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A novel approach to nonlinear variable-order fractional viscoelasticity

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This paper addresses nonlinear viscoelastic behavior of fractional systems with variable time-dependent fractional order. In this case, the main challenge is that the Boltzmann linear superposition principle, i.e. the theoretical basis on which linear viscoelastic fractional operators are formulated, does not apply in standard form because the fractional order is not constant with time. Moving from this consideration, the paper proposes a novel approach where the system response is derived by a consistent application of the Boltzmann principle to an equivalent system, built at every time instant based on the fractional order at that instant and the response at all the previous ones. The approach is readily implementable in numerical form, to calculate either stress or strain responses of any fractional system where fractional order may change with time.

1. Introduction

Fractional calculus is a well-established and effective tool for material modelling, with a remarkable number of applications in the last decades [1–8]. A field where fractional calculus is widely applied to model material behavior is certainly viscoelasticity, with a great deal of analytical, numerical and experimental studies [9–21]. Typically, fractional calculus is used for linear viscoelasticity and, in this context, several examples of fractional operators have successfully reproduced material behavior with a limited number of parameters [22–26]. However, nonlinear viscoelastic effects were found in several materials, as polymers, asphalt concrete, plastics and rubber, with and without reinforcement, biological materials, soils

[27–31]. Nonlinear viscoelasticity may arise from various sources and developing pertinent models may be quite challenging. Fractional operators with kernel function depending nonlinearly on stress were proposed to model creep behavior of asphalt binders [32], where nonlinearity arises from high levels of stress/strain induced by heavy traffic or additives causing microstructural changes. Strain-rate dependent behavior of polypropylene polymers was modelled by Maxwell elements governed by a nonlinear fractional differential equation, which involves a relaxation time depending on the average strain rate [33]. The model was validated by experiments under uniaxial loading, for long term (relaxation test) and short term (tensile tests) material behavior at multiple strain rates [33]; a multi-axial constitutive law was also derived on a theoretical basis and implemented in finite element models. The concept proposed in ref. [33] was further developed by some authors in a later work [34], assuming that the relaxation time involved in the nonlinear fractional differential equation is governed by the overstress associated with strain-rate dependent behavior of the material. Further, nonlinear viscoelastic models of rubbers undergoing large strains were formulated, with the rate equation given as a nonlinear Volterra equation containing a fractional integral [35]. Recently, variable-order fractional operators have been adopted in several applications where the order of differintegration depends upon some state variables such as stress, strain, temperature among others [36–43]. Specifically, variable-order fractional operators have been used to model nonlinear viscoelastic behavior associated with a time-dependent evolution of mechanical properties: in ductile metals and soils undergoing strain softening during constant strain-rate tension and compression tests [44]; in polymers depending on the strain rate (in small and large deformations) [45]; in polymers [45] and asphalt mixtures [46] depending on the strain level. Indeed, during a deformation process, the mechanical properties of polymers may vary as a result of microstructural modifications as stretching, orientation, as well as strengthening of the molecular chains [45]; likewise, changes in the aggregate skeleton of asphalt mixtures, as densification or dilatation, may influence the mechanism of internal load transmission and modify the mechanical properties of the material [46] In these cases, therefore, the variation of the order of the fractional operators mirrors the variation of mechanical parameters of the material.

When dealing with variable fractional orders, however, the main theoretical challenge is that the classical Boltzmann linear superposition principle holds true only for constant fractional order and is not applicable, as such, for variable fractional order. Therefore, the question arises whether meaningful variable-order fractional operators can be built as a generalization of the constant-order ones, i.e. by simply replacing the constant fractional order by a variable one. Building on this observation, this paper introduces a novel approach to address nonlinear viscoelastic behavior of fractional systems with variable fractional order where the response is obtained from an equivalent system for which the Boltzmann principle applies meaningfully. The approach is formulated to calculate the stress response corresponding to a given strain history or viceversa. In particular, assuming that the strain history is known, the equivalent system is to be built at every time instant t_n with two fundamental properties: (i) equivalent and original system exhibit the same stress at every time $t < t_n$; (ii) the equivalent system is given a constant fractional order $\alpha = \alpha(t_n)$, where $\alpha(t_n)$ is the variable fractional order of the original system at time t_n. The same reasoning applies to build the equivalent system if the stress history is known, with the difference that, in this case, equivalent and original system exhibit the same strain history until t_n. On these bases, it will be shown that the equivalent system provides the stress/strain response of the original system in a meaningful theoretical framework, fully consistent with the Boltzmann principle. The proposed approach can be readily implemented in numerical form and lends itself to calculate the response of any fractional system whose fractional order may vary with time.

2. Preliminaries on fractional calculus and fractional viscoelasticity

In order to derive the constitutive laws of a viscoelastic material two tests are usually performed: the creep test and/or the relaxation test. The former consists in applying on a specimen a constant stress ($\sigma(t) = U(t)$, being $U(t)$ the unit step function) and measuring the correspondent strain history which is the creep function denoted as $J(t)$. The latter consists in applying on the specimen a constant strain ($\varepsilon(t) = U(t)$) and measuring the corresponding stress history, i.e. the so-called relaxation function denoted as $\Phi(t)$. On the basis of creep and relaxation tests and under the assumption of linearity, the response of a viscoelastic material to a generic imposed strain or stress history can be found applying the Boltzmann linear superposition principle. In this way, the two fundamental laws of linear viscoelasticity are written as

$$\sigma(t)=\int_0^t \Phi(t-\tau)\dot{\varepsilon}(\tau)d\tau \tag{2.1a}$$

$$\varepsilon(t)=\int_0^t J(t-\tau)\dot{\sigma}(\tau)d\tau \tag{2.1b}$$

Eqs (2.1) hold provided that the system is quiescent at $t\leq 0$. If the initial condition is not zero, namely $\varepsilon(0)=\varepsilon_0$, the additional term $\varepsilon_0\Phi(t)$ shall be included in the right hand side (r.h.s) of Eq. (2.1*a*). Likewise if the initial stress is not zero, i.e. $\sigma(0)=\sigma_0$, then the term $\sigma_0 J(t)$ will be included in the r.h.s. of Eq. (2.1*b*). By taking the Laplace transform of both sides of Eqs. (2.1) the relation between creep and relaxation in Laplace domain is readily found in the form

$$\hat{J}(s)\hat{\Phi}(s)=s^{-2} \tag{2.2}$$

where $\hat{J}(s)$ and $\hat{\Phi}(s)$ are the Laplace transform of the creep and relaxation functions respectively, and s is the (complex) parameter of the Laplace transform. Eq. (2.2) shows that the relaxation function may be derived from the creep function and viceversa. Fractional linear viscoelastic behavior has been demonstrated in experimental creep tests on various materials performed by Nutting [47] in 1921 and confirmed in subsequent experimental tests carried out by several authors (see e.g. [22,24–26,31] and pertinent literature). The creep law for a given material in linear regime reads

$$J(t)=\frac{t^{\beta}}{E_{\beta}\Gamma(1+\beta)};\quad 0\leq\beta\leq 1 \tag{2.3}$$

where $\Gamma(\cdot)$ is the Euler Gamma function and β and E_{β} are coefficients evaluated by best fitting procedures on pertinent experimental data. Using Eq. (2.2) the corresponding relaxation function may easily derived in the form

$$\Phi(t)=\frac{E_{\beta}t^{-\beta}}{\Gamma(1-\beta)} \tag{2.4}$$

By inserting Eqs. (2.4) and (2.3) in Eqs. (2.1*a*) and (2.1*b*) respectively, the fractional constitutive laws of linear viscoelasticity are obtained in the form

$$\sigma(t)=E_{\beta}\left({}^{C}D^{\beta}\varepsilon\right)(t) \tag{2.5a}$$

$$\varepsilon(t)=E_{\beta}^{-1}\left(I^{\beta}\sigma\right)(t) \tag{2.5b}$$

where $\left({}^{C}D^{\beta}\cdot\right)(t)$ is the *Caputo's fractional derivative* of order β and $\left(I^{\beta}\cdot\right)(t)$ is the *Riemann-Liouville* (RL) *fractional integral* of order β, defined as

$$\left({}^{C}D^{\beta}\varepsilon\right)(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}\dot{\varepsilon}(\tau)d\tau \tag{2.6a}$$

$$\left(I^{\beta}\sigma\right)(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\sigma(\tau)d\tau \tag{2.6b}$$

It is remarked that, if $\beta=0$, then $E_{\beta=0}\equiv E$ and Eq.(2.5*a*) reduces to $\sigma=E\varepsilon$, which is the constituive law of the purely elastic material; while, as $\beta=1$, Eq.(2.5*a*) reduces to $\sigma=E_{\beta=1}\dot{\varepsilon}=c\dot{\varepsilon}$, which is the constitutive law of the Newton-Petrof purely viscous fluid, being c the viscosity of the fluid. It follows that any fractional material has an intermediate behavior between elastic and viscous ones.

Without loss of generality and for sake of simplicity it is assumed that $\varepsilon(t)$ and $\sigma(t)$ are continuous and differentiable functions within the time interval of observation and at $t=0$. Moreover, assume that $\varepsilon(t)$ and $\sigma(t)$ are zero $\forall\, t|-\infty<t\leq 0$. In this case, a Grunwald-Letnikov (GL) [48] step-by-step discretization scheme may be used to represent Eqs. (2.6). This will be briefly outlined below since will be useful to formulate the proposed approach to nonlinear fractional viscoelasticity with variable order. In this regard,

let the time axis be subdivided into small steps of equal length Δt and let $t_j = j\Delta t$ be the generic time instant. Further, be $\boldsymbol{\varepsilon}_n^T = [\varepsilon_1\ \varepsilon_2\ \ldots\varepsilon_n]$ and $\boldsymbol{\sigma}_n^T = [\sigma_1\ \sigma_2\ \ldots\sigma_n]$ the strain and the stress vectors at $t_n = n\Delta t$, whose j-th entries are the strain and the stress evaluated at the time instant t_j, namely $\varepsilon_j = \varepsilon(t_j) = \varepsilon(j\Delta t)$ and $\sigma_j = \sigma(t_j) = \sigma(j\Delta t)$. Denoting as ∇^β and $\nabla^{-\beta}$ the GL derivative and the GL integral operator respectively, Eqs. (2.6*a*)-(2.6*b*) are approximated in the form

$$\boldsymbol{\sigma}_n = E_\beta \nabla^\beta \boldsymbol{\varepsilon}_n = E_\beta \Delta t^{-\beta} \boldsymbol{U}_n(\beta)\boldsymbol{\varepsilon}_n \tag{2.7a}$$

$$\boldsymbol{\varepsilon}_n = E_\beta^{-1} \nabla^{-\beta} \boldsymbol{\sigma}_n = E_\beta^{-1} \Delta t^{\beta} \boldsymbol{V}_n(\beta)\boldsymbol{\sigma}_n \tag{2.7b}$$

where $\boldsymbol{U}_n(\beta)$ is a $n \times n$ lower band triangular strip matrix defined as [49]

$$\boldsymbol{U}_n(\beta) = \begin{bmatrix} \omega_1(\beta) & 0 & & \cdots & 0 \\ \omega_2(\beta) & \omega_1(\beta) & 0 & & \vdots \\ \omega_3(\beta) & \omega_2(\beta) & \omega_1(\beta) & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \omega_n(\beta) & \omega_{n-1}(\beta) & \cdots & \omega_2(\beta) & \omega_1(\beta) \end{bmatrix} \tag{2.8}$$

where $\omega_j(\beta)$ may be constructed recursively as

$$\omega_1(\beta) = 1; \quad \omega_2(\beta) = -\beta; \quad \omega_{j+1}(\beta) = \omega_j(\beta)\frac{j-1-\beta}{j} \tag{2.9}$$

Also, the following identities

$$\boldsymbol{V}_n(\beta) = \boldsymbol{U}_n(\beta)^{-1} = \boldsymbol{U}_n(-\beta) \tag{2.10}$$

hold true. Eq. (2.10) implies that $\boldsymbol{V}_n(\beta)$, as well as $\boldsymbol{U}_n(\beta)$, is a lower band triangular strip matrix. Therefore, the first column of both matrices $\boldsymbol{U}_n(\beta)$, $\boldsymbol{V}_n(\beta)$ is sufficient to completely populate them. Following Podlubny [49], these matrices may be indicated as $\boldsymbol{U}_n(\beta) = \{\omega_1(\beta)\ldots\omega_n(\beta)\}$ and $\boldsymbol{V}_n(\beta) = \{\omega_1(-\beta)\ldots\omega_n(-\beta)\}$.

It has been demonstrated that, as the integration time step $\Delta t \to 0$, Eqs. (2.7) converge to Eqs. (2.5).

At this stage, an important feature of Eq.(2.7*a*)-(2.7*b*) is illustrated. Essentially, Eq.(2.7*b*) states that in order to evaluate the strain value at time $t = t_n$, that is ε_n, the entire stress history $\boldsymbol{\sigma}_n$ is needed. Indeed, according to Eq. (2.7*b*) it can be written that

$$\varepsilon_n = E_\beta^{-1}\Delta t^\beta \left(\omega_n(-\beta)\sigma_1 + \omega_{n-1}(-\beta)\sigma_2 + \cdots + \omega_1(-\beta)\sigma_n\right) \tag{2.11}$$

On the other hand, notice that according to Eq. (2.7*a*), the stress values $\sigma_1, \sigma_2, \ldots\ \sigma_{n-1}$ can be expressed as

$$\begin{gathered} \sigma_1 = E_\beta \Delta t^{-\beta}\omega_1(\beta)\varepsilon_1; \qquad \sigma_2 = E_\beta \Delta t^{-\beta}\left(\omega_2(\beta)\varepsilon_1 + \omega_1(\beta)\varepsilon_2\right) \\ \sigma_{n-1} = E_\beta \Delta t^{-\beta}\left(\omega_{n-1}(\beta)\varepsilon_1 + \omega_{n-2}(\beta)\varepsilon_2 + \cdots + \omega_1(\beta)\varepsilon_{n-1}\right) \end{gathered} \tag{2.12}$$

Substituting Eq.(2.12) into Eq. (2.11), the following expression is obtained for ε_n

$$\varepsilon_n = a_1(\beta)\varepsilon_1 + a_2(\beta)\varepsilon_2 + \cdots + a_{n-1}(\beta)\varepsilon_{n-1} + E_\beta^{-1}\Delta t^\beta \sigma_n \tag{2.13}$$

where

$$\begin{gathered} a_1(\beta) = \sum_{j=1}^{n-1}\omega_j(\beta)\omega_{n-j+1}(-\beta); \qquad a_2(\beta) = \sum_{j=1}^{n-2}\omega_j(\beta)\omega_{n-j}(-\beta); \\ a_{n-1}(\beta) = \omega_1(\beta)\omega_2(-\beta) \end{gathered} \tag{2.14}$$

Eq. (2.13) shows that the strain value ε_n may be evaluated starting from the entire past history of the strain, that is from $\boldsymbol{\varepsilon}_{n-1}$, and the value of the stress σ_n at $t = t_n$. Notice that the strain history $\boldsymbol{\varepsilon}_{n-1}$ in Eq.(2.13) plays the role of initial condition at the time $t = t_{n-1}$ to predict the strain value at the time $t = t_n$, that is to predict ε_n.

Likewise, using Eq. (2.7*a*) the stress σ_n reads

$$\sigma_n = E_\beta \Delta t^{-\beta} \sum_{j=1}^{n} \omega_j(\beta)\varepsilon_{n-j+1} \tag{2.15}$$

from which the following expression can be finally obtained

$$\sigma_n = a_1(-\beta)\sigma_1 + a_2(-\beta)\sigma_2 + \cdots + a_{n-1}(-\beta)\sigma_{n-1} + E_\beta \Delta t^{-\beta}\varepsilon_n \tag{2.16}$$

Eq. (2.16) shows that the stress σ_n may be evaluated starting from the entire past history of the stress, that is from $\boldsymbol{\sigma}_{n-1}$, and the value of the strain ε_n at $t = t_n$. Notice that the stress history $\boldsymbol{\sigma}_{n-1}$ in Eq.(2.16) plays the role of initial condition at the time $t = t_{n-1}$ to predict the stress value at the time $t = t_n$, that is to predict σ_n.

Interestingly, a few manipulations lead to the following relations:

$$a_1(\beta) = -\omega_n(\beta); \qquad a_2(\beta) = -\omega_{n-1}(\beta); \qquad a_{n-1}(\beta) = -\omega_2(\beta); \tag{2.17a}$$

$$a_1(-\beta) = -\omega_n(-\beta); \qquad a_2(-\beta) = -\omega_{n-1}(-\beta); \qquad a_{n-1}(-\beta) = -\omega_2(-\beta); \tag{2.17b}$$

Eqs. (2.17) allow Eqs. (2.13) and (2.16) to be rewritten in the form

$$\varepsilon_n = -\left(\omega_n(\beta)\varepsilon_1 + \omega_{n-1}(\beta)\varepsilon_2 + \cdots + \omega_2(\beta)\varepsilon_{n-1}\right) + E_\beta^{-1}\Delta t^{\beta}\sigma_n = -\sum_{j=1}^{n-1} \omega_{n-j+1}(\beta)\varepsilon_j + E_\beta^{-1}\Delta t^{\beta}\sigma_n \tag{2.18a}$$

$$\sigma_n = -\left(\omega_n(-\beta)\sigma_1 + \omega_{n-1}(-\beta)\sigma_2 + \cdots + \omega_2(-\beta)\sigma_{n-1}\right) + E_\beta \Delta t^{-\beta}\varepsilon_n = -\sum_{j=1}^{n-1} \omega_{n-j+1}(-\beta)\sigma_j + E_\beta \Delta t^{-\beta}\varepsilon_n \tag{2.18b}$$

The concepts outlined above will be essential to understand the proposed approach to nonlinear fractional viscoelasticity with fractional order, developed in Section 4.

3. Non-linear power law viscoelasticity, existing models in literature

It is known that viscoelastic materials may exhibit strong deviations from linear behavior. For instance, in his early investigations, Nutting [47] proposed the following creep function, which depends nonlinearly on the stress level $\sigma(t) = \sigma U(t)$, with $\sigma > 0$:

$$\varepsilon(t) = \frac{\sigma^{\alpha} t^{\beta}}{E_\beta \Gamma(1+\beta)}; \qquad \alpha > 0; \quad 0 \le \beta \le 1 \tag{3.1}$$

For a general load history where the stress may be either positive or negative, Eq. (3.1) takes the general form:

$$\varepsilon(t) = \frac{|\sigma|^{\alpha}\,\mathrm{sgn}(\sigma) t^{\beta}}{E_\beta \Gamma(1+\beta)}; \qquad \alpha > 0; \quad 0 \le \beta \le 1 \tag{3.2}$$

Although the Boltzmann linear superposition principle does not hold in the space $\varepsilon - \sigma$, it is still applicable in the space $\varepsilon - p$, where $p = |\sigma|^{\alpha}\,\mathrm{sgn}(\sigma)$, yielding

$$\varepsilon(t) = \frac{E_\beta^{-1}}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} p(\tau)d\tau = \frac{E_\beta^{-1}}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} |\sigma(\tau)|^{\alpha}\,\mathrm{sgn}(\sigma(\tau))d\tau \tag{3.3}$$

Eq. (3.3) represents a quasi-linear viscoelastic model, where the classical fractional operator is consistently defined in a new space $\varepsilon - p$, where the Boltzmann linear superposition principle holds true. Recognize that, in this case, β is assumed to have a constant value for the whole time of observation and regardless of the value of the stress and/or strain.

In some studies constitutive laws involving time-dependent material parameters have been developed. For instance, the following constitutive law involving a fractional operator with variable order have been proposed to capture the nonlinear behavior of viscoelastic materials [45]:

$$\sigma(t)=\frac{E_\beta(t)}{\Gamma(1-\beta(t))}\int_0^t(t-\tau)^{-\beta(t)}\dot{\varepsilon}(\tau)d\tau=E_\beta(t)\left({}^CD^{\beta(t)}\varepsilon\right)(t) \tag{3.4}$$

where $\left({}^CD^{\beta(t)}\cdot\right)(t)$ is the Caputo's variable order fractional derivative [50] and $E_\beta(t)$ and $\beta(t)$ are functions representing the material parameters evolution in all the observation time. An alternative form of Eq. (3.4) was proposed in the literature [40], that is

$$\sigma(t)=\int_0^t\frac{E_\beta(\tau)}{\Gamma(1-\beta(\tau))}(t-\tau)^{-\beta(\tau)}\dot{\varepsilon}(\tau)d\tau \tag{3.5}$$

In the discretized form, Eq. (3.4) and Eq. (3.5) take the following general form

$$\boldsymbol{\sigma}_n=\bar{\boldsymbol{U}}_n\boldsymbol{\varepsilon}_n \tag{3.6}$$

where $\bar{\boldsymbol{U}}_n$ for the model of Eq.(3.4) is given as

$$\bar{\boldsymbol{U}}_n=\begin{bmatrix}\eta_1\omega_1(\beta_1) & 0 & \cdots & \cdots & 0\\ \eta_2\omega_2(\beta_2) & \eta_2\omega_1(\beta_2) & 0 & & \vdots\\ \eta_3\omega_3(\beta_3) & \eta_3\omega_2(\beta_3) & \eta_3\omega_1(\beta_3) & \ddots & \vdots\\ \vdots & & & \ddots & 0\\ \eta_n\omega_n(\beta_n) & \eta_n\omega_{n-1}(\beta_n) & \cdots & \cdots & \eta_n\omega_1(\beta_n)\end{bmatrix} \tag{3.7}$$

while $\bar{\boldsymbol{U}}_n$ for the model of Eq.(3.5) is

$$\bar{\boldsymbol{U}}_n=\begin{bmatrix}\eta_1\omega_1(\beta_1) & 0 & \cdots & \cdots & 0\\ \eta_1\omega_2(\beta_1) & \eta_2\omega_1(\beta_2) & 0 & & \vdots\\ \eta_1\omega_3(\beta_1) & \eta_2\omega_2(\beta_2) & \eta_3\omega_1(\beta_3) & \ddots & \vdots\\ \vdots & & & \ddots & 0\\ \eta_1\omega_n(\beta_1) & \eta_2\omega_{n-1}(\beta_2) & \cdots & \cdots & \eta_n\omega_1(\beta_n)\end{bmatrix} \tag{3.8}$$

being $\eta_j=E_{\beta_j}\Delta t^{-\beta_j}$. Eqs. (3.7) and (3.8) clarify that the two models (3.4) and (3.5) involve different assumptions to calculate the stress σ_n at $t=t_n$: the model in (3.4) "interprets" the past strain history of the material with the parameters at $t=t_n$ (see rows of matrix $\bar{\boldsymbol{U}}_n$ in Eq. (3.7)), while the model in (3.5) "interprets" the past strain history of the material with past parameters (see rows of matrix $\bar{\boldsymbol{U}}_n$ in Eq. (3.8)).

In recent literature the mechanical parameters β and E_β have been directly related to the level of strain (see e.g. [46] in large strain conditions) or to the strain rate (see e.g. [45,46]), where the dependence of the parameters from the strain/strain rate level is calibrated by means of experimental tests. If the parameters are assumed to be dependent on the strain level, the constitutive laws (3.4) and (3.5) may be written as

$$\sigma(t)=\frac{E_\beta(\varepsilon(t))}{\Gamma(1-\beta(\varepsilon(t)))}\int_0^t(t-\tau)^{-\beta(\varepsilon(t))}\dot{\varepsilon}(\tau)d\tau \tag{3.9a}$$

$$\sigma(t)=\int_0^t\frac{E_\beta(\varepsilon(\tau))}{\Gamma(1-\beta(\varepsilon(\tau)))}(t-\tau)^{-\beta(\varepsilon(\tau))}\dot{\varepsilon}(\tau)d\tau \tag{3.9b}$$

respectively. Note that, if the strain history is known a-priori, the dependence of the parameters on t may be directly explicited and then Eqs. (3.4) and (3.5) are equivalent to Eqs. (3.9). Analogous formulations to those of Eqs. (3.9) may be written for the model with strain rate dependent parameters. At this stage, recognize that both strain-dependent variable-order operators in Eqs. (3.9), as well as those in Eq.(3.4)-Eq.(3.5), rely on a generalization of constant-order fractional operators, which merely consists in replacing the constant order with a variable one. However, it is worth remarking that this generalization corresponds indeed to apply the Boltzmann linear superposition principle to a material with the following nonlinear, strain-dependent relaxation function

$$\sigma(t)=\frac{E_\beta(\varepsilon_0)t^{-\beta(\varepsilon_0)}}{\Gamma(1-\beta(\varepsilon_0))}\varepsilon_0 \tag{3.10}$$

On the other hand the same generalization, as applied to a fractional operator Eq. (3.7) where the role of stress and strain is reversed, would correspond to apply the Boltzmann linear superposition principle to a material with the following nonlinear, strain-dependent creep function

$$\varepsilon(t)=\frac{E_\beta^{-1}(\sigma_0)t^{\beta(\sigma_0)}}{\Gamma(1+\beta(\sigma_0))}\sigma_0 \tag{3.11}$$

On the basis of these observations, it is of interest to develop alternative fractional calculus approaches to nonlinear viscoelastic materials with time-dependent parameters, which may rely on a consistent application of the Boltzmann linear superposition principle. Same considerations about the applicability of the Boltzmann superposition principle may be drawn regarding the time-dependent order fractional operators of Eqs. (3.4) and (3.5). For this purpose, an original approach will be introduced in the next section.

To summarize, if β is constant and the nonlinearity involves only the stress level, as in Eq. (3.1), the Boltzmann linear superposition principle still holds in a different space, where the response may be properly represented by classical fractional operators with constant order. However, for the purpose of handling systems where β may change during the time interval of interest, new and pertinent fractional calculus tools should be considered rather than variable-order fractional operators, as obtained from classical fractional operators upon replacing the constant order with a variable one; indeed, these operators implicitly rely on the assumption that the Boltzmann linear superposition holds true, which is not a rigorous assumption in presence of the nonlinearity associated with changing values of β. In this respect, an original approach will be presented next.

4. Non-linear power law viscoelasticity - proposed approach

The proposed approach is a suitable and an effective way to build the stress (strain) response of a nonlinear viscoelastic material with time-dependent fractional order to a generic imposed strain (stress) history. The key concept is that the proposed approach is based on a consistent application of the Boltmann superposition principle, as explained in the following. Assume that during all the observation time, the evolution of materials parameters E_β and β is known whatever are the physical quantities they depend on. This assumption is based on the observation that β and E_β may be dependent either on the strain and/or the strain rate level [45,46]. Then, once the imposed strain history is given, the values of β and E_β are known at each time instant of interest. Let the time axis be subdivided into small steps of equal length Δt and let $t_j=j\Delta t$ be the generic time instant. Further, be Δt so small that E_β and β may be taken as constant in every time interval. Specifically, in the interval $t_{n-1}\div t_n$ they are denoted as $E_{\beta_n}=E_\beta(t_n)$, $\beta_n=\beta(t_n)$ respectively. Let ε_n be the imposed deformation at $t=t_n$. Assuming that the stress response history has been already evaluated in the time interval $0\div t_{n-1}$, that is $\boldsymbol{\sigma}_{n-1}=[\sigma_1\ \sigma_2\ldots\sigma_{n-1}]$ is a known vector, now, we want to evaluate the stress response at $t=t_n$, that is σ_n.

Now, recognize that the stress σ_n at $t=t_n$ shall be built on keeping consistency with the whole stress history $\boldsymbol{\sigma}_{n-1}$ up to $t=t_{n-1}$ and the system parameters E_{β_n} and β_n within the interval $t_{n-1}\div t_n$. Hence, consistently with the requirements above, the key idea of the proposed approach is that the stress σ_n at $t=t_n$ can be obtained from an equivalent system built with the following properties:

(i) it exhibits the same stress history of the original system $\boldsymbol{\sigma}_{n-1}$ up to $t=t_{n-1}$;
(ii) it exhibits the constant parameters E_{β_n}, β_n of the original system through the whole time interval $0\div t_n$;
(iii) it is subjected to the same deformation level at $t=t_n$, that is ε_n.

Such a system can be considered equivalent to the original one in the sense that the two systems do exhibit the same stress history $\boldsymbol{\sigma}_{n-1}$ up to $t = t_{n-1}$ as well as the same parameters within the interval $t_{n-1} \div t_n$. Thus, based on these observations, the equivalent system can be consistently used to calculate the stress σ_n of the original system at $t = t_n$; on the other hand, since its parameters are constant through the whole time interval $0 \div t_n$ (see property (ii)), the Boltzmann linear superposition principle holds true for the equivalent system providing the sought stress σ_n as:

$$\boldsymbol{\sigma}_{n-1} = \Delta t^{-\beta_n} E_{\beta_n} \boldsymbol{U}_{n-1}(\beta_n)\hat{\boldsymbol{\varepsilon}}_{n-1}^{(n)} \quad \Rightarrow \quad \hat{\boldsymbol{\varepsilon}}_{n-1}^{(n)} = \frac{\Delta t^{\beta_n}}{E_{\beta_n}} \boldsymbol{V}_{n-1}(\beta_n)\boldsymbol{\sigma}_{n-1} \tag{4.1}$$

$$\sigma_n = \Delta t^{-\beta_n} E_{\beta_n}\left(\omega_n(\beta_n)\hat{\varepsilon}_1^{(n)} + \omega_{n-1}(\beta_n)\hat{\varepsilon}_2^{(n)} + \cdots + \omega_2(\beta_n)\hat{\varepsilon}_{n-1}^{(n)} + \omega_1(\beta_n)\varepsilon_n\right) \tag{4.2}$$

In Eqs. (4.1) and (4.2), symbol $\hat{\boldsymbol{\varepsilon}}_{n-1}^{(n)}$ is used to distinguish the strain of the equivalent system from that of the original one and $\hat{\varepsilon}_k^{(n)}$, for $k = 1, 2, \ldots, n-1$, are its components. Now, using Eq. (2.13) for the equivalent system, it can be written that

$$\begin{gathered} \hat{\varepsilon}_1^{(n)} = \frac{\Delta t^{\beta_n}}{E_{\beta_n}}\omega_1(-\beta_n)\sigma_1; \qquad \hat{\varepsilon}_2^{(n)} = \frac{\Delta t^{\beta_n}}{E_{\beta_n}}\left(\omega_2(-\beta_n)\sigma_1 + \omega_1(-\beta_n)\sigma_2\right) \\ \hat{\varepsilon}_{n-1}^{(n)} = \frac{\Delta t^{\beta_n}}{E_{\beta_n}}\left(\omega_{n-1}(-\beta_n)\sigma_1 + \omega_{n-2}(-\beta_n)\sigma_2 + \cdots + \omega_1(-\beta_n)\sigma_{n-1}\right) \end{gathered} \tag{4.3}$$

Then, using Eq. (4.3) in Eq. (4.1) the following expression can be obtained for the stress σ_n at $t = t_n$

$$\sigma_n = a_1(-\beta_n)\sigma_1 + a_2(-\beta_n)\sigma_2 + \cdots + a_{n-1}(-\beta_n)\sigma_{n-1} + E_{\beta_n}\Delta t^{-\beta_n}\varepsilon_n \tag{4.4}$$

where, in view of Eqs. (2.17*b*), we have

$$\begin{gathered} a_1(-\beta_n) = \sum_{j=1}^{n-1}\omega_j(-\beta_n)\omega_{n-j+1}(\beta_n) = -\omega_n(-\beta_n); \qquad a_2(-\beta_n) = \sum_{j=1}^{n-2}\omega_j(-\beta_n)\omega_{n-j}(\beta_n) = -\omega_{n-1}(-\beta_n); \\ a_{n-1}(-\beta_n) = \omega_1(-\beta_n)\omega_2(\beta_n) = -\omega_2(-\beta_n) \end{gathered} \tag{4.5}$$

and hence

$$\sigma_n = -\left(\omega_n(-\beta_n)\sigma_1 + \omega_{n-1}(-\beta_n)\sigma_2 + \cdots + \omega_2(-\beta_n)\sigma_{n-1}\right) + E_{\beta_n}\Delta t^{-\beta_n}\varepsilon_n \tag{4.6}$$

Remarkably, Eq. (4.6) mirrors Eq. (2.18*b*) for linear systems and demonstrates that the stress σ_n at $t = t_n$ built by the proposed approach is fully consistent with the previous stress history $\boldsymbol{\sigma}_{n-1}$ and the parameters E_{β_n} and β_n of the system within the time interval $t_{n-1} \div t_n$. Indeed, in the integration scheme here proposed, and specifically according to Eq. (4.1) and Eq. (4.6), the initial conditions in terms of the stress history, i.e. $\boldsymbol{\sigma}_{n-1}$, are not violated. The procedure, shown above for a single time instant, can be extended to every time instant of interest. Assuming that an arbitrary strain history $\boldsymbol{\varepsilon} = [\varepsilon_1 \; \varepsilon_2 \; \ldots \varepsilon_n]$ is known on a given time domain, along with the evolution of the material parameters E_β and β, the proposed approach can be implemented as follows. Starting from $t = 0$, the unknown stress σ_1 at $t_1 = \Delta t$ is obtained as

$$\sigma_1 = E_{\beta_1}\Delta t^{-\beta_1}\varepsilon_1 \tag{4.7}$$

Next, the unknown stress σ_2 at time t_2 can be obtained based on the stress σ_1 at the previous time instant t_1 along with the known strain ε_2 at $t = t_2$.

$$\sigma_2 = a_1(-\beta_2)\sigma_1 + E_{\beta_2}\Delta t^{-\beta_2}\varepsilon_2 = -\omega_2(-\beta_2)\sigma_1 + E_{\beta_2}\Delta t^{-\beta_2}\varepsilon_2 \tag{4.8}$$

In this manner, the whole response stress history will be calculated step by step. The unknown stress σ_n at $t = t_n$ will be obtained via Eq. (4.4), i.e. using the previously built stress history σ_{n-1} and the known strain at $t = t_n$.

It is underlined that an identical procedure may be developed for the evaluation of the strain when the stress is known, as often happens in practical structural applications. Specifically, in the framework of the proposed approach the strain ε_n at time $t_n = n\Delta t$ is evaluated in the form

$$\varepsilon_n = -\left(\omega_n(\beta_n)\varepsilon_1 + \omega_{n-1}(\beta_n)\varepsilon_2 + \cdots + \omega_2(\beta_n)\varepsilon_{n-1}\right) + E_{\beta_n}^{-1}\Delta t^{\beta_n}\sigma_n \tag{4.9}$$

Notice that Eq.(4.9) mirrors Eq. (2.18*a*) for linear systems. Analogously to Eq. (4.6), the value of strain obtained with Eq.(4.9) is consistent with the previous strain history $\boldsymbol{\varepsilon}_{n-1}$ and with the parameters β_n and E_{β_n} within the time interval $t_{n-1} \div t_n$.

Finally, it is worth emphasizing that the quasi-linear viscoelastic model of Eq. (3.3) is a special case of the proposed approach in the form of Eq. (4.9). As a matter of fact, the discretization of Eq. (3.3) and some simple manipulations (as those defined in Section 2) lead to

$$\varepsilon_n = -\left(\omega_n(\beta_n)\varepsilon_1 + \omega_{n-1}(\beta_n)\varepsilon_2 + \cdots + \omega_2(\beta_n)\varepsilon_{n-1}\right) + E_{\beta_n}^{-1}\Delta t^{\beta_n}|\sigma_n|^{\alpha}\,\mathrm{sgn}(\sigma_n) \tag{4.10}$$

that is exactly the same result that one obtains if the proposed approach is applied directly to the quasi-linear viscoelastic model of Eq. (3.3).

In conclusion, the proposed approach is developed to appropriately compute the stress/strain response to a generic strain/stress history of a nonlinear fractional viscoelastic material with time-dependent parameters. In practice, for instance, the material parameters β and E_β could be obtained experimentally as functions of strain to calculate the stress response to a given strain history or, conversely, as functions of stress to calculate the strain response to a given stress history. Various relaxation tests, each at a given level of strain ε_0, could provide experimental data to be fitted by Eq.(3.10) in order to obtain the corresponding strain-dependent parameters $\beta(\varepsilon_0)$ and $E_\beta(\varepsilon_0)$. In this manner, once a generic strain history is imposed, the material parameters will be known at every time instant (as functions of the strain at that instant) and Eq.(4.6) could be used to compute the sought stress response. Likewise, various creep tests, each at a given level of stress σ_0, could provide experimental data to be fitted by Eq.(3.11) in order to derive the corresponding stress-dependent parameters $\beta(\sigma_0)$ and $E_\beta(\sigma_0)$. For an imposed stress history, the material parameters would be known at every time instant (as functions of the stress at that instant) and, in this case, Eq.(4.9) would provide the sought strain response.

Finally, it is noteworthy that the approach above may be applied also for the case in which the system experiences a change of the mechanical parameters due to external causes, such as temperature variations. In this case the parameters vary because they are very sensitive to the temperature variations, as it may be the case of some polymers or asphalt, where temperature effects may cause E_β and β to vary.

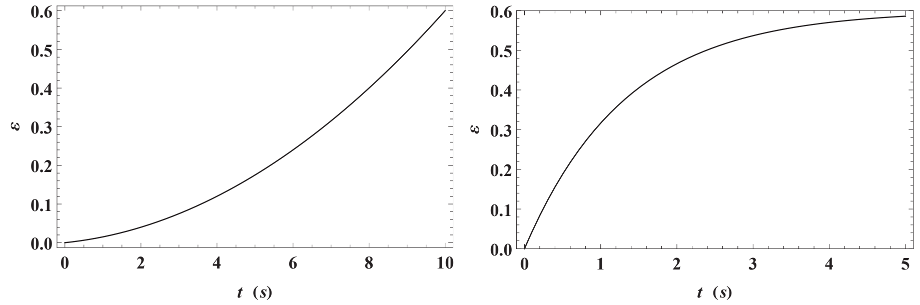
In the next section, a numerical application of the proposed method is shown.

5. Numerical Example

In order to elucidate the proposed approach, consider a Polyethylene Terephtalate (PET) specimen and assume that two test strain histories are known, as given in Figure 1. The two strain histories are built with the requirement that the associated strain rates take on values within the range of strain rates reported in Figure 2. Indeed, for the whole range of strain rates in Figure 2, a complete set of material parameters β and E_β can readily be constructed by fitting experimentally based values of β and E_β available in the literature for 3 values of the strain rate (see Table 1 and black dots in Figure 2), as obtained by Meng et

$\dot{\varepsilon}(1/\mathrm{s})$	β	$E_\beta(\mathrm{MPa\,s}^{\beta})$
0.01	0.5193	15.94
0.1	0.3784	6.70
1	0.3594	3.91

Table 1. Model parameters for PET at small strain for different strain rate values [45].

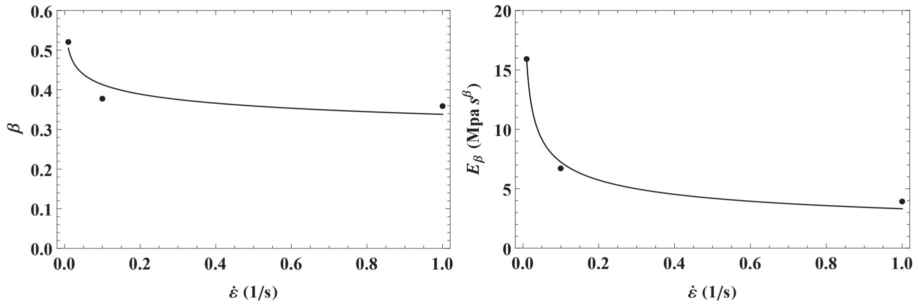


(a) (b)

Figure 1. Imposed deformation history on a PET specimen: a) $\varepsilon = 0.01t + 0.005t^2$, b) $\varepsilon = 0.6(1 - \exp(-0.75t))$.

al. [45] based on experimental data provided by Boyce in ref. [51]. Therefore, the proposed approach is implemented on assuming the strain histories in Figure 1, calculating at every time instant the associated strain rate and using the corresponding material parameters from Figure 2.

Figure 1 shows that the two strain histories are both monotonically increasing with time. The difference between the two strain histories is in the strain rate: while the strain rate is increasing with time in Figure 1a, it is decreasing in Figure 1b. This difference affects the time histories of the mechanical parameters β and E_β that are shown in Figure 3 and 4 for the applied strain histories of Figure 1: while in Figure 3 the mechanical parameters are both decreasing with time, in Figure 4 they are both increasing with time.

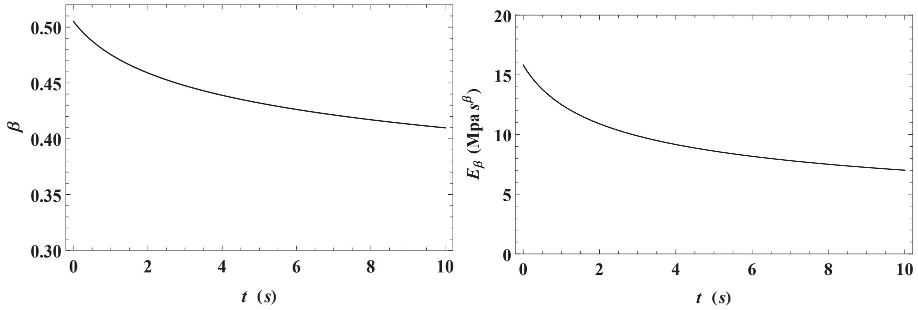


(a) (b)

Figure 2. Trend of the fractional order β (a) and E_β (b) vs. the strain rate $\dot{\varepsilon}$ for PET.

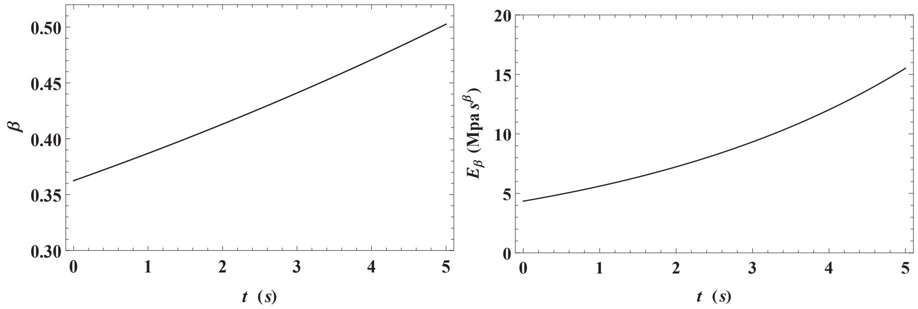
Figure 5 shows the resulting stress computed via the proposed method, represented via a continuous black line, and the stress computed via the variable order fractional viscoelastic model (Eq. (3.4)), represented via a dotted gray line. For both methods the time step is $\Delta t = 0.01$ s.

As can be seen from Figure 5, results from the two methods appear different for increasing time instants. Specifically, at the last time instant, for the strain history of Figure 1a the variable order fractional viscoelastic model predict a stress response almost by 6% larger than the proposed approach. However the behavior of the responses evaluated with the two approaches is comparable. As for the strain history of Figure 1b, the stress at the final instant evaluated with Eq. (3.4) is about 40% smaller than the stress predicted with the proposed approach. Moreover, a substantial difference in the behavior of the two responses may be observed. Indeed, as the time elapses, and the values of the parameters increase, the stress predicted by Eq. (3.4) grows more and more slowly. Differently, the stress predicted with the proposed



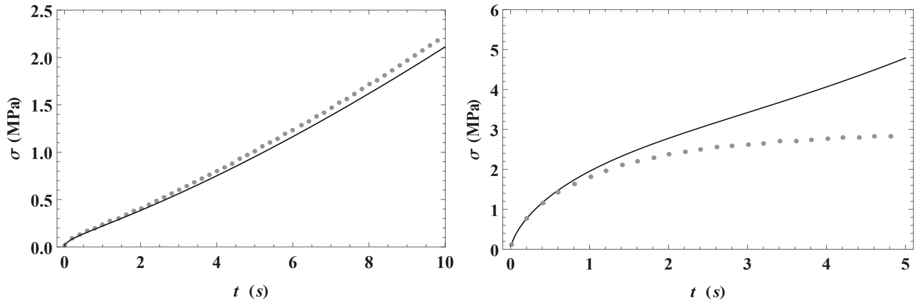
(a) (b)

Figure 3. Variation of β and E_β with t for the imposed strain history in Figure 1a.



(a) (b)

Figure 4. Variation of β and E_β with t for the imposed strain history in Figure 1b.



(a) (b)

Figure 5. The stress resulting from a) the imposed strain history in Figure 1a and b) for the imposed strain history in Figure 1b, obtained via the proposed method (continuous black line) and via Eq. (3.4) (dotted gray line).

approach continues to grow more quickly in comparison with the stress predicted by Eq. (3.4). Besides these general comments, it is apparent that any definitive conclusion on the accuracy of proposed approach and existing one based on Eq. (3.4) should be drawn based on appropriate and extensive experimental campaigns, possibly performed with various histories where strain and strain rates are not constant, in order to attain a full insight into material behavior under different test conditions. Nevertheless, a noteworthy feature of the proposed approach, in comparison with the existing one, is that the stress response at every time instant $t = t_n$ is fully consistent with the whole stress history until $t = t_{n-1}$ and with the material parameters E_{β_n} and β_n within the time interval $t_{n-1} \div t_n$, see Eq. (4.6).

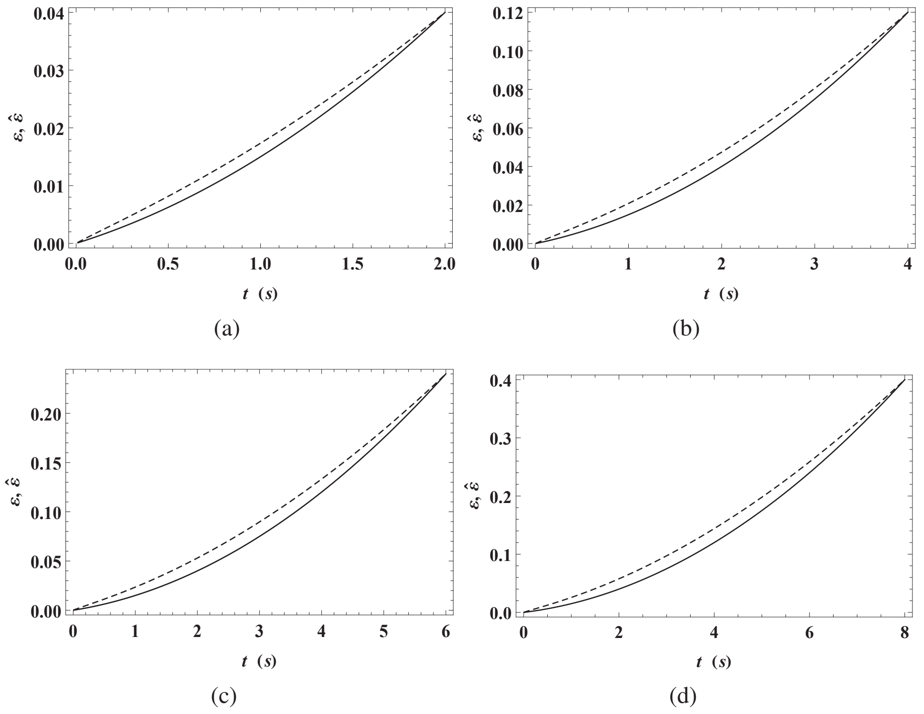


Figure 6. Imposed strain history (continuous black line) vs equivalent strain history (dashed black line) up to different time instants t, (a) $t = 2\,\mathrm{s}$, (b) $t = 4\,\mathrm{s}$, (c) $t = 6\,\mathrm{s}$, (d) $t = 8\,\mathrm{s}$ for the strain history of Figure 2a

Finally, an important feature of the proposed method is illustrated. At this regard, the equivalent strain, related to the imposed strain history in Figure 1a, is reported in Figure 6 at four different time instants. The imposed strain history, up to the same instants, is reported also. As can bee seen from Figure 6, at every time instants, the equivalent strain converges to the imposed strain, ensuring the continuity of the corresponding stress in view of Eq. (4.2).

6. Conclusions

In this paper, a novel step-by-step approach to predict the response of nonlinear viscoelastic fractional systems with time-dependent fractional order has been proposed. Based on the assumption that the time steps are selected small enough to consider the system parameters as constant within every time step, the system response at $t_n = n\Delta t$ is obtained from an equivalent system, which exhibits the same past stress history of the original system within the time interval $0 \div t_{n-1}$; moreover, the equivalent system is characterized by the same constant parameters E_{β_n}, β_n of the original system through the time interval $0 \div t_n$. The equivalent system behaves linearly within the time interval $0 \div t_n$ and, for this reason, the Boltzmann linear superposition principle can be consistently applied to calculate the sought response at $t = t_n$.

The proposed approach has been tested with two numerical simulations where parameters have been obtained fitting experimental data available in the literature. Results obtained by the proposed approach have been compared with results obtained by means of a variable-order fractional operator existing in the literature and substantial differences have been found. The results suggest that, in order to fully validate the proposed approach, ad-hoc designed experimental tests with various histories of non-constant strain and strain rates are necessary. Finally, it is to notice that the proposed approach remains valid if the system parameters depend on an external state variable, such as the temperature. Further effort is warranted by the authors to formulate the proposed approach for implementation within general solution methods for large-scale structures, either at a continuum or a discrete level.

Data Accessibility. None.

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