Well-posedness for the backward problems in time for general time-fractional diffusion equation

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Partial Differential Equations — *Well-posedness for the backward problems in time for general time-fractional diffusion equation*, by Giuseppe Floridia, Zhiyuan Li and Masahiro Yamamoto, communicated on 28 April 2020.

Abstract. — In this article, we consider an evolution partial differential equation with Caputo time-derivative with the zero Dirichlet boundary condition: $\partial_t^\alpha u + Au = F$ where $0 < \alpha < 1$ and the principal part $-A$, is a non-symmetric elliptic operator of the second order. Given a source F, we prove the well-posedness for the backward problem in time and our result generalizes the existing results assuming that $-A$ is symmetric. The key is a perturbation argument and the completeness of the generalized eigenfunctions of the elliptic operator A.

Key words: Fractional PDE, backward problem, well-posedness

Mathematics Subject Classification: 35R11, 34A12

1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. Henceforth let $L^2(\Omega)$ denote the real Lebesgue space with the scalar product $(\cdot\,,\cdot)$ and the norm $\|\cdot\|$, and let $H^1(\Omega)$, $H_0^1(\Omega)$, $H^2(\Omega)$ be the Sobolev spaces (e.g., Adams [1]). By $\|u\|_{H^2(\Omega)}$ we denote the norm in $H^2(\Omega)$ for example.

We consider a fractional partial differential equation:

$$\begin{cases} \partial_t^\alpha u(x,t) = -Au(x,t) + F(x,t), & x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega} = 0, \\ u(x,0) = a(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $-A$ is a uniformly elliptic operator and not necessarily symmetric. Throughout this article, we assume that $0 < \alpha < 1$, and the Caputo derivative $\partial_t^\alpha g$ is defined by

$$\partial_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dg}{ds}(s)\, ds,$$

where Γ denotes the gamma function. It is known that there exists a unique solution $u = u(x,t)$ to the initial boundary value problem (1.1) under suitable conditions on A, a and F, and we refer for example to Gorenflo, Luchko and Yamamoto [12], Kubica, Ryszewska and Yamamoto [17], Kubica and Yamamoto [18], Sakamoto and Yamamoto [23], Zacher [33], and also later as lemmata we will show the regularity.

Equation (1.1) describes slow diffusion which can be considered as anomalous diffusion in highly heterogeneous media and is different from the classical case of $\alpha = 1$. In particular, the Caputo derivative is involved with memory term which possesses some averaging effect, and so (1.1) has not strong smoothing property: for $a \in L^2(\Omega)$, we can expect only $u(\cdot, t) \in H^2(\Omega)$ with each $t > 0$. This is an essential difference from the case of $\alpha = 1$.

Now we will formulate our problem and results. For $v \in H^2(\Omega)$, we set

$$-Av(x) := \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v)(x) + \sum_{j=1}^{d} b_j(x)\partial_j v(x) + c(x)v(x), \tag{1.2}$$

where

$$a_{ij} = a_{ji} \in C^1(\overline{\Omega}), \quad b_j, c \in C^1(\overline{\Omega}), \quad 1 \le i, j \le d$$

and there exists a constant $\kappa > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ge \kappa \sum_{j=1}^{d} \xi_j^2, \quad x \in \overline{\Omega}, \ \xi_1, \ldots, \xi_d \in \mathbb{R}.$$

We consider

$$\begin{cases} \partial_t^\alpha u(x,t) = -Au(x,t), & x \in \Omega, \ 0 < t < T, \\ u|_{\partial\Omega} = 0, \\ u(\cdot, T) = b \end{cases} \tag{1.3}$$

with $b \in H^2(\Omega) \cap H_0^1(\Omega)$.

We state our first main result.

Theorem 1.1. *For each $b \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (1.3) such that $\partial_t^\alpha u \in C((0,T]; L^2(\Omega))$. Moreover we can choose constants $C_1, C_2 > 0$ depending on T such that*

$$C_1\|u(\cdot,0)\|_{L^2(\Omega)} \le \|u(\cdot,T)\|_{H^2(\Omega)} \le C_2\|u(\cdot,0)\|_{L^2(\Omega)}. \tag{1.4}$$

To the best knowledge of the authors, Sakamoto and Yamamoto [23] is the first work for the well-posedness of the backward problem in time for the case of symmetric A, that is, $b_j \equiv 0$ for $1 \le j \le d$. Moreover by a technical reason, [23] assumes that $c \le 0$. As for backward problems for time-fractional equations with symmetric A, we can refer to many works: Liu and Yamamoto [19], Tuan, Huynh, Ngoc, and Zhou [24], Floridia and Yamamoto [34]. In particular, as for numerical approaches, see Tuan, Long and Tatar [25], Tuan, Thach, O'Regan, and Can [26]. Wang and Liu [27, 28], Wang, Wei and Zhou [29], Wei and Wang [30], Xiong, Wang and Li [31], Yang and Liu [32] and the references therein. However, we do not find the results for non-symmetric A. Originally the backward well-posedness comes from the time fractional derivative ∂_t^α, and should not rely on the symmetry of the elliptic operator A, and Theorem 1.1 is a natural generalization of the existing results since [23] to the case of a general uniform elliptic operator A. As is seen by the proof, we can further prove

COROLLARY 1.2. *In Theorem* 1.1, *for each distinct* $T_1, T_2 > 0$, *there exist contants* $C_3 = C_3(T_1, T_2) > 0$ *and* $C_4 = C_4(T_1, T_2) > 0$ *such that*

$$C_3\|u(\cdot, T_2)\|_{H^2(\Omega)} \le \|u(\cdot, T_1)\|_{H^2(\Omega)} \le C_4\|u(\cdot, T_2)\|_{H^2(\Omega)}.$$

Furthermore we can show also the backward well-posedness with the presence of a non-homogeneous term F. For the formulation, we introduce some function spaces. Let

$$-A_0 v(x) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v), \quad \mathscr{D}(A_0) = H^2(\Omega) \cap H_0^1(\Omega). \tag{1.5}$$

Then it is known that the specrum $\sigma(A_0)$ consists entirely of eigenvalues with finite multiplicities and according to the multiplicities we number:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 < \cdots. \tag{1.6}$$

Also we know that we can choose eigenfunctions φ_n for λ_n, $n \in \mathbb{N}$ such that $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$. Then we can define the fractional power A_0^γ with $\gamma \ge 0$:

$$\begin{cases} A_0^\gamma v = \sum_{n=1}^\infty \lambda_n^\gamma (v, \varphi_n)\varphi_n, \\ \mathscr{D}(A_0^\gamma) = \{v \in L^2(\Omega);\ \sum_{n=1}^\infty \lambda_n^{2\gamma}|(v, \varphi_n)|^2 < \infty\}, \\ \|A_0^\gamma v\| = (\sum_{n=1}^\infty \lambda_n^{2\gamma}|(v, \varphi_n)|^2)^{\frac{1}{2}}. \end{cases} \tag{1.7}$$

We can refer for example to Pazy [20] and we can derive (1.7) directly from

$$A_0 v = \sum_{n=1}^\infty \lambda_n (v, \varphi_n)\varphi_n,$$

$$\mathscr{D}(A_0) = \left\{ v \in L^2(\Omega);\ \sum_{n=1}^\infty \lambda_n^2 |(v, \varphi_n)|^2 < \infty \right\}.$$

Moreover we know that $\mathscr{D}(A_0^{\frac{1}{2}}) = H_0^1(\Omega)$, $\mathscr{D}(A_0^\gamma) \subset H^{2\gamma}(\Omega)$. Henceforth we set $\|v\|_{\mathscr{D}(A_0^\gamma)} = \|A_0^\gamma v\|$.

Now we are ready to state the well-posedness with non-homogeneous term.

THEOREM 1.3. *Let* $F \in L^\infty(0, T; \mathscr{D}(A_0^\varepsilon))$ *with some* $\varepsilon > 0$. *For each* $b \in H^2(\Omega) \cap H_0^1(\Omega)$, *there exists a unique solution*

$$u \in C((0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$$

to

$$\begin{cases} \partial_t^\alpha u = -Au + F(x,t), & x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega} = 0, \\ u(\cdot, T) = b \end{cases}$$

and we can choose a constant $C > 0$ *such that*

$$\|u(\cdot, 0)\| \le C(\|u(\cdot, T)\|_{H^2(\Omega)} + \|F\|_{L^\infty(0,T;\mathscr{D}(A_0^\varepsilon))}).$$

The article is composed of three sections. In Section 2, we show fundamental properties of the fractional differential equations and Section 3 is devoted to the proofs of Theorems 1.1 and 1.3.

2. Preliminaries

Let us recall (1.5) and (1.6). For $0 < \alpha < 1$ and $\beta > 0$, by $E_{\alpha,\beta}(z)$ we denote the Mittag–Leffler function with two parameters:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

(e.g., Podlubny [21]). Then $E_{\alpha,\beta}(z)$ is an entire function in $z \in \mathbb{C}$. We set

$$S(t)a = \sum_{n=0}^{\infty} (a, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x), \quad t \ge 0$$

and

$$K(t)a = \sum_{n=0}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)(a, \varphi_n)\varphi_n(x), \quad t > 0$$

for $a \in L^2(\Omega)$.

Henceforth we write $u(t) = u(\cdot, t)$, etc., and we regard u as a mapping defined in $(0, T)$ with values in $L^2(\Omega)$. Moreover $u(t) \in H_0^1(\Omega)$ means $u(\cdot, t) = 0$ on $\partial\Omega$ in the trace sense (e.g., [1]). Then we can see the following.

Lemma 2.1. (i) *There exists a constant* $C > 0$ *such that*

$$\|S(t)a\| \le C\|a\|, \quad t \ge 0 \tag{2.1}$$

and

$$\|A_0 S(t)a\| \le Ct^{-\alpha}\|a\|, \quad t > 0. \tag{2.2}$$

For $0 \le \gamma \le 1$, there exists a constant $C(\gamma) > 0$ such that

$$\|A_0^\gamma K(t)a\| \le C(\gamma)t^{\alpha(1-\gamma)-1}\|a\|, \quad t > 0. \tag{2.3}$$

(ii) *Let $G \in L^\infty(0, T; \mathscr{D}(A_0^\varepsilon))$ with some $\varepsilon > 0$ and $a \in L^2(\Omega)$. Then*

$$u(t) = S(t)a + \int_0^t K(t-s)G(s)\,ds, \quad t > 0 \tag{2.4}$$

is in $C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ and satisfies $\partial_t^\alpha u \in L^1(0, T; L^2(\Omega))$,

$$\begin{cases} \partial_t^\alpha u(t) = -A_0 u(t) + G(t), \quad t > 0, \\ \lim_{t\to 0}\|u(\cdot, t) - a\| = 0, \\ u(\cdot, t) \in H_0^1(\Omega), \quad 0 < t < T. \end{cases} \tag{2.5}$$

(iii) *For each $t > 0$, there exists a constant $C > 0$ such that*

$$\|u(t)\|_{H^2(\Omega)} \le C(t^{-\alpha}\|a\| + \|A_0^\varepsilon G\|_{L^\infty(0,T;L^2(\Omega))}).$$

Remark 1. We can prove stronger regularity of $\partial_t^\alpha u$ but the lemma is sufficient for our purpose.

Proof (of Lemma 2.1). (i) We can refer to Gorenflo, Luchko and Yamamoto [12], and for completeness we give the proof. First we note

$$|E_{\alpha,1}(-\eta)| \le \frac{C}{1+\eta}, \quad \eta > 0 \tag{2.6}$$

(e.g., Theorem 1.6 (p. 35) in Podlubny [21]).

Since $\{\varphi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$, by (2.6) we have

$$\begin{aligned} \|S(t)a\|^2 &= \sum_{n=1}^\infty |(a, \varphi_n)|^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \\ &\le \sum_{n=1}^\infty |(a, \varphi_n)|^2 \left(\frac{C}{1 + |\lambda_n t^\alpha|}\right)^2 \le C\sum_{n=1}^\infty |(a, \varphi_n)|^2, \end{aligned}$$

that is, (2.1) follows.

Next, since

$$A_0 S(t)a = \sum_{n=1}^\infty (a, \varphi_n)\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n,$$

again by (2.6) we see

$$\begin{aligned}\|A_0S(t)a\|^2 &= t^{-2\alpha}\sum_{n=1}^{\infty}|(a,\varphi_n)|^2|\lambda_n t^\alpha|^2|E_{\alpha,1}(-\lambda_n t^\alpha)|^2\\ &\le Ct^{-2\alpha}\sum_{n=1}^{\infty}|(a,\varphi_n)|^2\Big(\frac{|\lambda_n t^\alpha|}{1+|\lambda_n t^\alpha|}\Big)^2, \quad t>0,\end{aligned}$$

which implies (2.2).

By (1.7), we have

$$A_0^\gamma K(t)a = \sum_{n=1}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\lambda_n^\gamma(a,\varphi_n)\varphi_n,$$

and so

$$\begin{aligned}\|A_0^\gamma K(t)a\|^2 &\le t^{2\alpha-2}\sum_{n=1}^{\infty}\frac{C}{(1+|\lambda_n t^\alpha|)^2}\lambda_n^{2\gamma}|(a,\varphi_n)|^2\\ &= Ct^{2\alpha-2}\sum_{n=1}^{\infty}\frac{\lambda_n^{2\gamma}t^{2\gamma\alpha}}{(1+|\lambda_n t^\alpha|)^2}t^{-2\alpha\gamma}|(a,\varphi_n)|^2\\ &\le Ct^{2(\alpha-\alpha\gamma)-2}\sup_{\xi\ge 0}\Big(\frac{\xi^\gamma}{1+\xi}\Big)^2\sum_{n=1}^{\infty}|(a,\varphi_n)|^2.\end{aligned}$$

By $0\le\gamma\le 1$, we see that $\sup_{\xi\ge 0}\frac{\xi^\gamma}{1+\xi}<\infty$, and so (2.3) can be seen. Thus the proof of Lemma 2.1 (i) is complete.

(ii) In terms of e.g., Theorem 4.1 in [12] and Theorems 2.1 and 2.2 in [23], we already know some regularity of $u(t)$.

By Theorem 2.1 (i) in [23] or by (2.1), we can verify that $S(t)a \in C([0,T]; L^2(\Omega))$ and $\lim_{t\to 0}\|S(t)a-a\|=0$. By (2.2), we see that

$$A_0\Big(\sum_{n=1}^{N}(a,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n\Big)$$

converges in $C([\delta,T];L^2(\Omega))$ as $N\to\infty$ with arbitrarily fixed $\delta>0$. Therefore $A_0S(t)a\in C([\delta,T];L^2(\Omega))$, which implies

$$S(t)a\in C([\delta,T];\mathscr{D}(A_0)) = C([\delta,T];H^2(\Omega)\cap H_0^1(\Omega)). \tag{2.7}$$

Moreover, we can directly prove that $\partial_t^\alpha(E_{\alpha,1}(-\lambda_n t^\alpha)) = -\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)$, and obtain

$$\partial_t^\alpha S(t)a = \sum_{n=1}^{\infty}\partial_t^\alpha(E_{\alpha,1}(-\lambda_n t^\alpha))(a,\varphi_n)\varphi_n = \sum_{n=1}^{\infty}-\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)(a,\varphi_n)\varphi_n.$$

Hence, by (2.6) we see that

$$\|\partial_t^\alpha S(t)a\|^2 = \sum_{n=1}^\infty \lambda_n^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 |(a,\varphi_n)|^2 \tag{2.8}$$

$$= t^{-2\alpha} \sum_{n=1}^\infty (\lambda_n t^\alpha)^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 |(a,\varphi_n)|^2$$

$$\le Ct^{-2\alpha} \sum_{n=1}^\infty |(a,\varphi_n)|^2 \Big(\frac{\lambda_n t^\alpha}{1+\lambda_n t^\alpha}\Big)^2 \le Ct^{-2\alpha}\|a\|^2$$

and

$$\partial_t^\alpha S(t)a \in C((0,T];L^2(\Omega)). \tag{2.9}$$

By (2.3) with $\gamma = 0$, we can easily verify that

$$\left\|\int_0^t K(t-s)G(s)\,ds\right\| \le C\int_0^t (t-s)^{\alpha-1}\|G(s)\|\,ds$$

$$\le C\|G\|_{L^\infty(0,T;L^2(\Omega))}\frac{t^\alpha}{\alpha} \to 0.$$

Hence, with $S(t)a \in C([0,T];L^2(\Omega))$, we see that $\lim_{t\to 0}\|u(t)-a\| = 0$.

Moreover by Theorem 2.2 (i) in [23], we see

$$\partial_t^\alpha\Big(\int_0^t K(t-s)G(s)\,ds\Big) \in L^2(\Omega\times(0,T)).$$

This with (2.8), we obtain $\partial_t^\alpha u \in L^1(0,T;L^2(\Omega))$.

Now we will prove

$$\int_0^t K(t-s)G(s)\,ds \in C((0,T];H^2(\Omega)\cap H_0^1(\Omega)).$$

For arbitrarily fixed $0<\delta_0<\delta$, we set

$$v_{\delta_0}(t) = \int_0^{t-\delta_0} A_0K(t-s)G(s)\,ds, \quad t\ge\delta.$$

By (2.3) we can see that $v_{\delta_0} \in C([\delta,T];L^2(\Omega))$. For $\delta\le t\le T$, by (2.3) we estimate

$$\left\|\int_0^t A_0K(t-s)G(s)\,ds - v_{\delta_0}(t)\right\| = \left\|\int_{t-\delta_0}^t A_0K(t-s)G(s)\,ds\right\|$$

$$= \left\|\int_{t-\delta_0}^t A_0^{1-\varepsilon}K(t-s)A_0^\varepsilon G(s)\,ds\right\|$$

$$\leq C \int_{t-\delta_0}^{t} (t-s)^{\alpha\varepsilon - 1} \|A_0^{\varepsilon} G(s)\| \, ds$$

$$\leq C \|A_0^{\varepsilon} G\|_{L^\infty(0,T;L^2(\Omega))} \frac{\delta_0^{\alpha\varepsilon}}{\alpha\varepsilon}.$$

Hence

$$v_{\delta_0} \to \int_0^t A_0 K(t-s) G(s) \, ds \quad \text{in } C([\delta, T]; L^2(\Omega))$$

as $\delta_0 \to 0$, and by $v_{\delta_0} \in C([\delta, T]; L^2(\Omega))$, we conclude that

$$\int_0^t K(t-s) G(s) \, ds \in C([\delta, T]; H^2(\Omega) \cap H_0^1(\Omega))$$

for any $\delta > 0$, and then

$$\int_0^t K(t-s) G(s) \, ds \in C((0, T]; H^2(\Omega) \cap H_0^1(\Omega)).$$

Consequently by (2.7), we obtain $u \in C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$.
Finally, by (2.3) we have

$$\begin{aligned} \left\| A_0 \int_0^t K(t-s) G(s) \, ds \right\| &= \left\| \int_0^t A_0^{1-\varepsilon} K(t-s) A_0^{\varepsilon} G(s) \, ds \right\| \\ &\leq C \int_0^t (t-s)^{\alpha\varepsilon - 1} \|A_0^{\varepsilon} G(s)\| \, ds \\ &\leq C \|A_0^{\varepsilon} G\|_{L^\infty(0,T;L^2(\Omega))} \frac{t^{\alpha\varepsilon}}{\alpha\varepsilon}. \end{aligned}$$

With (2.2), the proof of the part (iii) is complete. Thus the proof of Lemma 2.1 is complete. □

Henceforth we set

$$Bv(x) = \sum_{j=1}^{d} b_j(x) \partial_j v(x) + c(x) v(x), \quad v \in \mathscr{D}(B) = H^2(\Omega) \cap H_0^1(\Omega).$$

Next by Lemma 2.1, we can prove

LEMMA 2.2. *Let $F \in L^\infty(0, T; \mathscr{D}(A_0^{\varepsilon}))$ with some $\varepsilon > 0$ and $a \in L^2(\Omega)$. Then the solution u to* (1.1) *belongs to*

$$C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$$

and there exists a constant $C > 0$ *depending on* T, *such that*

$$\|u(T)\|_{H^2(\Omega)} \le C(t^{-\alpha}\|a\| + \|A_0^{\varepsilon}F\|_{L^\infty(0,T;L^2(\Omega))}), \quad t > 0.$$

PROOF (OF LEMMA 2.2). Without loss of generality, we can assume that $0 < \varepsilon < \frac{1}{4}$. By Lemma 2.1, we have

$$u(t) = S(t)a + \int_0^t K(t-s)F(s)\,ds + \int_0^t K(t-s)Bu(s)\,ds. \tag{2.10}$$

By Gorenflo, Luchko and Yamamoto [12] or Kubica, Ryszewska and Yamamoto [17], we know that there exists a unique solution $u \in C([0,T];L^2(\Omega))$ to (2.10). Applying A_0 to equation (2.10), we have

$$A_0u(t) = A_0S(t)a + \int_0^t A_0^{1-\varepsilon}K(t-s)A_0^{\varepsilon}F(s)\,ds + \int_0^t A_0^{1-\varepsilon}K(t-s)A_0^{\varepsilon}Bu(s)\,ds.$$

Then, applying Lemma 2.1 (i), we obtain

$$\begin{aligned}\|u(t)\|_{H^2(\Omega)} &\le Ct^{-\alpha}\|a\| + C\int_0^t (t-s)^{\alpha\varepsilon-1}\,ds\|A_0^{\varepsilon}F\|_{L^\infty(0,T;L^2(\Omega))}\\ &\quad + C\int_0^t (t-s)^{\alpha\varepsilon-1}\|u(s)\|_{H^2(\Omega)}\,ds\\ &\le C(t^{-\alpha}\|a\| + \|A_0^{\varepsilon}F\|_{L^\infty(0,T;L^2(\Omega))}) + C\int_0^t (t-s)^{\alpha\varepsilon-1}\|u(s)\|_{H^2(\Omega)}\,ds.\end{aligned}$$

Here we used the following: by $0 < \varepsilon < \frac{1}{4}$ we have $\|A_0^{\varepsilon}v\| \sim \|v\|_{H^{2\varepsilon}(\Omega)}$ for $v \in \mathscr{D}(A_0^{\varepsilon}) = H^{2\varepsilon}(\Omega)$ (e.g., Fujiwara [11]), and so

$$\|A_0^{\varepsilon}Bu(s)\| \le C\|Bu(s)\|_{H^{2\varepsilon}(\Omega)} \le C\|u(s)\|_{H^2(\Omega)}$$

because $Bu(s) \in H^1(\Omega) \subset \mathscr{D}(A_0^{\varepsilon})$ by $u(s) \in H^2(\Omega) \cap H_0^1(\Omega)$. The generalized Gronwall inequality (e.g., Henry [14] or Lemma A.2 in [17]) yields

$$\begin{aligned}\|u(t)\|_{H^2(\Omega)} &\le C(t^{-\alpha}\|a\| + \|A_0^{\varepsilon}F\|_{L^\infty(0,T;L^2(\Omega))})\\ &\qquad + Ce^{Ct}\int_0^t (t-s)^{\alpha\varepsilon-1}(s^{-\alpha}\|a\| + \|A_0^{\varepsilon}F\|_{L^\infty(0,T;L^2(\Omega))})\,ds\\ &\le C(t^{-\alpha}\|a\| + \|A_0^{\varepsilon}F\|_{L^\infty(0,T;L^2(\Omega))})\\ &\qquad + Ce^{Ct}\Big(t^{\alpha\varepsilon-\alpha}\frac{\Gamma(\alpha\varepsilon)\Gamma(1-\alpha)}{\Gamma(1-\alpha+\alpha\varepsilon)}\|a\| + \frac{t^{\alpha\varepsilon}}{\alpha\varepsilon}\|A_0^{\varepsilon}F\|_{L^\infty(0,T;L^2(\Omega))}\Big).\end{aligned}$$

Consequently

$$\|u(t)\|_{H^2(\Omega)} \le C(t^{-\alpha}\|a\| + \|A_0^\varepsilon F\|_{L^\infty(0,T;L^2(\Omega))}).$$

Thus the proof of Lemma 2.2 is complete. □

Finally we know

LEMMA 2.3. *For $T > 0$, the operator*

$$S(T) : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$$

is surjective and there exist constants $C_1, C_2 > 0$ such that

$$C_1\|S(T)a\|_{H^2(\Omega)} \le \|a\| \le C_2\|S(T)a\|_{H^2(\Omega)}.$$

Lemma 2.3 is proved as Theorem 4.1 in [23], whose proof is based on the representation of $S(T)a$ by the eigenfunction expansion and the complete monotonicity of $E_{\alpha,1}(-\lambda_n t^\alpha)$ (e.g., Gorenflo, Kilbas, Mainardi and Rogosin [13], pp. 46–47, Pollard [22]).

3. Proofs of Theorems 1.1 and 1.3

3.1. Proof of Theorem 1.1

In terms of the lower-order part B of the elliptic operator $-A$, we can rewrite (1.1) as

$$\begin{cases} \partial_t^\alpha u(t) = -A_0 u(t) + Bu(t), & t > 0, \\ u(0) = a, \\ u(t) \in H_0^1(\Omega), & 0 < t < T. \end{cases} \tag{3.1}$$

By Lemma 2.1 (ii), we have

$$b := u_a(T) = S(T)a + \int_0^T K(T-s)Bu_a(s)\, ds. \tag{3.2}$$

Here, by $u_a(t)$, we denote the solution to (3.1). Applying Lemma 2.3 to (3.2), we obtain

$$a = S(T)^{-1}b - S(T)^{-1}\int_0^T K(T-s)Bu_a(s)\, ds =: S(T)^{-1}b - La, \tag{3.3}$$

where

$$La = S(T)^{-1}\int_0^T K(T-s)Bu_a(s)\, ds. \tag{3.4}$$

First Step. We prove that $L : L^2(\Omega) \to L^2(\Omega)$ is a compact operator. We set

$$L_0 a = \int_0^T K(T-s)Bu_a(s)\,ds, \quad a \in L^2(\Omega).$$

Then $La = S(T)^{-1}L_0 a$.

We choose $0 < \delta_0 < \delta_1 < \frac{1}{4}$. We will estimate $\|A_0^{1+\delta_0}L_0 a\|$. We note that $A_0^\gamma K(t)a = K(t)A_0^\gamma a$ for $\gamma \ge 0$ and $a \in \mathscr{D}(A_0^\gamma)$, which can be directly verified. By (2.3), we have

$$\begin{aligned}
\|A_0^{1+\delta_0}L_0 a\| &= \left\| \int_0^T A_0^{1+\delta_0}K(T-s)Bu_a(s)\,ds \right\| \\
&= \left\| \int_0^T A_0^{1+\delta_0-\delta_1}K(T-s)A_0^{\delta_1}B(u_a(s))\,ds \right\| \\
&\le C\int_0^T (T-s)^{\alpha(\delta_1-\delta_0)-1}\|A_0^{\delta_1}Bu_a(s)\|\,ds \\
&\le C\int_0^T (T-s)^{\alpha(\delta_1-\delta_0)-1}s^{-\alpha}\|a\|\,ds.
\end{aligned}$$

For the last inequality, we used $0 < \delta_0 < \delta_1 < \frac{1}{4}$, and $b_j, c \in C^1(\overline{\Omega})$ and Lemma 2.2, and $\mathscr{D}(A_0^{\delta_1}) = H^{2\delta_1}(\Omega)$ (e.g., [11]) and

$$\begin{aligned}
\|A_0^{\delta_1}Bu_a(s)\| &\le C\|Bu_a(s)\|_{H^{2\delta_1}(\Omega)} \\
&\le C\|u_a(s)\|_{H^{1+2\delta_1}(\Omega)} \le C\|A_0 u_a(s)\| \le Cs^{-\alpha}\|a\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|A_0^{1+\delta_0}L_0 a\| &\le C\|a\|\int_0^T (T-s)^{\alpha(\delta_1-\delta_0)-1}s^{-\alpha}\,ds \\
&= CT^{\alpha(\delta_1-\delta_0-1)}\frac{\Gamma(\alpha(\delta_1-\delta_0))\Gamma(1-\alpha)}{\Gamma(1-\alpha+\alpha(\delta_1-\delta_0))}\|a\|
\end{aligned}$$

because $\delta_1 - \delta_0 > 0$.

Since $\mathscr{D}(A_0^{1+\delta_0}) \subset H^{2+2\delta_0}(\Omega)$ and the embedding $H^{2+2\delta_0}(\Omega) \to H^2(\Omega)$ is compact, the operator $L_0 : L^2(\Omega) \to H^2(\Omega)$ is compact. Moreover $S(T)^{-1} : H^2(\Omega) \to L^2(\Omega)$ is bounded by Lemma 2.3, we see that $L = S(T)^{-1}L_0 : L^2(\Omega) \to L^2(\Omega)$ is a compact operator.

Second Step. Since $b \in H^2(\Omega) \cap H_0^1(\Omega)$, by Lemma 2.3 we have $p := S(T)^{-1}b \in L^2(\Omega)$ and we rewrite (3.3) as

$$(1+L)a = p \quad \text{in } L^2(\Omega). \tag{3.5}$$

In the First Step, we already prove that $L : L^2(\Omega) \to L^2(\Omega)$ is compact. Hence if we will prove that

$$La = -a \quad \text{implies} \quad a = 0, \tag{3.6}$$

then the Fredholm alternative yields that $(1 + L)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator, and the proof can be finished.

Equation (3.6) implies

$$S(T)a + \int_0^T K(T-s)Bu_a(s)\,ds = 0 \quad \text{in } L^2(\Omega).$$

Then we have to prove $a = 0$. For it, by means of Lemma 2.1 (ii), it is sufficient to prove that if w satisfies

$$\begin{cases} \partial_t^\alpha w(t) = -Aw(t), \\ w(t) \in H_0^1(\Omega), \quad 0 < t < T \end{cases}$$

and $w(T) = 0$ in $L^2(\Omega)$, then $w(0) = 0$.

We recall that the operator A is defined by (1.2) with $\mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then it is known that the spectrum $\sigma(A)$ of A consists entirely of eigenvalues with finite multiplicities. We denote $\sigma(A)$ by $\{\mu_1, \mu_2, \ldots\}$. Here $\sigma(A)$ is a set and so μ_i and μ_j, $i \neq j$ are mutually distinct. Let P_n be the projection for μ_n, $n \in \mathbb{N}$ which is defined by

$$P_n = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma(\mu_n)} (z - A)^{-1}\,dz,$$

where $\gamma(\mu_n)$ is a circle centered at μ_n with sufficiently small radius such that the disc bounded by $\gamma(\mu_n)$ does not contain any points in $\sigma(A) \backslash \{\mu_n\}$. Then $P_n : L^2(\Omega) \to L^2(\Omega)$ is a bouned linear operator and $P_n^2 = P_n$ for $n \in \mathbb{N}$ (e.g., Kato [15]). Setting $m_n := \dim P_n L^2(\Omega)$, we have $m_n < \infty$.

The following is a fundamental fact.

Lemma 3.1. *If $y \in L^2(\Omega)$ satisfies $P_n y = 0$ for all $n \in \mathbb{N}$, then $y = 0$.*

Proof. First we note

$$-(A^* v)(x) = \sum_{i,j=1}^d \partial_i(a_{ij}\partial_j v) - \sum_{j=1}^d \partial_j(b_j v) + c(x)v, \quad \mathscr{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega),$$

where A^* is the adjoint operator of A. Let P_n^* be the adjoint operator of P_n: $(P_n\varphi, \psi) = (\varphi, P_n^*\psi)$ for each $\varphi, \psi \in L^2(\Omega)$.

Then it is known (e.g., [15]) that $\sigma(A^*) = \{\overline{\mu_n}\}_{n \in \mathbb{N}}$, where $\bar{\mu}$ denotes the complex conjugate of $\mu \in \mathbb{C}$ and P_n^* is the projection for the eigenvalue $\overline{\mu_n}$ of A^*, and dim $P_n^*L^2(\Omega) = \dim P_nL^2(\Omega) = m_n$. Then by Theorem 16.5 in Agmon [2], we have

$$\overline{\mathrm{Span}_{n\in\mathbb{N}}\, P_n^*L^2(\Omega)} = L^2(\Omega),$$

that is,

$$(y, P_n^*\psi) = 0, \quad n \in \mathbb{N}, \psi \in L^2(\Omega) \quad \text{imply } y = 0. \tag{3.7}$$

Now we can complete the proof of Lemma 3.1. Let $P_n y = 0$ for $n \in \mathbb{N}$. Then $(P_n y, \psi) = 0$ for all $\psi \in L^2(\Omega)$. Therefore $0 = (P_n y, \psi) = (y, P_n^*\psi)$ for all $n \in \mathbb{N}$ and $\psi \in L^2(\Omega)$, which yields $y = 0$ by (3.7). □

Third Step: Completion of the proof of Theorem 1.1. Let us note $\partial_t^\alpha(P_nu(t)) = P_n\partial_t^\alpha u(t)$ because $P_n : L^2(\Omega) \to L^2(\Omega)$ is a bounded operator. We set $u_n(t) = P_nu(t)$. Then

$$-P_nAu_n(t) = -Au_n(t) = -\mu_nu_n(t) + D_nu_n(t),$$

where D_n is an operator satisfying $D_n^{m_n} = O$, which corresponds to the Jordan canonical form. Then (3.1) yields

$$\begin{cases} \partial_t^\alpha u_n(t) = (-\mu_n + D_n)u_n(t), \\ u_n(0) = P_na, \quad n \in \mathbb{N}. \end{cases}$$

We can define an operator $E_{\alpha,1}((-\mu_n + D_n)t^\alpha)$ by the power series:

$$E_{\alpha,1}((-\mu_n + D_n)t^\alpha) = \sum_{k=0}^{\infty} \frac{(-\mu_n + D_n)^k t^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad t > 0.$$

Then we can directly verify

$$u_n(t) = E_{\alpha,1}((-\mu_n + D_n)t^\alpha)P_na, \quad t > 0. \tag{3.8}$$

Now we calculate the right-hand side of (3.8). Correspondingly to the Jordan canonical form, we can choose a suitable basis of $P_nL^2(\Omega)$:

$$\psi_j^k : k = 1, \ldots, \ell_n, \quad j = 1, \ldots, d_k$$

satisfying $\sum_{k=1}^{\ell_n} d_k = m_n$, and

$$\begin{cases} (A - \mu_n)\psi_1^k = 0, \\ (A - \mu_n)\psi_2^k = \psi_1^k, \\ \cdots\cdots\cdots, \\ (A - \mu_n)\psi_{d_k}^k = \psi_{d_k-1}^k, \quad 1 \le k \le \ell_n. \end{cases}$$

We expand $P_n a$ in terms of this basis in $P_n L^2(\Omega)$:

$$P_n a = \sum_{k=1}^{\ell_n} \sum_{j=1}^{d_k} a_j^k \psi_j^k.$$

Then

$$E_{\alpha,1}((-\mu_n + D_n)t^\alpha)(\psi_1^k \psi_2^k \cdots \psi_{d_k}^k)\begin{pmatrix} a_1^k \\ \vdots \\ a_{d_k}^k \end{pmatrix}$$

$$= \sum_{m=0}^{\infty} t^{\alpha m} \frac{(-\mu_n + D_n)^m}{\Gamma(\alpha m + 1)} (\psi_1^k \psi_2^k \cdots \psi_{d_k}^k) \begin{pmatrix} a_1^k \\ \vdots \\ a_{d_k}^k \end{pmatrix}$$

$$= (\psi_1^k \psi_2^k \cdots \psi_{d_k}^k)$$

$$\times \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m + 1)} \begin{pmatrix} (-\mu_n)^m & * & \cdots & * & * \\ 0 & (-\mu_n)^m & \cdots & * & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (-\mu_n)^m & * \\ 0 & 0 & \cdots & 0 & (-\mu_n)^m \end{pmatrix} \begin{pmatrix} a_1^k \\ \vdots \\ a_{d_k}^k \end{pmatrix}.$$

Since $u_n(T) = 0$, we see that each component of the above is equal to 0 at $t = T$, and so

$$\begin{cases} E_{\alpha,1}(-\mu_n T^\alpha) a_1^k + \sum_{p=2}^{d_k} \theta_{1p} a_p^k = 0, \\ E_{\alpha,1}(-\mu_n T^\alpha) a_2^k + \sum_{p=3}^{d_k} \theta_{2p} a_p^k = 0, \\ \cdots\cdots\cdots \\ E_{\alpha,1}(-\mu_n T^\alpha) a_{d_k-1}^k + \theta_{d_k-1, d_k} a_{d_k}^k = 0, \\ E_{\alpha,1}(-\mu_n T^\alpha) a_{d_k}^k = 0, \end{cases} \tag{3.9}$$

where θ_{jp} with $j+1 \le p \le d_k$ and $j = 1, \ldots, d_k - 1$, are some constants depending also on T. By the complete monotonicity (e.g., [13] and [22]), we see that $E_{\alpha,1}(-\mu_n T^\alpha) \neq 0$. Therefore by the backward substitution in (3.9), we can sequentially obtain $a_{d_k}^k = 0$, $a_{d_k-1}^k = 0, \ldots, a_1^k = 0$ for $k = 1, \ldots, \ell_n$. Hence $P_n a = 0$ for each $n \in \mathbb{N}$. Then we reach $a = 0$ in $L^2(\Omega)$. Thus the proof of Theorem 1.1 is complete. □

3.2. Proof of Theorem 1.3

Let $w = w(t)$ be the solution to

$$\begin{cases} \partial_t^\alpha w(t) = -Aw(t) + F, & t > 0, \\ w(0) = 0, \quad w(t) \in H_0^1(\Omega), & t > 0. \end{cases}$$

Since $F \in L^\infty(0, T; \mathscr{D}(A_0^\varepsilon))$, Lemma 2.2 proves that $w \in C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$. We consider

$$\begin{cases} \partial_t^\alpha v(t) = -Av(t), & t > 0, \\ v(T) = b - w(T), \quad v(t) \in H_0^1(\Omega), & t > 0. \end{cases} \tag{3.10}$$

By Theorem 1.1, for $b \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution $v \in C([0, T]; L^2(\Omega)) \cap C((0, T]; H^2(\Omega) \cap H_0^1(\Omega))$ such that $\partial_t^\alpha v \in C((0, T]; L^2(\Omega))$ to (3.10). Setting $u = v + w$, we see that $u(T) = b - w(T) + w(T) = b$. Then we can verify that u satisfies

$$\begin{cases} \partial_t^\alpha u(t) = -Au(t) + F(t), & t > 0, \\ u(T) = b, \quad u(t) \in H_0^1(\Omega), & t > 0. \end{cases}$$

The uniqueness of u is seen by Theorem 1.1. Thus the proof of Theorem 1.3 is complete. □

In future projects we would investigate similar problems where the principal part is an operator of order greater than 2, like in [9] and in [10], and in the case of applied systems like [3], [5], [7], [8], and [16]. Moreover we would study related inverse problems similarly to [4] and [6].

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Giuseppe Floridia
Department PAU
Università Mediterranea di Reggio Calabria
Via dell'Università 25
89124 Reggio Calabria, Italy
floridia.giuseppe@icloud.com

Zhiyuan Li
School of Mathematics and Statistics
Shandong University of Technology
Zibo
Shandong 255049, China
zyli@sdut.edu.cn

Masahiro Yamamoto
Graduate School of Mathematical Sciences
The University of Tokyo
Komaba, Meguro
Tokyo 153-8914, Japan
and

Honorary Member of Academy of Romanian Scientists
Splaiul Independentei Street, no. 54
050094 Bucharest, Romania
and
Peoples' Friendship University of Russia (RUDN University)
6 Miklukho-Maklaya St
Moscow 117198, Russia
myama@ms.u-tokyo.ac.jp