

Modified Neoclassical Growth Models with Delay: A Critical Survey and Perspectives

Massimiliano Ferrara

Department of Law and Economics
University Mediterranea of Reggio Calabria
Via dei Bianchi 2, 89127 Reggio Calabria, Italy
massimiliano.ferrara@unirc.it

Luca Guerrini

Department of Management
Polytechnic University of Marche
Piazza Martelli 8, 60121 Ancona, Italy
luca.guerrini@univpm.it

Roberto Mavilia

Foreigners University Dante Alighieri of Reggio Calabria
and MEDALics Research Centre for Mediterranean Relations
Via del Torrione 95, 89125 Reggio Calabria, Italy
roberto.mavilia@unistrada.it

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Abstract

This paper reviews the relationship between lags and cycles into classical structures of growth models (see Solow-Swan and Ramsey models). Some remarks and perspectives of research are determined.

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1 Introduction

All production takes time, that is the transformation of inputs into outputs does not occur instantaneously but exhibits a gestation lag. Jevons (1871), was one of the first economists which underlined the important property that "capital is concerned with time". In his work "The Theory of Political Economy" he quoted the following passage from Hearn's work (1864) entitled "Plutology":

"A vineyard is unproductive for at least three years before it is thoroughly fit for use. In gold mining there is often a long delay, sometimes even of five or six years, before gold is reached".

The time dimension of capital was further studied by Hayek (1931), who identified in the time of production one of the possible sources of aggregate fluctuation. Hayek's analysis was based on Böhm-Bawerk's work (1891) on concept of the "average period of production".

The Hayek's insight was formally confirmed for the first time by Kalecki (1935), who rigorously showed that lags in the production of capital goods produce cycles endogenously, rather than being driven by exogenous shocks. Other early works in this area were done by Frisch and Holme (1935), and by Belz and James (1938). Notice that Kalecki frequently referred to a paper by Tinbergen (1931) on shipbuilding as guiding his mode of analysis. This is not, however, direct quotation of Tinbergen, rather, Kalecki often expanded and explained that which in Tinbergen is unclear and non-obvious. For example, as Kalecki noted, Tinbergen did not carry out a full analysis of the stability of the system, which was of fundamental importance to the conclusions.

From a mathematical point of view, Kalecki's model of the economic cycle is a differential equation with a delay parameter, i.e. a delay differential equation (DDE). Kalecki significantly contributed to the development of mathematical techniques used to characterize stability in linear DDEs that were widely applicable across a number of scientific fields. We must strongly emphasize that the general stability properties of a linear one delay differential equation, such as Kalecki's model of business cycle, were not fully understood until a 1950's theorem of Hayes, although Kalecki did a quite thorough analysis of stability. Delay differential equations, and in general functional differential equations are very interesting but, at the same time, quite complicated mathematical objects. Since the first contributions of Kalecki (1935), Frisch and Holme (1935), and Belz and James (1938), very few authors have used this mathematical instrument for modeling the time structure of capital. This line of argument was revived by the seminal article of Kydland and Prescott (1982), who empirically analyzed how far time consuming investment, which they called time to build (capital goods need some time over which they require investments in order to be produced), could explain real business cycles. While Kydland and Prescott

(1982) argued that the time to build feature is essential to cyclical fluctuations in their model, this was doubted by Ioannides and Taub (1992). Rustichini (1989) and Asea and Zak (1999) showed, in simple delayed continuous time optimal control models with one capital good (but a different lag structure) that the time to build feature is the driving force for the oscillatory system dynamics. More recently, this issue has been extensively addressed in the literature, see, e.g., Szydlowski (2002),(2003), Krawiec and Szydlowski (2004), Brons and Jensen (2007), Bambi (2008), Collard, Licandro and Puch (2008), Matsumoto and Szidarovszky (2011), Yuksel (2011), Winkler (2011), Bambi, Fabbri and Gozzi (2012), Ferrara (2009,2010,2011).

This paper review the relationship between lags and cycles in general equilibrium. The general equilibrium cases considered are production lag augmented versions of the Solow-Swan (1956) growth model as well as the single-good optimal growth model of Ramsey (1928), Cass (1965) and Koopmans (1965).

2 Solow-Swan model with time delay

In a Kaleckian spirit, we consider a version of the 1956 model of Solow and Swan in which there is a delay of $T \geq 0$ periods before capital can be used for production. This delay captures the time it takes to produce and install capital goods. At time t , the productive capital stock is $k(t - T)$. By assumption, the saving rate s is constant, a proportion of the capital stock, δ , depreciates during production, and we have a constant population normalized to unity ($n = 0$). As a result the model of economic growth with a built-in delay parameter in delivering capital goods is given by the following dynamic equation with a delayed argument:

$$\dot{k}_t = sf(k_{t-T}) - \delta k_{t-T} \quad (1)$$

for some initial function $k_t = \phi, t \in [-T, 0]$. Instead of an initial point value for an ordinary differential equation, the initial function ϕ_t is required, which is defined over the range of time delimited by the delay. To analyse the properties of equation (1) it is useful to carry out a local stability analysis. Equation (1) has exactly the same equilibrium point of the standard Solow-Swan model ($T = 0$) since time delay does not change the equilibria of the equation. Linearizing the right hand-side of equation (1) at its unique non-trivial steady state k_* , and then making the change of variables $x_t = k_t - k_*$, we get

$$\dot{x}_t = Ax_{t-T} \quad (2)$$

where $A = sf'(k_*) - \delta < 0$. We consider the characteristic equation for equation (2) which can formally be obtained by the substitution of a probe

solution $k_t = e^{\lambda t}$ into equation (2). Then we obtain

$$\lambda - Ae^{-\lambda T} = 0 \quad (3)$$

Equation (3) is a quasi-polynomial, which exhibits an infinite number of (complex) roots. For roots of equation (3) that have real parts that are negative, the model has damped cycles, positive roots correspond to explosive cycles, while purely imaginary roots, if they exist, indicate that the model has periodic cycles. The Poincar-Andronov-Hopf (PAH) theorem states the conditions that are necessary to produce such a bifurcation and a limit cycle. There is the generalisation of the PAH theorem to the case of functional differential equations. Therefore it is sufficient to check all the assumptions of this theorem to prove the existence of endogenous cycles in equation (2). The first step is to find the value of the time delay parameter, $T = T_0$, for which the Hopf bifurcation takes place. The method employed is to posit that such a purely imaginary solution exists and to rule out contradictions that would show the supposition to be false. A pair of purely imaginary roots means that solutions to equation (3) have the form $\lambda = i\omega_0, \omega_0 > 0$. After some simple manipulation we obtain $\omega_0 = -A$ and $T_0 = \pi/(-2A)$. The next step in proving the existence of the Hopf bifurcation is checking that there is no other eigenvalues with $Re\lambda = 0$. The last step in proving the existence of Hopf cycles is to check the transversality condition $[dRe\lambda(T)/dT]_{T=T_0} \neq 0$. The confirmation of this property ends our proof of the existence of the Hopf cycle in this model.

We have thus shown that augmenting the Solow model with a production lag changes the interior steady state from being stable to being, in general, a saddle for most initial functions ϕ_T and values of the lag T . For a particular value of the production lag, $T = T_0$, there is a corresponding initial function so that only the dynamics of the model lie on the center manifold, a two-dimensional subspace of the eigenspace that contains purely imaginary roots to the characteristic equation. On the center manifold, we have proved that the model exhibits endogenous cycles.

A separate issue is the problem of the stability of the limit cycle itself. To prove this fact one has to use normal coordinates and the theorem of the center manifold. To our knowledge the only work for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions in the delayed Solow-Swan model is due to by Brons and Jensen (2007), but using a different approach: the Poincare-Lindstedt's perturbation method. However, their calculations contains some mistakes.

3 Ramsey model with time delay

Now we examine whether production-lag-induced cycles will emerge in a more complicated setting, that of the optimal growth model, where consumers choose

savings each period as a function of their labor income, wealth, and the interest rate. Consider an economy that is inhabited by a continuum of infinitely-lived individuals. The representative individual's preferences are represented by a continuous, strictly increasing and concave utility function $u(c_t) = (c_t^{1-\sigma} - 1)/(1 - \sigma)$, with $\sigma > 0$ and c_t consumption, satisfying the Inada conditions and a subjective discount rate $\rho > 0$. In this economy it takes $T > 0$ periods to produce and install new capital equipment. The infinite horizon planning problem for this economy is given by

$$\max \int_0^{\infty} u(c_t) e^{-\rho t} dt,$$

subject to

$$\dot{k}_t = f(k_{t-T}) - \delta k_{t-T} - c_t,$$

with initial conditions $k_t = \phi_t, t \in [-T, 0]$. The problem of optimising the functional can be solved by using standard Hamiltonian methods and the generalised version of Pontryagin's maximum principle (El-Hodiri et al., 1972, Kolmanovskii and Myshkis, 1999). Asea and Zak (1999) arrived to the following optimality conditions:

$$\begin{cases} \dot{k}_t = f(k_{t-T}) - \delta k_{t-T} - c_t, \\ \dot{c}_t = \frac{c_t}{\sigma} [\rho + \delta - f'(k_{t-T})]. \end{cases} \quad (4)$$

The first-order conditions are similar to the standard optimal growth model with the exception of the time delay in production. The interior steady state is exactly the same as the standard optimal growth model, again because the time delay is irrelevant at a steady state. As in the delayed Solow-Swan model, Asea and Zak (1999) analyzed the production-lag optimal growth model to see if it produces Hopf cycles. The analysis proceeds by linearizing system (4) about the interior steady state. Then, the dynamics near the steady state can be determined using the associated characteristic equation. They demonstrated that the steady state, though typically a saddle, may exhibit Hopf cycles. Furthermore, the optimal path to the steady state is oscillatory.

However, their result is puzzling because a time-to-build assumption should lead to a system of mixed functional differential equations:

$$\begin{cases} \dot{k}_t = f(k_{t-T}) - \delta k_{t-T} - c_t, \\ \dot{c}_t = \frac{c_t}{\sigma} \{ [f'(k_{t-T}) - \delta] (\frac{c_t}{c_{t+T}})^{\sigma} e^{-\rho T} - \rho \}. \end{cases} \quad (5)$$

Asea and Zak's mistake (1999) was in a wrong application of the Pontryagin's maximum principle. This is not a minor point, and makes the resolution of this problem an important open issue. A special case, the AK growth model, was analyzed in detail by Bambi (2008) and Winkler (2009). They showed that the optimal system dynamics converges in an oscillatory manner toward the long run balanced growth path or exhibits a limit cycle around it. Furthermore, the oscillatory behavior is the more pronounced the larger is the delay.

4 Open problems

In this frame of research it would be interesting to trying to solve the following open problems:

- Delayed neoclassical models where population growth $n \neq 0$. This will lead to a more complicated setting: an equation with delay dependent parameters.
- Delayed neoclassical models where population growth follows the logistic law.
- Delayed neoclassical models, where time delay is not constant but variable.
- Stochastic analogue of delayed neoclassical models.

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