A variational approach to multiplicity results for boundary value problems on the real line

Original
A variational approach to multiplicity results for boundary value problems on the real line / Barletta, G., Bonanno, G., O'Regan, D.. - In: PROCEEDINGS OF THE ROYAL SOCIETY OF EDINBURGH. SECTION A. MATHEMATICS. - ISSN 0308-2105. - 145:1(2015), pp. 13-29. [10.1017/S0308210513001200]

Published
DOI: http://doi.org/10.1017/S0308210513001200
The final published version is available online at:https://www.cambridge.org/core/journals/proceedings-of-

Proceedings of the Royal Society of Edinburgh, **144A**, 1–17, *2014*

A variational approach to multiplicity results for boundary-value problems on the real line

Gabriele Bonanno
Department of Civil, Computer, Construction, Environmental Engineering and Applied Mathematics, University of Messina, 98166 Messina, Italy

Giuseppina Barletta
Dipartimento di Meccanica e Materiali (MECMAT), University of Reggio Calabria, 89100 Reggio Calabria, Italy

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

Author: please check running head on odd pages – abbreviated form of title OK?

Author: email address required for (corresponding) author.

(MS received 26 July 2013; accepted 27 November 2013)

We study the existence and multiplicity of solutions for a parametric equation driven by the p-Laplacian operator on unbounded intervals. Precisely, by using a recent local minimum theorem we prove the existence of a non-trivial non-negative solution to an equation on the real line, without assuming any asymptotic condition neither at 0 nor at ∞ on the nonlinear term. As a special case, we note the existence of a non-trivial solution for the problem when the nonlinear term is sublinear at 0. Moreover, under a suitable superlinear growth at ∞ on the nonlinearity we prove a multiplicity result for such a problem.

1. Introduction

Boundary-value problems (briefly BVPs) on infinite intervals model many problems arising from physical phenomena, such as the flow of a gas through a semi-infinite porous medium or non-Newtonian fluid flows (see [20] and references therein), and, as a result, they are widely studied (see, for example, [15, 18, 25]). More generally, elliptic equations on the whole space were investigated and we refer the reader to [3, 11] and [24, ch. 6.4] for an overview on this subject; see also [12] for the non-smooth case.

Changes to sentence OK?

In order to aid clarity, here and in other parts of the paper I have moved some inline equations to display and also brought out of display text that runs on from the preceding paragraph. Also, in this equation I assume that you refer to the variable x and have assumed that 'a.e.' means 'almost every': all OK? Please mark any instances where 'a.e.' means 'almost everywhere'.

The aim of this paper is to investigate elliptic problems on the real line. To be precise, we are interested in the existence and multiplicity of non-negative solutions to the following problem. Find $u \in W^{1,p}(\mathbb{R})$ satisfying

(P_λ) $(|u'(x)|^{p-2}u'(x))' + B|u(x)|^{p-2}u(x) = \lambda\alpha(x)g(u(x))$ for almost every (a.e.) x in $\mathbb{R}$,

changes to display OK?

where λ is a real positive parameter, B is a real positive number, and $\alpha, g\colon \mathbb{R} \to \mathbb{R}$ are two functions such that

$$\alpha \in L^1(\mathbb{R}), \quad \alpha(x) \geqslant 0 \text{ for a.e. } x \in \mathbb{R}, \quad \alpha \not\equiv 0,$$

and g is continuous and non-negative. Many authors studied BVPs (parametric or otherwise) on unbounded intervals and approached the problem using different techniques (see, for example, [[illegible], 15, 17–20, 23, 25]). In particular, in [18] the authors studied the existence and uniqueness of positive solutions of a one-parameter family of logistic equations of the type

$$u'' + af(x)u - b(x)u^p = 0 \quad \text{in } \mathbb{R} \text{ or in } \mathbb{R}_+.$$

They obtained the solution as a minimum point of the energy functional associated with the previous equation in $\mathcal{D}^{1,2}(\mathbb{R})$ and $\mathcal{D}_0^{1,2}(\mathbb{R}_+)$, respectively, with $a \in (\lambda_1, \lambda_*)$ and they showed the non-existence of solutions for $a \geqslant \lambda_*$.

The method of upper and lower solutions was used in [15, 25] for two Sturm–Liouville value problems in $[0, +\infty[$. In [25] the authors looked for positive unbounded solutions. They gave necessary and sufficient conditions for the existence of positive solutions, with a sublinear growth assumption on the nonlinear term. Using a particular cone and a fixed-point theorem they also discussed the multiplicity. The method of unbounded upper and lower solutions of [25] was generalized in [15], where the authors used the Schauder fixed-point theorem to show the existence of a positive solution to their problem.

In our paper the structure of the problem, as well as the assumptions on the nonlinear term, are not comparable with the papers cited above. Our primary tool in proving the main result of this paper is a local minimum theorem recently established in [5] and, in order to obtain multiple solutions, we use a two critical points theorem presented in [6]. Our main result (theorem 3.1) ensures the existence of a non-trivial solution without requiring any asymptotic condition on g either at 0 or at ∞. Moreover, as a consequence, we point out a result where only the sublinearity of g at 0 is required in order to obtain the existence of a non-trivial solution (see corollary 3.4). Finally, we present a result where two non-trivial solutions are guaranteed under a suitable growth of g at ∞ (see theorem 3.10 and remark 3.11). Such a growth type was introduced and developed by Ambrosetti and Rabinowitz in [1] and it is worth noting that such an assumption is usually accompanied by the superlinearity of g at 0 to ensure the existence of only one non-trivial solution.

As an example, here we point out the following special cases of our results.

Theorem 1.1. *Assume that*

$$\int_0^{11} g(t)\,\mathrm{d}t < 11 \int_0^1 g(t)\,\mathrm{d}t.$$

Then, for each

$$\lambda \in \left] \frac{11}{\pi} \frac{1}{\int_0^1 g(t)\,\mathrm{d}t}, \frac{11}{\pi} \frac{11}{\int_0^{11} g(t)\,\mathrm{d}t} \right[$$

the problem

$$-u'' + u = \lambda \frac{g(u)}{1+x^2}, \quad x \in \mathbb{R},$$
$$u(-\infty) = u(+\infty) = 0,$$

admits at least one non-trivial and non-negative classical solution $u_{0,\lambda}$ *such that*

$$|u_{0,\lambda}(x)| < 11$$

for all $x \in \mathbb{R}$.

THEOREM 1.2. *Assume that* α *is continuous in* $\mathbb{R}$, $g(0) > 0$ *and*

$$0 < \mu \int_0^{\xi} g(s)\,\mathrm{d}s \leqslant \xi g(\xi)$$

for all $\xi \geqslant s$ *and for some* $s > 0$ *and* $\mu > p$.

Then there exists $\lambda^* > 0$ *such that, for each* $\lambda \in \,]0, \lambda^*[$, *the problem*

$$-(|u'(x)|^{p-2}u'(x))' + |u(x)|^{p-2}u(x) = \lambda\alpha(x)g(u(x)), \quad x \in \mathbb{R},$$
$$u(-\infty) = u(+\infty) = 0,$$

admits at least two non-trivial and non-negative classical solutions.

In theorem 1.1, no asymptotic conditions either at 0 or at ∞ are required, while theorem 1.2 ensures two non-trivial solutions under a suitable condition at ∞ of Ambrosetti–Rabinowitz type.

The paper is arranged as follows. In § 2 we establish all the preliminary results that we need, and in § 3 we present our main results.

2. Mathematical background

Let $(E, |\cdot|)$ be a real Banach space. We denote by E^* the dual space of E, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between E^* and E.

We denote by $|\cdot|$ and by $|\cdot|_t$ the usual norms on $\mathbb{R}$ and on $L^t(\mathbb{R})$, for all $t \in [1, +\infty]$, while $W^{1,p}(\mathbb{R})$ indicates the closure of $C_0^{\infty}(\mathbb{R})$ with respect to the norm

$$\|u\|_{1,p} := (|u'|_p^p + |u|_p^p)^{1/p}.$$

When $p = 2$ the norm is induced by the scalar product

$$(u, v) = (u', v')_{L^2} + (u, v)_{L^2}.$$

It is well known that $W^{1,p}(\mathbb{R}) \equiv W_0^{1,p}(\mathbb{R})$ and $W^{1,p}(\mathbb{R})$ is embedded in $L^t(\mathbb{R})$ for any $t \in [p, +\infty]$.

REMARK 2.1. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\mathbb{R})$, then it has a subsequence that pointwise converges to some $u \in W^{1,p}(\mathbb{R})$ and also weakly converges in $L^{\infty}(\mathbb{R})$. Indeed, it can be inferred from the compact embedding $W^{1,p}(\mathbb{R}) \hookrightarrow C([-R, R])$, $R > 0$, and the continuity of $W^{1,p}(\mathbb{R}) \to L^{\infty}(\mathbb{R})$.

In the following, we consider $W^{1,p}(\mathbb{R})$ endowed by the norm Change OK?

$$\|u\| = \left(\int_{\mathbb{R}} (|u'(x)|^p + B|u(x)|^p)\,\mathrm{d}x\right)^{1/p},$$

which is equivalent to the usual one. We have the following proposition. Equivalent to the usual norm? Please clarify.

Proposition 2.2. *One has*

$$|u|_\infty \leqslant c_B \|u\| \tag{2.1}$$

for all $u \in W^{1,p}(\mathbb{R})$, *where*

$$c_B = 2^{(p-2)/p} \left(\frac{p-1}{p}\right)^{(p-1)/p} \left(\frac{1}{B}\right)^{(p-1)/p^2}. \tag{2.2}$$

Proof. We follow the argument in [10, theorem 4, p. 138, formula (4.61)], taking the equivalence of the norms into account. For clarity, we give a sketch of the proof. Let $v \in W^{1,1}(\mathbb{R})$. From

$$v(z) - v(w) = \int_w^z v'(t)\,\mathrm{d}t,$$

taking into account that $\lim_{|y|\to+\infty} v(y) = 0$, one has

$$v(x) = \int_{-\infty}^x v'(t)\,\mathrm{d}t \quad \text{and} \quad -v(x) = \int_x^\infty v'(t)\,\mathrm{d}t.$$

Hence,

$$2|v(x)| \leqslant \int_{-\infty}^\infty |v'(t)|\,\mathrm{d}t,$$

that is,

$$|v(x)| \leqslant \frac{1}{2}\int_{\mathbb{R}} |v'(t)|\,\mathrm{d}t$$

for all $v \in W^{1,p}(\mathbb{R})$. Now, let $u \in W^{1,p}(\mathbb{R})$. By choosing $v(x) = B^{(p-1)/p}|u(x)|^p$ for all $x \in \mathbb{R}$, one has

$$B^{(p-1)/p}|u(x)|^p \leqslant \frac{1}{2}\int_{\mathbb{R}} B^{(p-1)/p} p|u(t)|^{p-1}|u'(t)|\,\mathrm{d}t.$$

From Hölder inequality one has 'the Hölder inequality' or 'Hölder's inequality'?

$$B^{(p-1)/p}|u(x)|^p \leqslant \frac{p}{2}(B^{1/p}|u|_p)^{p-1}|u'|_p,$$

that is,

$$|u(x)| \leqslant \left(\frac{1}{B}\right)^{(p-1)/p^2} \left(\frac{p}{2}\right)^{1/p} (B^{1/p}|u|_p)^{(p-1)/p}|u'|_p^{1/p}.$$

Noting that $x^\alpha y^{1-\alpha} \leqslant \alpha^\alpha(1-\alpha)^{(1-\alpha)}(x+y)$, $x, y \geqslant 0$, $0 < \alpha < 1$ (see [10, p. 130, formula (4.47)]), one has

$$|u(x)| \leqslant \left(\frac{1}{B}\right)^{(p-1)/p^2} \left(\frac{p}{2}\right)^{1/p} \left(\frac{p-1}{p}\right)^{(p-1)/p} \left(\frac{1}{p}\right)^{1/p} \times \left[\left(\int_{\mathbb{R}} |u'(x)|^p\,\mathrm{d}x\right)^{1/p} + \left(\int_{\mathbb{R}} B|u(x)|^p\,\mathrm{d}x\right)^{1/p}\right].$$

Therefore, taking into account the classical inequality

$$a^{1/p} + b^{1/p} \leqslant 2^{(p-1)/p}(a+b)^{1/p},$$

one has

$$|u(x)| \leqslant \left(\frac{1}{B}\right)^{(p-1)/p^2} \left(\frac{1}{2}\right)^{1/p} \left(\frac{p-1}{p}\right)^{(p-1)/p} 2^{(p-1)/p} \times \left[\left(\int_{\mathbb{R}} |u'(x)|^p \, \mathrm{d}x\right) + \left(\int_{\mathbb{R}} B|u(x)|^p \, \mathrm{d}x\right)\right]^{1/p},$$

that is,

$$|u|_\infty \leqslant c_B \|u\|,$$

and the proof is complete. □

We set

$$G(t) = \int_0^t g(\xi) \, \mathrm{d}\xi \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Our hypotheses on g guarantee that $G \in C^1(\mathbb{R})$ and $G'(t) = g(t) \geqslant 0$ for all $t \in \mathbb{R}$, so G is non-decreasing.

Now, we put

$$\Phi(u) = \frac{1}{p}\|u\|^p, \tag{2.4}$$

and we define $\Psi\colon W^{1,p}(\mathbb{R}) \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}} \alpha(x) G(u(x)) \, \mathrm{d}x = \int_{\mathbb{R}} \alpha(x) \left(\int_0^{u(x)} g(\xi) \, \mathrm{d}\xi\right) \mathrm{d}x, \quad \forall u \in W^{1,p}(\mathbb{R}). \tag{2.5}$$

It is clear that the assumptions on α and g guarantee that the functional Ψ is well defined. In fact, one sees that the following inequality holds for any $u \in W^{1,p}(\mathbb{R})$.

$$\begin{aligned} |\Psi(u)| &\leqslant \int_{\mathbb{R}} \alpha(x)|G(u(x))| \, \mathrm{d}x \leqslant \int_{\mathbb{R}} \alpha(x) \max_{x\in\mathbb{R}} |G(u(x))| \, \mathrm{d}x \leqslant \int_{\mathbb{R}} \alpha(x) \max_{|\xi|\leqslant|u|_\infty} |G(\xi)| \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \alpha(x) \max\{-G(-|u|_\infty), G(|u|_\infty)\} \, \mathrm{d}x \\ &= M_u|\alpha|_1. \end{aligned}$$

Our main tool is a local minimum theorem proved in [5] (see [5, theorem 3.1]). Here, we use the version as given in [6] (see theorem 2.6; see also [8,22]). Before stating it, we give some definitions. Let E be a real Banach space and let $\Phi, \Psi\colon E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals, put $I_\lambda = \Phi - \lambda\Psi$, $\lambda > 0$, and fix $r \in\,]-\infty, +\infty]$.

The list below has been bullet-pointed for clarity: OK? Also, since PS presumably indicates 'Palais–Smale', it has been moved to Roman type in order to avoid possible confusion with a product of variables P and S.

Word added – OK?

Definition 2.3. We say that a functional I_λ verifies the Palais–Smale condition cut-off upper at r (in short, the $(\mathrm{PS})^{[r]}$-condition) if any sequence $\{u_n\}$ such that

- $I_\lambda(u_n)$ is bounded,
- $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_* = 0$,
- $\Phi(u_n) < r$

has a convergent subsequence.

Journal style is to not introduce definition (or theorem, lemma etc.) environments with a fragment so I have completed this sentence: OK?

Word added – OK?

When $r = +\infty$ the previous definition recovers the classical definition of the Palais–Smale condition given below.

Definition 2.4. We say that the functional I_λ verifies the Palais–Smale condition (in short, the (PS)-condition) if any sequence $\{u_n\}$ such that

- $I_\lambda(u_n)$ is bounded,
- $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_* = 0$

has a convergent subsequence.

Definition 2.5. We say that $u \in E$ is a critical point of I_λ when $I'_\lambda(u) = 0_{E^*}$, that is, $I'_\lambda(u)(v) = 0$ for all $v \in E$.

Theorem 2.6 (Bonanno [6, theorem 2.2]). *Let E be a real Banach space and let $\Phi, \Psi \colon E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_E \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in E$ with $0 < \Phi(\tilde{u}) < r$ such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.6}$$

and, for each

$$\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)$^{[r]}$-condition. Then, for each

$$\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

there is a $u_\lambda \in \Phi^{-1}(]0,r[)$ (hence, $u_\lambda \neq 0$) such that $I_\lambda(u_\lambda) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(]0,r[)$ and $I'_\lambda(u_\lambda) = 0$.

We explicitly observe that, contrary to [22, theorem 2.5], in theorem 2.6 the sequential weak lower semi-continuity of I_λ is not required and, in addition, the local minimum is non-trivial.

Now we recall a multiple critical points result obtained in [6] that is based on the theorem of the local minimum [5, theorem 3.1] and on the classical theorem of Ambrosetti–Rabinowitz in [1].

Theorem 2.7 (Bonanno [6, theorem 3.2]). *Let E be a real Banach space and let $\Phi, \Psi \colon E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each

$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[,$$

the functional I_λ admits two distinct critical points.

In our situation, the space E coincides with $W^{1,p}(\mathbb{R})$, while $I_\lambda \colon W^{1,p}(\mathbb{R}) \to \mathbb{R}$ is the energy functional related to (P_λ), and is defined as

Changes to sentence OK?

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where Φ, Ψ are given in (2.4) and (2.5). It is well know that Φ, Ψ are continuously Gâteaux differentiable. If u is a critical point of I_λ, then $I'_\lambda(u) \equiv 0$, that is,

$$\int_{\mathbb{R}} (|u'(x)|^{p-2}u'(x)v'(x) + B|u(x)|^{p-2}u(x)v(x) - \lambda\alpha(x)g(u(x))v(x))\,\mathrm{d}x = 0$$

for all $v \in W^{1,p}(\mathbb{R})$, so u is a (weak) solution to (P_λ). Moreover, when α is, in addition, a continuous function on $\mathbb{R}$, the (weak) solutions of (P_λ) are actually classical, as standard computations show.

Lemma 2.8. *Let Φ and Ψ be defined as above and fix $\lambda > 0$. Then, $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for any $r > 0$.*

Proof. Let $\{u_n\} \subseteq W^{1,p}(\mathbb{R})$ be a sequence such that $\{I_\lambda(u_n)\}$ is bounded,

The equation below has been moved to display to avoid a bad line-break: OK?

$$\lim_{n\to+\infty} \|I_\lambda(u_n)\|_{W^{1,p}(\mathbb{R})^*} = 0$$

and $\Phi(u_n) < r$ for all $n \in \mathbb{N}$.

From $\Phi(u_n) < r$, taking into account that Φ is coercive, $\{u_n\}$ is bounded $W^{1,p}(\mathbb{R})$. Therefore, up to a subsequence, $u_n(x) \to u(x)$, $x \in \mathbb{R}$, and $\{u_n\}$ weakly converges to u in $L^\infty(\mathbb{R})$ (see remark 2.1).

'bounded in $W^{1,p}(\mathbb{R})$'?

Now, taking into account that $\{u_n\}$ is bounded in $L^\infty(\mathbb{R})$ (being weakly convergent in $L^\infty(\mathbb{R})$), there is an $M > 0$ such that $|u_n(x)| \leqslant M$ for all $n \in \mathbb{N}$, for a.e. $x \in \mathbb{R}$. It follows that $g(u_n(x)) \leqslant \max_{|\xi|\leqslant M} g(\xi)$ for which $\alpha g(u_n) \in L^1(\mathbb{R})$ for all $n \in \mathbb{N}$. Since $g(u_n(x)) \to g(u(x))$ for a.e. $x \in \mathbb{R}$ (g is a continuous function), from Lebesgue's theorem one has that $\{\alpha g(u_n)\}$ is strongly converging to $\alpha g(u)$ in $L^1(\mathbb{R})$. Since $u_n \rightharpoonup u$ in $L^\infty(\mathbb{R})$, $\alpha g(u_n), \alpha g(u) \in L^1(\mathbb{R}) \subseteq (L^\infty(\mathbb{R}))^*$ (see [9, p. 102]) and $\alpha g(u_n) \to \alpha g(u)$ in $L^1(\mathbb{R})$, the definition of weak convergence and [9, proposition III.5(iv)] leads to

Changes to sentence OK?

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \alpha(x)g(u_n(x))(u_n(x) - u(x))\,\mathrm{d}x = 0. \tag{2.7}$$

Now, from $\lim_{n\to+\infty} \|I_\lambda(u_n)\|_{W^{1,p}(\mathbb{R})^*} = 0$, there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, such that

$$\left| \int_{\mathbb{R}} (|u'_n|^{p-2}u'_n v' + B|u_n|^{p-2}u_n v - \lambda\alpha g(u_n)v)\,\mathrm{d}x \right| \leqslant \varepsilon_n$$

for all $n \in \mathbb{N}$, for all $v \in W^{1,p}(\mathbb{R})$ such that $\|v\| \leqslant 1$.

Setting $v = (u_n - u)/\|u_n - u\|$, one has

$$\int_{\mathbb{R}} (|u_n'|^{p-2} u_n'(u_n' - u') + B|u_n|^{p-2} u_n(u_n - u) \\ - \lambda\alpha g(u_n)(u_n - u))\,\mathrm{d}x \leqslant \varepsilon_n \|u_n - u\| \quad (2.8)$$

for all $n \in \mathbb{N}$.

Noting that

$$|a|^{p-1}|b| \leqslant \frac{p-1}{p}|a|^p + \frac{1}{p}|b|^p,$$

one has

$$\begin{aligned}\int_{\mathbb{R}} (|u_n'|^{p-2} u_n'(u_n' - u') &+ B|u_n|^{p-2} u_n(u_n - u))\,\mathrm{d}x \\ &= \int_{\mathbb{R}} (|u_n'|^p + B|u_n|^p)\,\mathrm{d}x - \int_{\mathbb{R}} (|u_n'|^{p-2} u_n' u' + B|u_n|^{p-2} u_n u)\,\mathrm{d}x \\ &\geqslant \|u_n\|^p - \int_{\mathbb{R}} \left(\frac{p-1}{p}|u_n'|^p + \frac{1}{p}|u'|^p + B\frac{p-1}{p}|u_n|^p + B\frac{1}{p}|u|^p\right)\mathrm{d}x \\ &= \|u_n\|^p - \frac{p-1}{p}\|u_n\|^p - \frac{1}{p}\|u\|^p \\ &= \frac{1}{p}\|u_n\|^p - \frac{1}{p}\|u\|^p.\end{aligned}$$

Thus, from (2.8), one has

$$\frac{1}{p}\|u_n\|^p - \frac{1}{p}\|u\|^p \leqslant \lambda \int_{\mathbb{R}} \alpha g(u_n)(u_n - u)\,\mathrm{d}x + \varepsilon_n \|u_n - u\|,$$

that is,

$$-\varepsilon_n \|u_n - u\| + \frac{1}{p}\|u_n\|^p \leqslant \lambda \int_{\mathbb{R}} \alpha g(u_n)(u_n - u)\,\mathrm{d}x + \frac{1}{p}\|u\|^p. \quad (2.9)$$

Taking into account (2.7), from (2.9) one has

$$\limsup_{n \to +\infty} \frac{1}{p}\|u_n\|^p \leqslant \frac{1}{p}\|u\|^p.$$

Hence, [9, proposition III.30] ensures that $\{u_n\}$ strongly converges to $u \in W^{1,p}(\mathbb{R})$ and the proof is complete. □

Now, if we assume in addition that g satisfies an Ambrosetti–Rabinowitz-type condition at ∞, then I_λ satisfies the classical (PS)-condition. To be precise, we have the following result.

LEMMA 2.9. *Assume that*

(AR) *there are $s > 0$ and $\mu > p$ such that $0 < \mu G(\xi) \leqslant \xi g(\xi)$ for all $\xi \geqslant s$.*

Then, I_λ satisfies the (PS)-condition and it is unbounded from below.

Proof. Let $\{u_n\}$ be a sequence such that

$$|I_\lambda(u_n)| \leqslant M \quad \text{for some } M > 0 \text{ for all } n \in \mathbb{N}, \tag{2.10}$$

and

$$I'_\lambda(u_n) \to 0 \quad \text{in } W^{1,p}(\mathbb{R})^* \text{ as } n \to \infty. \tag{2.11}$$

First, we claim that there is a $K \geqslant 0$ such that

$$u_n(x) \geqslant -K \tag{2.12}$$

for a.e. $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. To this end, setting u_n^- in the usual way, we prove that $\{u_n^-\}$ is bounded in $W^{1,p}(\mathbb{R})$. From (2.11) one has $|I'_\lambda(u_n)(v)| \leqslant \varepsilon_n \|v\|$ for all $v \in W^{1,p}(\mathbb{R})$ with $\varepsilon_n \to 0^+$. Thus, in particular, $|I'_\lambda(u_n)(u_n^-)| \leqslant \varepsilon_n \|u_n^-\|$, that is,

Changes to sentence OK?

$$\left| \int_{\mathbb{R}} (|u'_n|^{p-2} u'_n {u_n^-}' + B|u_n|^{p-2} u_n u_n^-)\,\mathrm{d}x - \lambda \int_{\mathbb{R}} \alpha(x) g(u_n(x)) u_n^-(x)\,\mathrm{d}x \right| \leqslant \varepsilon_n \|u_n^-\|,$$

hence

$$\|u_n^-\|^p + \lambda \int_{\mathbb{R}} \alpha(x) g(u_n(x)) u_n^-(x)\,\mathrm{d}x \leqslant \varepsilon_n \|u_n^-\|.$$

Therefore,

$$0 \leqslant \|u_n^-\|^p \leqslant \|u_n^-\|^p + \lambda \int_{\mathbb{R}} \alpha(x) g(u_n(x)) u_n^-(x)\,\mathrm{d}x \leqslant \varepsilon_n \|u_n^-\|.$$

Thus, $\{u_n^-\}$ strongly converges to 0 in $W^{1,p}(\mathbb{R})$, so it is bounded in $W^{1,p}(\mathbb{R})$.

Thus, in particular, it is bounded in $L^\infty(\mathbb{R})$ (see (2.1)) and one has $0 \leqslant u_n^-(x) \leqslant K$ for some $K \geqslant 0$ and for a.e. $x \in \mathbb{R}$, and our claim is proved.

Now, we prove that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R})$. Again, from (2.11), one has $|I'_\lambda(u_n)(u_n)| \leqslant \varepsilon_n \|u_n\|$. Then,

$$-I'_\lambda(u_n)(u_n) \leqslant \varepsilon_n \|u_n\| \tag{2.13}$$

for all $n \in \mathbb{N}$ and with $\varepsilon_n \to 0^+$.

On the other hand, one has

$$I'_\lambda(u_n)(u_n) = \|u_n\|^p - \lambda \int_{\mathbb{R}} \alpha(x) g(u_n(x)) u_n(x)\,\mathrm{d}x$$

and

$$\begin{aligned} \frac{1}{\mu} I'_\lambda(u_n)(u_n) &= \frac{1}{\mu}\|u_n\|^p - \frac{\lambda}{\mu} \int_{\mathbb{R}} \alpha(x) g(u_n(x)) u_n(x)\,\mathrm{d}x \\ &= \frac{1}{\mu}\|u_n\|^p - \frac{\lambda}{\mu} \int_{\mathbb{R}} \alpha(x)[g(u_n(x)) u_n(x) - \mu G(u_n(x))]\,\mathrm{d}x \\ &\quad - \lambda \int_{\mathbb{R}} \alpha(x) G(u_n(x))\,\mathrm{d}x. \end{aligned}$$

It follows that

$$I_\lambda(u_n) - \frac{1}{\mu}I'_\lambda(u_n)(u_n) = \frac{1}{p}\|u_n\|^p - \lambda\int_{\mathbb{R}}\alpha(x)G(u_n(x))\,\mathrm{d}x - \frac{1}{\mu}\|u_n\|^p + \frac{\lambda}{\mu}\int_{\mathbb{R}}\alpha(x)[g(u_n(x))u_n(x) - \mu G(u_n(x))]\,\mathrm{d}x + \lambda\int_{\mathbb{R}}\alpha(x)G(u_n(x))\,\mathrm{d}x,$$

that is,

$$I_\lambda(u_n)-\frac{1}{\mu}I'_\lambda(u_n)(u_n) = \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|^p+\frac{\lambda}{\mu}\int_{\mathbb{R}}\alpha(x)[g(u_n(x))u_n(x)-\mu G(u_n(x))]\,\mathrm{d}x.$$

Taking (AR) into account, one has

The equation below has been moved to display to avoid a bad line-break: OK?

$$\int_{u_n(x)\geqslant s}\alpha(x)[g(u_n(x))u_n(x) - \mu G(u_n(x))]\,\mathrm{d}x \geqslant 0.$$

Moreover, from (2.12), one has

$$\left|\int_{-K\leqslant u_n(x)<s}\alpha(x)[g(u_n(x))u_n(x) - \mu G(u_n(x))]\,\mathrm{d}x\right| \leqslant \int_{-K\leqslant u_n(x)<s}\alpha(x)\max_{\xi\in[-K,s]}|g(\xi)\xi - \mu G(\xi)|\,\mathrm{d}x \leqslant |\alpha|_1 T,$$

where $T = \max_{\xi\in[-K,s]}|g(\xi)\xi - \mu G(\xi)|$. Hence,

$$I_\lambda(u_n) - \frac{1}{\mu}I'_\lambda(u_n)(u_n) \geqslant \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|^p + \frac{\lambda}{\mu}\int_{K\leqslant u_n(x)<s}\alpha(x)[g(u_n(x))u_n(x) - \mu G(u_n(x))]\,\mathrm{d}x \geqslant \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|^p - \frac{\lambda}{\mu}|\alpha|_1 T.$$

From (2.10) and (2.13), it follows that

$$\left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|^p - \frac{\lambda}{\mu}|\alpha|_1 T \leqslant M + \frac{1}{\mu}\varepsilon_n\|u_n\|,$$

that is,

$$\left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|^p \leqslant M + \frac{1}{\mu}\varepsilon_n\|u_n\| + \frac{\lambda}{\mu}|\alpha|_1 T. \tag{2.14}$$

Hence, (2.14) ensures that $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R})$.

Now, arguing exactly as in the proof of lemma 2.8, $\{u_n\}$ admits a convergent subsequence, so I_λ satisfies (PS).

Finally, standard computations show that (AR) implies that

$$G(\xi) \geqslant a_1\xi^\mu - a_2$$

for all $\xi \geqslant 0$ and some positive constants a_1 and a_2, and hence I_λ is unbounded from below. The proof is complete. $\square$

3. Main results

Throughout this section we adopt the following notation for some constants that will appear often in the following. Put

$$\left.\begin{aligned} A &= \frac{\int_{-1}^{1} \alpha(x)\,\mathrm{d}x}{|\alpha|_1} = \frac{\alpha_0}{|\alpha|_1}, \\ l &= c_B\left(2^{2p-1} + \frac{B}{2(p+1)} + 2B\right)^{1/p}, \\ R &= \frac{A}{l^p}, \end{aligned}\right\} \tag{3.1}$$

where c_B is given in proposition 2.2.

We observe that if, for example, $p = 2$, $B = 1$ and $\alpha(x) = 1/(1+x^2)$, then $l = (\frac{61}{12})^{1/2}$ and $R = \frac{6}{61}$.

Our main result is the following.

THEOREM 3.1. *Assume that there exist two positive constants γ, κ, with $\kappa < \gamma$, such that*

$$\frac{G(\gamma)}{\gamma^p} < R\frac{G(\kappa)}{\kappa^p}. \tag{3.2}$$

Then, for each

$$\lambda \in \left]\frac{1}{pc_B^p|\alpha|_1}\frac{1}{R}\frac{\kappa^p}{G(\kappa)}, \frac{1}{pc_B^p|\alpha|_1}\frac{\gamma^p}{G(\gamma)}\right[,$$

problem (P_λ) admits at least one non-trivial and non-negative solution $u_{0,\lambda}$ such that $|u_{0,\lambda}|_\infty < \gamma$.

Proof. Our aim is to apply theorem 2.6. To this end, we take $E = W^{1,p}(\mathbb{R})$, and Φ, Ψ, I_λ are as in § 2. All of the assumptions on regularity required on Φ and Ψ are established and, from lemma 2.8, the functional I_λ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r > 0$. It is enough to prove (2.6). To this end, choose $r = (1/pc_B^p)\gamma^p$ and

Perhaps ‘all of the assumptions on the necessary regularity on...’? Change OK?

$$\tilde{u}(x) = \begin{cases} 4\kappa(x+1)+\kappa & \text{if } x \in [-5/4, -1[, \\ \kappa & \text{if } x \in [-1, 1], \\ 4\kappa(1-x)+\kappa & \text{if } x \in]1, 5/4], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{u} \in W^{1,p}(\mathbb{R})$. Moreover, one has

$$\begin{aligned}
\Phi(\tilde{u}) &= \frac{1}{p}\|\tilde{u}\|^p \\
&= \frac{1}{p}\left(\int_{\mathbb{R}} |\tilde{u}'(x)|^p \, \mathrm{d}x + B \int_{\mathbb{R}} |\tilde{u}(x)|^p \, \mathrm{d}x \right) \\
&= \frac{1}{p}\left(\frac{(4\kappa)^p}{2} + B\left(\frac{1}{2(p+1)} + 2 \right) \kappa^p \right) \\
&= \frac{\kappa^p}{p}\left(2^{2p-1} + \frac{B}{2(p+1)} + 2B \right) \\
&= \kappa^p \frac{1}{p} \frac{l^p}{c_B^p},
\end{aligned}$$

and

$$\Psi(\tilde{u}) = \int_{-5/4}^{5/4} \alpha(x) G(\tilde{u}(x)) \, \mathrm{d}x \geqslant \int_{-1}^{1} \alpha(x) G(\tilde{u}(x)) \, \mathrm{d}x = \alpha_0 G(\kappa).$$

Hence,

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geqslant |\alpha|_1 p c_B^p \frac{A}{l^p} \frac{G(\kappa)}{\kappa^p}. \tag{3.3}$$

Moreover, from $\kappa < \gamma$ one has $\kappa l < \gamma$. In fact, arguing by contradiction, if we assume that $\kappa < \gamma \leqslant l\kappa$, one has $R(G(\kappa)/\kappa^p) = A(G(\kappa)/l^p \kappa^p) \leqslant A(G(\kappa)/\gamma^p) \leqslant G(\kappa)/\gamma^p \leqslant G(\gamma)/\gamma^p$, which contradicts (3.2). Thus, from $\kappa l < \gamma$, one has

$$\Phi(\tilde{u}) = \kappa^p \frac{1}{p} \frac{l^p}{c_B^p} < \frac{1}{p c_B^p} \gamma^p = r,$$

so $0 < \Phi(\tilde{u}) < r$.

Moreover, for all $u \in W^{1,p}(\mathbb{R})$ such that $\|u\| < (pr)^{1/p}$, taking proposition 2.2 into account, one has

$$|u|_\infty \leqslant c_B \|u\| < c_B (pr)^{1/p} = \gamma. \tag{3.4}$$

Hence,

$$\begin{aligned}
\sup_{\Phi(u)<r} \Psi(u) &= \sup_{\|u\|<(pr)^{1/p}} \int_{\mathbb{R}} \alpha(x) G(u(x)) \, \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}} \alpha(x) \sup_{|\xi|<\gamma} G(\xi) \, \mathrm{d}x \\
&\leqslant |\alpha|_1 G(\gamma).
\end{aligned}$$

From this we deduce that

$$\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} \leqslant \frac{|\alpha|_1 G(\gamma)}{(1/p c_B^p)\gamma^p} = |\alpha|_1 p c_B^p \frac{G(\gamma)}{\gamma^p}. \tag{3.5}$$

Hence, from assumption (3.2), owing to (3.3) and (3.5), one has

$$\frac{\sup_{\Phi(u)<r}\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and (2.6) is proved. Moreover, taking into account that, again owing to (3.3) and (3.5), one has

$$\left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[\supseteq \left]\frac{1}{pc_B^p|\alpha|_1}\frac{1}{R}\frac{\kappa^p}{G(\kappa)}, \frac{1}{pc_B^p|\alpha|_1}\frac{\gamma^p}{G(\gamma)}\right[.$$

Therefore, theorem 2.6 ensures that, for all Word added – OK?

$$\lambda \in \left]\frac{1}{pc_B^p|\alpha|_1}\frac{1}{R}\frac{\kappa^p}{G(\kappa)}, \frac{1}{pc_B^p|\alpha|_1}\frac{\gamma^p}{G(\gamma)}\right[,$$

there is a $u_{0,\lambda} \in \Phi^{-1}(]0,r[)$ (hence, $u_{0,\lambda} \neq 0$) such that $I_\lambda(u_{0,\lambda}) \leqslant I_\lambda(u)$ for all $u \in \Phi^{-1}(]0,r[)$ and $I'_\lambda(u_{0,\lambda}) = 0$. It follows that $u_{0,\lambda}$ is a non-zero solution of problem (P_λ) and, from (3.4), one has $|u_{0,\lambda}|_\infty < \gamma$.

Finally, by standard computations, we have $u_{0,\lambda} \geqslant 0$. In fact, from $I'_\lambda(u_{0,\lambda})(v) = 0$ for all $v \in W^{1,p}(\mathbb{R})$, by choosing $v = u_{0,\lambda}^- \geqslant 0$, one has Changes to sentence OK?

$$-\|u_{0,\lambda}^-\|^p = \lambda\int_{\mathbb{R}} \alpha(x)g(u_{0,\lambda}(x))u_{0,\lambda}^-(x)\,\mathrm{d}x \geqslant 0, \tag{3.6}$$

hence $\|u_{0,\lambda}^-\| = 0$. The proof is complete. □

Remark 3.2. Theorem 3.1 ensures the existence of one non-trivial solution without requiring asymptotic conditions either at 0 or at ∞. Theorem 1.1 is an immediate consequence.

Example 3.3. Put

$$\tilde{g}(u) = \begin{cases} u^2 & \text{if } u \leqslant 1, \\ \dfrac{1}{u} & \text{if } 1 < u < 11, \\ h(u) & \text{if } u \geqslant 11, \end{cases}$$

where $h\colon \mathbb{R} \to \mathbb{R}$ is a completely arbitrary function. From theorem 1.1 the problem

$$-u'' + u = \frac{4\pi}{1+x^2}\tilde{g}(u), \quad x \in \mathbb{R},$$
$$u(-\infty) = u(+\infty) = 0,$$

admits at least one non-trivial and non-negative classical solution. Indeed, it is enough to apply theorem 1.1 to the continuous function

$$g^*(u) = \begin{cases} u^2 & \text{if } u \leqslant 1, \\ \dfrac{1}{u} & \text{if } 1 < u < 11, \\ \dfrac{1}{11} & \text{if } u \geqslant 11, \end{cases}$$

so that the solution $\bar{u}$ relative to g^*, with $|\bar{u}|_\infty < 11$, is also a solution to our problem; we note that

$$\int_0^{11} g^*(t)\,\mathrm{d}t = \frac{1}{3} + \ln 11 < \frac{11}{3} = 11 \int_0^1 g^*(t)\,\mathrm{d}t$$

and

$$\frac{11}{\pi} \frac{1}{\int_0^1 g^*(t)\,\mathrm{d}t} = \frac{33}{\pi} < 4\pi < \frac{121}{\pi} \frac{1}{(1/3) + \ln 11} = \frac{11}{\pi} \frac{11}{\int_0^{11} g^*(t)\,\mathrm{d}t}.$$

We now point out some consequences of theorem 3.1.

Corollary 3.4. *Assume that*

$$\lim_{t \to 0^+} \frac{g(t)}{t^{p-1}} = +\infty.$$

Then, for each $\gamma > 0$ *and for each*

$$\lambda \in \left]0, \frac{1}{pc_B^p |\alpha|_1} \frac{\gamma^p}{G(\gamma)}\right[,$$

problem (P_λ) *admits at least one non-trivial and non-negative solution* $u_{0,\lambda}$ *such that* $|u_{0,\lambda}|_\infty < \gamma$.

Proof. Let γ be an arbitrary positive real number and let

$$\lambda \in \left]0, \frac{1}{pc_B^p |\alpha|_1} \frac{\gamma^p}{G(\gamma)}\right[.$$

From our assumption, one has $\lim_{t \to 0^+} pc_B^p |\alpha|_1 R(G(t)/t^p) = +\infty$. Thus, corresponding to $M > 1/\lambda$ there exists $\kappa^* > 0$ such that for any $\kappa \in \,]0, \kappa^*[$ one has $pc_B^p |\alpha|_1 R(G(\kappa)/\kappa^p) > M$. Therefore, by choosing $\kappa < \min\{\kappa^*, \gamma\}$, we can apply theorem 3.1 and we obtain the conclusion. □

Remark 3.5. We explicitly observe that corollary 3.4 ensures the existence of a non-trivial solution under the condition that g is sublinear at 0, without requiring any condition at ∞. Easy examples that satisfy this assumption can be constructed, for example, $g(u) = \sqrt{|u|}$. We recall that in order to apply the mountain pass theorem, the superlinearity of g at 0 must be required as well as a suitable condition at ∞.

Corollary 3.6. *Assume that*

$$\lim_{t \to +\infty} \frac{g(t)}{t^{p-1}} = 0.$$

Then, for each $\kappa > 0$ *and for each*

$$\lambda \in \left] \frac{1}{pc_B^p |\alpha|_1} \frac{1}{R} \frac{\kappa^p}{G(\kappa)}, +\infty \right[,$$

the problem (P_λ) *admits at least one non-trivial and non-negative solution* $u_{0,\lambda}$.

Proof. Let κ be an arbitrary positive real number and let

Word added – OK?

$$\lambda \in \left] \frac{1}{pc_B^p|\alpha|_1} \frac{1}{R} \frac{\kappa^p}{G(\kappa)}, +\infty \right[.$$

From our assumption one has $\lim_{t\to+\infty} pc_B^p|\alpha|_1(G(t)/t^p) = 0$. Thus, corresponding to $\varepsilon > 0$ such that $\varepsilon < 1/\lambda$, there exists $\gamma^* > 0$ such that for any $\gamma > \gamma^*$ one has $pc_B^p|\alpha|_1(G(\gamma)/\gamma^p) < \varepsilon$. Therefore, by choosing $\gamma > \max\{\gamma^*, \kappa\}$, we can apply theorem 3.1 and the conclusion follows. □

Remark 3.7. We explicitly observe that the solution guaranteed by corollary 3.6 is non-trivial.

Corollary 3.8. *Assume that*

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty \quad \textit{and} \quad \lim_{t\to+\infty} \frac{g(t)}{t^{p-1}} = 0.$$

Then, for any $\lambda > 0$, (P_λ) admits at least one non-trivial and non-negative solution $u_{0,\lambda}$.

Proof. It follows by arguing as in the proofs of corollaries 3.4 and 3.6. □

Remark 3.9. Clearly, the conclusion of corollary 3.4 holds under the assumption that

$$\limsup_{t\to 0^+} \frac{G(t)}{t^p} = +\infty,$$

and corollary 3.6 holds under the assumption

Word added – OK?

$$\liminf_{t\to+\infty} \frac{G(t)}{t^p} = 0.$$

Now we point out a multiplicity result, where only a condition at ∞ on g is required.

Theorem 3.10. *Assume that*

(AR) *there are $s > 0$ and $\mu > p$ such that $0 < \mu G(\xi) \leqslant \xi g(\xi)$ for all $\xi \geqslant s$.*

Then, for each

$$\lambda \in \left] 0, \frac{1}{pc_B^p|\alpha|_1} \sup_{\gamma>0} \frac{\gamma^p}{G(\gamma)} \right[,$$

problem (P_λ) admits at least two distinct non-negative solutions $u_{0,\lambda}$ and $u_{1,\lambda}$.

Proof. Our aim is to apply theorem 2.7. To this end, we take $E = W^{1,p}(\mathbb{R})$, and Φ, Ψ, I_λ are as in § 2. All the assumptions on regularity required on Φ and Ψ are established and, from lemma 2.9, the functional I_λ satisfies the (PS)-condition and it is unbounded from below. Moreover, for a fixed λ as in the conclusion and γ such that

See earlier comment regarding this.

I'm unsure what conclusion this refers to. Please clarify.

$$\lambda < \frac{1}{pc_B^p|\alpha|_1} \frac{\gamma^p}{G(\gamma)},$$

arguing as in the proof of theorem 3.1 (see (3.5)), one has

$$\frac{1}{pc_B^p|\alpha|_1}\frac{\gamma^p}{G(\gamma)} \leqslant \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}.$$

Hence, from theorem 2.7, the functional I_λ admits at least two distinct critical points, which are, as seen in the proof of theorem 3.1, non-negative solutions of (P_λ), and the conclusion follows. □

Remark 3.11. If $g(0) \neq 0$, both the solutions guaranteed by theorem 3.10 are non-trivial. It follows that theorem 1.2 is an immediate consequence of theorem 3.10.

Example 3.12. From theorem 3.10, the problem

$$-u'' + u = \frac{1}{4}\frac{1+u^4}{1+x^2},$$
$$u(-\infty) = u(+\infty) = 0,$$

admits at least two non-trivial and non-negative classical solutions. Indeed, it is enough to verify that $0 < 3(\xi + (\xi^5/5)) \leqslant \xi(1+\xi^4)$ for all $\xi \geqslant 5^{1/4}$ and

$$\frac{1}{4} < \frac{1}{2c_B^2|\alpha|_1}\frac{1}{G(1)}.$$

References

1 A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Analysis* **14** (1973), 349–381.

2 G. Barletta. Existence results for semilinear elliptical hemivariational inequalities. *Nonlin. Analysis* **68** (2008), 2417–2430. Not cited in text!

3 T. Bartsch and Z.-Q. Wang. Existence and multiplicity results for some superlinear elliptic problems on ℕ. *Commun. PDEs* **20** (1995), 1725–1741.

4 T. Bartsch, A. Pankov and Z.-Q. Wang. Nonlinear Schrödinger equations with steep potential well. *Commun. Contemp. Math.* **4** (2001), 549–569. Not cited in text!

5 G. Bonanno. A critical point theorem via the Ekeland variational principle. *Nonlin. Analysis* **75** (2012), 2992–3007.

6 G. Bonanno. Relations between the mountain pass theorem and local minima. *Adv. Nonlin. Analysis* **1** (2012), 205–220.

7 G. Bonanno and D. O'Regan. A boundary value problem on the half-line via critical point methods. *Dynam. Syst. Applic.* **15** (2006), 395–408.

8 G. Bonanno and A. Sciammetta. An existence result of one nontrivial solution for two point boundary value problems. *Bull. Austral. Math. Soc.* **84** (2011), 288–299.

9 H. Brézis. *Analyse fonctionnelle: théorie et applications* (Paris: Masson, 1983).

10 V. I. Burenkov. *Sobolev spaces on domains*, vol. 137 (Leipzig: Teubner, 1998).

11 F. H. Clarke. *Optimization and nonsmooth analysis*, Classics in Applied Mathematics, vol. 5 (Philadelphia, PA: SIAM, 1990). Not cited in text!

12 F. Gazzola and V. Radulescu. A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbb{R}^N$. *Diff. Integ. Eqns* **13** (2000), 47–60.

13 N. Ghoussoub. *Duality and perturbation methods in critical point theory*, Cambridge Tracts in Mathematics, vol. 107 (Cambridge University Press, 1993). Not cited in text!

14 A. Kristály. A double eigenvalue problem for Schrödinger equations involving sublinear nonlinearities at infinity. *Electron. J. Diff. Eqns* **42** (2007), 1–11. Not cited in text!

15 H. Lian, P. Wang and W. Ge. Unbounded upper and lower solutions method for Sturm–Liouville boundary value problem on infinite intervals. *Nonlin. Analysis* **70** (2009), 2627–2633.

16 R. Livrea and S. A. Marano. Existence and classification of critical points for non-differentiable functions. *Adv. Diff. Eqns* **9** (2004), 961–978. Not cited in text!

17 R. Ma. Existence of positive solutions for second-order boundary value problems on infinity intervals. *Appl. Math. Lett.* **16** (2003), 33–39.

18 L. Ma and X. Xu. Positive solutions of a logistic equation on unbounded intervals. *Proc. Am. Math. Soc.* **130** (2002), 2947–2958.

19 T. Moussaoui and K. Szymanska-Dębowska. Resonant problem for a class of BVPs on the half-line. *Electron. J. Qual. Theory Diff. Eqns* **53** (2009), 1–10.

20 R. P. Agarwal, O. G. Mustafa and Y. V. Rogovchenko. Existence and asymptotic behavior of solutions of a boundary value problem on an infinite interval. *Math. Computer Modelling* **41** (2005), 135–157.

21 P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, vol. 65 (Providence, RI: American Mathematical Society, 1986). Not cited in text!

22 B. Ricceri. A general variational principle and some of its applications. *J. Computat. Appl. Math.* **113** (2000), 401–410.

23 K. Szymanska. Resonant problem for some second-order differential equation on the half-line. *Electron. J. Diff. Eqns* **160** (2007), 1–9.

24 M. Willem. *Minimax theorems* (Birkhäuser, 1996).

25 B. Yan, D. O'Regan and R. P. Agarwal. Unbounded solutions for singular boundary value problems on the semi-infinite interval: upper and lower solutions and multiplicity. *J. Computat. Appl. Math.* **197** (2006), 365–386.

(*Issued Publication date 2014*)