non-homogeneous granules distribution, given by $\tilde{\varphi}$.

Here, we compare our results with [92] where, again, the Principle of Equipresence of Truesdell were substituted by that of the Phase Separation, while, furthermore, the kinematic terms on the left hand side of their micromomentum balances (33) and (34) remain ambiguous, because the full Lagrangian derivatives of the kinetic energy, due to micromotions, do not seem complete: in fact, *e.g.*, it is easily to demonstrate that, for a material with rigid grains, the kinetic coefficient μ_2 is proportional to $\beta^{-\frac{2}{3}}$ (see equation $(6)_1$ of [43]) and so the second term on the left hand side of our equation (64), for $i = 2$, misses in the balance (33) of [92].

However, even now, we easily retrieve from our equations (109)-(112) the essence of the constitutive analysis in [92] of a fluid-saturated granular material with incompressible thermoelastic constituents, when, there, the mass conservations (28) and (29) and the saturation constraint (30) are used in the balance equations (31)-(34), the viscous coefficients are null places and the pressures are identified as it follows: $\pi = \phi^{+r}$, $p_i = \zeta_i^r$ and $\beta_i = \zeta_i^a - (-1)^i \phi^{+a} = \pi_i^c$.

8.3 Poroelastic materials

The last application of our theory concerns an isothermal flow of a fluid component through the channels of a solid skeleton with large pores, namely a part of soil [38,94,26,77,28], so the model is in some sense complementary to the previous one. Even now every constituent is considered incompressible (so $\gamma_i = \text{constant}_i$), and therefore also the immiscible mixture is; moreover, the mixing itself is regarded as if it happened without the creation of voids, and so the saturation constraint is applied again. Following Biot [8], we consider virtual mass effects due to diffusion and also introduce the microinertia associated with the rates of change of the volume fraction of the fluid, as well as that due to the deformation of the lacunae in the vicinity of their boundaries.

We wish to observe that the model for the solid constituent here considered is thought to have an affine microstructure with kinematic parameters independent from the macro motion, in order to include all other models. In particular, the presence of the saturating fluid constituent support this hypothesis, when the pores are very large and constitute a structure of sparse trabeculae, as it is the case of porous materials with evolving microstructure [78,33] or of cancellous bone [24]: Cowin itself [22] pointed out the importance of the shape of the pores in the description of bone canaliculi or of lacunae containing osteocytes, *e.g.*, in the human bone the lacunae are roughly ellipsoidal with mean values along the axes of about 9 μm, 22 μm and 4 μm. Moreover, 'Voids theories' are recovered in the case that pores are small and finely dispersed, as in [75], because our microstructure reduces to a spherical tensor depending upon a single additional variable [16]. Furthermore, a very interesting intermediate model, in which the ellipsoidal microstructure is partially constrained by the macro-motion, the isotropic part being free, falls within the theory presented here and permit us to offer an interpretation of constitutive prescriptions, involving the displacement gradients of higher order than the first, which allows one to circumvent certain apparent inconsistencies with the second law of thermodynamics (see, also, [49,47]), but is too elaborate and lengthy to be mentioned here explicitly: it will be presented in a forthcoming work.

The microstructural kinematic variable ν_1 for the skeleton is a 2^{nd} order symmetric tensor with positive determinant V ($\in \mathrm{Sym}^+$), that is the left microstretch, which takes into account for contractions or expansions of the large pores in the material (see [39–41]). Instead the fluid variable ν_2 is the volume fraction β, *i.e.*, the proportion of space occupied by the two-phase fluid constituent of the body. Therefore, we have that the infinitesimal generator $\mathcal{A}_1$ of the local action of the proper orthogonal group SO(3) over Sym^+, defined in (11), has the following components

$$(\mathcal{A}_1)_{\alpha\beta\iota} = V_{\alpha\gamma}\varepsilon_{\gamma\beta\iota} - \varepsilon_{\alpha\gamma\iota}V_{\gamma\beta}, \tag{115}$$

while $\mathcal{A}_2 = 0$, because a proportion does not change for a rotation. Consequently, the corotational time derivative of the microstretch $\breve{V}$ in (30) represents the Oldroyd's time derivative [72] of V with rate the spin tensor $W_1 = -\varepsilon r_1$:

$$\breve{V} = V'^1 + (\varepsilon r_1)V - V(\varepsilon r_1). \tag{116}$$

The kinetic co-energy χ_1 for materials with affine microstructure is usually assumed to be a quadratic form in V'^1 with a costant coefficient μ_1 [11,39], thus

$$\kappa_1 = \chi_1 := 2^{-1}\mu_1(V'^1)^2, \tag{117}$$

while χ_2 is as in $(61)_2$. We suppose further that the tensorial micromomentum growth for the porous phase is isotropic of coefficient $-\phi^+$, the opposite of that of the fluid.

We insert the hypotheses made in the mechanical balance equations (51), (20), (21), (66) and reduce them to (100), (63), $(64)_2$, $(68)_2$ and the following ones:

$$\rho_1\mu_1 V''^1 = \operatorname{div}\Sigma - Z + \rho_1 C - \phi^+ I, \tag{118}$$

$$\operatorname{skw} T_1 + 2^{-1}\varepsilon\, z^+ = \operatorname{skw}\,(VZ + \operatorname{grad} V \odot \Sigma)\,, \tag{119}$$

$$\rho\,\dot{\epsilon} = w + z^+ \cdot (r_1 - r_2) + m^+ \cdot (u_1 - u_2) + \phi^+ \left(\operatorname{tr} V'^1 - \beta'^2\right). \tag{120}$$

In these equations the fields $\operatorname{div}\Sigma, Z$ and C for the porous constituent are all 2nd order symmetric tensors: the 3rd order microstress tensor Σ is normally related to boundary microtractions, even if, in some cases, it could express weakly non-local internal effects, Z includes interactive forces between the gross and fine structures and C is interpreted as an externally controlled pore pressure. Further, we used the balances for the whole mixture (37) and introduced the following tensor product '$\odot$' of components $(\mathrm{grad}\, V \odot \Sigma)_{ij} := V_{ih,k}\, \Sigma_{jhk}$.

From equation (63), for $i = 2$, we could obtain the Darcy's law, if we neglect the inertial terms and make suitable constitutive proposals on fields m, b_2 and T_2. The balance of micromomentum $(64)_2$ for the volume fraction β generalizes the classical Langmuir's evolution equation, while the balance (118) for the microstretch V includes the Wilmanski's porosity balance as well as the equation which rules the changes of internal surfaces area of the pores (see [59,93,2], respectively).

Finally, we apply the saturation constraint by differentiating it with respect to time and using the identity $(80)_2$ and the definition $(59)_1$:

$$\begin{aligned} 0 = \dot{\beta}_1 + \dot{\beta}_2 = {} & \beta_1' + \beta_2' - u_1 \cdot \mathrm{grad}\, \beta_1 - u_2 \cdot \mathrm{grad}\, \beta_2 = \\ & = -\beta'^1 + \beta'^2 + u_1 \cdot \mathrm{grad}\, \beta - u_2 \cdot \mathrm{grad}\, \beta; \end{aligned} \tag{121}$$

thus, for equation $(100)_1$, the differential link is:

$$\beta'^2 = (1-\beta) I \cdot D_1 + (u_2 - u_1) \cdot \mathrm{grad}\, \beta. \tag{122}$$

In the actual context with $\theta_i = \bar{\theta}$ constant and k null, the perfect constraint condition (58) becomes

$$\begin{aligned} 0 = {} & \mathrm{sym}\, T_1^r \cdot D_1 + \mathrm{sym}\, T_2^r \cdot D_2 + Z^r \cdot [V'^1 + 2(\varepsilon r_1) V] + \zeta_2^r \beta'^2 + \\ & + \Sigma^r \cdot [\mathrm{grad}\, V'^1 + 2(\varepsilon r_1)(\mathrm{grad}\, V)] + \mathcal{S}_2^r \cdot \mathrm{grad}\, \beta'^2 + z^{+r} \cdot (r_1 - r_2) + \\ & + m^{+r} \cdot (u_1 - u_2) + \phi^{+r} (I \cdot V'^1 - \beta'^2) = \\ & = [\mathrm{sym}\, T_1^r + (1-\beta)(\zeta_2^r - \phi^{+r})\, I] \cdot D_1 + \mathrm{sym}\, T_2^r \cdot D_2 + \\ & + (Z^r + \phi^{+r} I) \cdot V'^1 + \Sigma^r \cdot \mathrm{grad}\, V'^1 + \mathcal{S}_2^r \cdot \mathrm{grad}\, \beta'^2 + \\ & + [z^{+r} - 2\varepsilon (V Z^r + \mathrm{grad}\, V \odot \Sigma^r)] \cdot r_1 - z^{+r} \cdot r_2 + \\ & + [m^{+r} - (\zeta_2^r - \phi^{+r})\, \mathrm{grad}\, \beta] \cdot (u_1 - u_2), \end{aligned} \tag{123}$$

where the symmetry properties of V, Z and Σ were used. Therefore, we have that the reactions have to satisfy the following relations:

$$\begin{aligned} & \mathrm{sym}\, T_1^r = -(1-\beta)\, \pi^{in}\, I, \quad \mathrm{sym}\, T_2^r = 0, \quad Z^r = -\phi^{+r} I, \\ & \Sigma^r = 0, \quad \mathcal{S}_2^r = 0, \quad m^{+r} = \pi^{in} \mathrm{grad}\, \beta, \quad z^{+r} = 0, \end{aligned} \tag{124}$$

where π^{in} is the interface pressure between constituents (74).

At this point, as regards the constitutive equations, in order to avoid a lot of specialization, we consider a very special case in which the set of kinetic variables is the following one: $\mathcal{P} = \{\beta, d := \operatorname{grad}\beta, F_1, V, \mho := \operatorname{grad} V\}$; moreover, we observe that, inserting results (124) in the purely mechanical energy equation (120), the mixture internal energy ϵ replaces the free energy as potential function. If we now differentiate $\epsilon(\mathcal{P})$ and substitute the result along with (31), (116) and (122) into (120), by performing standard calculations as described above, the results can be written as

$$
\begin{aligned}
&\mathrm{sym}\; T_1^a = (1-\beta)\pi_1^c\, I + \rho\, \mathrm{sym}\left[\frac{\partial \epsilon}{\partial F_1} F_1^T - \mho^T \odot \left(\frac{\partial \epsilon}{\partial \mho}\right)^T\right],\\
&\mathrm{sym}\; T_2^a = \beta(\sigma - \pi_2^c) I - \rho\, \mathrm{sym}\left(d \otimes \frac{\partial \epsilon}{\partial d}\right), \quad \mathcal{S}_2^a = \rho\, \frac{\partial \epsilon}{\partial d},\\
&Z^a + \phi^{+a} I = \rho\, \frac{\partial \epsilon}{\partial V}, \quad \Sigma^a = \rho\, \frac{\partial \epsilon}{\partial \mho}, \quad z^{+a} = -\rho\, \varepsilon \left(d \otimes \frac{\partial \epsilon}{\partial d}\right),\\
&m^{+a} = \mathrm{div}\left(\rho\; \mathrm{d} \otimes \frac{\partial \epsilon}{\partial d}\right) - \left[\mathrm{div}\left(\rho\, \frac{\partial \epsilon}{\partial d}\right)\right] d,
\end{aligned}
\tag{125}
$$

where

$$
\pi_i^c = \rho_i \frac{\partial \epsilon}{\partial \beta}, \quad \text{for } 1 = 1, 2, \quad \text{and} \quad \sigma = \zeta_2^a - \phi^{+a}
\tag{126}
$$

are the configuration pressures of the i-constituents and the hydrostatic pressure acting on the mixture, respectively; in addition, two compatibility conditions are valid on z^{+a} and m^{+a} due to the balance laws for the mixture $(37)_{2,3}$, as well as internal constraints:

$$
\begin{aligned}
\varepsilon \Bigg\{ d \otimes \frac{\partial \epsilon}{\partial d} + F_1 \left(\frac{\partial \epsilon}{\partial F_1}\right)^T + \mho^T \odot \left(\frac{\partial \epsilon}{\partial \mho}\right)^T + \\
+2\left[V \left(\frac{\partial \epsilon}{\partial V}\right)^T + \mho \odot \frac{\partial \epsilon}{\partial \mho}\right]\Bigg\} = 0,
\end{aligned}
\tag{127}
$$

$$
\begin{aligned}
\mathrm{div}\left\{\rho\left[\mathrm{d} \otimes \frac{\partial \epsilon}{\partial d} + \mho^T \odot \left(\frac{\partial \epsilon}{\partial \mho}\right)^T\right]\right\} + \rho\left[\frac{\partial \epsilon}{\partial F_1}(\mathrm{grad}\, F_1) + \frac{\partial \epsilon}{\partial \mho}\, \mho\right] = \\
= \mho^T \left[\mathrm{div}\left(\rho\, \frac{\partial \epsilon}{\partial \mho}\right)\right] + \left[\mathrm{div}\left(\rho\, \frac{\partial \epsilon}{\partial d}\right)\right] d.
\end{aligned}
\tag{128}
$$

Now, by inserting relations (124-125) in the balance equations (63), (118), $(64)_2$, (119) and $(68)_2$, then using conditions (127-128) and substituting (119) and $(68)_2$ into equations (63), we obtain

$$\gamma_1(1-\beta)v_1' = \operatorname{div}\tilde{T}_1 + \gamma_1(1-\beta)\,b_1 - f, \tag{129}$$

$$\gamma_2\beta v_2' = \operatorname{div}\tilde{T}_2 + \gamma_2\beta\,b_2 + f \tag{130}$$

$$\gamma_1(1-\beta)\,\mu_1\,V^{\prime\prime 1} = \operatorname{div}\left(\rho\frac{\partial\epsilon}{\partial\mho}\right) - \rho\frac{\partial\epsilon}{\partial V} + \gamma_1(1-\beta)\,C, \tag{131}$$

where the pure symmetric Cauchy stresses $\tilde{T}_i$ for the solid and the fluid constituents are defined by

$$\begin{aligned}\tilde{T}_1 &:= (1-\beta)(\pi_1^c - \pi_{in})\,I + \rho\,\mathrm{sym}\left(\frac{\partial\epsilon}{\partial F_1}F_1^T - E\right),\\ \tilde{T}_2 &:= \beta(\sigma - \pi_2^c)I,\end{aligned} \tag{132}$$

while the interaction force f between the phases is

$$f := \left[\pi_{in} - \operatorname{div}\left(\rho\frac{\partial\epsilon}{\partial d}\right)\right]d, \tag{133}$$

with

$$\pi_{in} = \operatorname{div}\left(\rho\frac{\partial\epsilon}{\partial d}\right) - \sigma + \gamma_2\beta\left\{\delta_2 - \beta\left[\mu_2(\operatorname{div}v_2)'^2 - \left(\frac{1}{2}\beta\frac{d\mu_2}{d\beta} + \mu_2\right)(\operatorname{div}v_2)^2\right]\right\}.$$

The effects of the microstructure describing the large pores appear only in the extended stress of Ericksen's type for the solid phase, defined by

$$E := -\rho\,\mathrm{sym}\left[d\otimes\frac{\partial\epsilon}{\partial d} + \mho^T\odot\left(\frac{\partial\epsilon}{\partial\mho}\right)^T\right], \tag{134}$$

and in the micromomentum equation (131).

Finally, we can still propose a system of mechanical equations of motion for an incompressible saturated poroelastic material, which satisfies the constitutive principle of equipresence in addition to the other axioms usually considered for mixtures. There are 15 scalar unknowns, $\beta, v_1, v_2, V, \pi_{in}$ and σ, and 15 scalar differential equations $(100)_1$, (122), (129-131) and the one after (133) for π_{in}, for which we would also expect that the appropriate initial and boundary conditions, as well as the surface tractions associated with $\tilde{T}_i$ and Σ^a, are provided to resolve them.

The comparison in the poro-elastic mechanics for this last example is with classical mixture theories of Bowen [9] and Pence [77] for the isothermal flow. In particular, the Cauchy stress tensor for the solid constituent of the transversely isotropic case (see equation (82) of [77]) is directly recovered, if one binds, even partially, the microstructural parameter V to the macro-deformation and represents the Ericksen tensor (134) appropriately. Instead, Bowen replaced the solid equation of linear momentum by that of the mixture (see equation (3.32) of [9]) that, in our notations, is as follows

$$\sum \gamma_i \beta_i v_i' = \operatorname{div} \tilde{T} + \sum \gamma_i \beta_i b_i; \tag{135}$$

therefore, by supposing that the microstructure is absent, and so the potential ϵ does not depend on d and $\mho$, $\mu_2 = 0$ and $\delta_2 = 0$, we have

$$\tilde{T} \, (:= \tilde{T}_1 + \tilde{T}_2) = [-\pi_{in} + (1-\beta)\pi_1^c - \beta \pi_2^c] \; I + \rho \operatorname{sym} \left(\frac{\partial \epsilon}{\partial F_1} F_1^T \right), \tag{136}$$

where we can easily recognize his stress tensor T_I, when we put $\Psi_I = \rho\epsilon$ and $\lambda = \pi_{in} - \frac{\partial(\rho_1 \epsilon)}{\partial \beta}$, and insert definitions $(126)_1$ for the configuration pressures π_i^c (see (3.22) of [9]); moreover, the expression of our Cauchy tensor $\tilde{T}_2$ for the fluid phase coincides with (3.29) of [9].

9 Peculiar solutions

9.1 Micro-rotations in a quasi-linear dilatant granular material

The example of dilatant granular media that we are dealing with here, continua which are models of suspensions of rigid spheres in a fluid, is, in a sense, complementary to that considered in §8.3; rather, it could be thought of as an enriched sample of the concentrated suspensions of §8.2: in fact, now, the peculiar microstructure of the solid phase is spherical, that is, the rotations of the individual granules must be taken into account, in addition to the use of volume fraction β (see [16,43]).

In particular, we consider the flow of a large number of discrete inelastic particles ($\gamma_2 =$ const.) at relatively high concentrations and with interstices filled by a gas or a fluid of neglectable mass ($\gamma_1 \approx 0$), as it is for cohesionless soils, such as sand with rough surface grains, or fluidized particulate beds. The admissible micro-motions of the body consist of either the rotation R of the same granules rigid with respect to each other, as well as the homogeneous dilatations, or contractions, of the macro-elements, *i.e.*, radial motions due to the displacements of the grains relative to the center of mass of the macro-element itself, as introduced by Reynolds [80]; the material macro-element of the granular material has a fine structure and, in a mental magnification, we think of it as a sort of quasi–particle, which consists of a rigid grain and its immediate rigid neighbours [43].

Therefore, the microstructural kinematic variable ν_2 is now the rotation R, while the infinitesimal generator $\mathcal{A}_2$, defined in (11), has the following components

$$(\mathcal{A}_2)_{\alpha\beta\iota} = \varepsilon_{\alpha\iota\gamma} R_{\gamma\beta}, \tag{137}$$

for which the angular momentum balance $(119)_2$ becomes now

$$\operatorname{skw} T_2 = \gamma_2 \operatorname{skw} \left[R \left(\frac{\partial \epsilon}{\partial R} \right)^T + \operatorname{grad} R \odot \frac{\partial \epsilon}{\partial (\operatorname{grad} R)} \right], \tag{138}$$

while the pure mechanical balance equations of interest in the problem are similar to those of the previous section, only by exchanging the subscript 1 with 2 (here the complementarity), with the exception of the negligible equation (129).

In particular, they are equations $(100)_2$, (130) and (131):

$$\beta'^2 + \beta \operatorname{div} v_2 = 0, \qquad \gamma_2 \beta v_2' = \operatorname{div} \hat{T}_2 + \gamma_2 \beta\, b_2, \tag{139}$$

$$\mu_2\, \beta\, R''^2 = \operatorname{div} \left[\beta \, \frac{\partial \epsilon}{\partial (\operatorname{grad} R)} \right] - \beta \frac{\partial \epsilon}{\partial R}, \tag{140}$$

where we suppose to zero all the growth rates, as well as the micro-viscosities in the granular phase in absence of external microspin actions C.

A general treatment of the constitutive relations of $\hat{T}_2$ and (the partial derivatives of) ϵ for dilatant granular assemblies with rotating grains is given in [43,44], while their quasi-linear expression has recently appeared in [3], where the classical model of Coulomb granular material [51] is also obtained as a particular case.

We consider, in detail, cohesionless granular media in which mutual granular fluctuations are negligible, the dissipative part of the Cauchy's stress $\hat{T}_2$ behaves like a viscous fluid with viscosity coefficients $\hat{\lambda}$ and $\hat{\mu}$ [50], and the conservative stress component depends on $\{\beta, \operatorname{grad} \beta, \operatorname{grad} R\}$ (see, in general, [3]); therefore,

$$\begin{aligned} \hat{T}_2 = & \Big[2\, \hat{\alpha} \beta\, \Delta \beta + \hat{\alpha}\, |\operatorname{grad} \beta|^2 - \hat{\beta}\, \beta^2 + 2\, \hat{\delta}\, |\operatorname{grad} R|^2 + \hat{\lambda} (\operatorname{tr} D) \Big] I + \\ & + 2\, \hat{\mu}\, D - 2\, \hat{\alpha} \operatorname{grad} \beta \otimes \operatorname{grad} \beta - 4\, \hat{\delta}\, (\operatorname{grad} R)^T \odot (\operatorname{grad} R)^T \end{aligned} \tag{141}$$

and

$$\hat{\Sigma}^a := \beta \frac{\partial \epsilon}{\partial (\operatorname{grad} R)} = 2\, \hat{\delta} \operatorname{grad} R, \quad \hat{Z}^a := \beta \frac{\partial \epsilon}{\partial R} = 0. \tag{142}$$

The micro-elastic constant $\hat{\delta}$ of the hyperstress tensor Σ^a depends on the macro-elastic ones. In [27] it has been shown that “the (three) elastic constants of the granular micropolar theory (modeled as a Cosserat’s material) are strongly correlated and that only two of them can be identified independently”, and thus, it has been suggested a strict correlation of $\hat{\delta}$ with the Coulomb’s constant $\hat{\alpha}$ related to $\hat{T}_2$. In particular, $\hat{\delta}$ is proportional to $\hat{\alpha}$ through the square of an intrinsic length scale that we put equal to the initial grain radius $(r_g/\sqrt{5})$.

Moreover, the constant micro-kinetic coefficient $\mu_2 := \frac{1}{5} r_g^2 \gamma_2$ for rigid grains appears in the Appendix A of [37], where we assumed a special initial geometrical configurations of the grains, supposed all spherical of constant radius r_g in the reference placement.

For our numerical purposes, we specify balance equations (139) and (140) to two-dimensional setting, for which the velocity v_2 has only two components $(\hat{v}_1, \hat{v}_2)$ in the Oxy plane, the rotation R is around the z-axis of angle Θ only and b_2 is the gravity force g directed along Oy. Thus, inserting (141) and (142), using the definition $(12)_2$, for $i = 2$, and placing the expressions now indicated for $\hat{\delta}$, μ_2 and b_2, we write the two-dimensional system of four mechanical equations for dilatant granular material as

$$\frac{\partial \beta}{\partial \tau} + \operatorname{div}\,(\beta\,\hat{v}_1, \beta\,\hat{v}_2) = 0, \tag{143}$$

$$\begin{aligned}
&\frac{\partial(\beta\hat{v}_1)}{\partial \tau} + \operatorname{div}\,\Bigg\{\beta\hat{v}_1^2 + \bar{\beta}\beta^2 + \bar{\alpha}\left[\left(\frac{\partial \beta}{\partial x}\right)^2 - \left(\frac{\partial \beta}{\partial y}\right)^2\right] - \bar{\lambda}\frac{\partial \hat{v}_2}{\partial y} + \\
&\quad +2\,\bar{\delta}\left[\left(\frac{\partial \Theta}{\partial x}\right)^2 - \left(\frac{\partial \Theta}{\partial y}\right)^2\right] - 2\,\bar{\alpha}\beta\left(\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2}\right) - \left(\bar{\lambda} + 2\bar{\mu}\right)\frac{\partial \hat{v}_1}{\partial x}, \\
&\quad \beta\hat{v}_1\,\hat{v}_2 + 2\bar{\alpha}\frac{\partial \beta}{\partial x}\frac{\partial \beta}{\partial y} + 4\,\bar{\delta}\,\frac{\partial \Theta}{\partial x}\frac{\partial \Theta}{\partial y} - \bar{\mu}\left(\frac{\partial \hat{v}_1}{\partial y} + \frac{\partial \hat{v}_2}{\partial x}\right)\Bigg\} = 0,
\end{aligned} \tag{144}$$

$$\begin{aligned}
&\frac{\partial(\beta\hat{v}_2)}{\partial \tau} + \operatorname{div}\,\Bigg\{\beta\hat{v}_1\,\hat{v}_2 + 2\bar{\alpha}\frac{\partial \beta}{\partial x}\frac{\partial \beta}{\partial y} + 4\,\bar{\delta}\,\frac{\partial \Theta}{\partial x}\,\frac{\partial \Theta}{\partial y} - \bar{\mu}\left(\frac{\partial \hat{v}_1}{\partial y} + \frac{\partial \hat{v}_2}{\partial x}\right), \\
&\quad \beta\hat{v}_2^2 + \bar{\beta}\beta^2 - \left(\bar{\lambda} + 2\bar{\mu}\right)\frac{\partial \hat{v}_2}{\partial y} - \bar{\lambda}\frac{\partial \hat{v}_1}{\partial x} - \bar{\alpha}\left[\left(\frac{\partial \beta}{\partial x}\right)^2 - \left(\frac{\partial \beta}{\partial y}\right)^2\right] - \\
&\quad -2\bar{\alpha}\beta\left(\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2}\right) - 2\,\bar{\delta}\left[\left(\frac{\partial \Theta}{\partial x}\right)^2 - \left(\frac{\partial \Theta}{\partial y}\right)^2\right]\Bigg\} = -\beta g,
\end{aligned} \tag{145}$$

$$\frac{\partial(\beta\omega)}{\partial \tau} + \operatorname{div}\left[\beta\omega\hat{v}_1 - 2\,\bar{\alpha}\,\frac{\partial \Theta}{\partial x}, \beta\omega\hat{v}_2 - 2\,\bar{\alpha}\,\frac{\partial \Theta}{\partial y}\right] = 0, \tag{146}$$

where $\omega := \Theta^{'2}$ is grain angular velocity normal to the xy-plane, while coefficients with overbar are those with the over hat divided by the true mass density of the granules, *e.g.*, $\bar{\alpha} = \frac{\hat{\alpha}}{\gamma_2}$.

The constant parameters are chosen for a closely packed system composed by rigid spheres with equal radii $r_g = 0.4 \times 10^{-3}$ m, of true mass density $\gamma_2 = 1.5 \times 10^3$ Kg·m^{-3}, suspended in air at room temperature of 20°C. The reference volume fraction β_* of the solid is set at 70 per cent, while the elasticity parameters of Coulomb were calculated in [27], *i.e.*, the first is $\hat{\beta} = 4.94 \times 10^4$ N·m^{-2} and the second $\hat{\alpha} = 9.1 \times 10^4$ N, so the micro-elastic parameter is, at the end, $\hat{\delta} = 2.912 \times 10^{-3}$ N. These parameters refer to dry sand.

Due to the presence of air, the mixture is compressible and since it is subject to small variations in pressure, temperature and velocity gradients, both the shear (or kinematic) and the bulk (or volume) viscosity $\hat{\mu}$ and $\hat{\mu}_b := \hat{\lambda} + \frac{2}{3}\hat{\mu}$, respectively, can be treated as constants. When the distribution of granules is fairly uniform throughout the air, in order to obtain the kinematic viscosity $\hat{\mu}$ we can use the semi-empirical formula

$$\frac{\hat{\mu}}{\mu^0} = 1 + \hat{\phi}\beta_*, \tag{147}$$

where $\mu^0 = 1.83 \times 10^{-5}$ Pa·s is the shear viscosity of air and $\hat{\phi} \approx 67,81$ is an interaction factor which depend on the grain incompressibility other than the geometry of the involved particles (see equations (9-4.22) and (9-5.3) of [54]), whereas the bulk to shear viscosity ratio is 1 [81,73], for which we have, finally, the value $\hat{\lambda}\,(= \hat{\mu}_b - \frac{2}{3}\hat{\mu}) = 2.96 \times 10^{-4}$ Pa·s. All the coefficients of this example are summarized in Table 1.

Table 1
Parameters for the model of dilatant granular material

Par.	Value	Description
γ_2	1.5×10^3 Kg·m^{-3}	true mass density of dry sand
β_*	0.7	reference volume fraction
r_g	4×10^{-4} m	reference granule radius
μ_2	1.2×10^{-4} Kg·m^{-1}	micro-kinetic coefficient
μ^0	1.83×10^{-5} Pa·s	shear viscosity coefficient of air at 20°C
$\hat{\mu}$	8.87×10^{-4} Pa·s	shear viscosity coefficient of the packed system
$\hat{\lambda}$	2.96×10^{-4} Pa·s	second viscosity coefficient of the packed system
$\hat{\beta}$	4.94×10^4 N·m^{-2}	first Coulomb's elasticity parameter
$\hat{\alpha}$	9.1×10^4 N	second Coulomb's elasticity parameter
$\hat{\delta}$	2.912×10^{-3} N	micro-elastic coefficient

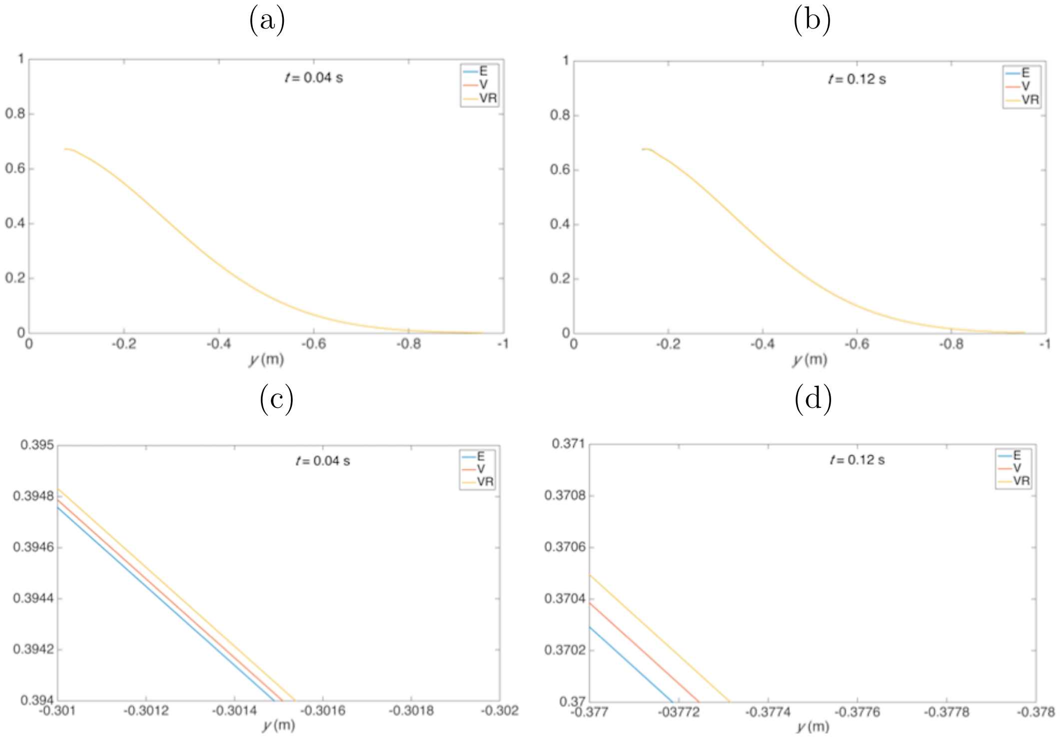

Fig. 1. Parallel between the distributions of the volume fraction β, for the E-, V- and VR-models, along the vertical cut line taken at: (a) $\tau = 0.04$ s; (b) $\tau = 0.12$ s; (c) and (d): 10^3 magnification of arbitrary portions of boxes (a) and (b), respectively.

In developing numerical tests, we limit ourselves to analyze the influence of micro-rotations on macroscopic motion. We consider a squared section $\Lambda = [0, 1] \times [-1, 0]$ m^2 of a three dimensional vertical channel, perpendicular to two delimiting vertical plates, spaced 1 m apart, along the horizontal direction x, in which the granular material can flow in the vertical direction. We adopt a uniform mapped mesh of [28 × 28] grid points and a time step $\Delta\tau = 0.0005$ s, $\tau \in (0, \tau_0]$ and $\tau_0 = 0.25$ s. The dependent variables are volume fraction β, mass fluxes $\beta\hat{v}_1, \beta\hat{v}_2$ and couple flux $\beta\,\omega$, all depending on $(x, y, \tau) \in \Lambda \times (0, \tau_0]$. Initial conditions are $\beta\hat{v}_1(0) = 0$ m·s^{-1}, $\beta\hat{v}_2(0) = 0$ m·s^{-1} and $\beta\omega(0) = 0.01$ s^{-1}, while, for the initial volume fraction, we suppose that $\beta(0) = \exp[-(0.15)^{-1}|y|^2]$, *i.e.*, we admit that an initial chunk of granular material is settled along the x direction, decreasing along the y direction with a half-gaussian-like shape. The granular flow is supposed entirely confined inside the Λ domain. Consequently, we impose for the volume fraction β zero-flux boundary conditions on all sides of the domain. We impose zero-flux boundary conditions on the top side for the velocity, and Dirichlet boundary conditions on all other sides with $\beta\hat{v}_1 = \beta\hat{v}_2 = 0$ m·s^{-1}. Finally, concerning the angular velocity of the grain, zero-flux boundary conditions are imposed on both top and bottom sides of the integration domain, and Dirichlet conditions with $\beta\omega = 0.6$ s^{-1} on left and right sides. In order to give a quantitative description of the granular dynamics, we consider a ‘vertical cut line’, that is a vertical straight line cutting the integration domain at $x = 0.5$ m, splitting the domain up into two symmetric sub-domains.

In Figs. 1 and 2 we compare the profiles of β and $\beta\hat{v}_2$, respectively, obtained from our 'complementary' microstretching model (VR) with the voids theory [50] in the purely elastic case (E), with zero $\hat{\mu}, \hat{\lambda}, \hat{\delta}$, and the dissipative one (V), with $\hat{\delta} = 0$ only.

At each time step, the profiles appear almost identical in boxes (a)-(b) of Figs. 1 and 2, hence a magnification of them is shown in boxes (c)-(d), respectively. The magnifications in boxes (c) and (d) put in evidence differences among E, V and VR profiles, in which the contribution coming from E model is always weaker with respect to the contribution coming from V model, in turn weaker with respect to VR model contribution, at least till $\tau = 0.12$ s (and taking into account the absolute tolerance of 10^{-4} imposed to the numerical procedure).

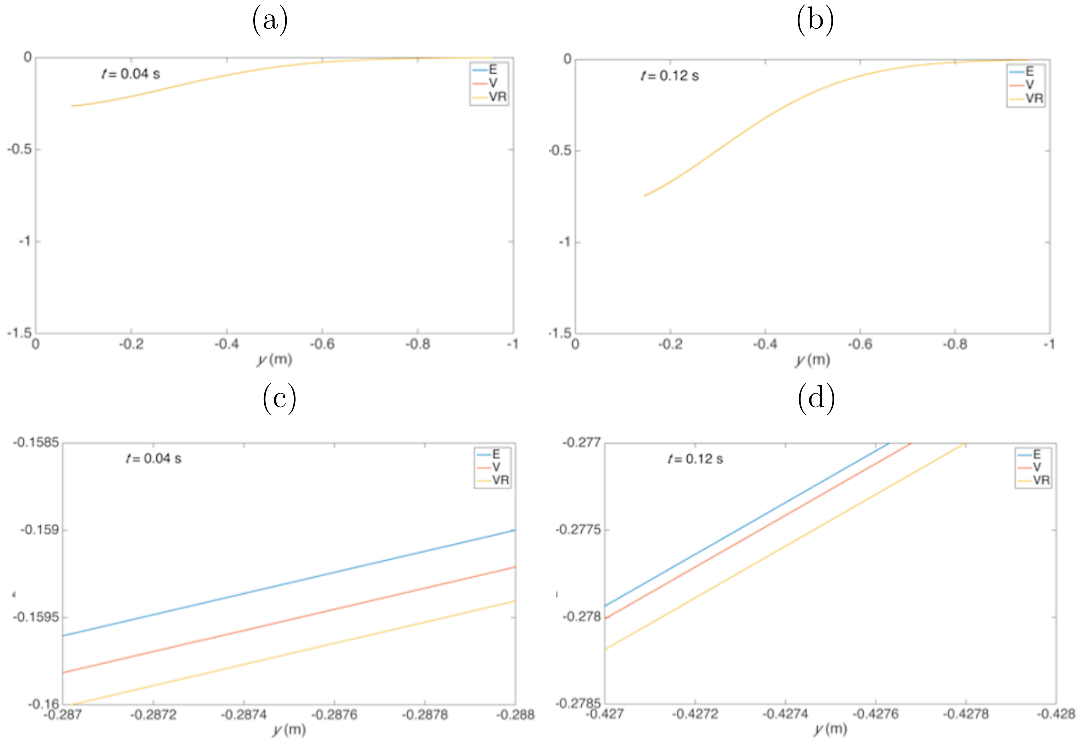

Fig. 2. Parallel between the distributions of the mass flux $\beta\hat{v}_2$, for the E-, V- and VR-models, along the vertical cut line taken at: (a) $\tau = 0.04$ s; (b) $\tau = 0.12$ s; (c) and (d): 10^3 magnification of arbitrary portions of boxes (a) and (b), respectively.

In Figure 3 the profile of the $\beta\omega$ variable along the vertical cut line is shown at different time steps. In box (a), until $\tau = 0.12$ s, the $\beta\omega$ profiles fall in the 10^{-3} s^{-1} range, but if compared to the profiles obtained at the higher time steps, as in box (b), the former appear flatted due to the different scale of the profiles for $\tau = 0.16$ s and $\tau = 0.20$ s. (see, also, [3] from where figures are taken).

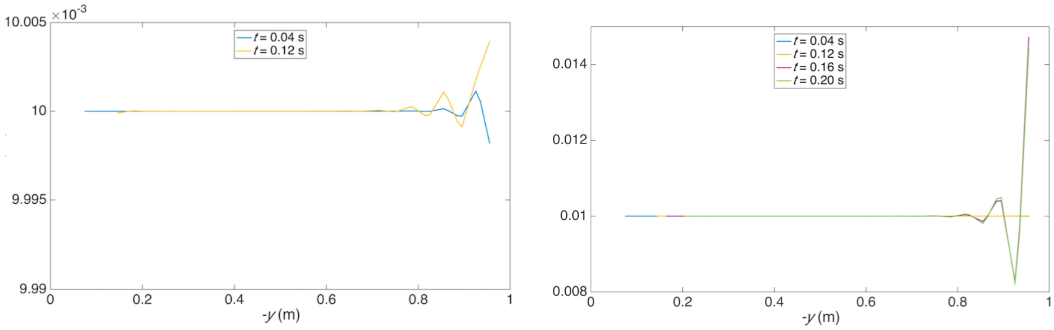
(a) (b)

Fig. 3. Profiles of the mass flux $\beta\omega$ along the vertical cut line: (a) parallel between the profiles taken at $\tau = 0.04$ s and $\tau = 0.12$ s; (b) parallel among the profiles taken at $\tau = 0.04\,s, \tau = 0.12\,s, \tau = 0.16$ s and $\tau = 0.20$ s.

Such results are consistent with the initial conditions imposed to the model simulation: in fact, while $\beta\hat{v}_1(0) = \beta\hat{v}_2(0) = 0$ m·s^{-1}, instead $\beta\omega(0) = 0.01$ s^{-1}, then, at $t = 0$ s, a non-zero rotational contribution is already present, which is relevant in the first stage of the granular dynamics. During the time evolution, the $\beta\omega$ contribution is weak and confined within the $10^{-5} - 10^{-2}$ s^{-1} range (see Figs. 3), while the growing $\beta\hat{v}_2$ contribution makes the differences between V and VR dynamics to vanish, even if a major contribution to the granular dynamics coming from the V model cannot be excluded, for $\tau > 0.20$ s, on the basis of physical considerations.

9.2 Micro-vibrations in a linear thermo-elastic porous solid

Consolidate soils can be considered as porous continua with a fabric determined by the way grains are in contact, possibly linked by cohesive forces. Under dynamic solicitations, such as those induced by trains on a railway, vehicles on pounding roads or earthquakes, we may recognize interplay between macroscopic and microscopic vibrations. The phenomenon has its counterpart in the dynamics of foams, the scheme developed in the previous sections allows us to describe such a phenomenon with a certain detail.

We remark again that pore volume fraction is insufficient to describe the microdeformations of the holes when they are large [23]. In fact, the linear theory of porous materials (as well as classical Cauchy's one) does not predict size effects in torsion as they occurs in the mechanics of bones [21] or rod-shaped specimens of dense polyurethane foams [58].

Instead the linear theories obtained from Cosserat brothers' theory [30] and/or the model of porous media with ellipsoidal microstructure [35], later renamed as microstrain continua [31], were used to study numerous applications to the model of media with affine microstructure. The latter is surely preferable when we consider soils or cellular solids consisting of a solid matrix with large pores filled by gas, because the effects of micro-rotations are trivial. As an example of this complex microstructure, we show in Fig. 4 (see also [5]) a graphene foam sheet (spongy graphene, 2" x 2" x 1.2 mm).

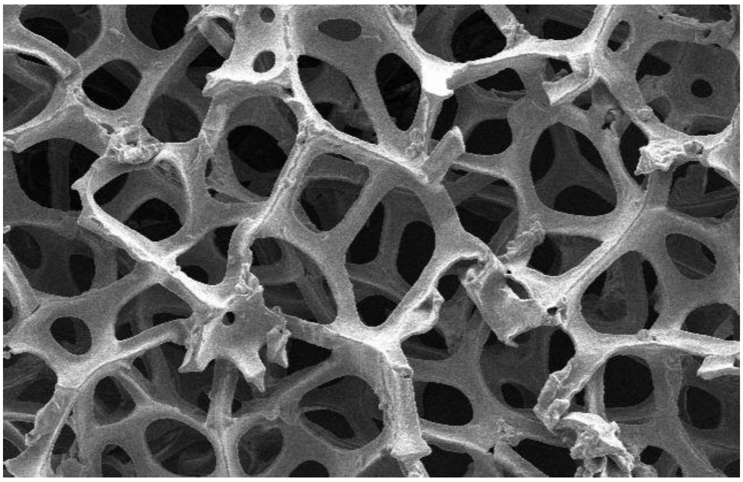

Fig. 4. Scanning electron micrograph of cellular structures of graphene foam.

In order to compare the solutions of micro-vibrations in different models (with voids, or microstretch or ellipsoidal microstructure), we linearize the system of differential balance equations in the case in which the saturating fluid is a gas of negligible mass ($\rho_2 \approx 0$). Also we neglect macroscopic dynamics in the mixture, *i.e.*, impose

$$\bar{u}_i(X_i, \tau) := x_i(X_i, \tau) - X_i = 0 \quad \text{for} \quad i = 1, 2. \tag{148}$$

Then, we follow also analyses in [36,45]. Particularly, we consider a homogeneous isotropic, thermoelastic porous solid initially undisturbed and at uniform temperature θ_0. We suppose further that all the external volume contributions vanish, as well as the micromomentum growth ϕ^{+a}, *i.e.*, $b_i = 0$, $C = O$, $\delta_2 = 0$, $\lambda = 0$ and $\phi^{+a} = 0$. Therefore, the only pure equations of interest in the problem are micro-momentum balance (118) and thermal evolution (99),

$$\rho_1\, \mu_1\, V''^{1} = \operatorname{div} \Sigma^a - Z^a \quad \text{and} \quad \rho_1\, \theta \dot{\eta}^a = \operatorname{div} q^a, \tag{149}$$

respectively. For them we refer to constitutive relations $(82)_1$ and $(124)_{3,4}$.

The linear representations of constitutive fields, depending on the set $\{V_1 := V - I,\ \operatorname{grad} V_1,\ \vartheta := \theta - \theta_0,\ \operatorname{grad} \vartheta\}$, have been already obtained in [36] (equations $(28)_{2,3}$ and (30)). Together with the Fourier law for q^a, they reduce the balances (149) to

$$\begin{aligned}
\frac{\partial^2 V_1}{\partial \tau^2} = {} & v_{sm}^2\,\Delta V_1 + 2(v_{tm}^2 - v_{sm}^2)\,\mathrm{sym}\,[\mathrm{grad}\,(\mathrm{div}\,V_1)] + \bar{\lambda}_1 \mathrm{grad}^{\,2}(\mathrm{tr}\ V_1) + \\
& + \left[\bar{\lambda}_1 \mathrm{div}^{\,2} V_1 + \bar{\lambda}_2 \Delta(\mathrm{tr}\,V_1) - \bar{\lambda}_3 \mathrm{tr}\,V_1 - \bar{\gamma}_3 \vartheta\right] I - 2\bar{\lambda}_4 V_1, \qquad (150) \\
0 = {} & \bar{\gamma}_1 \frac{\partial \vartheta}{\partial \tau} + \bar{\gamma}_2 \Delta \vartheta + \mu_1 \bar{\gamma}_3 \mathrm{tr}\,\dot{V}_1, \qquad (151)
\end{aligned}$$

where $\mathrm{tr}\,(\cdot)$ denotes once again the trace, *i.e.*, in a flat metric, $\mathrm{tr}\,V_1 := V_1 \cdot I$; moreover, v_{sm}^2, v_{tm}^2 and $\bar{\lambda}_i$, for $i = 1, \ldots, 4$, are micro-elastic constants, $\bar{\gamma}_3$ depends on micro-mechanical and thermal properties of the porous body; $(\bar{\gamma}_1 \theta_0)$ is the specific heat at constant strain, while $\bar{\gamma}_2 := \xi_*(\rho_{1*}\theta_0)^{-1} \geq 0$, with ξ_* the coefficient of thermal conductivity (see, also, [33]).

We can uncouple the spherical and deviatoric components of the linear balance of micromomentum (150) to obtain, respectively,

$$\begin{aligned}
\frac{\partial^2 \nu}{\partial \tau^2} = {} & \left(\frac{1}{3} v_{sm}^2 + \frac{2}{3} v_{tm}^2 + 2\bar{\lambda}_1 + 3\bar{\lambda}_2\right) \Delta \nu - 3\bar{\gamma}_3 \vartheta + \\
& + \left[2(v_{tm}^2 - v_{sm}^2) + 3\bar{\lambda}_1\right] \mathrm{div}\,(\mathrm{div}\,V^D) - (3\bar{\lambda}_3 + 2\bar{\lambda}_4)\,\nu \quad \text{and} \qquad (152) \\
\frac{\partial^2 V^D}{\partial \tau^2} = {} & v_{sm}^2\,\Delta V^D + 2(v_{tm}^2 - v_{sm}^2)\,\left\{\mathrm{sym}\,\left[\mathrm{grad}\,(\mathrm{div}\,V^D)\right]\right\}^D + \\
& + \left[\frac{2}{3}(v_{tm}^2 - v_{sm}^2) + \bar{\lambda}_1\right] (\mathrm{grad}^{\,2}\nu)^D - 2\bar{\lambda}_4 V^D, \qquad (153)
\end{aligned}$$

where ν is the trace of V_1, while its deviatoric part is defined by $V^D := V_1 - \frac{1}{3}\nu I$.

Consider solutions of the form

$$\nu = \hat{\nu}\, e^{ib\tau}, \;\; V^D = \hat{V}\, e^{ib\tau}, \;\; \vartheta = \hat{\vartheta}\, e^{ib\tau}, \qquad (154)$$

where $\hat{\nu}$, $\hat{V}$ and $\hat{\vartheta}$ are constant amplitudes, b is the frequency and i is the imaginary unit. By inserting these expressions into equations, we get the system of algebraic equations

$$\left(b^2 - 3\bar{\lambda}_3 - 2\bar{\lambda}_4\right) \hat{\nu} = 3\bar{\gamma}_3 \hat{\vartheta}, \;\; \left(b^2 - \bar{\lambda}_4\right) \hat{V}^D = O, \;\; \bar{\gamma}_1 \hat{\vartheta} + \mu_1 \bar{\gamma}_3 \hat{\nu} = 0, \qquad (155)$$

with the restriction on the free energy density ψ^a to be positive definite which implies

$$\bar{\gamma}_1 \left(3\bar{\lambda}_3 + 2\bar{\lambda}_4\right) > 3\mu_1 \bar{\gamma}_3^2, \quad \bar{\lambda}_4 > 0 \qquad (156)$$

(see equations (30) and (31) of [36]).

Eventually, we get admissible values of frequency b for different types of waves.

- ◦) *Dilatational modes*:

$$b_d = \sqrt{3\bar{\lambda}_3 + 2\bar{\lambda}_4 - 3\mu_1\bar{\gamma}_1^{-1}\gamma_3^2}, \quad \text{with} \quad \hat{\nu} = 3\,\hat{V}_{11}, \qquad (157)$$
$$\hat{V}_{22} = \hat{V}_{33} = \hat{\nu}/3, \quad \hat{V}_{ij} = 0,\ \forall\, i \neq j, \quad \hat{\vartheta} = -\mu_1\bar{\gamma}_1^{-1}\bar{\gamma}_3\hat{\nu}:$$

  the cutoff frequency b_d of this micro-thermal oscillation is real, for the restriction $(156)_1$, and we may expect one longitudinal acoustical wave in the three-dimensional porous medium to couple with the micro-modes to form four optical branches.

- ◦) *Extensional modes with a constant volume*:

$$b_e = \sqrt{\bar{\lambda}_4}, \text{ with } \hat{V}_{11} = -\hat{V}_{22} - \hat{V}_{33}, \hat{\nu} = \hat{\vartheta} = 0,\ \hat{V}_{ij} = 0,\ \forall\, i \neq j : \qquad (158)$$

  the cutoff frequency b_e of the micro-oscillations is real for inequality $(156)_2$, while no thermal vibration is present; we may also expect one transverse acoustical wave to couple with the micro-modes to form three optical mechanical branches.

- ◦) *Pure shear modes*:

$$b_s = \sqrt{\bar{\lambda}_4}, \text{ with } \hat{V}_{ij} \neq 0,\ \forall\, i \neq j,\ \hat{V}_{ii} = 0,\ \forall\, i,\ \hat{\vartheta} = 0: \qquad (159)$$

  their cutoff frequency b_s coincides with the real frequency b_e of the extensional modes and even here micro-oscillations are purely mechanical; the transverse acoustic wave in the microstructured medium couple again the micro-modes to form three optic branches.

When we neglect thermic phenomena, our oscillating solutions recover three of the mechanical micro-vibrations obtained for general microstructure in [67].

The characteristic values of the frequencies given in equations (157-159) are fixed, once the material parameters $\bar{\lambda}_3, \bar{\gamma}_3$, $\tilde{\gamma} := \mu_1\bar{\gamma}_1^{-1}\bar{\gamma}_3$, $\bar{\lambda}_4$, and θ_0 in system (151-153) are specified. Here, we want to compare micro-oscillations for some different materials for which they have been calculated experimentally: in particular, in Table 2 they refer to porous materials with nano-pores, modeled linear thermoelastic solids with ellipsoidal microstructure with vanishing Cosserat's couple modulus: high density rigid polyurethane closed-cell foam (PO) [58], nickel foams (NI) [5,70] and Berea sandstone saturated with air (SA) (or similar)[83,56].

Figures 5 and 6 report dimensionless micro-vibration (154) for the three porous solids in Table 2 in the longitudinal and transverse cases (equations (157) and (158)-(159)), respectively. Wave frequencies in foam materials are higher than seismic micro-vibrations in the sandstone, more in the polymeric matrix than

Table 2
Dimensional parameters for porous solids with ellipsoidal microstructure.

Parameter	PO	NI	SA
$\bar{\lambda}_3$	$2{,}16 \times 10^{12}$ s^{-2}	$3{,}70 \times 10^{11}$ s^{-2}	$5{,}00 \times 10^{10}$ s^{-2}
$-\bar{\gamma}_3$	$2{,}71 \times 10^{10}$ s^{-2} °C^{-1}	$4{,}61 \times 10^{9}$ s^{-2} °C^{-1}	$4{,}61 \times 10^{9}$ s^{-2} °C^{-1}
$\tilde{\gamma}$	$3{,}46 \times 10^{0}$ °C	$5{,}34 \times 10^{-1}$ °C	$6{,}67 \times 10^{-1}$ °C
$\bar{\lambda}_4$	$3{,}23 \times 10^{12}$ s^{-2}	$5{,}57 \times 10^{11}$ s^{-2}	$3{,}00 \times 10^{10}$ s^{-2}
θ_0	$2{,}20 \times 10^{1}$ °C	$2{,}00 \times 10^{1}$ °C	$2{,}50 \times 10^{1}$ °C
b_d	$3{,}64 \times 10^{6}$ s^{-1}	$1{,}49 \times 10^{6}$ s^{-1}	$4{,}68 \times 10^{5}$ s^{-1}
$b_e = b_s$	$2{,}54 \times 10^{6}$ s^{-1}	$1{,}06 \times 10^{6}$ s^{-1}	$2{,}45 \times 10^{5}$ s^{-1}

in the metallic one; as noted above, transverse waves proceed unaffected by thermal properties.

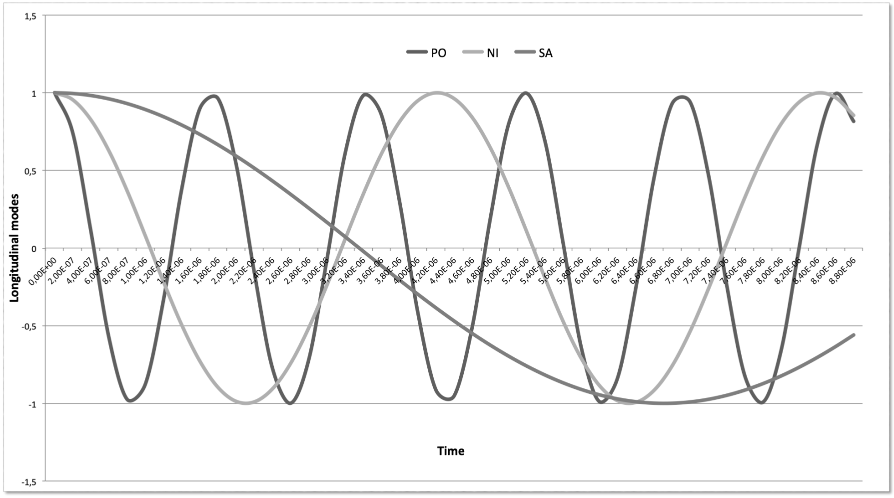

Fig. 5. Dilatational modes of coefficient b_d.

Now, we want to compare micro-vibrations for porous media described in theory of voids, the microstretched model and ours, the one with ellipsoidal microstructure, *i.e.*, the general affine microstructure internally constrained to have null micro-rotation. In particular, only for the third example we still observe transverse waves with frequencies that do not disappear, because, obviously, the parameter $\bar{\lambda}_4$ vanishes in the first two.

Therefore, in Table 3 we summarize all significant parameters of the mentioned models, at uniform room temperature θ_0: for the theory of voids, we consider the magnesium crystal like material (MA) [25,82]; for the thermo-microstretch-

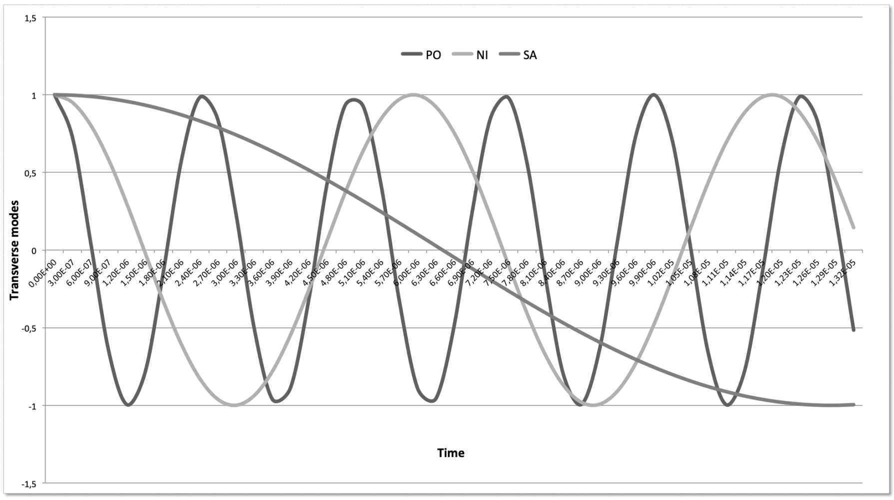

Fig. 6. Extensional and shear modes with the same coefficient $b_e = b_s$.

elastic solid (with vanishing Cosserat's couple modulus vanishing), aluminum matrix with randomly distributed epoxy spheres (AL) [32,57], a sheet molding compound (SMC) of randomly oriented chopped glass fibers in polyester matrix [4,57], and an electro-microelastic solid (EM), with zero dielectric coefficients [85]. The physical dimensions of the parameters of Table 3 are shown in Table 2, as well as the pertinent values for the (SA) material.

Table 3
Dimensional parameters for different models of porous solids.

Parameter	MA	AL	SMC	EM
$\bar{\lambda}_3$	5,37 $\times 10^{20}$	5,97 $\times 10^{6}$	9,88 $\times 10^{5}$	5,52 $\times 10^{9}$
$-\bar{\gamma}_3$	2,19 $\times 10^{17}$	8,62 $\times 10^{5}$	1,03 $\times 10^{7}$	9,19 $\times 10^{4}$
$\tilde{\gamma}$	2,76 $\times 10^{1}$	4,72 $\times 10^{-2}$	5,60 $\times 10^{-2}$	5,45 $\times 10^{-2}$
θ_0	2,50 $\times 10^{1}$	2,00 $\times 10^{1}$	2,00 $\times 10^{1}$	2,00 $\times 10^{1}$
b_d	4,02 $\times 10^{10}$	4,24 $\times 10^{3}$	1,88 $\times 10^{3}$	1,29 $\times 10^{5}$

Finally, the cutoff frequency b_d for the dilatational mode in the magnesium-crystal-like-material, viewed as a thermoelastic medium with voids, results to be very high with respect to one computed on the basis of microstretch and microstrain models, while the frequency obtained for electro-microstretch solids is of the same order of the previous microstrain and micromorphic samples. Instead, the detected microstretch solids with zero Cosserat's couple constant, *i.e.*, the aluminum-epoxy composite and the polyester matrix with glass fibers, give lower frequencies than microstrain ones. All the models presented in Table 3 do not show transverse micro-vibrations.

10 Concluding remarks

In this paper the balance principles for an immiscible mixture of continua with microstructure in the presence of chemical-physical phenomena are presented by generalizing previous thermodynamic theories of multiphase mixtures. A new formulation for the balances of moment of momentum is proposed and additional terms are also included in the peculiar energy equations, corresponding to the work done by the respective terms in the micro-momentum balances, introduced, for each constituent of mixture, to accommodate for the dynamical effects played by the respective microstructural descriptors. Moreover, we assume that the entropy flux of each constituent is not egual to the heat flux divided by the temperature of the constituent itself.

Furthermore, a new procedure is presented to incorporate internal constraints, such as the saturation condition or the incompressibility of the constituents, in the balance equations for immiscible mixtures of interest in the thermomechanics of the soils, and some applications are considered. In particular, a first example is studied in which a saturated fluid suspension is considered in detail by imposing the general principles, that govern the constitutive equations, including that of equipresence, usually substituted, in this field, by that of phase separation. Secondly, the fluid suspension is specified as incompressible to comprise concentrated granular materials in the fluids and the complete Lagrangian derivative of the kinetic energies due to micromotions, even if constrained. Finally, an isothermal flow of a fluid component through the big pores of a solid skeleton is examined, where the model for the solid constituent is thought to have an ellipsoidal microstructure with the fluid that fills all the interstices. The proposed model is proved to be perfectly consistent with previous known theories, derived from theories of the voids or from the classical ones, even if, also now, it satisfies all Truesdell's metaphysical principles, while those theories have failed in some respects. It had already been used to describe the transport of pollutants with rainwater in the soils.

It is remarkable that, for all the applications, it was possible to obtain a set of reaction-free thermodynamic equations; moreover, it has been found that the internal microactions are always coupled with the micromomentum growths, and therefore it remains very difficult to separate the relative influences on the micromotions.

Besides, to perform numerical simulations on simplified models, firstly it was quasi-linearized a continuum theory for a cohesionless viscous dilatant granular material with rotating grains, which generalizes the voids theory and the

Cosserat brothers' medium, and was tested on a vertical granular gravity flow to obtain numerical results showing that the effects due to the rotation of granules are relevant in a first stage of the granular dynamics, as well as those due to shear and bulk viscosities. Secondly, a linear theory of a thermoelastic solid with nano-pores was used to study the propagation of micro-waves, with three admissible results: a dilatational micro-thermal oscillation and two solutions, both with no thermal vibrations, with the same frequency and with null trace: a shear mode and an extensional mode with constant volume. Comparisons of wave frequencies were also made revealing that, in foam materials, they are higher than seismic micro-vibrations in the sandstone, more in the polymeric matrix than in the metallic one.

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