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GUM-based Decisional Criteria to Make Decisions in Presence of Measurement Uncertainty

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Abstract- This paper deals with the issue of decision-making involving measurement data. Often, measurement results must be put in comparison with limits so to take a decision on specification conformity or limit overcoming. This task is made complex due to measurement uncertainty. In fact, a measurement result is represented by an interval of values that could reasonably be attributed to the measurand. Additional information concerning probability density function and level of confidence may be available. As a consequence, the comparison concerns a limit and an interval of values having a probability distribution. This becomes more difficult when the limit also comes from a measurement, so even it is affected by uncertainty. In this case, two intervals of values have to be compared. The author aims to propose simple and intuitive decisional criteria to guide user during the decision-making process. Three main cases have been faced: the case of comparison with a simple limit; the case of conformity assessment with a conformance interval; and the case of comparison between two measurement results. The suggested criteria are based on the theory of the uncertainty evaluation reported in the Guide to the Expression of Uncertainty in Measurement. In detail, since a measurement result can be described by a statistical distribution, this information is used to compute the probability of specification conformity or the probability of limit overcoming. Different probability density functions have been considered. The final aim of this work is to provide a full overview on decisional problems where measurement results are compared in order to overcome the current lack of the state of the art. At the same time, the author wants to provide intuitive and easy to use decisional criteria to encourage their use even by users not experienced with decision-making and with the effects of uncertainty on the decisional process.

Index Terms – **Measurement uncertainty, decision-making, limit overcoming, conformity assessment, decisional criteria.**

I. Introduction

Several applications require to put in comparison the result of a measurement with a specification value, a control limit or a reference measure. For example, environmental monitoring activities need to compare the measured value concerning a polluting parameter with control or warning limits fixed by laws or regulations [1]. In biomedical field or healthcare, screening or clinical parameters have to be put in comparison with reference values to diagnose a specific pathology [2]-[4]. Another common application field concerns the process control where it is needed to assess the conformity of specific control parameters to the project specifications. Moreover, in conformity assessment, measurement results are typically used to decide if an item of interest conforms to a specified requirement [5]-[9]. So in daily life, each of us experiences or faces the assessment and comparison between data coming from a measurement process and a threshold or reference value which has to not be overcome. Often, the limit overcoming is index of an alert, warning or risky situations. In other cases, the conformity assessment is necessary to take a specific decision which has influence on the process evolution so entailing specific consequences. In all these cases, the main issue concerns the comparison process when data are affected by measurement uncertainty [10]-[12]. In such a circumstance, the comparison is not just a simple mathematical assessment between a value and a limit. In fact, the measurement result is represented by an interval of values that could reasonably be attributed to the measurand [13]-[15]. So, in the simplest case, the comparison regards an interval and a limit value. The matter becomes more and more complex when the same limit is the result of a measurement process. Consequently, the comparison would concern two intervals of values. In addition, under certain conditions such intervals could even be overlapping so increasing the indecision about the comparison result. In such a scenario, the user has to take a decision without having a standardised and unique decisional procedure. It must be said that a multitude of decisional criteria have been proposed in literature. Neural networks, fuzzy theory, Bayes' theorem are some of the most common approaches used to solve decision-making problems [8], [9], [11], [12], [16]-[24].

The *Joint Committee for Guides in Metrology* (JCGM) tries to solve the question by issuing the document *JCGM 106:2012* "Evaluation of measurement data – The role of measurement uncertainty in conformity assessment". This document clarifies its scope and application field. It states that *"conformity assessment, as broadly defined, is any activity undertaken to determine, directly or indirectly, whether a product, process, system, person or body meets relevant standards and fulfils specified requirements. In a particular kind of conformity assessment, sometimes called inspection, the determination that a product fulfils a specified requirement relies on measurement as a principal source of information. ISO 10576-1:2003 sets out guidelines for checking conformity with specified limits in the case where a quantity is measured and a resulting coverage interval is compared with a tolerance interval. The JCGM 106:2012 extends this approach to include explicit consideration of risks, and develops general procedures for deciding conformity based on measurement results, recognizing the central role of probability distributions as expressions of uncertainty and* *incomplete information"*. In detail, *JCGM 106:2012* proposes as an approach the balance between the risks of incorrect accept/reject decision associated with measurement uncertainty so to minimize the costs associated with such incorrect decisions. This document provides guidelines to compute the conformance probability and the probability of incorrect decision by considering the probability density function (PDF) associated to the measurand, the tolerance limits and the limits of the acceptance interval. The user has to define an interval, called acceptance interval, of permissible measured values. This interval is chosen in order to balance the risks of accepting non-conforming items (*consumer's risk*) or rejecting conforming items (*producer's risk*).

The criteria proposed in *JCGM 106:2012* are based on a Bayesian approach using prior and posterior PDFs. The preliminary weakness is that this document and others are focused mainly on the conformity assessment of a product [5]-[9], [25]-[28], so disregarding several application cases where measured data are put in comparison with an alert or warning limit [1]. In addition, the *JCGM* approach fails in its complexity. The author is experienced with decision-making processes and decisional criteria. He has proposed in the last years several decisional criteria based on fuzzy and Bayesian approaches, where the probabilities of incorrect decisions have been evaluated and widely discussed, [1]-[4]. Even the risks associated to an incorrect decision have been considered in these works. However, the main issue of such approaches concerns the complexity and computational burden of the decisional criteria which are not always easy to understand and to apply. The increase of the computational complexity and of the implementation costs could discourage the use of such decision-making algorithms or make complex their use. In addition, the document *JCGM 106:2012* does not take into account the case of two measurement results to be compared in presence of uncertainty. Therefore, the aim of the present paper is to provide an overview of the most common decisional problems that may be encountered in the practice. To solve them, the author proposes intuitive and simple criteria based only on the assumptions used to evaluate the standard uncertainty and reported in the *Guide to the Expression of Uncertainty in Measurement* (GUM) [14]. The strength of the *GUM* is due to its basic theory which is commonly understandable although it faces such a complex issue. Therefore, a model based on the only uncertainty definition could meet the same dissemination so encouraging the use of decision-making even by users not experienced on this issue. As a consequence, the paper scope is to suggest alternative criteria to be used in those cases where it is difficult or impossible to use the Bayesian approach proposed in *JCGM 106:2012* or where the user is not experienced with such mathematical models and with the conformity issues. Although *JCGM 106:2012* is not in conflict with uncertainty theory included in the *GUM*, the former fails in its scope by introducing new concepts without considering all application cases expected in the practice. Therefore, the decision-making criteria proposed in the following are based on the measurement uncertainty definition. In addition, they intended for all cases where a measurement result is put in comparison with a simple limit or threshold or where it is compared with another measurement result.

The remaining of this paper is structured as follows. The next Section describes the measurement uncertainty evaluation according to the guidelines of *GUM*. Section III describes the proposed decisional criteria and the main application cases are discussed. Section IV reports a discussion on the comparison with other methods existing in literature so to highlight weaknesses and strengths of the proposed criteria. Then conclusions complete the paper.

II. Measurement Uncertainty Evaluation

To understand the assumptions of the decisional criteria proposed in the following, reference to the *Guide to the expression of Uncertainty in Measurement* (*GUM*) is here made [14]. The GUM allows user to evaluate the standard uncertainty associated to a measurement result. The measurement uncertainty is a *non-negative parameter characterizing the dispersion of the quantity values being attributed to a measurand, based on the information used*, [15]. It allows to express the measurement result by a *set of quantity values being attributed to the measurand together with any other available relevant information*, [15]. According to available information, the *GUM* suggests two approaches to the evaluation of standard uncertainty: *type A* and *type B* methods. Both methods are based on a statistical approach where the measurement uncertainty is estimated by means of a standard deviation, whereas the expected value of the measurand is evaluated by a mean. These two simple assumptions allow to characterize the measurement result by a PDF which describes the distribution and dispersion of the values that can reasonably be attributed to the measurand [29], [30]. In other words, this PDF contains relevant information about the set of measurand values such that some may be more representative of the measurand or more probable than others.

II.A Type A evaluation of standard uncertainty

If a priori knowledge on the measurand is not available, the *type A* approach has to be chosen. In detail, the first method is based on statistical analysis of series of observations obtained by repeated measures of the measurand. Repeated measures of the measurand have to be performed under the hypothesis that the measurand is constant during the measurements. Once that *n* independent observations have been obtained, according to the guidelines in JCGM 2008, the best estimate of the expectation of the measurand *X* is obtained by the arithmetic mean of the observations x_i repeated under the same conditions of measurement:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1}$$

This value represents the best estimation of the expectation or expected value *x* of the measurand *X*, so it is $x=\bar{X}$. The *type A* evaluation of standard uncertainty is got by the positive square root of the variance of the mean:

$$s^2(\bar{x}) = \frac{s^2(x_k)}{n} \tag{2}$$

where x_k is the individual observation, while $s^2(x_k)$ is estimated by the experimental variance of the observations:

$$s^2(x_k) = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{X})^2 \tag{3}$$

As a consequence, the *Type A* standard uncertainty is obtained by the experimental standard deviation of the mean:

$$u(\bar{X}) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \bar{X})^2} \tag{4}$$

This value represents the best estimation of the standard uncertainty $u(x)$ associated with the measurand expectation x, so it is $u(x)=u(\bar{X})$.

It is important to notice that the number of observations n should be large enough to ensure that $\bar{X}$ provides a reliable estimate of the measurand expectation x of the random variable X and that $s(\bar{x})$ provides a reliable estimate of the standard uncertainty.

II.B Type B evaluation of standard uncertainty

Differently, when complementary a priori knowledge is available, such as when a PDF is assumed, the *GUM* suggests to use the second method [14]. Typically, the latter is founded on previous measurement data, manufacturer's specifications, general knowledge of the behaviour and properties of relevant materials and instruments or data provided in calibration and other certificates. This method can be used when the estimation of the measurand cannot be obtained from repeated observations or when a *Type A* evaluation is based on a comparatively small number of statistically independent observations. Then the measurand expectation and the standard uncertainty are obtained by means of the available information. So let us state that the measurand value lies within the interval Δ^- to Δ^+. These values represent the upper and lower bounds of values that can reasonably be attributed to the measurand according to a known statistical distribution. In this case, the expectation or expected value of X, is obtained by the midpoint of the interval, so it can be estimated by:

$$x = (\Delta^+ + \Delta^-)/2 \tag{5}$$

The *Type B* standard uncertainty can be evaluated by calculating the standard deviation of the statistical distribution associated to the interval Δ^- to Δ^+. If Δ is the half-width of the interval:

$$\Delta = (\Delta^+ - \Delta^-)/2 \tag{6}$$

The standard uncertainty is evaluated by the equation:

$$u(x)=\Delta / k \tag{7}$$

where k depends on the statistical distribution type and on the confidence level associated to the interval, so it is:

$$k = \begin{cases} 1 & for\ Gaussian\ distribution\ and\ 68.27\%\ confidence\ level \\ 2 & for\ Gaussian\ distribution\ and\ 95.45\%\ confidence\ level \\ 3 & for\ Gaussian\ distribution\ and\ 99.73\%\ confidence\ level \\ \sqrt{3} & for\ rectangular\ distribution \\ \sqrt{6} & for\ triangular\ distribution \end{cases}$$

The above expression reports the main statistical distributions that mostly occur in the common practice. Consequently, only these distributions will be considered in the investigated application cases reported in the next Section. For other distribution types, it is possible to compute the value *k* by referring to the statistical tables and properties of the specific distribution. To guide the user during the choice of the best evaluation method, the *GUM* states that *Type B* method can be as reliable as *Type A* method, especially in a measurement situation where *Type A* evaluation is based on a comparatively small number of statistically independent observations [14].

II.C Indirect measurements and combined uncertainty

Sometimes, a measurand Y is not measured directly, but it is determined from N other quantities $X_1, X_2, ..., X_N$ through a functional relationship f, such as:

$$Y = f(X_1, X_2, ..., X_N) \tag{8}$$

The measurand value y can be estimated from (8) replacing the input estimates $x_1, x_2, ..., x_N$ for the values of the N quantities $X_1, X_2, ..., X_N$. The combined standard uncertainty of y, denoted by $u_c(y)$, can be obtained by combining the standard uncertainties of the input estimates $x_1, x_2, ..., x_N$. The combined standard uncertainty $u_c(y)$ is given by the *law of propagation of uncertainty* [14]:

$$u_c(y)=\sqrt{\sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)} \tag{9}$$

The above relation is valid only if the input quantities X_i are independent or uncorrelated. When the input quantities are correlated, the appropriate expression for the combined uncertainty is:

$$u_c(y)=\sqrt{\sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right) u(x_i, x_j)} \tag{10}$$

However, when the nonlinearity of the function f is significant, higher-order terms in the Taylor series expansion in (10) must be included [14].

III. The proposed GUM-based Decisional Criteria

The previous Section shows that, both for *Type A* and *Type B* evaluations, any measurement result is represented by an interval of values $x \pm u(x)$ or $y \pm u_c(y)$ for direct and indirect measurements, respectively. Such an interval can be characterized by a probability density function which provides further information on the distribution and dispersion of the values that can reasonably be attributed to the measurand. In addition, such PDF contains relevant information about the values more representative of the measurand or more probable than others [29], [30]. The mean and standard deviation of the PDF are the measurand expectation and the standard uncertainty (or the combined standard uncertainty), respectively. For further details concerning the choice of the probability density function associated to the measurement result makes reference to sub-Section III.D. With these only assumptions, it is possible to define simple and intuitive decisional criteria which are merely based on the *GUM* guidelines [14]. Three different application cases are here considered:

III.A the comparison between a measurement result and a single limit or threshold;
III.B the comparison between two measurement results;
III.C the comparison between a measurement result and a conformity interval.

Each application case is described item by item in the following sub-Sections. Examples occurring in the common practice are reported as use cases of the proposed criteria.

III.A Comparison between a measurement result and a limit

This sub-Section describes a common application case occurring when a measure must be put in comparison with a limit, a threshold or a specification maximum level. It happens, for example, in the environmental monitoring field where environmental parameters have to comply with warning or alert limits reported in laws or regulations [1]. Another possible use case concerns the medical diagnosis, where clinical data have to be put in comparison with reference levels so to diagnose a possible pathology [2]-[4]. Even the conformity assessment in industrial or control processes involves to compare measurement results with a specification limit [5]-[9].

Let X be the measurand, that is the quantity to be measured, and $x \pm u(x)$ the relative measurement result according to *GUM*. No matter what method has been used to evaluate the standard uncertainty. In addition, the present procedure can be applied both to direct and indirect measures. If L is the limit, that is the threshold or the reference level, it is possible to calculate the probability P_O that the measurand X overcomes the limit L. It is the level of reliability associated to the limit overcoming decision. Please, note that the terminology concerning the limit depends on the specific case study, i.e. tolerance limit, specification limit, reference threshold, conformance limit, and so on. Let be $f_u(X)$ the PDF of the random variable X, that is the measurand, associated to the standard measurement uncertainty $u(x)$. By considering the most common statistical distribution types, it is possible to assume, for example, that $f_u(X)$ is a normal or Gaussian distribution. Consequently, the function $f_u(X-x)$ in Figure 1 represents the probability distribution of the measurand X associated to the measurement result. This distribution is centred on the measurand expectation, i.e. its nominal value x. As it is well known, the normal distribution is extended from $-\infty$ to $+\infty$. As a consequence, by a statistical point of view, this PDF would depict an infinite interval of values being attributed to the measurand with a level of confidence equal to 100%. However, to be compliant with GUM and in order to take a decision on the overcoming or non-overcoming of the limit L, it is possible to delimit the distribution in Figure 1 by fixing the lower and upper end a and b, respectively, according to the definition of measurement result reported in GUM. By referring to [14], it is possible to define an expanded uncertainty by choosing a level of confidence p. The expanded uncertainty is obtained by multiplying the standard uncertainty by a coverage factor k, so the obtained measurement result with expanded uncertainty is $x \pm ku(x)$. The choice of k is based on the desired coverage probability or level of confidence required of the interval. For example, by considering the statistical Gaussian distribution properties, an interval with a level of confidence equal to 99.73% can be chosen by assuming $k=3$, as a consequence it is $a= x-3u(x)$ and $b=x+3u(x)$. A coverage factor equal to 2 entails a level of confidence $p=95.45\%$, whereas a coverage factor equal to 1 entails a level of confidence $p=68.27\%$. These represent the most common levels of confidence used in the practice, however other levels of confidence are possible by choosing a different coverage factor according to statistical properties of the Gaussian distribution.

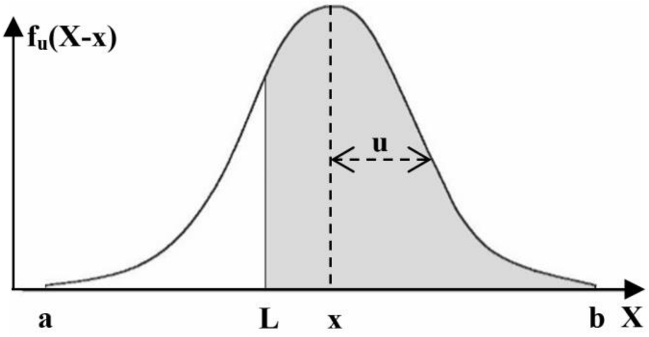

Figure 1. Gaussian probability density function associated to the measurement result.

To avoid misunderstandings and to better understand the mathematical meaning of the probability density function associated to the measurement result in Figure 1, consider that:

- X is a random variable whose value is unknown;
- x is the expected value assigned to X as result of a measurement outcome;
- the measurement result $x \pm u(x)$ provides an interval of values being attributed to the measurand, in particular, such values are reported along the horizontal axis of Figure 1;
- the function $f_u(X-x)$ assigns to each value (that can reasonably be attributed to the measurand) its probability to represent the true value of the measurand, as a consequence, the distribution is centred on the measurand expectation or its nominal value x.

With these considerations, the overcoming probability P_O is estimated by the following integral:

$$P_O = \int_L^b f_u(X-x)dx \tag{11}$$

Equation (11) provides quantitative information on the reliability of the limit overcoming. In particular, the previous integral is reduced to the evaluation of the area in grey colour as in Figure 1. Since L could be an upper or a lower bounds, it is possible to define the non-overcoming probability P_{NO} by integrating the PDF in the interval from a to L. This quantity can be easily estimated by subtracting P_O from the value of the chosen level of confidence normalized to 1. This new quantity provides the probability, in the range from 0 to 1, that the measurand is compliant with the reference level or with the warning/alert threshold or in conformity with the control value, depending on the specific use case. The main advantage of the proposed method is to get quantitative information on the reliability of the comparison result for all cases where a measurement result is compared with a reference value of any kind.

This quantity can be used to define a decisional criterion in order to guide the user to make a decision on the conformity or non-conformity with the limit. Depending on the specific case study, the user has to fix the minimum decision acceptance level D (in percentage value) according to the risk associated to the decision. For example, for risky cases where human life is put at direct risk, D can be set equal to 100% (please note that this value is not applicable to Gaussian distribution, but only to delimited distributions). In other cases, the user can define smaller value (80%, 90% …) depending on the maximum tolerable risk to make a wrong decision. As a consequence, if $P_O\% \geq D$ then the measurand overcomes the limit, or in other words, the measurand is not in compliance with the reference value. When a non-conformity decision is taken, the value $P_{NO}\%$ represents the probability to have made a wrong decision. Differently, if $P_{NO}\% \geq D$ then the measurand does not overcome the limit, or in other words, the measurand is in compliance with the reference value. So, when the conformity decision is taken, the value $P_O\%$ represents the probability to have made a wrong decision.

In the same way, it is possible to consider different probability density functions. In fact, the above described decisional criteria can be used for any statistical distribution representing the measurement result. So, let $f_u(X)$ be a uniform or rectangular distribution, see Figure 2, where a and b are the lower and the upper end of the distribution, respectively. Since the standard uncertainty $u(x)$ represents the standard deviation of the considered distribution, as a consequence, a and b are not equal to x-$u(x)$ and x+$u(x)$, respectively. In fact, this distribution does not represent the distribution of the values of the interval $x \pm u(x)$, but it is the distribution associated to the measurement result and to its uncertainty. By considering the statistical properties of the rectangular distribution, it is $a=x-\sqrt{3}u(x)$ and $b=x+\sqrt{3}u(x)$. With this assumption, the present distribtion depicts the values that can reasonably be attributed to the measurand with a level of confidence equal to 100%.

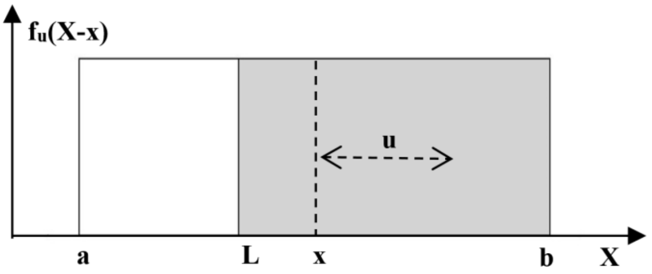

Figure 2. Rectangular probability density function associated to the measurement result.

If $L \leq b$, the overcoming probability P_O is estimated by the following integral:

$$P_O = \int_L^b f_u(X - x)dx \tag{12}$$

In this case it is possible to evaluate direclty the integral value by calculating the area of the rectangle in grey colour, obtaining:

$$P_O = \begin{cases} 1 & if\ L \leq a \\ (b - L)/(b - a) & if\ a < L < b \\ 0 & if\ L \geq b \end{cases} \tag{13}$$

Also in this case it is $P_{NO}=1-P_O$. So, if $L>b$, $P_O=0$ and $P_{NO}=1$. In particular, if x is the result of a direct measure and if its standard uncertainty $u(x)$ has been evaluated by using the *Type B* evaluation as described in the sub-Section II.B, a is equal to Δ^- and b is equal to Δ^+.

Finally, let $f_u(X)$ be a triangular distribution, see Figure 3, where a and b are the lower and the upper end of the distribution, respectively. By considering the statistical properties of the triangular distribution, it is $a=x-\sqrt{6}u(x)$ and $b=x+\sqrt{6}u(x)$. Even in this case, the present distribtion depicts the values that can reasonably be attributed to the measurand with a level of confidence equal to 100%.

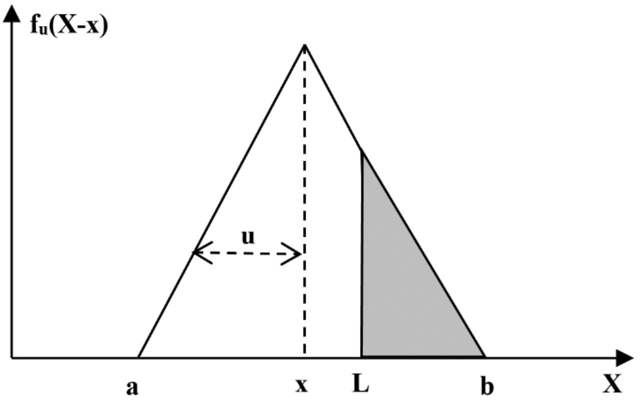

Figure 3. Triangular probability density function associated to the measurement result.

In this case, if $L \leq b$, the overcoming probability P_O can be estimated by using the same expression in (12). If $L>b$, $P_O=0$ and $P_{NO}=1$. Also in this case, if x is the result of a direct measure and if its standard uncertainty $u(x)$ has been evaluated by using the *Type B* evaluation, a is equal to Δ^{-} and b is equal to Δ^{+}.

Since, both the rectangular and triangular distributions are delimited functions compared to the Gaussian one, as a consequence they have to be preferred in those cases where the measurement result allows to get a level of confidence equal to 100% by using the appropriate coverage factor k which is equal to $\sqrt{3}$ and $\sqrt{6}$ for the rectangular and triangular distributions, respectively, (refer to JCGM 2008 for further details).

III.B Comparison between two measurement results

The case described in this sub-Section is the most complex to face since there is not a simple limit to be compared with the measurement result. In detail, two measurement results are here compared in order to make a decision. In this case, two intervals of values have to be put in comparison in order to decide which of the two measurement results overcomes the other. Suppose to have two measures concerning the same quantity X. In fact, the comparison between two measurement results is meaningful only if the two intervals of values are congruent and refer to the same measurement quantity. Their measurement results are $x_1 \pm u(x_1)$ and $x_2 \pm u(x_2)$, respectively. Assume that their probability distributions are uniform or triangular. The first and easiest case occurs when the two measurement intervals and the associated PDFs are disjoint as in Figure 4. This Figure reports three different sub-cases: comparison between two rectangular PDFs; comparison between two triangular PDFs; comparison between one triangular PDF and one rectangular PDF. A decision has to be made on the overcoming of the first measurement result by the second one.

To avoid misunderstandings and to better understand the mathematical meaning of Figure 4 and of the following equations, consider that:

- X is a random variable associated with the two measurement results;
- x_i is the expected value assigned to the *i-th* measurement result with $i=\{1; 2\}$;
- $f_{u,i}(X-x_i)$ is the probability density function associated to the *i-th* measurement result with $i=\{1; 2\}$.

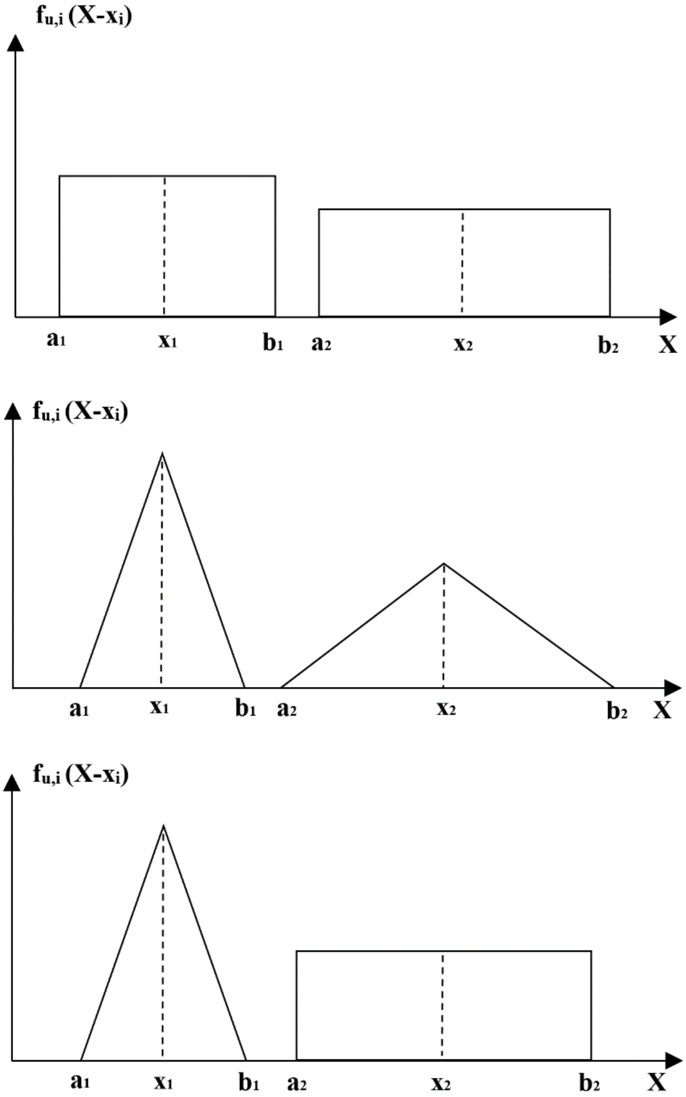

Figure 4. Comparison between two measurement results (case #1).

Although, in all these cases the decision is clear, let P_{NO} be the non-overcoming probability. It can be evaluated by summing the areas of the two PDFs which respect the non-overcoming condition. In other words, it needs to evaluate the sum of: 1) the part of the area of the PDF associated to the first measurement result which is entirely at the left side of the PDF associated to the second measurement result; 2) and the part of the area of the PDF associated to the second measurement result which is entirely at the right side of the PDF associated to the first measurement result. Possible parts of the two PDFs overlapping have not to be considered to compute P_{NO}, these parts will be used in order to get information on the probability of indecision P_{IND} (see grey areas in Figures 5, 6, 7 and 8 and eq. (16) for further reference). So the equation computing P_{NO} is given by:

$$P_{NO} = \left(\int_{a_1}^{b_1} f_{u,1}(X - x_1)dx_1 + \int_{a_2}^{b_2} f_{u,2}(X - x_2)dx_2\right)/2 \tag{14}$$

where the normalization to 1 of the probability value has been made by dividing by 2 the sum. The reader has to pay attention that, in this specific case, each integral in the previous formula can be replaced by calculating the area of the associated geometrical figure. In the above case #1, since a confidence level equal to 100% has been considered for both PDFs, by computing the previous equation, the value of P_{NO} is equal 1, whereas P_{IND} is equal to 0.

Similarly, the overcoming probability P_O can be estimated by summing the areas of the two PDFs which respect the overcoming condition. In other words, it needs to evaluate the sum of: 1) the part of the area of the PDF associated to the first measurement result which is entirely at the right side of the PDF associated to the second measurement result; 2) and the part of the area of the PDF associated to the second measurement result which is entirely at the left side of the PDF associated to the first measurement result. By considering the specific case reported in Figure 4, it is P_O=0.

Now, to avoid confusion, it is opportune to focus attention on only one sub-case. So let suppose that both the two measurement results have a rectangular distribution. The following case #2 refers to partially overlapping distributions, see Figure 5. To avoid any misunderstanding, the PDF associated to the second measurement result is here depicted in dashed line.

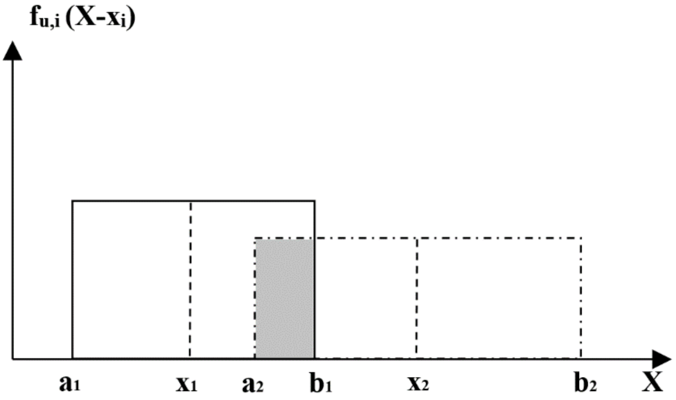

Figure 5. Comparison between two measurement results (case #2).

According to the above rule, it is possible to define the non-overcoming probability by the equation:

$$P_{NO} = (\int_{a_1}^{a_2} f_{u,1}(X - x_1)dx_1 + \int_{b_1}^{b_2} f_{u,2}(X - x_2)dx_2)/2 \tag{15}$$

In this case, the overcoming probability P_O is null, whereas the indecision probability P_{IND} can be obtained by the joint probability [29], [30] associated to the grey area. Since the measurands are independent, it is equal to:

$$P_{IND} = (\int_{a_2}^{b_1} f_{u,1}(X - x_1)dx_1 * \int_{a_2}^{b_1} f_{u,2}(X - x_2)dx_2) \tag{16}$$

This probability provides further information on the indecision affecting the decisional process. This value can be considered as a further discriminating parameter in order to define specific decisional criteria according to the application case as described at the end of this sub-Section. By increasing the overlapping area, it is obtained the case #3 in Figure 6.

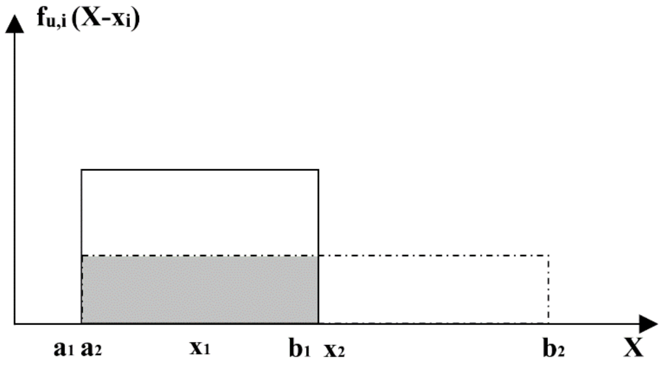

Figure 6. Comparison between two measurement results (case #3).

It is possible to compute the non-overcoming probability by the equation:

$$P_{NO} = (\int_{b_1}^{b_2} f_{u,2}(X - x_2)dx_2)/2 \tag{17}$$

The overcoming probability P_O is null, whereas the indecision probability is equal to:

$$P_{IND} = (\int_{a_1}^{b_1} f_{u,1}(X - x_1)dx_1 * \int_{a_1=a_2}^{b_1} f_{u,2}(X - x_2)dx_2) \tag{18}$$

By shifting furthermore on the left the PDF associated to the second measurement result, it is obtained the case #4 in Figure 7.

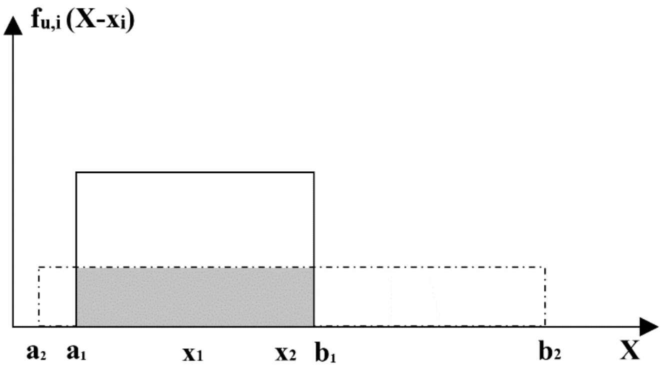

Figure 7. Comparison between two measurement results (case #4).

It is possible to compute the non-overcoming probability by the equation:

$$P_{NO} = (\int_{b_1}^{b_2} f_{u,2}(X - x_2)dx_2)/2 \tag{19}$$

The overcoming probability P_O is obtained by the expression:

$$P_O = (\int_{a_2}^{a_1} f_{u,2}(X - x_2)dx_2)/2 \tag{20}$$

whereas the indecision probability is equal to:

$$P_{IND} = (\int_{a_1}^{b_1} f_{u,1}(X - x_1)dx_1 * \int_{a_1}^{b_1} f_{u,2}(X - x_2)dx_2) \tag{21}$$

By shifting furthermore on the left the PDF associated to the second measurement result, it is obtained the case #5 in Figure 8.

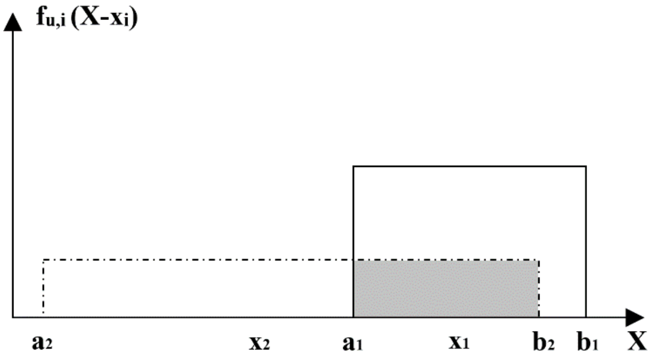

Figure 8. Comparison between two measurement results (case #5).

The non-overcoming probability is null, because no part of the PDF associated to the first measurement result is entirely at the left side of the PDF associated to the second measurement result, and at the same time no part of the PDF associated to the second measurement result is entirely at the right side of the PDF associated to the first measurement result.

The overcoming probability P_O is obtained by the expression:

$$P_O = (\int_{a_2}^{a_1} f_{u,2}(X - x_2)dx_2 + \int_{b_2}^{b_1} f_{u,1}(X - x_1)dx_1)/2 \tag{22}$$

whereas the indecision probability is equal to:

$$P_{IND} = (\int_{a_1}^{b_2} f_{u,1}(X - x_1)dx_1 * \int_{a_1}^{b_2} f_{u,2}(X - x_2)dx_2) \tag{23}$$

A further case #6 is obtained by shifting the PDF associated to the second measurement result entirely on the left of the PDF associated to the first measurement result as in Figure 9. The non-overcoming probability and the indecision probability are null. Since, in this case, the first measurement result clearly overcomes the second measurement result, the overcoming probability P_O is equal to 1 and can be obtained by the expression:

$$P_O = (\int_{a_2}^{b_2} f_{u,2}(X - x_2)dx_2 + \int_{a_1}^{b_1} f_{u,1}(X - x_1)dx_1)/2 \tag{24}$$

Also in this case, the decision is obvious, however the mathematical result proves the consistency of the criteria.

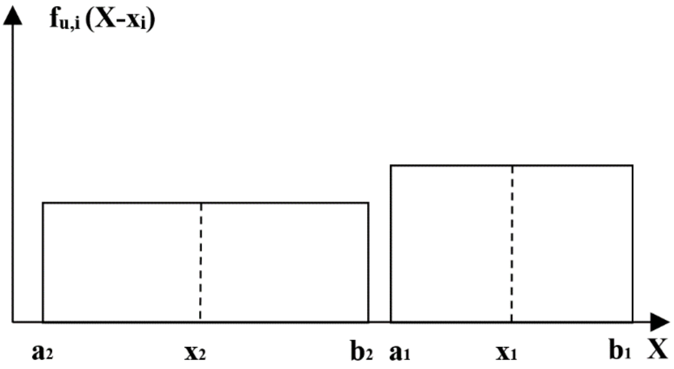

Figure 9. Comparison between two measurement results (case #6).

In the last considered case #7, two different statistical distributions are put in comparison so to complete the possible cases which can be found in the practice. In detail, a rectangular distribution is associate to the first measurement result, whereas a Gaussian distribution is associated to the second measurement result, see Figure 10. As in the sub-Section III.A, it is possible to delimit the Gaussian distribution by fixing the end values a_2 and b_2. So, if a level of confidence $p= 99.73\%$ is considered, it is $a_2=x_2-3u(x_2)$ and $b_2=x_2+3u(x_2)$.

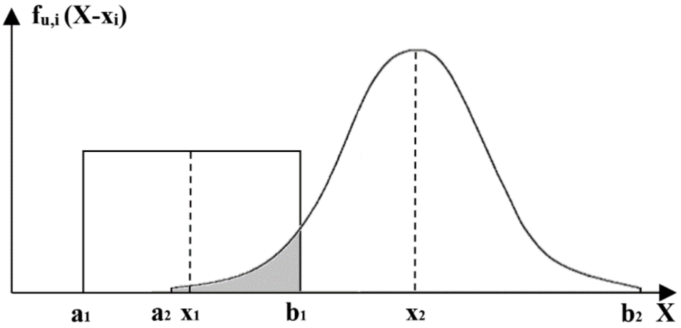

Figure 10. Comparison between two measurement results (case #7).

With this assumption, both the distributions are delimited, so this case can be faced by using the above described procedure. In particular, this case is similar to the case #2. As a consequence, it is possible to compute the non-overcoming probability by the equation:

$$P_{NO} = (\int_{a_1}^{a_2} f_{u,1}(X-x_1)dx_1 + \int_{b_1}^{b_2} f_{u,2}(X-x_2)dx_2)/2 \tag{25}$$

The overcoming probability P_O is null, whereas the indecision probability is equal to:

$$P_{IND} = (\int_{a_2}^{b_1} f_{u,1}(X-x_1)dx_1 * \int_{a_2}^{b_1} f_{u,2}(X-x_2)dx_2) \tag{26}$$

The above equations allow to cover all possible cases resulting from the combinations between the considered probability density functions. Even though the probability distributions here considered, as an example, are the Gaussian, triangular and rectangular ones, any other probability distribution can be used if applicable.

Also in presence of two measurement results, it is possible to define suitable decisional criteria. The user has to fix the minimum decision acceptance level D (in percentage value) according to the risk associated to the decision. Consequently, if $P_O\%\geq D$ then the first measurement result overcomes the second measurement result. Differently, if $P_{NO}\%\geq D$ then the first measurement result does not overcome the second measurement result. P_{IND} provides additional information about the indecision affecting the decisional process. In specific application cases where the risks associated to a wrong decision are high as in the industrial, health or safety fields, the indecision probability could provide a further discriminating decisional criterion. The user can fix a maximum indecision level IND (in percentage value) which has to not be overcome. So, if $P_{IND}\%>IND$, any decision has to be postponed. Further measurements or additional information on the process are required to make a final decision since the consequent risks are greater than the advantages achievable in taking a decision.

III.C Conformity interval

In this sub-Section, it is considered the case covered by the document *JCGM 106:2012*. In detail, a conformity interval is fixed. So two specification limits are obtained. Let L_L and L_U be the lower tolerance limit and the upper tolerance limit, respectively. It is possible to assume that $f_u(X)$ is a normal or Gaussian distribution. Consequently, the function $f_u(X-x)$ in Figure 11 represents the probability distribution of the measurement result around the measurand expectation, i.e. around its nominal value x, where $a=x\pm ku(x)$ and $b=x\pm ku(x)$.

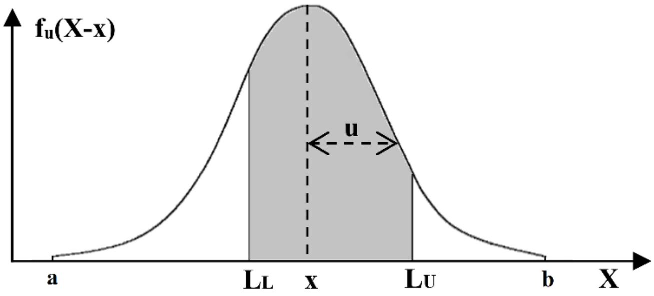

Figure 11. Gaussian probability density function associated to the measurement result.

The conformity probability P_C provides quantitative information on the conformity of the measurement result with the conformity interval. It can be estimated by the following integral:

$$P_C = \int_{L_L}^{L_U} f_u(X-x)dx \tag{27}$$

The value of the above integral is equal to the area in grey colour depicted in Figure 11. In particular, if the tolerance limits do not fall into the interval *[a; b]*, then the integral has to be computed from the maximum value between L_L and *a* to the minimum value between L_U and *b*.

Equation (27) can be used even to compute the conformity probability for the rectangular and the triangular distributions reported in Figures 12 and 13, respectively.

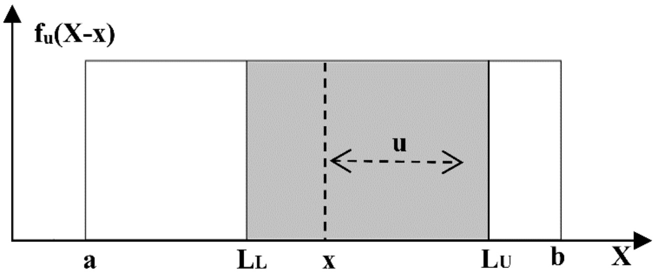

Figure 12. Rectangular probability density function associated to the measurement result.

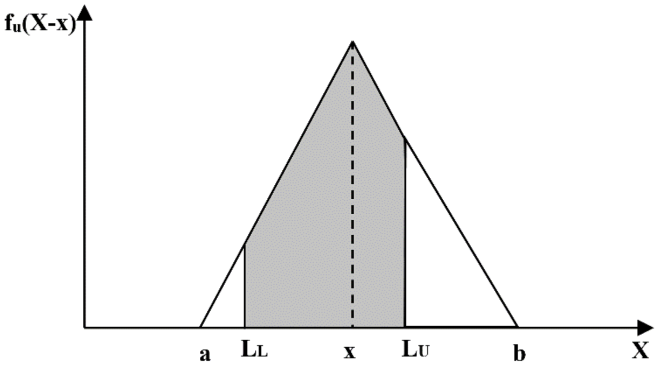

Figure 13. Triangular probability density function associated to the measurement result.

If *D* is the minimum decision acceptance level (in percentage value) defined according to the risk associated to the decision, it is possible to state that if $P_C\%\geq D$ then the measurand is compliant with the conformity interval. When the conformity decision is taken, the complementary value of $P_C\%$ (it is $P_{NC}\%=100-P_C\%$) represents the percentage probability to have made a wrong decision, or in other words it is the percentage level of the non-conformity probability with the conformity interval.

III.D Guide to Probability Density Function Assignment

The assignment of the probability density function is a crucial issue for the right application of the proposed decisional criteria. Therefore, to guide the reader in properly choosing the best fitting probability distribution, this sub-Section aims to provide practical recommendations. This specific issue is addressed by the *Annex G* of the GUM (pag. 70 in [14]).

Although this may appear to be a fortuitous accident to know the probability distribution, it is a direct consequence of the available information associated to the measurement. In the practice, this is a common task performed by the same user who makes the measurement, however an underestimation of these aspects could entail critical consequences.

Typically, when a measurement is performed, it is supposed that the user knows the observed physical phenomenon and the laws that regulate the measurand. In fact, according to the GUM, "when reporting the result of a measurement, and when the uncertainty evaluation is performed, the user should give a full description of how the measurand is defined". In this view, the PDF assignment encode the knowledge and expertise of the metrologist who formulates the model and who is ultimately responsible for the quality of the final results. The PDFs are typically assigned based on an analysis of a series of indications (*Type A* evaluation of uncertainty) or based on scientific judgement using available information, such as historical data, calibrations and expert judgement (*Type B* evaluation of uncertainty).

In detail, if the measurement result comes from a *Type A* evaluation, the analysis of the frequencies distribution of the repeated observations is suggested in order to study the type of distribution (unimodal, bimodal etc…). The observations are classified into mutually exclusive classes. The frequency of repetition of each class is estimated by counting the number of observations that fall into the corresponding class. The obtained histogram gives the number of repetitions of each class. By observing this histogram, it is possible to make a preliminary hypothesis on the kind of probability distribution. In addition, a chi-squared test allows user to determine if there is a significant difference between the expected distribution and the observed frequencies of the observations. The purpose of the test is to evaluate how likely the observations that are made would be, assuming the null hypothesis is true.

If the measurement result comes from a *Type B* evaluation, the available information provides directly knowledge of the type of distribution. Therefore, in this case, no further task is required to the user.

If the measurement result comes from an indirect measurement, according to the GUM, for many practical measurements in a broad range of fields, the following conditions prevail (see pag. 77 in [14]):

1. the estimate y of the measurand Y is obtained from estimates x_i of a significant number of input quantities X_i that are describable by well-behaved probability distributions, such as the normal, the triangular and rectangular distributions;
2. the standard uncertainties $u(x_i)$ of these estimates, which may be obtained from either *Type A* or *Type B* evaluations, contribute comparable amounts to the combined standard uncertainty $u_c(y)$ of the measurement result y;
3. the linear approximation implied by the law of propagation of uncertainty is adequate;
4. the uncertainty of $u_c(y)$ is reasonably small because its effective degree of freedom has a significant magnitude.

Under these circumstances, the probability distribution characterized by the measurement result and its combined standard uncertainty can be assumed to be normal because of the *Central Limit Theorem*. This consequence is surely right if the quantities X_i are characterized by normal distributions, then the resulting convolved distribution of Y will also be normal. However, even if the distributions of the X_i are not normal, the distribution of Y may often be approximated by a normal distribution because of the *Central Limit Theorem* (refer to pag. 71 in [14]).

In other cases, if the probability distributions of the input quantities X_i upon which the measurand Y depends are known, and if Y is a linear function of the input quantities, then the probability distribution of Y may be obtained by convolving the individual probability distributions. If the functional relationship between Y and its input quantities is nonlinear and a first-order Taylor series expansion of the relationship is not an acceptable approximation, then the probability distribution of Y cannot be obtained by convolving the distributions of the input quantities. In such cases, other analytical or numerical methods are required.

Further considerations are included in the supplement OIML G 1-101:2008 “Evaluation of measurement data - Supplement 1 to the “Guide to the expression of uncertainty in measurement - Propagation of distributions using a Monte Carlo method” in [31], at the Section 6 “Probability density functions for the input quantities”. This clause gives guidance on the assignment, in some common circumstances, of PDFs to the input quantities X_i in the formulation stage of uncertainty evaluation. Such an assignment can be based on Bayes’ Theorem or the Principle of Maximum Entropy.

The determination of the final measurement interval is a direct consequence of the assigned probability distribution and of the chosen level of confidence. Extensive experience with a full knowledge of the uses to which a measurement result is put can make more easy the assignment of the proper distribution and the selection of the right coverage factor.

IV. Application Cases: Strengths and Weaknesses

Scientific literature includes a lot of models and methods which assess the influence of measurement uncertainty in decision-making process. Some of them provide even decisional criteria in order to reduce the risk to make a wrong decision. The complexity of this specific topic is due to the different aspects which characterize the decision-making. Specific attention is here focused on the decision-making problems concerning the comparison of a measurement result affected from uncertainty with a limit or another measurement result. In this scenario, neural networks, fuzzy theory and Bayesian methods are the most explored solutions. Although, a comprehensive comparison by a mathematical point of view with the other methods reported in literature could be complex and dispersive, so overcoming the specific aim of the present work, this sub-Section aims to provide further guidelines about the application of the proposed criteria.

EURO Working Group Multicriteria Decision Aiding [32] and International Society on Multiple Criteria Decision Making [33] provide an interesting context of real-world application cases addressing the decision-making issue. By a point of view at-large, multiple objective decision problems are a largely investigated topic that can be considered part of the category including the decision-making problems. Fuzzy theory is used in [34] to provide a compromise solution for mathematical programming problems. To reduce computational burden and time, even evolutionary algorithms are used for solving multi objective optimization problems as in [35]. The main issue faced by these solutions concerns the large number of function evaluations needed. Nevertheless, such problems fall outside the scope of this work because they do not consider specific measurement results to be compared.

This work starts from the measurement uncertainty definition and its main reference is the theory included in the GUM [14]. To guide the user during the choice of the application case, as a general rule, the above described decisional criteria can be applied to all case studies or application cases where the result of a measurement affected by uncertainty, that is an interval of values being attributed to the measurand, is to be compared with a reference limit or a warning/alert threshold. Even conformity assessment problems fall into the scope of the present work. Therefore, for example, in the environmental monitoring field, often measurement data are put in comparison with thresholds which must be not overcome according to laws or regulations. The monitoring of the human exposure to electromagnetic field in [1] is a possible application case of the proposed decisional criteria. Also quantities coming from a generic physical phenomenon, such as climatic parameters, are regularly compared with reference values in order to classify weather conditions.

In the health field, data provided by diagnostic tools, such as eco-Doppler based ultrasound systems, are put in comparison with threshold values to diagnose specific pathologies as in [4]. The ECG signals represent a further example of application case. Amplitude and time measures of the ECG wave components are compared with reference values to assess the occurrence of a cardiac disease as in [2].

In the industrial world, measures coming from tests or process control activities are used for making decisions on conformity assessment and on possible drift phenomena or out-of-control conditions regulated by limits, [37]-[39]. When information on measurement uncertainty is available, the criteria reported in Section III can guide the decision-maker by taking into account the effects due to uncertainty so improving the reliability of the final decision.

In everyday life, user can face the problem to put in comparison the result of a measurement with a limit or to take a decision on what quantity is greater than other. Although, this application case falls into the scope of the present work, the user has to evaluate the risks and consequences associated to a wrong decision. In fact, the main weakness of such decisional criteria is due to the necessity to evaluate the measurement uncertainty. As a consequence, if the user is not experienced with such issue, the proposed criteria could appear complex or inapplicable. In addition, in some cases, uncertainty evaluation is not performed because user does not know sufficiently the measurand and the investigated phenomenon, or if the accuracy of the used instrumentation is unknown, or if the uncertainty evaluation is considered time-consuming because the benefits in using a decision-making procedure do not overcome the costs due to the procedure application. However, note that all the previous cases are index of unavailable information or of low-risk processes.

As a general rule, the use of the proposed criteria is suggested when information on measurement uncertainty is available or when risks and consequences associated to a possible wrong decision are significant. In all these cases, it is strongly recommended the evaluation of the uncertainty associated to the measurement results according to the GUM. Consequently, since the proposed criteria are GUM-based, their use will be a direct consequence of the uncertainty evaluation. They can guide the user in order to make the proper decision by taking into account the uncertainty influence so reducing the risks to make a wrong decision. The main strengths of the proposed criteria are: i) their general assumptions; ii) the theory based on the uncertainty definition; iii) their application to any problem where a measurement result has to be put in comparison with a limit or with another measurement result; iv) their applicability to a large number of cases can be found in the practice; v) their capability in addressing the issue of the comparison between two measurement results having uncertainty.

The Joint Committee on Guides in Metrology (JCGM) has issued in 2012 the document *JCGM 106* to assess the role of measurement uncertainty in Conformity Assessment, [28]. A Bayesian approach is proposed in that document in order to balance the risks of accepting non-conforming items (*consumer's risk*) or rejecting conforming items (*producer's risk*). The main weaknesses of such method are due to the use of Bayes' Theorem, although this is not in contrast with the GUM, it introduces further new theory and mathematical concepts to address the matter. In addition, even though its greater complexity, the *JCGM 106* concerns only the conformity assessment problem discussed in III.C, so disregarding all cases falling into III.A and III.B. In addition, about the use of the Bayesian approach, R. Willink and R. White in [40] alert metrologists about the differences between the frequentist approach of the GUM and Bayesian approaches and the consequences of those differences. Although, it is often claimed that Bayesian approaches are philosophically consistent and are able to tackle problems beyond the reach of classical statistics, according to the authors, the value to science of any statistical analysis is in the long-term success rates and on this point, classical methods perform well and Bayesian analyses can perform poorly. Thus the purpose of this note is to highlight some of the weaknesses of the Bayesian approach. Willink and White argue that moving away from well-established, easily taught frequentist methods that perform well, to computationally expensive and numerically inferior Bayesian analyses recommended by the GUM supplements (such as *JCGM 106*) is ill-advised. In this scenario, the authors recommend that whatever Bayesian methods are adopted, the metrology community should insist on proven long-term numerical performance.

Fuzzy theory is extensively used to address decision-making problems as, for example, in [17]-[19], [24] and [41]. The main weaknesses of this approach concern the greater computational burden and the implementation complexity of such algorithms. In addition, the new introduced concepts and fuzzy theory are modelled and adapted on the definition of measurement uncertainty. However, the fuzzy approach is often based on the vagueness assumption about the measurement process. This assumption clashes with the foundation of the GUM theory which assumes the knowledge on the measurand and on the laws regulating the observed physical phenomenon. In contrary case, the same uncertainty evaluation is discouraged. As a consequence, in most of the cases reported in literature, the two theories (GUM vs Fuzzy) seem to collide in their fundaments.

V. Conclusions

In the present paper, the author has proposed possible decisional criteria to cover the most common cases where a measurement result has to be put in comparison with a reference value, a specification limit, a conformity interval or with another measurement result. The intention is to consider the role of the measurement uncertainty in making decisions. The comparison involving a measurement result is not a simple mathematical comparison between two quantity values. In fact, the measurement uncertainty is cause of vagueness in the measurement result. So, a measurement result is represented by an interval of values which are reasonably attributed to the measurand. As a consequence, uncertainty could be cause of wrong decisions. The proposed decisional criteria provide a guide to users who have to face with this issue. The criteria are simple and intuitive and are based on the *Guide to the expression of Uncertainty in Measurement* [14]. A multitude of decisional criteria have been proposed in literature. Neural networks, fuzzy theory, Bayes' theorem are some of the most common approaches used to solve decision-making problems. Even the *Joint Committee for Guides in Metrology* has proposed possible decisional criteria for the case of conformity assessment [28]. However, no all cases occurring in the practice have been contemplated. So, for example, the comparison between two measurement results is not taken into consideration. The present paper aims to provide a full overview on the decisional problems where measurement results are considered. At the same time, the author wants to provide intuitive and easy to use decisional criteria to encourage their use even by users not experienced with measurement uncertainty and its effects on the decisional process. No further concepts or knowledge are required except for the theory to evaluate uncertainty in measurements reported in *GUM*.

The final aim of the present work is to propose general criteria so to cover all the cases which can be met in the practice, first and foremost the comparison between two measurement results. The last specific issue (which is faced in III.B) makes the proposed criteria original with respect to the others contributions reported in literature.

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