Finite-element formulation of a nonlocal hereditary fractional-order Timoshenko beam

FINITE ELEMENT FORMULATION OF A NON-LOCAL HEREDITARY FRACTIONAL-ORDER TIMOSHENKO BEAM

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ABSTRACT

A mechanically-based non-local Timoshenko beam model, recently proposed by the authors, hinges on the assumption that non-local effects can be modeled as elastic long-range volume forces and moments mutually exerted by non-adjacent beam segments, which contribute to the equilibrium of any beam segment along with the classical local stress resultants. Long-range volume forces/moments linearly depend on the product of the volumes of the interacting beam segments, and on pure deformation modes of the beam, through attenuation functions governing the space decay of non-local effects.

This paper investigates the response of this non-local beam model when viscoelastic long-range interactions are included, modeled by Caputo's fractional derivatives. The finite element method is used to discretize the pertinent fractional-order equations of motion. Closed-form solutions are

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obtained for creep tests by typical tools of fractional calculus. Numerical results are presented for various non-local parameters.

KEYWORDS

Non-local Viscoelasticity; Non-local Damping; Fractional Calculus; Long-range Interactions; Timoshenko Beam.

INTRODUCTION

In the last few decades, much effort has been devoted to develop non-local beam theories. Certainly, one of the reasons is the need for adequate and computationally-efficient modeling of microstructural effects in beam-like micro- and nano-devices (Lakes 1991; Aifantis 1994; Qian et al. 2002; Arash and Wang 2012). Indeed, these effects, which have been revealed by experimental tests on materials such as graphite (Tang 1983), copper (Poole et al. 1996), epoxy (Lam et al. 2003) and polypropylene (McFarland and Colton 2005), cannot be described by the intrinsically free-scale classical continuum approach while, on the other hand, could be captured only at the expense of computationally intensive and, in some cases, almost prohibitive atomistic/molecular simulations (Wang and Hu 2005). A further important application of non-local beam theories is at a macroscopic scale, whenever an intrinsic dependence exists between the response at a given point and the response at surrounding points of a beam. Such a dependence may arise as a result of external patches, long adhesive joints in composites, surface treatments using fluids, or fibers in fiber-reinforced composites. In these cases, instead of modeling all components of the system, as beam and external patch, or composite matrix and embedded fibers, a simpler yet accurate solution can be obtained from 1D equilibrium equations of the beam, where coupling between responses at non-adjacent points is accounted for by appropriate non-local terms. Non-local beam theories are also suitable for modeling effects produced, at a given point, by the complex deformations of non-adjacent beam cross sections, as these effects cannot be captured by classical beam models where cross section remain planes (Lei et al. 2006; Challamel 2011, 2013).

In general, non-local beam theories rely on introducing non-local terms in a classical continuum, which is formulated within the framework of classical Euler-Bernoulli (EB), Timoshenko (TM) or higher-order beam theories. Following this approach, early non-local beam models have been built using the non-local Eringen's integral law for normal and shear stress (Eringen 1972, 1983) in EB and TM beam models (e.g., see the study by Lu et al. (2007) and references therein), and higher-order beam models (Reddy 2007; Aydogdu 2009). Also, several non-local theories alternative to Eringen's integral theory have been used to build non-local beam models. Among the many, there exist non-local EB beam models based on modified couple stress theories (Park and Gao 2006; Kong et al. 2008), general strain gradient elasticity theory (Lam et al. 2003), gradient elasticity theory and integral elasticity theory with a constitutive relation combining local and non-local curvatures (Challamel and Wang 2008), micropolar elasticity constitutive law (McFarland and Colton 2005), and a hybrid approach involving a strain energy functional with local and non-local curvatures (Zhang et al. 2010). Non-local TM beam models have been built by Wang et al. (2010) in conjunction with the strain gradient elasticity theory presented by Lam et al. (2003), by Ma et al. (2008) based on a modified couple stress theory. Also, non-local EB and TM beam models have been developed based on a stress gradient elasticity theory (Pradhan 2012; Yang and Lim 2012). Very recently, non-local EB and TM beam models have been proposed by fractional generalizations of gradient elasticity theories, based on a new fractional variational principle for Lagrangians with Riesz fractional derivatives (Tarasov and Aifantis 2015).

In the non-local beam models briefly recalled above, non-locality affects the stiffness terms. However, an interesting and challenging task is a non-local modeling of damping effects. In fact, non-local damping models could be of interest to capture damping effects at a microstructural level that, as recent studies show, may play an in important role in image acquisition via high-speed atomic force microscopes as the scan rates increase (Payton et al., 2012), or may significantly affect the frequency measurements of vibrating nano-sensors (Murmu and Adhikari, 2012) detecting the mass of small particles based on shifts in measured frequencies (Calleja et al., 2012). Also, damping effects in nanostructures have been detected as a result of external magnetic forces (Lee and Lin, 2010), humidity or thermal effects (Chen et al., 2011). At a macroscopic level, on the other hand, non-local damping may be produced when responses at non-adjacent points are coupled by external patches, adhesive joints or surface treatments, or by embedded fibers in composites (Lei et al. 2006; Friswell et al. 2007).

In order to model damping effects in nano-beams, EB beam models with non-local viscoelastic behavior have been proposed by Lei et al. (2013), including multi-parameter time-dependent viscoelastic terms in the standard Eringen's law for normal stress. Non-local Kelvin-Voigt and three-parameter viscoelastic models have been discussed in detail. Applications have been presented on single walled carbon nanotubes, using a transfer function approach for free vibration analysis. The torsional behavior of functionally-graded nano-beams, including non-local viscoelasticity by suitable modifications of Eringen's law for shear stress, has been studied by Barretta et al. (2015) and, in particular, a closed-form response has been obtained for a viscoelastic model including a Maxwell model connected in series with a Voigt model.

As for non-local damping effects at a macroscopic level, an early EB beam model with non-local damping has been proposed by Russell (1992). Equations of motion include a viscous long-range moment per unit length, given by an integral depending on the relative rate of rotation between the beam segment and non-adjacent ones, through an appropriate attenuation function. An additional moment is also involved in the natural boundary conditions (B.C.). The model was conceived for EB composite beams with longitudinal fibers, such as fiberglass, boron and graphite composites, to account for the dissipation that may occur at the fiber-matrix interface due to imperfect bonding. Russell (1992) modeled the effects of such a dissipation as a damping moment, which depends on the relative rate of rotation between non-adjacent beam segments, in recognition of the fact that, while a differential rotation takes place along the beam axis, opposite motions of the fibers relative to the matrix occur above and below the elastic axis, and the resulting dissipation forces transmitted from the fibers to the matrix produce indeed a damping moment, within any beam segment which the fibers pass through. Obviously, the attenuation function reflects that coupling due to fibers progressively decays with distance. Experimental evidence for this non-local damping model has been found by Russell (1992) in the free vibrations of a boron-epoxy composite beam and, later, by Banks and coworkers (Banks and Inman 1991; Banks et al. 1994). A similar damping mechanism has been proposed by Russell (1992) also for the longitudinal vibrations of a fiber-reinforced composite bar.

More recently, Friswell and coworkers (Lei et al. 2006; Friswell et al. 2007) have proposed a non-local EB beam model where non-local damping terms are built as a weighted average of a velocity field over the beam domain, with appropriate attenuation functions taken as weighting functions. External and internal non-local damping models have been considered, depending on the transverse displacement and its fourth-order derivative, respectively. While the external damping model is seen as the result of external damping patches, long adhesive joints in composites or surface damping treatments using fluids, the internal non-local damping model has been thought as a homogenized model of an intrinsic dependence between the response at a given point and the response at the surrounding points of the medium (Flugge 1975). Such a dependence may be associated with effects produced at a given point by the complex deformations of non-adjacent beam cross sections, not adequately described by the plane-section assumption of classical beam models (Lei et al. 2006) or, alternatively, may compensate for uncertainties in spatial location of the damping sources, and dependence of damping mechanism on the material microstructure (Friswell et al. 2007). In their work, Friswell and coworkers have considered either viscous or viscoelastic non-local damping, the latter with time-dependent exponential forms (Friswell et al. 2007).

In the last few years, the authors have proposed non-local EB and TM beam models (Di Paola et al. 2013, Di Paola et al. 2014, Alotta et al. 2014, Failla et al. 2015), within a mechanically-based approach to non-locality, which treats non-local effects as long-range interactions resulting from relative motion of non-adjacent volume elements (Di Paola et al. 2009, 2010a, 2010b; Failla et al. 2010, 2013). In these non-local beam models, in particular, long-range interactions are volume forces/moments resulting from a differential motion of non-adjacent beam segments, measured by the pure deformation modes of the beam (Fuchs 1991, 1997), i.e. a "pure axial" symmetric mode, a "pure bending" symmetric mode and a "pure shear" asymmetric mode. The analytical form of the long-range volume forces/moments is built as linearly depending on the product of the volumes of the interacting beam segments, and the pure deformation modes, through pertinent attenuation functions governing the space decay of the non-local effects. In previous studies, the authors have considered elastic and viscous long-range interactions, either separately or simultaneously (Di Paola et al. 2013, Di Paola et al. 2014, Alotta et al. 2014, Failla et al. 2015).

In this paper, the purpose is to re-formulate the non-local TM beam model previously proposed by the authors, in order to include fractional-order viscoelastic long-range interactions. Fractional derivatives are indeed well-recognized mathematical tools for modeling long-memory effects (Tarasov and Zaslavsky 2007, 2008), and have already proved particularly suitable for modeling viscoelastic behavior (Rabotnov 1980, Bagley and Torvik 1983a, 1983b, 1985, 1986; Mainardi 2010; Meral et al. 2010; Di Paola et al. 2011; Di Paola et al. 2012; Di Paola et al. 2014; Failla and Pirrotta 2012, Sapora et al. 2014, Scimemi and Ponte 2014; Di Lorenzo et al. 2014). Here, in particular, the Caputo's fractional derivative will be used (Podlubny 1999) to model the fractional-order long-range interactions. On deriving the equations of motion, a corresponding discrete form will be obtained by the finite element (FE) method. Then, it will be shown that closed-form solutions may readily be derived for creep tests by simple rules of fractional calculus.

The paper is organized as follows. After a brief description of fractional operators in Section 2, the non-local TM beam model is introduced in Section 3, and related equations of motion in Section 4. FE discretization is described in Section 5, while closed-form solutions for creep tests are presented in Section 6. Numerical applications are discussed in Section 7.

FRACTIONAL-ORDER HEREDITARINESS

Preliminary investigations on fractional derivatives as applied to viscoelasticity modeling trace back to the work of Gemant (1938) and Bosworth (1946), who were the first to propose a fractional derivative model for viscoelasticity, and the studies by Scott-Blair and Gaffyn (1949) and Caputo (1974), who fitted fractional derivatives to experimental data. Later, Bagley and Torvik (1983a, 1983b, 1985, 1986) framed a fractional derivative viscoelastic model in the context of molecular theory, showing that, in order to capture the frequency-dependence of damping properties in viscoelastic materials, fractional derivatives are more appropriate than classical linear models such as the Kelvin–Voigt model. In the last three decades a considerable number of studies (Rogers 1983; Koeller 1984; Pritz 1996; Galucio et al. 2004; Adolfsson et al. 2005) have substantiated the capability of fractional derivatives to describe complex viscoelastic material behavior, in form of equations involving a small number of parameters (Di Paola et al. 2011).

Several viscoelastic models are based on the following Caputo's definition of fractional derivative (Podlubny 1999):

$$ {}_{C}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha}} \frac{df(\tau)}{d\tau} d\tau \qquad 0<\alpha<1 \tag{1} $$

It can readily be seen that, for systems at rest at $t = 0$ the Caputo's fractional derivative coincides with the Riemann-Liouville fractional derivative, defined as (Podlubny 1999):

$$ {}_{RL}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{1}{(t-\tau)^{\alpha}} f(\tau) d\tau \qquad 0<\alpha<1 \tag{2} $$

Time-domain discretization of the Caputo's fractional derivative (1) can be made by the well-known Grunwald-Letnikov algorithm (Spanos and Evangelatos 2010):

$$\left(D_C^{\alpha}x\right)(t)=\lim_{\Delta t\to 0}\Delta t^{-\alpha}\sum_{k=0}^{i}GL_k x\left(t_i-k\Delta t\right) \tag{3}$$

where GL_k are coefficients to be computed in the recursive form

$$GL_k=\frac{k-\alpha-1}{k}GL_{k-1}, \qquad GL_0=1.0 \tag{4}$$

BEAM MODEL

Figure 1 shows a beam of arbitrary cross section, referred to a Cartesian (orthogonal) coordinate system *Oxyz*, where axis *x* coincides with the centroidal axis, axes *y* and *z* are principal axes of the cross section, and *xz* is the bending plane. Be $\mathbf{x}=\left[x \quad y \quad z\right]^T$ the position vector and *L* the length of the beam. For simplicity, a uniform cross section is considered. It is assumed that the material is isotropic and linearly elastic.

Be $\mathbf{u}(\mathbf{x},t)$ the displacement vector, $\mathbf{u}^T(\mathbf{x},t)=\left[u_x \quad u_y \quad u_z\right]$. According to the TM beam theory, the small displacement components of a given point $P(\mathbf{x})$ in the beam can be cast in the form

$$u_x(\mathbf{x},t)=u(x,t)-z\varphi(x,t), \qquad u_z(\mathbf{x},t)=v(x,t), \qquad u_y(\mathbf{x},t)=0 \tag{5a,b,c}$$

where, for a cross section at x, $u(x,t)$, $v(x,t)$ and $\varphi(x,t)$ denote the *x*-, *z*-displacement and the rotation about the *y*-axis, the latter taken as positive if clockwise. The corresponding axial, bending and shear strain components, as given by the small strain equations, are

$$\varepsilon(x,t)=\frac{\partial u(x,t)}{\partial x}, \qquad \gamma(x,t)=\frac{\partial v(x,t)}{\partial x}-\varphi(x,t), \qquad \chi(x,t)=-\frac{\partial \varphi(x,t)}{\partial x} \tag{6a,b,c}$$

Local stress resultants

Be $\boldsymbol{\sigma}^{(l)}(\mathbf{x},t)=\begin{bmatrix}\sigma_x^{(l)} & \sigma_y^{(l)} & \sigma_z^{(l)} & \tau_{yz}^{(l)} & \tau_{xz}^{(l)} & \tau_{xy}^{(l)}\end{bmatrix}^T$ the vector of six components of the Cauchy stress tensor and be $N^{(l)}(x,t)$, $T^{(l)}(x,t)$ and $M^{(l)}(x,t)$ the classical local stress resultants, i.e. normal stress, shear stress and bending moment given by

$$N^{(l)}(x,t)=\int_A \sigma_x^{(l)}(x,t)\,dA\,,\quad T^{(l)}(x,t)=\int_A \tau_{xz}^{(l)}(x,t)\,dA\,,\quad M^{(l)}(x,t)=\int_A \sigma_x^{(l)}(x,t)\,z\,dA \qquad \text{(7a,b,c)}$$

The local stress resultants in Eqs.(7) are related to the corresponding axial, shear and bending strain by the constitutive laws of the TM beam:

$$N^{(l)}(x,t)=E^{*}A\varepsilon(x,t),\qquad T^{(l)}(x,t)=K_s G^{*}A\gamma(x,t),\qquad M^{(l)}(x,t)=E^{*}I\chi(x,t) \qquad \text{(8a,b,c)}$$

where A and I are the area and the moment of inertia of the cross section, K_S is the shear correction factor, $E^{*}=\beta_1 E$ and $G^{*}=\beta_1 G$, being E and G the Young and the shear modulus; β_1 is a dimensionless coefficient, $0\le\beta_1\le 1$, that weighs the amount of local effects (Di Paola et al. 2010b). In this respect, note that β_1 is introduced here as in those non-local theories where the non-local material is conceived as a two-phase elastic material (Altan, 1989; Polizzotto, 2001).

Long-range interactions

Long-range interactions are modeled on a mechanical basis. The fundamental assumption is that two non-adjacent beam segments of volume $\Delta V(x_i)$ and $\Delta V(\xi_k)$ located, respectively, at $x=x_i$ and $x=\xi_k$ on the beam axis, mutually exert long-range volume forces/moments as a result of their relative motion measured in terms of the "pure axial", "pure bending" and "pure shear" deformation modes of a TM beam (Fuchs 1991, 1997). It is assumed that the long-range volume forces/moments are self-equilibrated interactions, which counteract the relative motion of the beam segments. The analytical form is built as linearly depending on the product of the volumes of the interacting beam segments, through appropriate attenuation functions governing the space decay of non-local effects. Purely elastic and fractional-order viscoelastic long-range volume forces/moments are considered, the latter modeled by the Caputo's fractional derivative introduced in Section 2. A mechanical description of the long-range interactions is shown in Figure 2.

In the pure axial deformation mode, two non-adjacent beam segments of volume $\Delta V\left(x_i\right)$ and $\Delta V\left(\xi_k\right)$ exchange long-range volume axial forces as a result of the relative axial displacement:

$$\eta\left(x_i,\xi_k,t\right)=u\left(\xi_k,t\right)-u\left(x_i,t\right) \tag{9}$$

The specific volume axial forces exchanged by unit volumes $\Delta V\left(x_i\right)=1$ and $\Delta V\left(\xi_k\right)=1$, due to the pure axial deformation (9), are given by

$$q_x\left(x_i,\xi_k,t\right)=r_x\left(x_i,\xi_k,t\right)+d_x\left(x_i,\xi_k,t\right) \tag{10}$$

$$r_x\left(x_i,\xi_k,t\right)=g_x\left(x_i,\xi_k\right)\eta\left(x_i,\xi_k,t\right)\Delta V\left(x_i\right)\Delta V\left(\xi_k\right) \tag{11}$$

$$d_x\left(x_i,\xi_k,t\right)=\tilde{g}_x\left(x_i,\xi_k\right)\,{}_C D_{0^+}^{\alpha}\left(\eta\left(x_i,\xi_k,t\right)\right)\Delta V\left(x_i\right)\Delta V\left(\xi_k\right) \tag{12}$$

Likewise, in the pure bending mode, two non-adjacent beam segments of volume $\Delta V\left(x_i\right)$ and $\Delta V\left(\xi_k\right)$ exchange long-range volume moments as a result of the relative rotation:

$$\theta\left(x_i,\xi_k,t\right)=\varphi\left(\xi_k,t\right)-\varphi\left(x_i,t\right). \tag{13}$$

In this case, the specific volume moments exchanged by $\Delta V(x_i)=1$ and $\Delta V(\xi_k)=1$ are given as

$$q_{\varphi\varphi}(x_i,\xi_k,t)=r_{\varphi\varphi}(x_i,\xi_k,t)+d_{\varphi\varphi}(x_i,\xi_k,t) \tag{14}$$

$$r_{\varphi\varphi}(x_i,\xi_k,t)=g_{\varphi}(x_i,\xi_k)\theta(x_i,\xi_k,t)\Delta V(x_i)\Delta V(\xi_k) \tag{15}$$

$$d_{\varphi\varphi}(x_i,\xi_k,t)=\tilde{g}_{\varphi}(x_i,\xi_k)\,{}_{C}D_{0^+}^{\alpha}\left(\theta(x_i,\xi_k,t)\right)\Delta V(x_i)\Delta V(\xi_k) \tag{16}$$

Finally, in the pure shear mode, two non-adjacent beam segments of volume $\Delta V(x_i)$ and $\Delta V(\xi_k)$ exchange volume transverse forces and moments, as a result of their rotations with respect to the line given by the relative transverse displacement, that is

$$\psi(x_i,\xi_k,t)=\left[\frac{v(\xi_k,t)-v(x_i,t)}{\xi_k-x_i}-\varphi(\xi_k,t)\right]+\left[\frac{v(\xi_k,t)-v(x_i,t)}{\xi_k-x_i}-\varphi(x_i,t)\right] \tag{17}$$

The specific volume transverse forces and moments exchanged by $\Delta V(x_i)=1$ and $\Delta V(\xi_k)=1$ are given by

$$q_z(x_i,\xi_k,t)=r_z(x_i,\xi_k,t)+d_z(x_i,\xi_k,t) \tag{18}$$

$$r_z(x_i,\xi_k,t)=\frac{2\,\mathrm{sgn}(\xi_k-x_i)}{|x_i-\xi_k|}g_z(x_i,\xi_k)\psi(x_i,\xi_k,t)\Delta V(x_i)\Delta V(\xi_k) \tag{19}$$

$$d_z(x_i,\xi_k,t)=\frac{2\,\mathrm{sgn}(\xi_k-x_i)}{|x_i-\xi_k|}g_z(x_i,\xi_k)\,{}_{C}D_{0^+}^{\alpha}\left(\psi(x_i,\xi_k,t)\right)\Delta V(x_i)\Delta V(\xi_k) \tag{20}$$

where obviously $\dfrac{\mathrm{sgn}(\xi_k-x_i)}{|x_i-\xi_k|}=\dfrac{1}{\xi_k-x_i}$ and

$$q_{\varphi z}\left(x_i,\xi_k,t\right)=r_{\varphi z}\left(x_i,\xi_k,t\right)+d_{\varphi z}\left(x_i,\xi_k,t\right) \tag{21}$$

$$r_{\varphi z}\left(x_i,\xi_k,t\right)=g_z\left(x_i,\xi_k\right)\psi\left(x_i,\xi_k,t\right)\Delta V\left(x_i\right)\Delta V\left(\xi_k\right) \tag{22}$$

$$d_{\varphi z}\left(x_i,\xi_k,t\right)=\tilde{g}_z\left(x_i,\xi_k\right)\,{}_C D_{0^+}^{\alpha}\left(\psi\left(x_i,\xi_k,t\right)\right)\Delta V\left(x_i\right)\Delta V\left(\xi_k\right) \tag{23}$$

In Eqs.(12)-(16)-(20)-(23), ${}_C D_{0^+}^{\beta}\left(\cdot\right)$ is the Caputo fractional derivative operator (1) as applied to pure axial, pure bending and pure shear deformation modes.

Remarks on the proposed model of long-range interactions

In Eqs.(11)-(12) for the axial mode, Eqs.(15)-(16) for the bending mode, and Eqs.(19)-(20)-(22)-(23) for the shear mode, $g_s\left(x,\xi\right)$ and $\tilde{g}_s\left(x,\xi\right)$, for $s=x,\varphi,z$, are attenuation functions governing the space decay of purely elastic and purely viscoelastic long-range interactions. They shall be positive definite and must be taken as symmetric with respect to arguments x and ξ, to ensure that the long-range resultants exchanged by the interacting beam segments are mutual, according to Newton's third law. Further, notice that they are introduced as independent functions. That is, by $g_s\left(x,\xi\right)\neq\tilde{g}_s\left(x,\xi\right)$ for $s=x,\varphi,z$, a different spatial decay can be considered for purely elastic and purely viscoelastic long-range interactions, while $g_x\left(x,\xi\right)\neq g_\varphi\left(x,\xi\right)\neq g_z\left(x,\xi\right)$ and $\tilde{g}_x\left(x,\xi\right)\neq\tilde{g}_\varphi\left(x,\xi\right)\neq\tilde{g}_z\left(x,\xi\right)$ mean that spatial decay may vary depending on pure axial, pure bending and pure shear effects. This choice is made for the model to be as versatile as possible for experimental data fitting. A possible choice could be adopting the same mathematical model for the attenuation functions but with different parameters. Some examples of experimental data fitting have been given by the authors, assuming the same exponential attenuation function for non-local bending and shear effects in a non-local TM beam model (Alotta et al. 2014), or the same exponential form but with different parameters for non-local bending and shear effects in a non-local EB beam model (Di Paola et al. 2014). In both cases, purely elastic long-range interactions proved capable of reproducing stiffening size effects in epoxy beams, measured experimentally by Lam et al. (2003). Attenuation functions alternative to exponential ones could be fractional power-law or Gaussian functions, for both elastic and viscoelastic non-local effects (Di Paola et al., 2009; Failla et al., 2011; Friswell et al. 2007). Notice that power-law decay of non-local effects is, indeed, the basic assumption of the fractional calculus approach to non-local elasticity (Tarasov and Zaslavsky 2007, 2008; Atanackovic and Stankovic 2009; Sapora et al. 2013; Carpinteri et al. 2014; Tarasov 2014; Tarasov and Aifantis 2015; Sumelka and Blaszczyk 2014).

The volume forces/moments (11)-(12), (15)-(16), (19)-(20) and (22)-(23) may model non-local effects of various nature, triggered by a differential motion. At a microstructural level, they could be thought as homogenized measures of inter-atomic interactions arising from bond-stretching and angle variation (Li and Chou 2003; Wan and Delale 2010). In this context, fractional viscoelastic long-range interactions could be suitable for modeling damping effects, as revealed by recent experiments (Payton et al. 2012, Murmu and Adhikari 2012, Calleja et al. 2012, Lee and Lin 2010, Chen et al. 2011). At a macroscopic scale, they could reflect viscoelastic forces transmitted from the fibers to the matrix in a composite beam with multi-oriented fiber reinforcements, with the viscoelastic modeling of the dissipation mechanism at the fiber-matrix interface (Gosz et al.,1991) accounting for imperfect bonding due to mechanical imperfections, unreacted polymer components, fiber treatments or, in some cases, for the presence of an “engineered” interphase between fibers and matrix, to optimize composite performances (Matzenmiller and Gerlach, 2004; Fisher and Brinson, 2001). Certainly, a quite interesting feature of the non-local model is the fact that separate pure axial, pure bending and pure shear long-range interactions can be accounted for. This makes the model suitable for those applications where it can be assumed that non-local effects result only in long-range moments but not in long-range transverse forces. This may be the case of particular microstructures or composite beams where longitudinal fibers passing through a material matrix are placed only at the upper and lower surface of the beam.

It is apparent that, when non-local stiffness terms (11)-(15)-(19)-(22) and non-local fractional-order viscoelastic terms (12)-(16)-(20)-(23) are considered simultaneously in the model, the long-range volume forces/moments can be interpreted as the result of a non-local fractional Kelvin-Voigt connection between the interacting beam segments, see Figure 2. Obviously, when no viscoelastic terms are considered, the model reverts to that presented by the authors in previous publications on elastic non-local TM beams (Di Paola et al. 2014, Alotta et al. 2014, Failla et al. 2015).

NON-LOCAL BEAM MODEL EQUATIONS OF MOTION

Next, on dividing the beam in N segments of length Δx, the equations of motion of the beam segment of volume $\Delta V\left(x_i\right)=A\Delta x$ at $x=x_i=i\Delta x$, for $i=0,1,\dots N-1$ $\left(x_0=0,\, x_N=x_L\right)$, are written in the form (see Figure 3):

$$N^{(l)}\left(x_i+\Delta x\right)-N^{(l)}\left(x_i\right)+Q_x\left(x_i,t\right)+F_x\left(x_i,t\right)\Delta x-m\left(x_i\right)\ddot{u}\left(x_i,t\right)\Delta x=0 \tag{24a}$$

$$T^{(l)}\left(x_i+\Delta x\right)-T^{(l)}\left(x_i\right)+Q_z\left(x_i,t\right)+F_z\left(x_i,t\right)\Delta x-m\left(x_i\right)\ddot{v}\left(x_i,t\right)\Delta x=0 \tag{24b}$$

$$M^{(l)}\left(x_i+\Delta x\right)-M^{(l)}\left(x_i\right)-T^{(l)}\left(x_i\right)\Delta x-Q_\varphi\left(x_i,t\right)+I_\rho\left(x_i\right)\ddot{\varphi}\left(x_i,t\right)\Delta x=0 \tag{24c}$$

In Eqs.(24), dots mean differentiation with respect to time, $F_x\left(x,t\right)$ and $F_y\left(x,t\right)$ are introduced as generalized measures per unit length of the external forces on the beam, $m\left(x\right)=\rho\left(x\right)A$ and $I_\rho\left(x\right)=\int_A \rho\left(x\right)z^2 dA$, being $\rho\left(x\right)$ the mass per unit volume. Eqs.(24) state that the equilibrium of the beam segment of volume $\Delta V\left(x_i\right)$, at $x=x_i$, is attained due to the local stress resultants (7) exerted by the adjacent beam segments, and the resultants Q_x, Q_y and Q_φ of the volume forces/moments exerted by all the non-adjacent beam segments of volume $\Delta V\left(\xi_k\right)$ at $x=\xi_k$, $\xi_k \neq x_i$, given as

$$
\begin{aligned}
&Q_x\left(x_i,t\right)=\sum_{k=0,\,k\neq i}^{N-1} q_x\left(x_i,\xi_k,t\right)\\
&Q_z\left(x_i,t\right)=\sum_{k=0,\,k\neq i}^{N-1} q_z\left(x_i,\xi_k,t\right)\\
&Q_\varphi\left(x_i,t\right)=\sum_{k=0,\,k\neq i}^{N-1} q_{\varphi\varphi}\left(x_i,\xi_k,t\right)+q_{\varphi z}\left(x_i,\xi_k,t\right)
\end{aligned}
\tag{25a-c}
$$

For brevity, Q_x, Q_y and Q_φ will be referred to as *long-range resultants*.

On replacing Eq.(10) for q_x, Eq.(14) for $q_{\varphi\varphi}$, Eq.(18) for q_z and Eq.(21) for $q_{\varphi z}$, dividing Eqs.(24) by Δx and taking the limit $\Delta x \to 0$ lead to the following equations:

$$
\begin{aligned}
&E^* A\frac{\partial^2 u\left(x,t\right)}{\partial x^2}+F_x\left(x,t\right)+\\
&A^2\int_0^L\left[g_x\left(x,\xi\right)\eta\left(x,\xi,t\right)+\tilde{g}_x\left(x,\xi\right)\,{}_C D_{0^+}^{\alpha}\left(\eta\left(x,\xi,t\right)\right)\right]d\xi=m\left(x\right)\ddot{u}\left(x,t\right)
\end{aligned}
\tag{26a}
$$

$$
\begin{aligned}
&K_s G^* A\left[\frac{\partial^2 v\left(x,t\right)}{\partial x^2}-\frac{\partial\varphi\left(x,t\right)}{\partial x}\right]+F_z\left(x,t\right)+\\
&A^2\int_0^L\frac{2}{\xi-x}\left[g_z\left(x,\xi\right)\psi\left(x,\xi,t\right)+\tilde{g}_z\left(x,\xi\right)\,{}_C D_{0^+}^{\alpha}\left(\psi\left(x,\xi,t\right)\right)\right]d\xi=m\left(x\right)\ddot{v}\left(x,t\right)
\end{aligned}
\tag{26b}
$$

$$E^* I \frac{\partial^2 \varphi(x,t)}{\partial x^2} + K_s G^* A \left[\frac{\partial v(x,t)}{\partial x} - \varphi(x,t)\right] +$$

$$A^2 \int_0^L \left[g_\varphi(x,\xi)\theta(x,\xi,t) + \tilde{g}_\varphi(x,\xi)\, {}_C D_{0^+}^{\alpha}\left(\theta(x,\xi,t)\right)\right] d\xi + \qquad (26c)$$

$$A^2 \int_0^L \left[g_z(x,\xi)\psi(x,\xi,t) + \tilde{g}_z(x,\xi)\, {}_C D_{0^+}^{\alpha}\left(\psi(x,\xi,t)\right)\right] d\xi = I_\rho(x)\ddot{\varphi}(x,t)$$

where the constitutive local laws (8) have been introduced, and $\Delta V(x) = A\Delta x$, $\Delta V(\xi) = A\Delta\xi$ for the volumes of the interacting beam segments.

As for the boundary conditions (B.C.), it can readily be seen that the mechanical B.C. hold the classical form of local theory. This is true because, in the equilibrium equations at the beam ends, the long-range resultants (25) are infinitesimal of higher order with respect to the local stress resultants (e.g., see Di Paola et al. 2009). Also, time independent kinematic B.C. are considered. Therefore, the B.C. are given as

$$E^* A \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=x_i} = \mp N_i(t), \qquad \text{or} \qquad u(x_i,t) = u_i$$

$$K_s G^* A \left.\left[\frac{\partial v(x,t)}{\partial x} - \varphi(x,t)\right]\right|_{x=x_i} = \mp T_i(t), \qquad \text{or} \qquad v(x_i,t) = v_i \qquad (27a\text{-}c)$$

$$E^* I \left.\frac{\partial \varphi(x,t)}{\partial x}\right|_{x=x_i} = \mp M_i(t), \qquad \text{or} \qquad \varphi(x_i,t) = \varphi_i$$

where N_i, M_i and T_i, u_i, v_i and φ_i, denote the external forces/moments, displacements/rotations at the beam ends, i.e. at $x_0 = 0$ and $x_L = L$.

The equilibrium equations (26) clearly show that the non-local beam model is a displacement-based model, with long-range volume forces/moments that arise from relative displacements/rotations between non-adjacent beam segments, as given by the pure deformation modes (9)-(13)-(17). On the contrary, if the long-range volume transverse forces/moments were taken as depending on the relative transverse displacement and not on the pure shear deformation (17), long-range volume transverse forces/moments would erroneously arise from a relative transverse displacement induced, for instance, by a rigid rotation of the beam. That is, the non-local beam model is invariant with respect to rigid body motion and axial, bending and shear non-local behaviors are mechanically consistent.

The integral terms on the l.h.s. of Eqs.(26) are the long-range resultants per unit length. Interestingly, the viscoelastic long-range axial force in Eq.(26a) and moment in Eq.(26b), specifically the part due to the pure bending deformation mode (13), correspond to those introduced by Russell (1992) in his non-local damping model for a bar and a EB composite beam with longitudinal embedded fibers. Unlike the model proposed by Russell (1992), however, the proposed model includes long-range transverse forces/moments due to the asymmetric "pure shear" deformation mode between non-adjacent beam segments, and mechanical B.C. identical to those of classical local theory.

Finally, recognize that the non-local damping model is not proportional, as the fractional-order viscoelastic terms do not have the analytical form of the elastic ones, to which contribute both local and non-local terms.

FINITE ELEMENT FORMULATION

Following a standard approach of the FE method, consider a mesh with n disjointed elements of the same length, along the beam axis. Points shared by contiguous elements are *mesh nodes*. Abscissas of the nodes of the i^{th} element are denoted as $\hat{x}_i$ and $\hat{x}_{i+1}$, with $\hat{x}_1 = 0$ and $\hat{x}_{n+1} = L$ (symbol " ^ " is introduced to avoid confusion with abscissas x_i 's used in Sections 3-4), and l denotes the length of the i^{th} element. The displacement field within the i^{th} element is given the following form

$$\mathbf{u}_i(x,t) = \mathbf{N}_i(x)\mathbf{d}_i(t) \qquad i = 1,2,...,n \tag{28}$$

In Eq.(28), $\mathbf{u}_i(x,t)=\begin{bmatrix}u(x,t) & v(x,t) & \varphi(x,t)\end{bmatrix}^T$ is the vector of displacements/rotation within the i^{th} element, $\mathbf{d}_i(t)$ is the vector of the unknown nodal displacements of the i^{th} element, i.e.

$$\mathbf{d}_i(t)=\begin{bmatrix}u_{(i)1}(t) & v_{(i)1}(t) & \varphi_{(i)1}(t) & u_{(i)2}(t) & v_{(i)2}(t) & \varphi_{(i)2}(t)\end{bmatrix}^{\mathrm{T}} \tag{29}$$

where subscript "(i)" indicates the i^{th} element, while subscripts 1-2 denote first and second node of the element. In Eq.(28), $\mathbf{N}_i(x)$ is the matrix collecting the shape functions taken, in this paper, as the standard 1st order and 3rd order polynomial shape functions of the two-node TM beam element, for the axial and flexural response respectively. That is, $\mathbf{N}_i(x)$ is given as

$$\mathbf{N}_i^T(x)=\begin{bmatrix} \frac{\hat{x}_{i+1}-x}{l} & 0 & 0 \\ 0 & \frac{(l-y_i)(l^2(1+12\Omega)+(l-2y_i)y_i)}{l^3(1+12\Omega)} & \frac{6y_i(-l+y_i)}{l^3(1+12\Omega)} \\ 0 & \frac{(l-y_i)(l+6l\Omega-y_i)y_i}{l^2(1+12\Omega)} & \frac{(l+12l\Omega-3y_i)(l-y_i)}{l^2(1+12\Omega)} \\ \frac{x-\hat{x}_i}{l} & 0 & 0 \\ 0 & \frac{y_i(12l^2\Omega+3ly_i-2y_i^2)}{l^3(1+12\Omega)} & \frac{6(l-y_i)y_i}{l^3(1+12\Omega)} \\ 0 & -\frac{(l-y_i)y_i(6l\Omega+y_i)}{l^2(1+12\Omega)} & \frac{y_i(2l(-1+6\Omega)+3y_i)}{l^2(1+12\Omega)} \end{bmatrix} \tag{30}$$

where $y_i=x-\hat{x}_i$ and $\Omega=E^*I/G^*Al^2$.

Being $\mathbf{d}=\begin{bmatrix}u_1 & v_1 & \varphi_1 & u_2 & v_2 & \varphi_2 & .... & u_n & v_n & \varphi_n\end{bmatrix}^T$ the vector collecting all nodal displacements of the mesh, the nodal displacements of the i^{th} element are written as

$$\mathbf{d}_i(t)=\mathbf{C}_i\mathbf{d}(t) \tag{31}$$

being $\mathbf{C}_i$ the connectivity matrix. Next, following a standard Galerkin approach, the following equations can be derived

$$\mathbf{M}\ddot{\mathbf{d}}(t)+\mathbf{C}^{(nl)}\left({}_{C}D_{0^+}^{\alpha}\mathbf{d}(t)\right)+\mathbf{K}\mathbf{d}(t)=\mathbf{F}(t) \tag{32}$$

In Eq.(32), $\mathbf{K}$ is the 3(n+1)×3(n+1) global stiffness matrix, given as

$$\mathbf{K}=\mathbf{K}^{(l)}+\mathbf{K}^{(nl)}=\sum_{i=1}^{n}\mathbf{K}_i^{(l)}+\sum_{i=1}^{n}\mathbf{K}_i^{(nl)} \tag{33}$$

where $\mathbf{K}_i^{(l)}$ and $\mathbf{K}^{(nl)}$ are local and non-local stiffness matrices, respectively. The first is given as

$$\mathbf{K}_i^{(l)}=A\int_{\hat{x}_i}^{\hat{x}_{i+1}}\left(\mathbf{B}_i(x)\mathbf{C}_i\right)^T\mathbf{D}^*\ \mathbf{B}_i(x)\mathbf{C}_i dx \tag{34}$$

where $\mathbf{D}^*=Diag\left[E^*A \quad E^*I \quad G^*K_S A\right]$, $\mathbf{B}_i(x)$ is the 3×6 matrix

$$\mathbf{B}_i^T(x)=\begin{bmatrix} -\frac{1}{l} & 0 & 0 \\ 0 & \frac{6(-2l^2\Omega-ly_i+y_i^2)}{l^3(1+12\Omega)} & -\frac{6(l-2y_i)}{l^3(1+12\Omega)} \\ 0 & \frac{l^2(1+6\Omega)-4(l+3l\Omega)y_i+3y_i^2}{l^2(1+12\Omega)} & \frac{-4(l+3l\Omega)+6y_i}{l^2(1+12\Omega)} \\ \frac{1}{l} & 0 & 0 \\ 0 & \frac{6(2l^2\Omega+(l-y_i)y_i)}{l^3(1+12\Omega)} & \frac{6(l-2y_i)}{l^3(1+12\Omega)} \\ 0 & \frac{-6l^2\Omega+2l(-1+6\Omega)y_i+3y_i^2}{l^2(1+12\Omega)} & \frac{2(l(-1+6\Omega)+3y_i)}{l^2(1+12\Omega)} \end{bmatrix} \tag{35}$$

being $y_i = x - \hat{x}_i$. The second is given as

$$\mathbf{K}_i^{(nl)} = \mathbf{K}_i^{(nl,\eta)} + \mathbf{K}_i^{(nl,\theta)} + \mathbf{K}_i^{(nl,\psi)} = \sum_{j=1}^{n} \mathbf{K}_{ij}^{(nl,\eta)} + \sum_{j=1}^{n} \mathbf{K}_{ij}^{(nl,\theta)} + \sum_{j=1}^{n} \mathbf{K}_{ij}^{(nl,\psi)} \tag{36}$$

In Eq.(36), matrices $\mathbf{K}_{ij}^{(nl,\eta)}$, $\mathbf{K}_{ij}^{(nl,\theta)}$, $\mathbf{K}_{ij}^{(nl,\psi)}$ include the non-local stiffness contributions due to the long-range interactions between the between the differential volumes $dV(x) = Adx$ inside the i^{th} element $(\hat{x}_i \leq x \leq \hat{x}_{i+1})$, and the differential volumes $dV(\xi) = Adx$ inside the j^{th} element $(\hat{x}_j \leq \xi \leq \hat{x}_{j+1})$, namely

$$\mathbf{K}_{ij}^{(nl,\eta)} = \frac{A^2}{2} \int_{\hat{x}_i}^{\hat{x}_{i+1}} \int_{\hat{x}_j}^{\hat{x}_{j+1}} \left(\mathbf{N}_j^{(u)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(u)}(x)\mathbf{C}_i \right)^T g_x(x,\xi) \left(\mathbf{N}_j^{(u)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(u)}(x)\mathbf{C}_i \right) dx d\xi \tag{37a}$$

$$\mathbf{K}_{ij}^{(nl,\theta)} = \frac{A^2}{2} \int_{\hat{x}_i}^{\hat{x}_{i+1}} \int_{\hat{x}_j}^{\hat{x}_{j+1}} \left(\mathbf{N}_j^{(\varphi)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(\varphi)}(x)\mathbf{C}_i \right)^T g_\varphi(x,\xi) \left(\mathbf{N}_j^{(\varphi)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(\varphi)}(x)\mathbf{C}_i \right) dx d\xi \tag{37b}$$

$$\begin{aligned} \mathbf{K}_{ij}^{(nl,\psi)} = \frac{A^2}{2} \int_{\hat{x}_i}^{\hat{x}_{i+1}} \int_{\hat{x}_j}^{\hat{x}_{j+1}} & \left(2\frac{\mathbf{N}_j^{(v)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(v)}(x)\mathbf{C}_i}{\xi - x} - \mathbf{N}_j^{(\varphi)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(\varphi)}(x)\mathbf{C}_i \right)^T \\ & g_z(x,\xi) \left(2\frac{\mathbf{N}_j^{(v)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(v)}(x)\mathbf{C}_i}{\xi - x} - \mathbf{N}_j^{(\varphi)}(\xi)\mathbf{C}_j - \mathbf{N}_i^{(\varphi)}(x)\mathbf{C}_i \right) dx d\xi \end{aligned} \tag{37c}$$

In Eqs.(37), $\mathbf{N}_i^{(u)}$, $\mathbf{N}_i^{(v)}$ and $\mathbf{N}_i^{(\varphi)}$ are row vectors of the shape functions matrix $\mathbf{N}_i$, i.e.

$$\mathbf{N}_i^{(u)T}(x) = \frac{1}{l} \begin{bmatrix} \hat{x}_{i+1} - x \\ 0 \\ 0 \\ x - \hat{x}_i \\ 0 \\ 0 \end{bmatrix} \tag{38a}$$

$$\mathbf{N}_i^{(v)T}(x) = \frac{1}{l^3(1+12\Omega)} \begin{bmatrix} 0 \\ (l-y_i)(l^2(1+12\Omega)+(l-2y_i)y_i) \\ l(l-y_i)(l+6l\Omega-y_i)y_i \\ 0 \\ y_i(12l^2\Omega+3ly_i-2y_i^2) \\ -l(l-y_i)y_i(6l\Omega+y_i) \end{bmatrix} \tag{38b}$$

$$\mathbf{N}_i^{(\varphi)T}(x) = \frac{1}{l^3(1+12\Omega)} \begin{bmatrix} 0 \\ 6y_i(-l+y_i) \\ l(l+12l\Omega-3y_i)(l-y_i) \\ 0 \\ 6(l-y_i)y_i \\ ly_i(2l(-1+6\Omega)+3y_i) \end{bmatrix} \tag{38c}$$

being $y_i = x - \hat{x}_i$. Further, in Eq.(32) matrix $\mathbf{C}^{(nl)}$ is the 3(n+1)×3(n+1) global viscoelastic matrix. It is easy to recognize that $\mathbf{C}^{(nl)}$ has the same mathematical form as the non-local stiffness matrix $\mathbf{K}^{(nl)}$ where, however, $g_s(x,\xi)$ are replace by $\tilde{g}_s(x,\xi)$, for $s = x, \varphi, z$. Further, in Eq.(32) matrix $\mathbf{M}$ is the 3(n+1)×3(n+1) global consistent mass matrix (Reddy, 2006), while vector $\mathbf{F}(t)$ is the load vector given as

$$\mathbf{F}(t) = \sum_{i=1}^{n} \mathbf{F}_i(t) \tag{39}$$

with

$$\mathbf{F}_i(t) = \int_{V_i} (\mathbf{N}_i(x)\mathbf{C}_i)^T \,\bar{\mathbf{F}}(x,t)\, dV_i(x) + (\mathbf{N}_i(0)\mathbf{C}_i)^T \bar{\mathbf{F}}_0(t) + (\mathbf{N}_i(L)\mathbf{C}_i)^T \bar{\mathbf{F}}_L(t) \tag{40}$$

being $\bar{\mathbf{F}}(x,t) = [F_x(x,t) \quad F_z(x,t) \quad 0]^T$, $\bar{\mathbf{F}}_i(t) = [N_i(t) \quad T_i(t) \quad M_i(t)]^T$, $i = 0, L$.

Finally, two important remarks are in order. Unlike the local stiffness matrix $\mathbf{K}^{(l)}$, the non-local stiffness matrix $\mathbf{K}^{(nl)}$ and the viscoelastic matrix $\mathbf{C}^{(nl)}$ are fully-populated. Also, closed-form solutions for the elements of $\mathbf{K}^{(nl)}$ and $\mathbf{C}^{(nl)}$ can be obtained for attenuation functions $g_s(x,\xi)$ and $\tilde{g}_s(x,\xi)$ of common use in non-local theories, such as exponential or power-law functions. Details can be found in a previous study by the authors (Alotta et al. 2014) and are not reported here, for brevity.

TIME-DOMAIN SOLUTION

Given an arbitrary input $\mathbf{F}(t)$, Eq.(32) can be solved in the time domain following a general approach (Bagley and Torvik 1985; Bagley and Calico 1991; Di Paola and Pinnola 2014), which is based on the complex eigensolution of a multi-degree-of-freedom companion system, obtained from Eq.(32) by including a suitable number of additional state variables. However, load cases of particular interest, such as creep tests, can be tackled by closed-form solutions, as explained in the following.

In a typical creep test, it can be assumed that the load vector $\mathbf{F}(t)$ in Eq.(32) attains a constant value $\mathbf{F}(t)=\mathbf{F}$ at a given time after $t=t_0$, with a slow initial loading rate. Under this assumption, inertial terms in Eq.(32) can be neglected and, being $\mathbf{\Phi}$ the eigenvectors matrix of $\mathbf{A}=\mathbf{K}^{-1}\mathbf{C}_{NL}$, Eq.(32) can be recast as follows

$$\mathbf{\Lambda}\left({}_{C}D_{0^+}^{\alpha}\mathbf{z}\right)(t)+\mathbf{\Omega}\mathbf{z}(t)=\mathbf{\Phi}^T\mathbf{F}(t) \tag{41}$$

where $\mathbf{z}(t)=\mathbf{\Phi}^{-1}\mathbf{d}(t)$, while $\mathbf{\Lambda}=\mathbf{\Phi}^T\mathbf{C}^{(nl)}\mathbf{\Phi}$ and $\mathbf{\Omega}=\mathbf{\Phi}^T\mathbf{C}^{(nl)}\mathbf{\Phi}$ are diagonal matrices. System (41) is uncoupled. For instance, if the load vector $\mathbf{F}(t)$ is given the analytical form

$\mathbf{F}(t)=\mathbf{F}\cdot t/t_0+\mathbf{F}\cdot U(t-t_0)(1-t/t_0)$, with $U(t-t_0)$ denoting the unit-step function, exact closed-form solutions for components can be obtained for load cases of particular interest in viscoelasticity. For a typical creep test under a constant load distributed over the beam, $\mathbf{F}(t)=\mathbf{F}\cdot U(t)$, where $U(t)$ is the unit-step function, it yields

$$\lambda_j\left({}_C D_{0^+}^{\alpha} z_j(t)\right)+\omega_j z_j(t)=f_j\cdot t/t_0+f_j\cdot U(t-t_0)(1-t/t_0) \tag{42}$$

where $f_j=\mathbf{\Phi}_j^T\mathbf{F}$, λ_j and ω_j denote the *j*th elements of matrices $\mathbf{\Lambda}$ and $\mathbf{\Omega}$, while $z_j(t)$ is given by a Mittlag-Leffler function, as follows

$$z_j(t)=\frac{t}{t_0\omega_j}\left[1-E_{\alpha,2}\left(-\frac{\omega_j}{\lambda_j}t^{\alpha}\right)\right]-\frac{(t-t_0)U(t-t_0)}{t_0\omega_j}\left[1-E_{\alpha,2}\left(-\frac{\omega_j}{\lambda_j}(t-t_0)^{\alpha}\right)\right] \tag{43}$$

for

$$E_{\alpha,\gamma}(w)=\sum_{k=0}^{\infty}\frac{w^k}{\Gamma(\alpha k+\gamma)} \tag{44}$$

NUMERICAL APPLICATIONS

The behavior of the non-local TM beam model is illustrated focusing on the flexural response. Theoretical creep response of a simply-supported epoxy micro-beam with rectangular cross section will be presented, for the following parameters: Young's modulus $E=1.40\,\text{GPa}$, Poisson's coefficient $\nu=0.35$; *L*=300 μm, *b*=30 μm and *h*=15 μm are length, width and thickness of the cross section. In the local constitutive equations (8), $\beta_1=1$ is selected. As for the long-range interactions, it is assumed that pure bending and shear behaviors are governed by the same attenuation functions, i.e. $g_s(x,\xi)=g(x,\xi)$, $\tilde{g}_s(x,\xi)=\tilde{g}(x,\xi)$ for $s=x,z,\varphi$, with the following exponential forms:

$$g(x,\xi)=\frac{C}{h^2}\exp\left(-|x-\xi|/\lambda\right) \tag{45a}$$

$$\tilde{g}(x,\xi)=\frac{C_\alpha}{h^2}\exp\left(-|x-\xi|/\lambda_\alpha\right) \tag{45b}$$

where λ and λ_β are internal lengths. The non-local parameters (C, λ) and (C_α, λ_α) in Eqs.(45) are set in order to enhance non-local effects and assess how they affect the response. The larger is the internal length, the wider is the so-called influence distance, i.e. the maximum distance beyond which the attenuation functions and therefore the non-local effects become negligible. Notice that, as a result of the choice $\beta_1=1$, the non-local solution will tend to the solution obtained by the classical local TM theory, as $\lambda\to 0$ in Eq.(45a) and $\lambda_\alpha\to 0$ in Eq.(45b).

Using Eqs.(45) for the attenuation functions $g_s(x,\xi)$ and $\tilde{g}_s(x,\xi)$, for $s=x,z,\varphi$, terms in the non-local stiffness matrix $\mathbf{K}^{(nl)}$ and viscoelastic matrix $\mathbf{C}^{(nl)}$ can be built in a closed form (Alotta et al. 2014). Upon discretizing the equations of motion by the FE method, time-domain closed-form solutions are built based on Eqs.(41)-(42) in Section 6.

Consider the uniformly-distributed load

$$p(t)=p_0\cdot t/t_0+p_0\cdot U(t-t_0)(1-t/t_0);\quad p_0=1\ \mathrm{Nm}^{-1},\ t_0=10\ \mathrm{s} \tag{46}$$

Figure 4 through Figure 7 show the beam deflection as time elapses, normalized to the midspan deflection of the classical TM beam theory, $v^{(l)}(L/2)$, when 40 FEs are used. The first relevant observation is that the proposed model is capable of providing a large variety of viscoelastic behaviors as the fractional order α varies. This is a typical feature of fractional viscoelastic models, representing a significant advantage compared to classical viscoelastic models that combine multiple Maxwell or Kelvin-Voigt elements, as they generally involve a large number of parameters. It is also seen that, regardless of α, the deflection tends to the purely elastic non-local one, more rapidly as the fractional order α increases. It is worth noticing that the non-local deflection is stiffer than the corresponding classical local one, as a result of the stiffening effects due to the elastic long-range interactions, which counteract the relative motion between non-adjacent beam segments and, as such, provide additional stiffness with respect to the stiffness of the classical local TM terms (indeed, $\beta_1 = 1$ has been set in the local constitutive equations (8)). Solutions with a larger number of FEs do not differ from the ones shown in Figures 4-7, and are not reported for clarity.

Figure 8 shows the midspan deflection at given time instants, normalized to the midspan deflection of the classical TM beam theory, $v^{(l)}\left(L/2\right)$. Consistently with Figures 4-7, the midspan deflection tends to the purely elastic non-local counterpart as time elapses, and more rapidly as the fractional order α increases.

For a further insight into the proposed model, Figure 9 and Figure 10 show the midspan deflection for $\alpha = 0.5$, as parameters $\left(C_\alpha, \lambda_\alpha\right)$ in Eq.(45b) vary (again, normalized to the midspan deflection of the classical TM beam theory, $v^{(l)}\left(L/2\right)$). In particular, $\lambda_\alpha = \text{cost}$ and C_α varies in Figure 9, while $C_\alpha = \text{cost}$ while λ_α varies in Figure 10. It is clear that viscoelastic effects do increase with increasing C_α in Figure 9 and increasing λ_α in Figure 10. These results are consistent with the fact that, while C_α governs the magnitude of the viscoelastic long-range interactions, λ_α governs the distance beyond which such interactions are negligible and, consequently, the number of beam segments interacting with a given one.

CONCLUDING REMARKS

A non-local TM beam with fractional-order viscoelastic long-range interactions has been presented, within the theoretical framework of a recent mechanically-based approach to non-local theory (Di Paola et al. 2009, 2010a, 2010b; Failla et al. 2010, 2013). The key assumption is that classical stress resultants and long-range resultants contribute to the equilibrium of every beam segment. Classical stress resultants are exerted by adjacent segments, and long-range resultants are exchanged with all non-adjacent beam segments, as a result of relative motion measured by the pure deformation modes of TM beam kinematics, i.e. pure axial, bending and shear deformation modes. The long-range resultants are constructed as volume forces/moments, linearly-depending on the product of the volumes of the interacting beam segments through space-dependent attenuation functions, as is typical in non-local continua. Elastic and fractional-order viscoelastic long-range interactions are considered in the model. While the first depend on the pure deformation modes, the second depend on Caputo's fractional derivatives (Podlubny 1999) of the pure deformation modes. The resulting equilibrium equations are fractional differential equations. Since the long-range resultants are built as volume forces/moments, it is found that the related B.C. coincide with those of classical local theory (Di Paola et al. 2009). The model can be considered as a generalization of previous non-local models proposed by the authors (Di Paola et al. 2013, Di Paola et al. 2014, Alotta et al. 2014), which included purely elastic or Kelvin-Voigt viscoelastic long-range interactions.

The FE method has been applied to discretize the equilibrium equations. FE equations involve the classical local stiffness matrix $\mathbf{K}^{(l)}$, and non-local stiffness and viscoelastic matrices $\mathbf{K}^{(nl)}$ and $\mathbf{C}^{(nl)}$ associated with the long-range resultants. For typical creep tests, Mittag-Leffler power series closed-form solutions (Podlubny 1999) have been built based on a suitable representation of the displacement response, based on the eigenvectors of the matrix given as the product between the inverse of the global stiffness matrix $\mathbf{K}=\mathbf{K}^{(l)}+\mathbf{K}^{(nl)}$ and the non-local viscoelastic matrix $\mathbf{C}^{(nl)}$. Numerical applications have investigated the creep response of a simply-supported beam under a uniform load, assuming a typical exponential form for the spatial decay of elastic and viscoelastic long-range interactions (Friswell et al. 2007). Parameters and attenuation functions have been set on a theoretical basis, to enhance non-local effects. Results have shown that the model is quite versatile, and potentially capable of providing a large variety of viscoelastic responses with a limited number of parameters, as is typical in fractional modeling of viscoelasticity as compared with traditional viscoelastic models combining Maxwell and Kelvin-Voigt models.

The proposed model shares with alternative non-local beam models the idea of a continuum enriched with non-local terms. The mathematical form assumed for the non-local terms, i.e. long-range volume forces/moments acting on every beam volume as a result of its interaction with non-adjacent beam volumes, appears consistent with the typical approach of engineering beam theories, where the equilibrium of a beam segment is set in an average (weak) sense based on the stress resultants on the cross section (normal and shear forces, bending moment). Applications may be envisaged to capture non-local damping effects due to micro-structural effects, as well as those arising, for instance, at the fiber-matrix interface in fiber-reinforced composite beams.

Further developments will focus on appropriate mathematical treatment of the fractional viscoelastic response when uncertainty is considered (Muscolino et al. 2013).

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