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L^∞ boundedness of the solutions to the Schrödinger equation on noncompact Riemannian manifolds

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Abstract

We are interested in the boundedness of the solutions to a Schrödinger type equation, with an integrable, sign changing potential. The sufficient condition for the boundedness relies on the integrability of a function involving both the isocapacitary function of the domain and the decreasing rearrangement of the negative part of the potential.

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1. Introduction and main results

Let Ω be a connected open set in an n-dimensional Riemannian manifold $\mathbb{M}$, which will be assumed to be without boundary throughout. Suppose that $n \geq 2$, and

$$\mathcal{H}^n(\Omega) < \infty. \tag{1.1}$$

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We are concerned with the boundedness of the solutions to a Schrödinger type equation in Ω, subject to homogeneous Neumann boundary conditions on $\partial\Omega$, if $\Omega \neq \mathbb{M}$. Namely, we deal with solutions to the equation

$$-\mathrm{div}(A(x)|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = 0 \quad \text{in } \Omega, \tag{1.2}$$

with the boundary condition

$$A(x)|\nabla u|^{p-2}\nabla u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

if $\partial\Omega \neq \emptyset$. Here, $p > 1$, and $\mathbf{n}$ stands for the normal unit vector on $\partial\Omega$. $A : \Omega \to \mathbb{R}^{n\times n}$ is a matrix-valued function, with essentially bounded coefficients, satisfying

$$A(x)\xi \cdot \xi \geq |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n. \tag{1.4}$$

There is a rich literature on the existence of solutions to Neumann problems for elliptic PDE's ([1,9,11,13,14,22–25,27]), in different kind of domains, even irregulars. Many boundedness results for such problems, in regular domains, are also available ([6,12,17,19–21,28]). Regarding the boundedness of the solutions for Neumann problems in irregular domains, the situation changes drastically, because at the moment there are few papers on this topic. Some results in this direction can be found in [3,5,7,8,10,18].

We are interested to the boundedness of the solutions to (1.2) in (possibly) irregular sets Ω. To guarantee the boundedness of the solutions, we take into account a sort of balance between the irregularity of Ω (via the isocapacitary function) and the decreasing rearrangement of V_-, namely V_-^*. This is expressed via two integrals, one for $p \geq 2$, and the other for $1 < p < 2$ (see equations in (3.2)).

The research was motivated by the results in [3,7]. We prove a result that incorporates both Theorem 2.1 of [7] and Theorem 1.3, part (ii) of [3]. The methods used in [7] don't work in our context, due to the nonlinearity of (1.2). We follow the spirit of [3], even if the techniques of [3] are not directly applicable to study (1.2): this now is due to the presence of a sign changing potential. Nevertheless they are used to show the boundedness of the solutions to an auxiliary problem, that is one of the key steps in our proof.

The paper is organized as follows. In Section 2 we introduce everything necessary for the sequel. Section 3 is devoted to the statement and the proof of the main result. Finally, in Section 4 we present some examples involving a class of functions V, with a prescribed behaviour for V_-^* near 0. We couple these functions V with specific domains (Lipschitz and Hölder domains, cusp-shaped domains, γ-John domains, manifold of revolution). We also extend some examples of [7], for $p = 2$, to our case.

2. Background and preliminaries

Let Ω be an open set in an n-dimensional Riemannian manifold $\mathbb{M}$. The choice $\Omega = \mathbb{M}$ is allowed too.

We denote by ∇ the gradient operator, namely covariant differentiation on $\mathbb{M}$.

The Sobolev space $W^{1,p}(\Omega)$ is defined, for $p \in [1, \infty]$, as

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \ u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega)\},$$

and is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The space $W_0^{1,p}(\Omega)$ is the closure in $W^{1,p}(\Omega)$ of the set of continuously differentiable compactly supported functions in Ω.

The homogeneous Sobolev space $V^{1,p}(\Omega)$ is defined as

$$V^{1,p}(\Omega) = \left\{u : u \text{ is weakly differentiable in } \Omega \text{ and } |\nabla u| \in L^p(\Omega)\right\}.$$

If the set Ω is connected, then $V^{1,p}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{V^{1,p}(\Omega)} = \|u\|_{L^p(\omega)} + \|\nabla u\|_{L^p(\Omega)}. \tag{2.1}$$

For any open set ω such that $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$. Different choices of ω result in an equivalent norm for $V^{1,p}(\Omega)$.

Let $E \subset \mathbb{M}$. The standard p-capacity of E is defined, for $p \geq 1$, as

$$C_p(E) = \inf\left\{\int\limits_{\mathbb{M}} |\nabla u|^p \, d\mathcal{H}^n : u \in W_0^{1,p}(\mathbb{M}), u \geq 1 \text{ in some neighbourhood of } E\right\}. \tag{2.2}$$

Each function $u \in V^{1,p}(\Omega)$ has a representative $\widetilde{u}$, such that for every $\varepsilon > 0$, there exists a set $E \subset \Omega$, with $C_p(E) < \varepsilon$, such that $\widetilde{u}$ is continuous in $\Omega \setminus E$. The function $\widetilde{u}$ is called the precise representative, and it is unique, up to subsets of p-capacity zero. A property is said to hold p-quasi everywhere in Ω, p-q.e. for short, if it is fulfilled outside a set of p-capacity zero.

If $E \subset G \subset \Omega$. Then the p-capacity of E relative to G is

$$\operatorname{cap}_p(E, G) = \inf\left\{\int\limits_{\Omega} |\nabla\widetilde{u}|^p \, d\mathcal{H}^n : \widetilde{u} \in V^{1,p}(\Omega), \widetilde{u} \geq 1 \text{ in } E, \widetilde{u} = 0 \text{ in } \Omega \setminus G \;\; p\text{-q.e.}\right\}. \tag{2.3}$$

We can now define the capacity of a condenser

$$\operatorname{cap}_p(E) = \inf\left\{\operatorname{cap}_p(E, G) : E \subset G, \mathcal{H}^n(G) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\} \tag{2.4}$$

for every measurable set $E \subset \Omega$ such that $\mathcal{H}^n(E) \leq \frac{\mathcal{H}^n(\Omega)}{2}$.

We introduce the isocapacitary function of Ω, $\nu_{\Omega,p} : \left[0, \frac{\mathcal{H}^n(\Omega)}{2}\right] \to [0, \infty)$, defined as

$$\nu_{\Omega,p}(s) = \inf\left\{\operatorname{cap}_p(E) : E \subset \Omega, s \leq \mathcal{H}^n(E) \leq \tfrac{\mathcal{H}^n(\Omega)}{2}\right\} \qquad \text{for } s \in \left[0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right]. \tag{2.5}$$

Given a measurable function $u : \Omega \to \mathbb{R}$, we denote by $\mu_u : \mathbb{R} \to [0, \infty)$ its distribution function defined as

$$\mu_u(t) = \mathcal{H}^n(\{x \in \Omega : u(x) \geq t\}) \qquad \text{for } t \in \mathbb{R}.$$

The decreasing rearrangement $u^* : (0, |\Omega|) \to [0, \infty)$ of u is defined as

$$u^*(s) = \sup\{t : \mu_{|u|}(t) \geq s\} \qquad \text{for } s \in (0, \mathcal{H}^n(\Omega)),$$

and the signed decreasing rearrangement $u^\circ : (0, |\Omega|) \to \mathbb{R}$ as

$$u^\circ(s) = \sup\{t : \mu_u(t) \geq s\} \qquad \text{for } s \in (0, \mathcal{H}^n(\Omega)).$$

We will use also the Hardy-Littlewood inequality

$$\int_\Omega |u(x)v(x)| \, d\mathcal{H}^n \leq \int_0^{|\Omega|} u^*(s)v^*(s) \, ds \tag{2.6}$$

for any measurable functions u , $v : \Omega \to \mathbb{R}$.

Given $u \in V^{1,p}(\Omega)$, with median $\operatorname{med}(u) = u^\circ(\mathcal{H}^n(\Omega)/2) = 0$, we define the functions ψ_u, $\psi_{u,A} : \mathbb{R} \to \mathbb{R}$ as

$$\psi_u(t) = \int_0^t \left(\int_{\{u=\tau\}} |\nabla u|^{p-1} \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{1-p}} d\tau \qquad \text{for } t \in \mathbb{R}\,, \tag{2.7}$$

and

$$\psi_{u,A}(t) = \int_0^t \left(\int_{\{u=\tau\}} A(x)|\nabla u|^{p-3}\nabla u \cdot \nabla u \, d\mathcal{H}^{n-1}(x) \right)^{\frac{1}{1-p}} d\tau \qquad \text{for } t \in \mathbb{R}\,, \tag{2.8}$$

respectively. In (2.8), and in what follows, $A(x)$ denotes a Borel representative of $A(x)$. Such a representative exists by a standard result in measure theory (see e.g. [2, Exercise 1.3]). By condition (1.4), one has that

$$\psi_{u,A}(t) \leq \psi_u(t) \qquad \text{for } t \geq 0. \tag{2.9}$$

The assumption $\operatorname{med}(u) = 0$ and a variant of [26, Lemma 2.2.2/1] guarantee that

$$\nu_{\Omega,p}(\mu_u(t)) \leq \frac{1}{(\psi_u(t))^{p-1}} \quad \text{for } t > 0, \tag{2.10}$$

and hence, owing to (2.9),

$$\nu_{\Omega,p}(s) \leq \frac{1}{\psi_u(u^\circ(s))^{p-1}} \leq \frac{1}{\psi_{u,A}(u^\circ(s))^{p-1}} \qquad \text{for } s \in (0, \mathcal{H}^n(\Omega)/2). \tag{2.11}$$

Note that $u^\circ(s) > 0$ if $s \in (0, \mathcal{H}^n(\Omega)/2)$, since $\operatorname{med}(u) = 0$.

3. Main result

Given a function $V \in L^1(\Omega)$, and denote by $W_V^{1,p}(\Omega)$ the weighted Sobolev space

$$W_V^{1,p}(\Omega) = \{u : u \text{ is weakly differentiable and } \int\limits_\Omega |\left(\nabla u|^p + |V(x)||u|^p\right) d\mathcal{H}^n < \infty\}$$

A function $u \in W_V^{1,p}(\Omega)$ is said to be a weak solution to (1.2) if

$$\int\limits_\Omega A(x)|\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, d\mathcal{H}^n + \int\limits_\Omega V(x)|u|^{p-2}u\phi \, d\mathcal{H}^n = 0 \tag{3.1}$$

for every test function $\phi \in W_V^{1,p}(\Omega)$.

Let V_+ and V_- be respectively the positive and the negative part of V, so that $V = V_+ - V_-$. If $V \geq 0$ a.e. in Ω, then $u \equiv 0$ is the unique solution to (3.1): this can be seen testing equation (3.1) with $\varphi \equiv u$ and using (1.4). Thus the problem is significant only for those functions V for which $V_- \not\equiv 0$, according to the fact that the classical eigenvalue problem has only positive eigenvalues (namely $V \equiv -\gamma$, being γ an eigenvalue). The negative part of the function V comes into play in the main result, via its decreasing rearrangement V_-^*. In fact, the integral condition that guarantees the boundedness of the solutions, involves V_-^* and the isocapacitary function $\nu_{\Omega,p}$ of Ω.

Theorem 3.1. *Let $A : \Omega \to \mathbb{R}^{n\times n}$ be a matrix-valued function, with essentially bounded coefficients, satisfying (1.4), and let $V \in L^1(\Omega)$, with $V_- \not\equiv 0$. Assume that*

$$\int\limits_0 \left(\frac{sV_-^*(s)}{\nu_{\Omega,p}(s)}\right)^{\frac{1}{p-1}} \frac{1}{s}\, ds < \infty \qquad \text{if } p \geq 2, \tag{3.2}$$

$$\int\limits_0 \left(\frac{\int_0^s V_-^*(\sigma)d\sigma}{\nu_{\Omega,p}(s)}\right)^{\frac{1}{p-1}} \frac{V_-^*(s)}{\int_0^s V_-^*(\sigma)d\sigma}\, ds < \infty \qquad \text{if } p < 2\,.$$

Then any weak solution u to (1.2) is essentially bounded, and there exists a constant $C = C(V_-^*, \nu_{M,p})$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^p(\Omega)}\,. \tag{3.3}$$

In particular, inequality (3.3) holds with

$$C = \frac{\varepsilon^{-\frac{1}{p}}}{1-\delta}\,,$$

where δ and ε are such that $\frac{1}{p-1}\int_0^\varepsilon \frac{V_-^*(\varrho)\left(\int_0^\varrho V_-^*(\sigma)\,d\sigma\right)^{\frac{1}{p-1}-1}}{\nu_{\Omega,p}(\varrho)^{\frac{1}{p-1}}} d\varrho < \delta$.

Remark 3.2. Since $V_- \not\equiv 0$, conditions in (3.2) implies that $\lim_{s\to 0^+} \frac{s}{\nu_{\Omega,p}(s)} = 0$. Thus, the embedding $W^{1,p}_V(\Omega) \to L^p(\Omega)$ holds, as a consequence of [4, Theorem 2.1]. Therefore, the norm on the right-hand side of (3.3) is finite. Following the proof of Theorem 2.1 of [4], we can also deduce the compactness of the embedding.

When V_- is essentially bounded, then the conditions in (3.2) are equivalent to $\int_0 \left(\frac{s}{\nu_{\Omega,p}(s)}\right)^{\frac{1}{p-1}} \frac{1}{s}\, ds < \infty$, that is the condition used in [3].

Proof of Theorem 3.1. Note that if u is a solution to (1.2), then $-u$ is a solution too. So, it is not restrictive to assume $u^+ \not\equiv 0$. Given $s \in (0, \mathcal{H}^n(\Omega))$ and $h > 0$, choose the test functions ϕ defined as

$$\phi(x) = \begin{cases} 0 & \text{if } u(x) < u^\circ(s+h) \\ u(x) - u^\circ(s+h) & \text{if } u^\circ(s+h) \le u(x) \le u^\circ(s) \\ u^\circ(s) - u^\circ(s+h) & \text{if } u^\circ(s) < u(x)\,, \end{cases} \tag{3.4}$$

in equation (3.1).

Then, arguing as in [3], namely, dividing by $h > 0$ in (3.1) and passing to the limit as $h \to 0^+$, yields

$$\begin{aligned} &- u^{\circ\prime}(s) \int\limits_{\{u=u^\circ(s)\}} A(x)|\nabla u|^{p-3}\nabla u \cdot \nabla u d\mathcal{H}^{n-1}(x) \\ &\quad = -u^{\circ\prime}(s)\Bigg(-\int\limits_{\{u>u^\circ(s)\}} |u(x)|^{p-2}u(x)V(x)d\mathcal{H}^n(x)\Bigg) \quad \text{for a.e. } s \in (0, \mathcal{H}^n(\Omega)). \end{aligned} \tag{3.5}$$

Let $\Phi(s) = -\frac{d}{ds}\left(\int_{\{u>u^\circ(s)\}} V(x)d\mathcal{H}^n(x)\right)$. This function appears in Lemma 3.1 of [8], but with functions u and V are defined in subsets of $\mathbb{R}^n$, with the standard Lebesgue measure. A straightforward computation shows that all the conclusions of that Lemma hold also for manifolds. From (2.8), and equation (3.13) in [8], used with $g(t) = -|t|^{p-2}t$, equation (3.5) takes the form

$$-u^{\circ\prime}(s) \int\limits_{\{u=u^\circ(s)\}} A(x)|\nabla u|^{p-3}\nabla u \cdot \nabla u d\mathcal{H}^{n-1}(x) = -u^{\circ\prime}(s) \int\limits_0^s |u^\circ(\varrho)|^{p-2}u^\circ(\varrho)\Phi(\varrho)\, d\varrho \tag{3.6}$$

for a.e. $s \in (0, \mathcal{H}^n(\Omega))$. From (1.4) and (3.6) one has

$$\int\limits_0^s |u^\circ(\varrho)|^{p-2}u^\circ(\varrho)\Phi(\varrho)\, d\varrho > 0 \;\; \text{for a.e. } s \in (0, \mathcal{H}^n(\Omega)) \text{ such that } u^{\circ\prime}(s) \neq 0. \tag{3.7}$$

Due to the structure of u°, we know that $u^{\circ\prime}(s) = 0$ at most in countable numbers of (possibly) degenerate intervals in $(0, \mathcal{H}^n(\Omega))$. If $u^{\circ\prime}(s) = 0$ in $(0, \sigma)$ then u° is bounded from above and we pass to the proof of the boundedness of $-u$. Otherwise, we choose $\overline{s} > 0$ such that $u^\circ(\varrho) > 0$ for $\varrho \in (0, \overline{s})$, and $u^{\circ\prime}(\varrho) < 0$ for a.e. $\varrho \in (0, \overline{s})$. Note that, if $v = u - medu$, then

$$\frac{-u^{\circ\prime}(s)}{\left(\int_{\{u=u^\circ(s)\}} A(x)|\nabla u|^{p-3}\nabla u \cdot \nabla u d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p-1}}} = \big(-\psi_{v,A}(v^\circ(s))\big)'.$$

To simplify the notations, we set

$$\overline{\omega}(r) = \big(-\psi_{v,A}(v^\circ(r))\big)'. \tag{3.8}$$

Let $s \in (0, \overline{s})$ be such that $u^{\circ\prime}(s) < 0$. We divide (3.6) with $-u^{\circ\prime}(s)\int_{\{u=u^\circ(s)\}} A(x)|\nabla u|^{p-3}\nabla u \cdot \nabla u d\mathcal{H}^{n-1}(x)$, and rise to the power $\frac{1}{p-1}$. Then, we multiply with $-u^{\circ\prime}(s)$

$$-u^{\circ\prime}(s) = \overline{\omega}(s)\left(\int_0^s |u^\circ(\varrho)|^{p-2}u^\circ(\varrho)\Phi(\varrho)\,d\varrho\right)^{\frac{1}{p-1}} \qquad \text{for } s \in (0, \overline{s}). \tag{3.9}$$

Note that equality (3.9) holds in the whole $(0, \overline{s})$: when $u^{\circ\prime}(s) < 0$ it follows from the arguments above, while it is trivial when $u^{\circ\prime}(s) = 0$. We integrate (3.9) in $(s, \overline{s})$. Then

$$u^\circ(s) = u^\circ(\overline{s}) + \int_s^{\overline{s}} \overline{\omega}(r)\Big(\int_0^r u^\circ(\varrho)^{p-1}\Phi(\varrho)d\varrho\Big)^{\frac{1}{p-1}} dr \qquad \text{for } s \in (0, \overline{s}). \tag{3.10}$$

We prove that for every $\delta > 0$ there exists $\overline{\varepsilon} > 0$ such that

$$\int_0^s \overline{\omega}(r)\left(\int_0^r \Phi^*(\varrho)\,d\varrho\right)^{\frac{1}{p-1}} dr < \delta \qquad \text{for all } s \in (0, \overline{\varepsilon}). \tag{3.11}$$

Let $g(r) = \left(\int_0^r V_-^*(\varrho)\,d\varrho\right)^{\frac{1}{p-1}}$. Then $g'(r) = \frac{1}{p-1}V_-^*(r)\left(\int_0^r V_-^*(\varrho)\,d\varrho\right)^{\frac{1}{p-1}-1}$. From equation (3.12) in [8] one has $\int_0^r \Phi^*(\varrho)\,d\varrho \le \int_0^r V_-^*(\varrho)\,d\varrho$ for all $r \in (0, \mathcal{H}^n(\Omega))$. Using Fubini's Theorem and (2.11)

$$\begin{aligned}\int_0^s \overline{\omega}(r)\left(\int_0^r \Phi^*(\varrho)\,d\varrho\right)^{\frac{1}{p-1}} dr &\le \int_0^s \overline{\omega}(r)g(r)\,dr = \int_0^s \overline{\omega}(r)\left(\int_0^r g'(\varrho)d\varrho\right) dr \\ &= \int_0^s g'(\varrho)\left(\int_\varrho^r \overline{\omega}(r)dr\right) d\varrho \le \frac{1}{p-1}\int_0^s \frac{V_-^*(\varrho)\left(\int_0^\varrho V_-^*(\sigma)\,d\sigma\right)^{\frac{1}{p-1}-1}}{\nu_{\Omega,p}(\varrho)^{\frac{1}{p-1}}} d\varrho\,.\end{aligned} \tag{3.12}$$

Thus, when $1 < p < 2$, (3.11) follows from (3.2). If $p \ge 2$, the monotonicity of V_-^* guarantees that

$$\left(\int_0^{\varrho} V_-^*(\sigma)\,d\sigma\right)^{\frac{2-p}{p-1}} \leq (\varrho V_-^*(\varrho))^{\frac{2-p}{p-1}} \tag{3.13}$$

and (3.11) follows from (3.12), (3.13) and (3.2).

We can so fix $\delta, \varepsilon > 0$, $\varepsilon = min\{\overline{\varepsilon}, \overline{s}\}$, so that (3.10) and (3.11), hold for $s \in (0, \varepsilon)$. Next, we observe that the function $v(x) = \frac{u(x)}{u^\circ(\varepsilon)}$ is also a solution to (1.2). Thus, we can assume $u^\circ(\varepsilon) = 1$. Then (recall that $u^\circ(\varrho) > 0$ in $(0, \varepsilon)$) (3.10) reads as

$$v^\circ(s) = 1 + \int_s^{\varepsilon} \overline{\omega}(r)\left(\int_0^r v^\circ(\varrho)^{p-1}\Phi(\varrho)d\varrho\right)^{\frac{1}{p-1}} dr \qquad \text{for } s \in (0, \varepsilon). \tag{3.14}$$

We consider now an auxiliary problem. Let $K_+ := \{u \in L^\infty(0, \varepsilon),\ u \geq 0\ a.e. \text{ in } (0, \varepsilon)\}$. Define the operator $T_u : K_+ \to K_+$

$$T_u f(s) = 1 + \int_s^{\varepsilon} \overline{\omega}(r)\left(\int_0^r f(\varrho)^{p-1}|\Phi(\varrho)|d\varrho\right)^{\frac{1}{p-1}} dr \qquad \text{for } s \in (0, \varepsilon)\,. \tag{3.15}$$

Let now construct, by induction, a sequence $\{f_k\}_{k\in N}$, $f_k \in L^\infty(0, \varepsilon)$ for all $k \in N$.

$$\begin{cases} f_0(s) \equiv 1 \\ f_k(s) = T_u f_{k-1}(s) \text{ for } s \in (0, \varepsilon)\,. \end{cases} \tag{3.16}$$

We prove that

$$\|f_k\|_{L^\infty(0,\varepsilon)} \leq \sum_{h=0}^{k} \delta^h \qquad \text{for } k \in N, \quad \text{being } \delta \text{ fixed above.} \tag{3.17}$$

Using (2.6) and (3.11)

$$f_1(s) = 1 + \int_s^{\varepsilon} \overline{\omega}(r)\left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p-1}} dr < 1 + \delta \text{ for } s \in (0, \varepsilon)\,. \tag{3.18}$$

Thus $\|f_1\|_{\|L^\infty(0,\varepsilon)} \leq 1 + \delta$. Assume now that (3.17) holds for $k - 1$. Then, taking into account (2.6) and (3.11)

$$\begin{aligned} f_k(s) &= 1 + \int_s^{\varepsilon} \overline{\omega}(r)\left(\int_0^r f_{k-1}^{p-1}|\Phi(\varrho)|d\varrho\right)^{\frac{1}{p-1}} dr \\ &< 1 + \delta\sum_{h=0}^{k-1} \delta^h = \sum_{h=0}^{k} \delta^h \text{ for } s \in (0, \varepsilon)\,, \end{aligned} \tag{3.19}$$

whence (3.17) follows.

When $p \geq 2$ one has that

$$\|f_k - f_{k-1}\|_{L^\infty(0,\varepsilon)} \leq \delta^k \quad \text{for } k \in \mathbb{N}. \tag{3.20}$$

We prove inequality (3.20) by induction again. Inequalities (2.6) and (3.11) guarantee that

$$\|f_1 - f_0\|_{L^\infty(0,\varepsilon)} = \int_0^\varepsilon \overline{\omega}(r)\Big(\int_0^r |\Phi(\varrho)|d\varrho\Big)^{\frac{1}{p-1}} dr \leq \delta\,. \tag{3.21}$$

Inequality (3.20) is thus verified for $k = 1$. Suppose that it holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned}\|f_{k+1} - f_k\|_{L^\infty(0,\varepsilon)} &\leq \int_0^\varepsilon \overline{\omega}(r)\left|\|f_k|\Phi|^{\frac{1}{p-1}}\|_{L^{p-1}(0,r)} - \|f_{k-1}|\Phi|^{\frac{1}{p-1}}\|_{L^{p-1}(0,r)}\right| dr \\ &\leq \int_0^\varepsilon \|(f_k - f_{k-1})|\Phi|^{\frac{1}{p-1}}\|_{L^{p-1}(0,r)}\overline{\omega}(r)\, dr \\ &\leq \|f_k - f_{k-1}\|_{L^\infty(0,r)} \int_0^\varepsilon \overline{\omega}(r)\Big(\int_0^r |\Phi(\varrho)|d\varrho\Big)^{\frac{1}{p-1}} dr \leq \delta^{k+1},\end{aligned} \tag{3.22}$$

namely inequality (3.20).

Assume now that $1 < p < 2$. We shall make use of the inequalities

$$\left|r^{\frac{1}{p-1}} - s^{\frac{1}{p-1}}\right| \leq c_1|r - s|\big(r^{\frac{2-p}{p-1}} + s^{\frac{2-p}{p-1}}\big) \quad \text{for } r, s > 0, \tag{3.23}$$

$$\left|r^{p-1} - s^{p-1}\right| \leq c_2\, \frac{|r - s|}{r^{2-p} + s^{2-p}} \quad \text{for } r, s > 0 \tag{3.24}$$

for some constants $c_1 = c_1(p)$ and $c_2 = c_2(p)$.

In this case, one has that

$$\|f_k - f_{k-1}\|_{L^\infty(0,\varepsilon)} \leq \frac{(c_1c_2)^{k-1}\delta^k}{(1-\delta)^{(2-p)(k-1)}} \tag{3.25}$$

for $k \in \mathbb{N}$. Inequality (3.25) holds with $k = 1$ thanks to (3.21), which is still valid even if $1 < p < 2$. Arguing by induction again, assume now that inequality (3.25) holds for some $k \in \mathbb{N}$. Then

$$\begin{aligned}&\|f_{k+1} - f_k\|_{L^\infty(0,\varepsilon)} \\ &\leq \int_0^\varepsilon \overline{\omega}(r)\left|\left(\int_0^r f_k(\varrho)^{p-1}|\Phi(\varrho)|d\varrho\right)^{\frac{1}{p-1}} - \left(\int_0^r f_{k-1}(\varrho)^{p-1}|\Phi(\varrho)|d\varrho\right)^{\frac{1}{p-1}}\right| dr\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&\le c_1 \int_0^{\varepsilon} \overline{\omega}(r) \left| \int_0^{r} \left(f_k^{p-1}(\varrho) - f_{k-1}^{p-1}(\varrho) \right) |\Phi(\varrho)| d\varrho \right| \left(\left(\int_0^{r} f_k(\varrho)^{p-1} |\Phi|(\varrho) d\varrho \right)^{\frac{2-p}{p-1}} \right. \\
&\quad \left. + \left(\int_0^{r} f_{k-1}(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \right)^{\frac{2-p}{p-1}} \right) dr \\
&\le c_1 \int_0^{\varepsilon} \overline{\omega}(r) \left| \int_0^{r} \left(f_k^{p-1}(\varrho) - f_{k-1}^{p-1}(\varrho) \right) |\Phi(\varrho)| d\varrho \right| \\
&\quad \times \left(\| f_k \|^{2-p}_{L^\infty(0,\varepsilon)} + \| f_{k-1} \|^{2-p}_{L^\infty(0,\varepsilon)} \right) \left(\int_0^{r} |\Phi(\varrho)| d\varrho \right)^{\frac{2-p}{p-1}} dr \\
&\le 2c_1 \left(\frac{1}{1-\delta} \right)^{2-p} \int_0^{\varepsilon} \overline{\omega}(r) \left(\int_0^{r} \frac{c_2 | f_k(\varrho) - f_{k-1}(\varrho)|}{f_k(\varrho)^{2-p} + f_{k-1}(\varrho)^{2-p}} |\Phi(\varrho)| d\varrho \right) \left(\int_0^{r} |\Phi|(\varrho) d\varrho \right)^{\frac{2-p}{p-1}} dr \\
&\le 2c_1 c_2 \left(\frac{1}{1-\delta} \right)^{2-p} \frac{\| f_k - f_{k-1} \|_{L^\infty(0,\varepsilon)}}{2} \int_0^{\varepsilon} \overline{\omega}(r) \left(\int_0^{r} |\Phi(\varrho)| d\varrho \right)^{\frac{2-p}{p-1}+1} dr \\
&\le \frac{c_1 c_2}{(1-\delta)^{(2-p)}} \frac{(c_1 c_2)^{k-1} \delta^k}{(1-\delta)^{(2-p)(k-1)}} \int_0^{\varepsilon} \overline{\omega}(r) \left(\int_0^{r} |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p-1}} dr \le \frac{(c_1 c_2)^k \delta^{k+1}}{(1-\delta)^{(2-p)k}} \,,
\end{aligned}$$

where the second inequality holds by (3.23), the third by (3.17), the fourth by (3.24), the fifth by the fact that $f_k(\varrho) \ge 1$, the sixth by (3.25), and the last one by (3.11).

Inequality (3.20) when $p \ge 2$ and inequality (3.25) when $1 < p < 2$ ensure that, if ε is sufficiently small, then the sequence $\{f_k\}$ converges in $L^\infty(0,\varepsilon)$ to some function f, which solves the equation (3.27).

$$T_u f(s) = f(s) \qquad \text{for } s \in (0,\varepsilon). \tag{3.27}$$

Thus

$$f(s) = 1 + \int_s^{\varepsilon} \overline{\omega}(r) \Big(\int_0^{r} f(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \Big)^{\frac{1}{p-1}} dr \qquad \text{for } s \in (0,\varepsilon)\,. \tag{3.28}$$

We now assume by contradiction that $\|v^0\|_{L^\infty(0,\varepsilon)} = +\infty$. Then there exists $s_0 \in (0,\varepsilon]$ such that $v^0(s) \ge \|f\|_{L^\infty(0,\varepsilon)} \ge f(s)$ for all $s \in (0,s_0]$. Due to the continuity of v^0 and f, it is not restrictive to choose $s_0 = \sup\{s \in (0,\varepsilon]$ such that $v^0(s) \ge \|f\|_{L^\infty(0,\varepsilon)} \ge f(s)\}$. Thus $v^0(s_0) = f(s_0)$, and from (3.14) and (3.28)

$$0 = v^0(s_0) - f(s_0) = \int_{s_0}^{\varepsilon} \overline{\omega}(r)\Big[\Big(\int_0^r v^0(\varrho)^{p-1}\Phi(\varrho)d\varrho\Big)^{\frac{1}{p-1}} - \Big(\int_0^r f(\varrho)^{p-1}|\Phi(\varrho)|d\varrho\Big)^{\frac{1}{p-1}}\Big]dr \tag{3.29}$$

and, as consequence

$$v^0(s) - f(s) = \int_s^{s_0} \overline{\omega}(r)\Big[\Big(\int_0^r v^0(\varrho)^{p-1}\Phi(\varrho)d\varrho\Big)^{\frac{1}{p-1}} - \Big(\int_0^r f(\varrho)^{p-1}|\Phi(\varrho)|d\varrho\Big)^{\frac{1}{p-1}}\Big]dr \quad \text{for all } s \in (0, s_0]. \tag{3.30}$$

Our next, and final, step is to evaluate $\|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)}$, being $L^p_{|\Phi|}(0, s_0)$ the Lebesgue space of all measurable functions $u : (0, s_0) \to \mathbb{R}$ such that $\int_0^{s_0} |u(t)|^p|\Phi(t)|dt < +\infty$. Note that, due to equation (3.12) in [8] and bennet, we know that $u^0 \in L^p_{|\Phi|}(0, s_0)$. For $s \in (0, s_0)$ and $p \geq 2$, and using Holder's inequality

$$\begin{aligned}0 \leq |v^0(s) - f(s)| &\leq \int_s^{s_0} \overline{\omega}(r)\left(\|v_0|\Phi|^{\frac{1}{p-1}}\|_{L^{p-1}(0,r)} - \|f|\Phi|^{\frac{1}{p-1}}\|_{L^{p-1}(0,r)}\right) dr \\ &\leq \int_s^{s_0} \overline{\omega}(r)\|(v^0 - f)|\Phi|^{\frac{1}{p-1}}\|_{L^{p-1}(0,r)}\, dr \leq \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \int_s^{s_0} \overline{\omega}(r)\left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p(p-1)}} dr\,.\end{aligned}$$

Now, invoking Minkowski's integral inequality and (3.11)

$$\begin{aligned}0 &\leq \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \\ &\leq \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \left(\int_0^{s_0}\left(\int_s^{s_0} \overline{\omega}(r)\left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p(p-1)}} dr\right)^p |\Phi(s)|ds\right)^{\frac{1}{p}} \\ &\leq \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \int_0^{s_0} \overline{\omega}(r)\left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p(p-1)}} \left(\int_0^r |\Phi(s)|ds\right)^{\frac{1}{p}} dr \\ &= \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \int_0^{s_0} \overline{\omega}(r)\left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p-1}} dr < \delta\|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)}\,.\end{aligned}$$

Being $\delta < 1$, we deduce that $\|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} = 0$ and this contradicts $\|v^0\|_{L^\infty(0,s_0)} = +\infty$, because $f \in L^\infty(0, s_0)$.

When $1 < p < 2$ the proof is a little bit more delicate. We need some preliminary considerations. We begin with an estimate, via the Minkowski's integral inequality, of $\|v^0\|_{L^p_{|\Phi|}(0,r)}$ and of $\|f\|_{L^p_{|\Phi|}(0,r)}$, for $r \in (0, s_0)$.

$$
\begin{aligned}
\|v^0\|_{L^p_{|\Phi|}(0,r)} &\le \left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p}} \\
&+ \left(\int_0^r \left(\int_s^\varepsilon \overline{\omega}(\varrho)\|v^0\|_{L^p_{|\Phi|}(0,\varrho)} \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p(p-1)}} d\varrho\right)^p |\Phi(s)|ds\right)^{\frac{1}{p}} \\
&\le \left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p}} + \int_0^r \overline{\omega}(\varrho)\|v^0\|_{L^p_{|\Phi|}(0,\varrho)} \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p(p-1)}} \left(\int_0^\varrho |\Phi(s)|\,ds\right)^{\frac{1}{p}} d\varrho \\
&+ \int_r^\varepsilon \overline{\omega}(\varrho)\|v^0\|_{L^p_{|\Phi|}(0,\varrho)} \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p(p-1)}} \left(\int_0^r |\Phi(s)|\,ds\right)^{\frac{1}{p}} d\varrho \\
&\le \left(\int_0^r |\Phi(\varrho)|d\varrho\right)^{\frac{1}{p}} + \|v^0\|_{L^p_{|\Phi|}(0,r)} \int_0^r \overline{\omega}(\varrho) \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p-1}} d\varrho \\
&+ \left(\int_0^r |\Phi(s)|\,ds\right)^{\frac{1}{p}} \int_r^\varepsilon \overline{\omega}(\varrho)\|v^0\|_{L^p_{|\Phi|}(0,\varrho)} \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p(p-1)}} d\varrho
\end{aligned}
\tag{3.31}
$$

Moreover, taking into account the monotonicity of v^0

$$
\|v^0\|_{L^p_{|\Phi|}(0,\varrho)} \le \|v^0\|_{L^p_{|\Phi|}(0,r)} + \|v^0\|_{L^p_{|\Phi|}(r,\varrho)} \le \|v^0\|_{L^p_{|\Phi|}(0,r)} + v^0(r) \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p}}
\tag{3.32}
$$

for $\varrho \in (r, \varepsilon)$. Thus,

$$
\begin{aligned}
&\left(\int_0^r |\Phi(s)|\,ds\right)^{\frac{1}{p}} \int_r^\varepsilon \overline{\omega}(\varrho)\|v^0\|_{L^p_{|\Phi|}(0,\varrho)} \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p(p-1)}} d\varrho \\
&\qquad \le \|v^0\|_{L^p_{|\Phi|}(0,r)} \int_r^\varepsilon \overline{\omega}(\varrho) \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p(p-1)}+\frac{1}{p}} d\varrho \\
&\qquad + v^0(r) \left(\int_0^r |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p}} \int_r^\varepsilon \overline{\omega}(\varrho) \left(\int_0^\varrho |\Phi(\sigma)|\,d\sigma\right)^{\frac{1}{p(p-1)}+\frac{1}{p}} d\varrho
\end{aligned}
\tag{3.33}
$$

$$\le \|v^0\|_{L^p_{|\Phi|}(0,r)} \int_r^{\varepsilon} \overline{\omega}(\varrho) \left(\int_0^{\varrho} |\Phi(\sigma)|\, d\sigma \right)^{\frac{1}{p-1}} d\varrho + \delta v^0(r) \left(\int_0^{r} |\Phi(\sigma)|\, d\sigma \right)^{\frac{1}{p}} .$$

Coupling inequality (3.31) with (3.33) yields

$$\begin{aligned} \|v^0\|_{L^p_{|\Phi|}(0,r)} &\le \left(\int_0^r |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p}} \left(1 + \delta v^0(r)\right) + \|v^0\|_{L^p_{|\Phi|}(0,r)} \left(\int_0^{\varrho} |\Phi(\sigma)|\, d\sigma \right)^{\frac{1}{p}} \int_0^{\varepsilon} \overline{\omega}(\varrho)\, d\varrho \\ &\le \left(\int_0^r |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p}} \left(1 + \delta v^0(r)\right) + \delta \|v^0\|_{L^p_{|\Phi|}(0,r)} \end{aligned} \tag{3.34}$$

From the inequality (3.34) one infers that

$$\|v^0\|_{L^p_{|\Phi|}(0,r)} \le \frac{1 + \delta v^0(r)}{1 - \delta} \left(\int_0^r |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p}} \le \frac{1 + \delta}{1 - \delta} v^0(r) \left(\int_0^r |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p}} .$$

The same estimate holds for f.

Arguing as for equation (3.37) of [3], but taking into account that for $r \in (0, s_0)$ one has $v^0(r) - f(r) \ge 0$, we obtain

$$\begin{aligned} &\left(\int_0^r v^0(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p-1}} - \left(\int_0^r f(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p-1}} \\ &\le c_1 \left| \int_0^r \left(v^0(\varrho)^{p-1} - f(\varrho)^{p-1} \right) |\Phi(\varrho)|\, d\varrho \right| \\ &\times \left[\left(\int_0^r v^0(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \right)^{\frac{2-p}{p-1}} + \left(\int_0^r f(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \right)^{\frac{2-p}{p-1}} \right] \\ &\le c_1 c_2 \int_0^r \frac{(v^0(\varrho) - f(\varrho)) |\Phi(\varrho)|^{\frac{1}{p}} |\Phi(\varrho)|^{\frac{p-1}{p}}}{v^0(\varrho)^{2-p} + f(\varrho)^{2-p}}\, d\varrho \\ &\times \left(\|v^0\|^{2-p}_{L^p_{|\Phi|}(0,r)} + \|f\|^{2-p}_{L^p_{|\Phi|}(0,r)} \right) \left(\int_0^r |\Phi(\varrho)|\, d\varrho \right)^{\frac{2-p}{p(p-1)}} \end{aligned} \tag{3.35}$$

$$\begin{aligned}
&\le c_1 c_2 \|v^0 - f\|_{L^p_{|\Phi|}(0,r)} \left(\int_0^r |\Phi(\varrho)\, d\varrho \right)^{\frac{p-1}{p} + \frac{2-p}{p(p-1)}} \frac{\left(\|v^0\|^{2-p}_{L^p_{|\Phi|}(0,r)} + \|f\|^{2-p}_{L^p_{|\Phi|}(0,r)} \right)}{v^0(r)^{2-p} + f(r)^{2-p}} \\
&\le c_1 c_2 \|v^0 - f\|_{L^p_{|\Phi|}(0,r)} \left(\int_0^r |\Phi(\varrho)\, d\varrho \right)^{\frac{p-1}{p} + \frac{2-p}{p(p-1)}} \\
&\times \left(\frac{1+\delta}{1-\delta} \right)^{2-p} \left(\int_0^r |\Phi(\varrho)\, d\varrho \right)^{\frac{2-p}{p}} \frac{v^0(r)^{2-p} + f(r)^{2-p}}{v^0(r)^{2-p} + f(r)^{2-p}} \\
&= c_1 c_2 \left(\frac{1+\delta}{1-\delta} \right)^{2-p} \|v^0 - f\|_{L^p_{|\Phi|}(0,r)} \left(\int_0^r |\Phi(\varrho)\, d\varrho \right)^{\frac{1}{p(p-1)}} .
\end{aligned}$$

Finally, we evaluate $\|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)}$.

$$\begin{aligned}
\|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \le{} & \Bigg[\int_0^{s_0} \Bigg(\int_s^{s_0} \Bigg| \left(\int_0^r v^0(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p-1}} \\
& - \left(\int_0^r f(\varrho)^{p-1} |\Phi(\varrho)| d\varrho \right)^{\frac{1}{p-1}} \Bigg| \overline{\omega}(r)\, dr \Bigg)^p |\Phi(s)|\, ds \Bigg]^{\frac{1}{p}} \\
\le{} & c_1 c_2 \left(\frac{1+\delta}{1-\delta} \right)^{2-p} \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \\
& \times \left[\int_0^{s_0} \left(\int_s^{s_0} \left(\int_0^r |\Phi(\varrho)\, d\varrho| \right)^{\frac{1}{p(p-1)}} \overline{\omega}(r)\, dr \right)^p |\Phi(s)|\, ds \right]^{\frac{1}{p}} \\
\le{} & c_1 c_2 \left(\frac{1+\delta}{1-\delta} \right)^{2-p} \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} \\
& \times \left(\int_0^{s_0} \left(\int_0^r |\Phi(\varrho)|\, d\varrho \right)^{\frac{1}{p(p-1)} + \frac{1}{p}} \overline{\omega}(r)\, dr\, dr \right) \\
\le{} & c_1 c_2 \left(\frac{1+\delta}{1-\delta} \right)^{2-p} \delta \|v^0 - f\|_{L^p_{|\Phi|}(0,s_0)} .
\end{aligned} \tag{3.36}$$

We can choose δ sufficiently small, so we get a contradiction.

We have so proved that any solution to (3.14) is essentially bounded in $(0,\varepsilon)$, thus u° is essentially bounded too in $(0,\varepsilon)$. The same arguments guarantee that $(-u)^0$ is essentially bounded too. Also $\|u^\circ\|_{L^\infty(0,|\Omega|)} = \max\{u^\circ(0), |u^\circ(|\Omega|)|\}$ and from (3.10)

$$u^\circ(0) \le \frac{1}{1-\delta}, \qquad |u^\circ(|\Omega|)| = (-u)^\circ(0) \le \frac{(-u)^\circ(\varepsilon)}{1-\delta} = \frac{|u^\circ(|\Omega|-\varepsilon)|}{1-\delta}, \tag{3.37}$$

and

$$\|u\|_{L^\infty(\Omega)} \le \frac{max\{u^\circ(\varepsilon), (-u)^\circ(\varepsilon)\}}{1-\delta}. \tag{3.38}$$

Also, in view of the assumption (3.2) and of Theorem 2.1 of [4] $u \in L^p(\Omega)$, and

$$\begin{aligned} |u^\circ(\varepsilon)| &= \frac{1}{\varepsilon^{\frac{1}{p}}}\left(\int_0^\varepsilon u^\circ(\varepsilon)^p ds\right)^{\frac{1}{p}} \le \frac{1}{\varepsilon^{\frac{1}{p}}}\left(\int_0^\varepsilon u^\circ(s)^p ds\right)^{\frac{1}{p}} \\ &\le \frac{1}{\varepsilon^{\frac{1}{p}}}\|u^\circ\|_{L^p(0,|\Omega|)} = \frac{1}{\varepsilon^{\frac{1}{p}}}\|u\|_{L^p(\Omega)}. \end{aligned} \tag{3.39}$$

The same estimate holds for $(-u)^\circ(\varepsilon)$, thus (3.3) follows. □

4. Applications

In this Section we present some examples of domains and manifolds for which our result guarantees the boundedness of the solutions to (1.2). We assume that A is any function satisfying (1.4). In some the examples we show that if $V \in L^q(\Omega)$, for suitable $q > 1$, then our hypotheses are satisfied. We also consider, functions $V \in L^1(\Omega)$, such that

$$V_-^*(s) \approx s^\beta(-\lg s)^\delta \ \text{ near } 0, \tag{4.1}$$

for some $\beta \in (-1,0)$ and $\delta \in \mathbb{R}$ or $\beta = -1$ and $\delta < -1$, or $\beta = 0$ and $\delta \ge 0$, and we show that, for suitable β and δ, our hypotheses are satisfied, even in the borderline case ($\beta = -1$), when $V_- \notin L^q(\Omega)$ for all $q > 1$. It holds

$$\int_0^s V_-^*(\sigma)d\sigma \approx \begin{cases} s^{\beta+1}(-\lg s)^\delta & \text{if } \beta \in (-1,0) \text{ and } \delta \in \mathbb{R}, \text{ or } \beta = 0 \text{ and } \delta \ge 0, \\ (-\lg s)^{\delta+1} & \text{if } \beta = -1 \text{ and } \delta < -1. \end{cases} \tag{4.2}$$

This choice of V_-^* makes the two integrals in (3.2) equivalent, except that for the critical value $\beta = -1$. Thus (3.2) reads as

$$\int_0 \frac{s^{\frac{\beta+1}{p-1}-1}(-\lg s)^{\frac{\delta}{p-1}}}{\nu_{\Omega,p}(s)^{\frac{1}{p-1}}} ds \quad \text{if } \beta \in (-1,0) \text{ and } \delta \in \mathbb{R}, \text{ or } \beta = 0 \text{ and } \delta \ge 0,\ p > 1, \tag{4.3}$$

$$\int_0 \frac{(-\lg s)^{\frac{\delta}{p-1}}}{s\nu_{\Omega,p}(s)^{\frac{1}{p-1}}} ds \quad \text{if } \beta=-1,\, \delta<-1,\, p\ge 2,$$

$$\int_0 \frac{(-\lg s)^{\frac{\delta+1}{p-1}-1}}{s\nu_{\Omega,p}(s)^{\frac{1}{p-1}}} ds \quad \text{if } \beta=-1,\, \delta<-1,\, 1<p<2,$$

4.1. Lipschitz domains

Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then

$$\nu_{\Omega,p}(s) \approx \begin{cases} s^{\frac{n-p}{n}} & \text{if } 1<p<n \\ (-\log s)^{1-n} & \text{if } p=n \\ c & \text{if } p>n \end{cases} \qquad \text{near } 0. \tag{4.4}$$

Thus, (3.2) is satisfied with any $V \in L^1(\Omega)$, such that $V_- \in L^q(\Omega)$, with $q > \max\left\{\frac{n}{p}, 1\right\}$. When V satisfies (4.1), then Theorem 3.1 implies that any solution of problem (1.2) is bounded in Ω provided

$$\begin{cases} (\beta,\delta) \in \left(-\frac{p}{n}, 0\right) \times \mathbb{R},\ \text{or } \beta=0, \delta\ge 0,\ \text{or } \beta=-\frac{p}{n}(>-1),\ \delta<1-p & \text{if } 1<p<n, \\ (\beta,\delta) \in (-1,0) \times \mathbb{R},\ \text{or } \beta=0, \delta\ge 0,\ \text{or } \beta=-1 \text{ and } \delta<-2(p-1) & \text{if } p=n, \\ (\beta,\delta) \in (-1,0) \times \mathbb{R},\ \text{or } \beta=0, \delta\ge 0,\ \text{or } \beta=-1 \text{ and } \delta<1-p & \text{if } p>n. \end{cases}$$

4.2. Hölder domains

Consider the problem (1.2) in a connected bounded open set $\Omega \subset \mathbb{R}^n$, $n\ge 2$, whose boundary is Hölder continuous for some exponent $\alpha \in (0,1)$. Then,

$$\nu_{\Omega,p}(s) \ge \begin{cases} cs^{1-\frac{\alpha p}{n-1+\alpha}} & \text{if } 1<p<\frac{n-1}{\alpha}+1 \\ c\left(\log\frac{1}{s}\right)^{\frac{1-n}{\alpha}} & \text{if } p=\frac{n-1}{\alpha}+1 \\ c & \text{if } p>\frac{n-1}{\alpha}+1 \end{cases} \qquad \text{near } 0, \tag{4.5}$$

for some positive constant c (see the Sobolev-Poincaré embedding of [16], and [26, Theorem 6.4.3/2]).

When V satisfies (4.1), then Theorem 3.1 implies that any solution of problem (1.2) is bounded in Ω provided

$$\begin{cases} (\beta,\delta) \in \left(-\frac{\alpha p}{n-1+\alpha}, 0\right) \times \mathbb{R},\ \text{or } \beta=0, \delta\ge 0,\ \text{or } \beta=-\frac{\alpha p}{n-1+\alpha}(>-1),\ \delta<1-p \\ \quad \text{if } 1<p<\frac{n-1}{\alpha}+1, \\ (\beta,\delta) \in (-1,0) \times \mathbb{R},\ \text{or } \beta=0, \delta\ge 0,\ \text{or } \beta=-1 \text{ and } \delta<-2(p-1) \\ \quad \text{if } p=\frac{n-1}{\alpha}+1(>2), \\ (\beta,\delta) \in (-1,0) \times \mathbb{R},\ \text{or } \beta=0, \delta\ge 0,\ \text{or } \beta=-1 \text{ and } \delta<1-p \\ \quad \text{if } p>\frac{n-1}{\alpha}+1(>2). \end{cases}$$

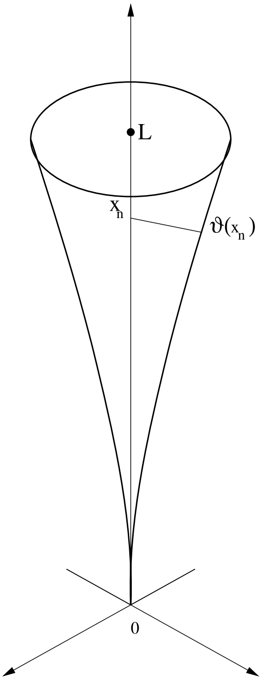

Fig. 1. A cusp shaped domain.

For a function $V \in L^q(\Omega)$, $q > 1$, equations (3.2) guarantee that any solution to (1.2) is bounded, provided

$$\begin{cases} q > \frac{n-1+\alpha}{\alpha p} & \text{if } p < \frac{n-1}{\alpha} + 1 \\ q > 1 & \text{if } p \geq \frac{n-1}{\alpha} + 1. \end{cases}$$

This situation covers that treated in [8]. When $V \equiv 1$ then we obtain the result in Example 5.1 of [3]

4.3. Cusp shaped domains

We consider the following cusp shaped domain

$$\Omega = \{x \in \mathbb{R}^n : |x'| < \vartheta(x_n), 0 < x_n < L\}$$

(Fig. 1), where, $x = (x', x_n)$ and $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $L > 0$ and $\vartheta : [0, L] \to [0, \infty)$ is a differentiable convex function such that $\vartheta(0) = 0$ and there exists the right derivative $\vartheta^{(h)}(0)$, up to the order $m \in N$, $m \geq 2$. Let $k := \min\{h \in N, \text{ such that } \vartheta^{(h)}(0) \neq 0\}$. The function $\Theta :$ $[0, L] \to [0, \infty)$, given by

$$\Theta(\rho) = n\omega_n \int_0^\rho \vartheta(r)^{n-1} dr \qquad \text{for } \rho \in [0, L],$$

where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$, is needful for the isocapacitary function. In fact, by [26, 4.3.5/1],

$$\nu_{\Omega,p}(s) \approx \left(\int_{\Theta^{-1}(s)}^{\Theta^{-1}(\mathcal{H}^n(\Omega))} \vartheta(r)^{\frac{1-n}{p-1}} dr \right)^{1-p} \qquad \text{for } s \in \left(0, \tfrac{\mathcal{H}^n(\Omega)}{2}\right). \tag{4.6}$$

Our conditions on ϑ guarantee that

$$\nu_{\Omega,p}(s)^{-\frac{1}{p-1}} \approx s^{\frac{p}{(p-1)[k(n-1)+1]} - \frac{1}{p-1}} \qquad \text{for } s \text{ near } 0. \tag{4.7}$$

Theorem 3.1 implies that any solution of problem (1.2) is bounded in Ω provided

$$\begin{cases} (\beta,\delta) \in]\max\{-\frac{p}{k(n-1)+1}, -1\}, 0[\times\mathbb{R}, \ \text{ or } \beta = 0, \ \delta \geq 0 \\ \beta = -\frac{p}{k(n-1)+1}, \ \text{ and } \delta < 1-p. \end{cases} \tag{4.8}$$

For a function $V \in L^1(\Omega)$, such that $V_- \in L^q(\Omega)$, equations (3.2) are satisfied provided $q > \frac{k(n-1)+1}{p}$.

4.4. γ-John domains

Next, we consider a γ-John domain, namely a bounded open set $\Omega \subset \mathbb{R}^n$ for which there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parametrized by arclenght, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq c r^\gamma \qquad \text{for } r \in [0,l].$$

If $1 < p < n$ and Ω is γ-John domain with

$$1 \leq \gamma < \frac{p}{n-1} + 1, \tag{4.9}$$

then [15, Theorem 2.3] and [26, Theorem 6.4.3/2] ensure that

$$\nu_{\Omega,p}(s) \approx s^{\frac{\gamma(n-1)+1-p}{n}} \qquad \text{near } 0. \tag{4.10}$$

Theorem 3.1 then guarantees that every solution of problem (3.1) in Ω is bounded, under these conditions on β and δ

$$\begin{cases} \beta \in \left(\frac{\gamma(n-1)+1-p}{n} - 1, 0\right), \ \delta \in \mathbb{R} & \text{and } p > 1, \\ \beta = \frac{\gamma(n-1)+1-p}{n} - 1, \ \delta < \min\{-1, 1-p\} & \text{and } p > 1, \\ \beta = 0 \text{ and } \delta \geq 0 & \text{and } p > 1. \end{cases} \tag{4.11}$$

Also, for a function V, such that $V^- \in L^q(\Omega)$, Theorem 3.1 guarantees that every solution of (3.1) is bounded provided $q > \frac{n}{n+p-1-\gamma(n-1)}$.

4.5. *A family of manifolds of revolution with borderline decay*

Consider a one-parameter family of manifolds of revolution $\mathbb{M}$ (see [3] for more details on this topic), whose profile $\varphi : [0, \infty) \to [0, \infty)$ is such that

$$\varphi(r) = e^{-r^\alpha} \qquad \text{for large } r, \tag{4.12}$$

and fulfils the assumptions of [3, Theorem 4.2]. This theorem guarantees that

$$\nu_{\mathbb{M},p}(s) \approx s\big(\log(1/s)\big)^{p-p/\alpha} \qquad \text{near } 0. \tag{4.13}$$

Due to the fast decay of $\nu_{\mathbb{M},p}(s)$ near zero, we can guarantee the boundedness of the solutions to (1.2) only when $V_-^* \approx (-\lg s)^\delta$ near 0, provided $\delta \in [0, 1)$, and

$$\alpha(1-\delta) > p. \tag{4.14}$$

This condition covers the one $\alpha > p$ obtained in [3, Example 5.4] for $V \equiv 0$.

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