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PROPAGATION OF UNCERTAIN STRUCTURAL PROPERTIES DESCRIBED BY IMPRECISE PROBABILITY DENSITY FUNCTIONS VIA RESPONSE SURFACE METHOD

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Abstract

The present study addresses the analysis of structures with uncertain properties modelled as random variables characterized by *imprecise Probability Density Functions* (*PDF*s), namely *PDF*s with interval basic parameters (mean-value, variance, etc.). Due to imprecision in the probabilistic model, the statistics of the response and the *failure probability* are described by interval quantities. An efficient procedure for evaluating the bounds of such quantities is developed. The proposed method stems from the application of a ratio of polynomial *response surface* [1],[2] in conjunction with the classical probabilistic analysis and the so-called *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*) [3]. Interval response statistics are derived as approximate explicit functions of the interval parameters describing *imprecise probabilities*. The range of the *interval failure probability* is estimated in terms of the *interval reliability index* once the bounds of the interval mean-value and variance of the response are evaluated.

Numerical results concerning a frame structure and a grid structure with uncertain Young's moduli characterized by *imprecise PDF*s are presented. The accuracy of the proposed method

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along with the influence of randomness and imprecision of the input parameters on response statistics and reliability assessment are investigated.

Keywords: *Imprecise Probability Density Function*; *Improved Interval Analysis*; *Response surface* method; Response statistics; *Reliability index*; *Failure probability*.

1. INTRODUCTION

Uncertainty quantification and propagation are gaining increasing importance as fundamental aspects in the analysis, design and optimization of engineering systems (see e.g., [4][5]). Indeed, several studies have demonstrated that uncertainties affecting structural parameters (e.g., material and/or geometric properties, fabrication details, etc.), models, environmental conditions and external loads have a substantial influence on structural safety assessment and decision-making. In this context, the selection of appropriate mathematical models for the representation of the non-deterministic input, based on available empirical information, plays a crucial role.

Uncertainty is commonly classified as either *aleatory* or *epistemic* [6]. *Aleatory* uncertainty is a property of the system associated with fluctuations/variability, also referred to as *irreducible* uncertainty to state that, even when all information on the considered quantity is available, it cannot be deterministically defined [7]. On the other hand, *epistemic* uncertainty, also referred to as *reducible* uncertainty, is due to lack of knowledge. Basically, *aleatory* uncertainty can be modelled as random variables or random fields using probabilistic methods, while *epistemic* uncertainty requires specific models relying on available information. Non-probabilistic or possibilistic methods, such as convex models [8], interval models [9],[10] and fuzzy sets [11] are commonly adopted for handling *epistemic* uncertainties. Both probabilistic and non-probabilistic methods, however, have some limitations [12]. In spite of its popularity, the traditional probabilistic approach fails to provide reliable results when available information is not sufficient to define a probabilistic model of uncertainties with adequate confidence. On the other hand, non-probabilistic methods are applicable for handling uncertainties described by incomplete or fragmentary data, but are unable to provide probabilistic information on the output. A deep insight into the main features of probabilistic and non-probabilistic models leads to the conclusion that they should not be think of as necessarily distinct and mutually exclusive [13]. Indeed, in practical applications input parameters may be affected simultaneously by *aleatory* and *epistemic* uncertaintiy.

The theory of *imprecise probability* (see e.g., [14],[15]) has been developed as a generalized uncertainty model intermediate between probabilistic and non-probabilistic approaches. In the framework of *imprecise probability*, a unified treatment of *epistemic* and *aleatory* uncertainty is provided. Specifically, *epistemic* uncertainties, typically arising when available data are quite limited and of poor quality as well as imprecise, diffuse, fluctuating, incomplete, fragmentary, vague or ambiguous, are handled by non-probabilistic methods and included in the traditional probabilistic model which thus turns out to be *imprecise*. In this respect, the theory of *imprecise probability* may be viewed as a generalization of the traditional probability theory (see e.g., [16]-[19]). The distinctive feature of *imprecise probabilities* is the characterization of the uncertainty of an event by two measure values, namely a lower probability and an upper probability [18]. Conversely, a single probability measure is assigned to possible values of the uncertain quantity in the context the conventional probabilistic model. A rich variety of concepts and mathematical representations of *imprecise probabilities* have been proposed in the literature, such as *evidence theory* [16],[17], *interval probability* [19], *probability bounds analysis* [20],[21], *fuzzy probability* [22], etc. The *evidence theory*, formulated by Dempster [16] and Shafer [17], can be considered as a variant of probability theory in which the elements of the sample space are not single points but sets of values. The *interval probability* theory has been introduced by Weichselberger [19] as a generalization of Kolmogorov’s classical probability achieved through the use of lower and upper probabilities. In the framework of *probability bounds analysis* [20],[21], also known as *p-box*, the standard interval analysis [9],[10] and the traditional probability theory are unified by representing an imprecise random variable by upper and lower bounds of its *cumulative density function* (*CDF*). *Fuzzy probability* [22] may be viewed as the result of fuzzy set theory and probability theory wherein imprecision in the probabilistic model specification is described with fuzzy sets [15].

Several efficient procedures are available in the literature to propagate uncertainties modelled as random variables or random fields. The analysis of real engineering problems under pure probabilistic uncertainties can be afforded relying on advanced algorithms and modern high-power computers. Uncertainty propagation in the framework of *imprecise probability* is more challenging since it involves expensive computations. Recently, Wei et al. [23] proposed a general framework, termed as *non-intrusive imprecise stochastic simulation*, for uncertainty propagation under the background of *imprecise probability*. Meta-models or surrogate models, traditionally used within a probabilistic framework, are becoming increasing popular tools to reduce the computational burden in *imprecise probability* propagation. Recent applications include, the evaluation of *failure probability* using the Kriging model [24], the propagation of multiple imprecise random field uncertainties through polynomial chaos expansion [25], a two-level meta-modelling approach developed using non-intrusive sparse polynomial chaos expansions to propagate uncertainties modeled as *p-boxes* [26], and squeal analysis of uncertain disc brakes based on *evidence theory* [27]. The first two authors [28] proposed an efficient procedure for the static analysis of structures with random axial stiffness described by *imprecise PDF* based on the use of the *Rational Series Expansion* (*RSE*) [29] in conjunction with the classical probabilistic analysis and the so-called *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*) [3]. The *RSE* provides an approximate analytical expression of the response which can be used as a surrogate of the expensive numerical model to efficiently propagate *imprecise probability*.

Over the last decades, much research efforts have been devoted to the development of efficient procedures to perform reliability and risk assessment under *imprecise probabilities* (see e.g., [30]-[37]). In this context, the aim of the analysis is the evaluation of an upper value for the *failure probability* as the worst case scenario [15] rather than a single value.

The present study addresses the static analysis of discretized structures with uncertain properties modeled as random variables characterized by *imprecise PDF*s. Imprecision is incorporated into an assigned probabilistic model by assuming that the basic parameters of the underlying *PDF* are described by intervals . Thus, a family of *PDF*s is obtained rather than a single distribution as in the context of the classical probability theory. This implies that response statistics and the *probability of failure* for an assigned limit state have an interval nature as well. In this context, the challenging task is to propagate simultaneously the random and the interval character of the uncertain parameters in order to obtain tight bounds of the output. To this aim, the method developed by the first two authors [28] for structures with uncertain axial stiffness is properly extended to the case of general discretized structures exhibiting fluctuations of material and/or geometrical properties described by imprecise *PDF*s. The key idea behind the proposed approach is to approximate the random displacements using a ratio of polynomial *response surface*, originally introduced in the context of classical probabilistic analysis [1] and recently extended to the interval framework [2]. Once the *response surface* is defined, the random and the interval character of uncertainty are propagated into two separate steps, according to the underlying idea of *imprecise probability*. First, by applying the classical probability theory, interval response statistics are obtained as approximate explicit functions of the interval parameters modelling imprecision. Then, the *IIA* via *EUI* [3] is applied to evaluate the bounds of response statistics. The main feature of the *IIA* via *EUI* is the capability to reduce overestimation affecting interval computations carried out in the context of the *classical interval analysis* [10].

Finally, to perform reliability assessment under imprecise random uncertainties, the *interval reliability index* and the *interval probability of failure* are defined and evaluated in explicit form for the case of structural response and resistance modelled as independent random variables characterized by *imprecise* and *precise* lognormal distributions, respectively.

The accuracy and efficiency of the proposed method are demonstrated by analyzing a frame structure and a grid structure with uncertain Young's moduli described by *imprecise PDF*s. For validation purposes, appropriate comparisons with the results obtained by applying *Monte Carlo Simulation* (*MCS*) in conjunction with the classical combinatorial procedure [28] are carried out.

The paper is organized as follows: Section 2 presents the problem statement for structures with uncertain parameters modeled as random variables within the framework of the traditional probability theory and introduces the ratio of polynomial *response surface*; in Section 3, first, the assumed imprecise probabilistic model of the uncertain properties involving interval basic parameters is defined, then the proposed method for the evaluation of the bounds of the interval mean-value and variance of the response is outlined; Section 4 focuses on reliability assessment under *imprecise probabilities*; finally, in Section 5, two numerical applications are presented; concluding remarks are given in Section 6.

2. LINEAR DISCRETIZED STRUCTURES WITH UNCERTAIN PROPERTIES

2.1 Problem formulation

Let us consider a n -DOF discretized structural system with p uncertain parameters, $\rho_i = \rho_{0,i}\left(1+X_i\right)$, ($i = 1, 2, \ldots, p$), whose dimensionless fluctuations X_i around the nominal value $\rho_{0,i}$ are modelled as zero-mean random variables collected into the vector

$\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \cdots & X_p \end{bmatrix}^{\mathrm{T}}$ where T means transpose operator. The equilibrium equations of the structure subjected to deterministic static loads can be written as follows:

$$\mathbf{K}(\mathbf{X})\mathbf{U}(\mathbf{X}) = \mathbf{f} \tag{1}$$

where $\mathbf{K}(\mathbf{X})$ is the $n \times n$ stiffness matrix, which depends on the random fluctuations X_i; $\mathbf{f}$ is the n-vector collecting the external nodal loads; and $\mathbf{U}(\mathbf{X})$ is the n-vector of the unknown random nodal displacements. In order to ensure physically meaningful values of the uncertain parameters, it is assumed that the fluctuations satisfy the conditions $|X_i| < 1$, ($i = 1, 2, \ldots, p$), with the symbol $|\bullet|$ meaning absolute value.

The random stiffness matrix can be expressed as sum of the nominal value plus a linear function of the fluctuations X_i (see e.g., [38]), i.e.:

$$\mathbf{K}(\mathbf{X}) = \mathbf{K}_0 + \sum_{i=1}^{p} \mathbf{K}_i X_i . \tag{2}$$

In the previous equation, $\mathbf{K}_0$ denotes the nominal stiffness matrix, which is a $n \times n$ positive definite symmetric matrix, while $\mathbf{K}_i$ is a $n \times n$ positive semi-definite symmetric matrix, defined, respectively, as

$$\mathbf{K}_0 = \mathbf{K}(\mathbf{X})\big|_{\mathbf{X}=\mathbf{0}}; \quad \mathbf{K}_i = \left.\frac{\partial \mathbf{K}(\mathbf{X})}{\partial X_i}\right|_{\mathbf{X}=\mathbf{0}} . \tag{3a,b}$$

Classical *Monte Carlo Simulation* (*MCS*) is a straightforward approach to perform the probabilistic characterization of the random displacement vector $\mathbf{U}(\mathbf{X})$. In spite of its generality, however, this method requires high computational times, especially for real engineering problems. To overcome this limitation, the knowledge of the explicit relationship between the response and the random variables X_i is highly desirable. As known, for real engineering problems, only the implicit dependence of the output on the input parameters is generally available. Several strategies have been proposed in the literature to derive the structural response as an approximate explicit function of the uncertain parameters. Within the static setting, special attention has been devoted to the evaluation of the inverse of the random stiffness matrix $\mathbf{K}(\mathbf{X})$ as an approximate explicit function of the random variables X_i. The Neumann series expansion (see e.g., [39]) has been widely used to achieve this aim. Recently, the so-called *Rational Series Expansion* (*RSE*) [29] has been derived as a modified explicit expression of the Neumann series for evaluating the inverse of a matrix with small modifications. More general approaches focus on the replacement of the real expensive numerical model with surrogates or meta-models which define an approximate functional dependence of the response on the input parameters. Orthogonal polynomial expansions of the response [40] and the so-called *response surface method* [41] are among the most popular approaches to define surrogate models for uncertainty propagation.

2.2 Response surface approach

In the present study, a ratio of polynomial *response surface*, originally derived in the context of classical probabilistic analysis [1] and recently extended to the interval framework [2], is applied.

Let the $j-$th component, $U_j(\mathbf{X})$, of the random displacement vector $\mathbf{U}(\mathbf{X})$ be approximated as sum of the nominal value, $U_{0,j}$, plus a deviation due to uncertainties, i.e.:

$$U_j(\mathbf{X}) = U_{0,j} + \sum_{i=1}^{p} U_{i,j}(X_i), \quad j = 1,2,\ldots,n. \tag{4}$$

Notice that the deviation is given by the superposition of the contributions, $U_{i,j}(X_i)$, ($i = 1,2,\ldots,p$), associated with the uncertain parameters separately taken. Assuming that the deviation, $U_{i,j}(X_i)$, due to the $i-$th uncertain parameter, can be approximated by a rational function of the fluctuation X_i, Eq. (4) can be recast as:

$$U_j(\mathbf{X}) = U_{0,j} + \sum_{i=1}^{p} \frac{X_i}{A_{i,j} + B_{i,j}\, X_i}, \quad j = 1,2,\ldots,n \tag{5}$$

where $A_{i,j}$ and $B_{i,j}$ are $2p$ unknown coefficients. Following the philosophy of the *response surface method* [41] and applying a *saturated design*, such coefficients can be estimated by fitting the approximate solution to the exact one at $2p$ selected sampling points. As shown in Ref. [1], the unknown coefficients $A_{i,j}$ and $B_{i,j}$ are almost insensitive to the choice of the sampling points. The $2p$ sampling points are herein selected as follows:

$$\begin{aligned} &X_i = X_i^{(1)},\, X_j = 0,\; i \neq j = 1,2,\ldots,p; \\ &X_i = X_i^{(2)},\, X_j = 0, i \neq j = 1,2,\ldots,p, \end{aligned} \tag{6a,b}$$

where $X_i^{(1)}$ and $X_i^{(2)}$ are two arbitrary realizations of the random variable X_i . Thus, taking into account Eq. (5), the coefficients $A_{i,j}$ and $B_{i,j}$, ($i = 1,2,\ldots,p$), can be obtained as solution of the following set of $2p$ linear algebraic equations:

$$\begin{aligned} &U_j(X_1 = 0,\ldots,X_i = X_i^{(1)},\ldots,X_p = 0) \equiv U_j^{(i,1)} = U_{0,j} + \frac{X_i^{(1)}}{A_{i,j} + B_{i,j} X_i^{(1)}}; \\ &U_j(X_1 = 0,\ldots,X_i = X_i^{(2)},\ldots,X_p = 0) \equiv U_j^{(i,2)} = U_{0,j} + \frac{X_i^{(2)}}{A_{i,j} + B_{i,j} X_i^{(2)}}, \qquad i = 1,2,\ldots,p \end{aligned} \tag{7a,b}$$

where $U_j^{(i,1)}$ and $U_j^{(i,2)}$ are the exact implicit displacements of the *j*-th DOF obtained setting all the fluctuations of the uncertain parameters equal to zero except the $i-\text{th}$ which is set to $X_i = X_i^{(1)}$ and $X_i = X_i^{(2)}$, respectively. Taking into account Eq. (2), the nominal displacement $U_{0,j}$ and the displacements $U_j^{(i,1)}$ and $U_j^{(i,2)}$ can be evaluated as follows:

$$\begin{aligned} &U_{0,j} = \left\{\mathbf{K}_0^{-1}\mathbf{f}\right\}_j; \\ &U_j^{(i,1)} = \left\{(\mathbf{K}_0 + \mathbf{K}_i X_i^{(1)})^{-1}\mathbf{f}\right\}_j; \\ &U_j^{(i,2)} = \left\{(\mathbf{K}_0 + \mathbf{K}_i X_i^{(2)})^{-1}\mathbf{f}\right\}_j, \qquad i = 1,2,\ldots,p \end{aligned} \tag{8a-c}$$

where $\{\bullet\}_j$ denotes the *j*-th element of the vector into parentheses.

Once the coefficients $A_{i,j}$ and $B_{i,j}$ are known, Eq. (5) provides an approximate explicit relationship between the displacement $U_j(\mathbf{X})$ and the random variables X_i which allows a straightforward evaluation of response statistics, based on the knowledge of the joint *Probability Density Function* (*PDF*) of the random fluctuations X_i. As known, a large amount of data is requested to define reliable probabilistic distributions of the uncertain physical properties. Traditional probabilistic methods rely on the assumption that available information is sufficient to accurately predict the *PDF* of the uncertain input parameters.

3. UNCERTAINTIES DESCRIBED BY IMPRECISE PROBABILITY DENSITY FUNCTION

3.1 Basic concepts and definitions

Let us assume now that the information on the uncertain parameters of the structure considered in the previous section is imprecise or vague, that is only a set of possible values that the quantity might take is known. Under this assumption, the credibility of the traditional probabilistic model becomes questionable and the zero-mean random variables X_i, collected into the vector $\mathbf{X}$, are more appropriately described by a family of joint *imprecise PDF*s. Such a family can be represented by the function $p_{\mathbf{X}}(\mathbf{x};\mathbf{a}_{\mathbf{X}}^I)$, which depends on the interval vector $\mathbf{a}_{\mathbf{X}}^I = \left[\mathbf{a}_{X_1}^{I\ \mathrm{T}}, \mathbf{a}_{X_2}^{I\ \mathrm{T}}, \ldots, \mathbf{a}_{X_p}^{I\ \mathrm{T}} \right]^{\mathrm{T}}$ collecting the set of s_i-order interval vectors $\mathbf{a}_{X_i}^I$ ($i = 1, 2, \ldots, p$) of basic parameters, with the apex I denoting interval variables. The set-interval vector $\mathbf{a}_{\mathbf{X}}^I \triangleq [\underline{\mathbf{a}}_{\mathbf{X}}, \overline{\mathbf{a}}_{\mathbf{X}}] \in \mathbb{IR}^{p \times s}$, where $\mathbb{IR}$ is the set of all closed real interval numbers, is constrained to belong to a $p \times s$-dimensional box with $s = s_1 + s_2 + \ldots + s_p$; the symbols $\underline{\mathbf{a}}_{\mathbf{X}}$ and $\overline{\mathbf{a}}_{\mathbf{X}}$ denote the *lower bound* (*LB*) and *upper bound* (*UB*) vectors [10], such that $\underline{\mathbf{a}}_{\mathbf{X}} \le \mathbf{a}_{\mathbf{X}} \le \overline{\mathbf{a}}_{\mathbf{X}}$. The *j*-th element of the *i*-th interval vector $\mathbf{a}^I_{X_i}$ can be defined as $a^I_{X_i,j} \triangleq \left[\underline{a}_{X_i,j}, \overline{a}_{X_i,j}\right] \in \mathbb{IR}$, where $\underline{a}_{X_i,j}$ and $\overline{a}_{X_i,j}$ are the *LB* and *UB* of the interval basic parameter $a^I_{X_i,j}$, respectively. The interval nature of the basic parameters $\mathbf{a}^I_{X_i}$ $(i = 1, 2, \ldots, p)$ reflects the imprecision of the joint *PDF* of the random variables X_i due to the lack of sufficient or accurate information.

From an engineering point of view, the p random variables X_i $(i = 1, 2, \ldots, p)$ can be assumed to be independent so that the joint *imprecise PDF*, $p_{\mathbf{X}}(\mathbf{x}; \mathbf{a}^I_{\mathbf{X}})$, can be written as:

$$p_{\mathbf{X}}(\mathbf{x}; \mathbf{a}^I_{\mathbf{X}}) = \prod_{i=1}^{p} p_{X_i}(x_i; \mathbf{a}^I_{X_i}) \tag{9}$$

where $p_{X_i}(x_i; \mathbf{a}^I_{X_i})$ is the marginal *imprecise PDF* of the random variable X_i which depends on the interval vector of basic parameters $\mathbf{a}^I_{X_i}$.

The set of *PDF*s describing the random variable X_i yields a set of statistical moments rather than a crisp value as in the context of the classical probability theory. This feature can be formally expressed by introducing the so-called *interval stochastic average operator* $\mathrm{E}^I\langle \bullet \rangle$ [28]. Specifically, the statistical moment of order k of the random variable X_i with *imprecise PDF* $p_{X_i}(x_i; \mathbf{a}^I_{X_i})$ is given by the following interval

$$\mathrm{E}^I\left\langle X_i^k \right\rangle = \int_{-\infty}^{+\infty} x_i^k \, p_{X_i}(x_i; \mathbf{a}^I_{X_i}) \, \mathrm{d}x_i \; = \left[\underline{\mathrm{E}}\left\langle X_i^k \right\rangle, \overline{\mathrm{E}}\left\langle X_i^k \right\rangle\right] \tag{10}$$

where $\underline{\mathrm{E}}\left\langle X^k \right\rangle$ and $\overline{\mathrm{E}}\left\langle X^k \right\rangle$ denote the *LB* and *UB*, defined as:

$$\underline{\mathrm{E}}\left\langle X_i^k \right\rangle = \min_{\mathbf{a}_{X_i} \in \mathbf{a}_{X_i}^I} \left\{ \int_{-\infty}^{+\infty} x_i^k \, p_{X_i}(x_i; \mathbf{a}_{X_i}) \, \mathrm{d}x_i \right\};$$
$$\overline{\mathrm{E}}\left\langle X_i^k \right\rangle = \max_{\mathbf{a}_{X_i} \in \mathbf{a}_{X_i}^I} \left\{ \int_{-\infty}^{+\infty} x_i^k \, p_{X_i}(x_i; \mathbf{a}_{X_i}) \, \mathrm{d}x_i \right\} \qquad \text{(11a,b)}$$

where the symbols $\min\{\bullet\}$ and $\max\{\bullet\}$ mean minimum (inferior) and maximum (superior) value of the quantity into parentheses under the condition that $\mathbf{a}_{X_i} \in \mathbf{a}_{X_i}^I$, respectively.

The assumed *imprecise PDF* model takes into account *epistemic* uncertainties affecting the input representation by means of the interval basic parameters. In agreement with the philosophy of the *imprecise probability* theory, *aleatory* and *epistemic* uncertainties are independent of each other. This feature allows a two-level propagation of uncertainties resorting to well-established tools in the framework of traditional probabilistic theory and interval uncertainty analysis.

3.2 Bounds of the interval mean-value and variance of the response

The statistics of the random displacement vector $\mathbf{U}(\mathbf{X})$ of the structure with uncertain properties described by *imprecise PDF* have an interval nature as well. In particular, by applying the interval stochastic average operator to Eq. (5) and taking into account that the random variables X_i are independent, the interval mean-value and variance of the $j-$th random displacement, $U_j(\mathbf{X})$, can be evaluated as:

$$\mu_{U_j}^I = \mathrm{E}^I \left\langle U_j(\mathbf{X}) \right\rangle = U_{0,j} + \sum_{i=1}^{p} \mu_{\chi_{i,j}}^I;$$
$$\sigma_{U_j}^{2I} = \mathrm{E}^I \left\langle U_j^2(\mathbf{X}) \right\rangle - (\mu_{U_j}^I)^2 = \sum_{i=1}^{p} \sigma_{\chi_{i,j}}^{2I} \qquad \text{(12a,b)}$$

where $\mu_{\chi_{i,j}}^I$ and $\sigma_{\chi_{i,j}}^{2I}$ are the interval mean-value and variance of the auxiliary random variables $\chi_{i,j}$ ($i = 1,2,\ldots,p;\ j = 1,2,\ldots,n$):

$$\chi_{i,j}(X_i) = \frac{X_i}{A_{i,j} + B_{i,j}\, X_i} \tag{13}$$

given by:

$$\begin{aligned} \mu^I_{\chi_{i,j}} &= \int_{-\infty}^{+\infty} \chi_{i,j}(x_i)\; p_{X_i}(x_i;\mathbf{a}^I_{X_i})\,\mathrm{d}x_i; \\ \sigma^{2I}_{\chi_{i,j}} &= \int_{-\infty}^{+\infty} \left(\chi_{i,j}(x_i) - \mu^I_{\chi_{i,j}}\right)^2 p_{X_i}(x_i;\mathbf{a}^I_{X_i})\,\mathrm{d}x_i \end{aligned} \tag{14a,b}$$

where $p_{X_i}(x_i;\mathbf{a}^I_{X_i})$ denotes the marginal *imprecise PDF* of the random variable X_i.

Once interval response statistics in Eqs. (14a,b) are defined according to the classical probabilistic analysis, *epistemic* uncertainties represented by the interval basic parameters $\mathbf{a}^I_{X_i}$ need to be propagated. Following the *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*) [3], the interval mean-value of the auxiliary random variables $\chi_{i,j}$ can be expressed in *affine form* as:

$$\mu^I_{\chi_{i,j}} = \mu_{0,\chi_{i,j}} + \Delta\mu_{\chi_{i,j}}\,\hat{e}^I_i \tag{15}$$

where $\hat{e}^I_i = [-1,+1]$ is the *EUI* [3] associated with the i-th random variable X_i; $\mu_{0,\chi_{i,j}}$ and $\Delta\mu_{\chi_{i,j}}$ are the midpoint value and deviation amplitude (or radius) of $\mu^I_{\chi_{i,j}}$ defined, respectively, as:

$$\mu_{0,\chi_{i,j}} = \frac{\underline{\mu}_{\chi_{i,j}} + \overline{\mu}_{\chi_{i,j}}}{2}; \quad \Delta\mu_{\chi_{i,j}} = \frac{\overline{\mu}_{\chi_{i,j}} - \underline{\mu}_{\chi_{i,j}}}{2} > 0 \tag{16a,b}$$

where $\underline{\mu}_{\chi_{i,j}}$ and $\overline{\mu}_{\chi_{i,j}}$ denote the *LB* and *UB* of $\mu^I_{\chi_{i,j}}$, given by:

$$\begin{aligned} \underline{\mu}_{\chi_{i,j}} &= \min_{\mathbf{a}_{X_i} \in \mathbf{a}^I_{X_i}} \left\{ \int_{-\infty}^{+\infty} \chi_{i,j}(x_i)\; p_{X_i}(x_i;\mathbf{a}^I_{X_i})\,\mathrm{d}x_i \right\}; \\ \overline{\mu}_{\chi_{i,j}} &= \max_{\mathbf{a}_{X_i} \in \mathbf{a}^I_{X_i}} \left\{ \int_{-\infty}^{+\infty} \chi_{i,j}(x_i)\; p_{X_i}(x_i;\mathbf{a}^I_{X_i})\,\mathrm{d}x_i \right\}. \end{aligned} \tag{17a,b}$$

Similarly, the interval variance of the auxiliary random variables $\chi_{i,j}$ can be expressed in *affine form* as:

$$\sigma^{2I}_{\chi_{i,j}} = \sigma^2_{0,\chi_{i,j}} + \Delta\sigma^2_{\chi_{i,j}} \hat{e}^I_i \tag{18}$$

where

$$\sigma^2_{0,\chi_{i,j}} = \frac{\underline{\sigma}^2_{\chi_{i,j}} + \overline{\sigma}^2_{\chi_{i,j}}}{2}; \quad \Delta\sigma^2_{\chi_{i,j}} = \frac{\overline{\sigma}^2_{\chi_{i,j}} - \underline{\sigma}^2_{\chi_{i,j}}}{2} \tag{19a,b}$$

are the midpoint and deviation amplitude of $\sigma^{2I}_{\chi_{i,j}}$ defined in terms of the *LB* and *UB*, given by:

$$\begin{aligned} \underline{\sigma}^2_{\chi_{i,j}} &= \min_{\mathbf{a}_{X_i} \in \mathbf{a}^I_{X_i}} \left\{ \int_{-\infty}^{+\infty} \left(\chi_{i,j} - \mu^I_{\chi_{i,j}} \right)^2 p_{X_i}(x_i; \mathbf{a}^I_{X_i}) \, \mathrm{d}x_i \right\}; \\ \overline{\sigma}^2_{\chi_{i,j}} &= \max_{\mathbf{a}_{X_i} \in \mathbf{a}^I_{X_i}} \left\{ \int_{-\infty}^{+\infty} \left(\chi_{i,j} - \mu^I_{\chi_{i,j}} \right)^2 p_{X_i}(x_i; \mathbf{a}^I_{X_i}) \, \mathrm{d}x_i \right\}. \end{aligned} \tag{20a,b}$$

Substituting Eq. (15) into Eq. (12a), the interval mean-value of the $j-$th random displacement, U_j, can be rewritten as sum of the midpoint value plus an interval deviation, i.e.:

$$\mu^I_{U_j} = \mathrm{mid}\left\{\mu^I_{U_j}\right\} + \mathrm{dev}\left\{\mu^I_{U_j}\right\} \tag{21}$$

where

$$\mathrm{mid}\left\{\mu^I_{U_j}\right\} = U_{0,j} + \sum_{i=1}^{p} \mu_{0,\chi_{i,j}}; \quad \mathrm{dev}\left\{\mu^I_{U_j}\right\} = \sum_{i=1}^{p} \Delta\mu_{\chi_{i,j}} \hat{e}^I_i . \tag{22a,b}$$

In the previous equations, $\mathrm{mid}\{\bullet\}$ and $\mathrm{dev}\{\bullet\}$ denote the midpoint and interval deviation of the quantity between curly brackets.

Based on Eq. (21) and following the philosophy of the *IIA* via *EUI*, the *LB* and *UB* of the interval mean-value of the $j-$th random displacement can be evaluated as:

$$\underline{\mu}_{U_j} = U_{0,j} + \sum_{i=1}^{p} \mu_{0,\chi_{i,j}} - \sum_{i=1}^{p} \Delta\mu_{\chi_{i,j}}; \quad \overline{\mu}_{U_j} = U_{0,j} + \sum_{i=1}^{p} \mu_{0,\chi_{i,j}} + \sum_{i=1}^{p} \Delta\mu_{\chi_{i,j}} . \tag{23a,b}$$

Similarly, substituting the *affine form* (18) into Eq. (12b), the interval variance of the $j-$th random displacement, U_j, can be rewritten as sum of the midpoint value plus an interval deviation, i.e.:

$$\sigma_{U_j}^{2I} = \mathrm{mid}\left\{\sigma_{U_j}^{2I}\right\} + \mathrm{dev}\left\{\sigma_{U_j}^{2I}\right\} \tag{24}$$

where:

$$\mathrm{mid}\left\{\sigma_{U_j}^{2I}\right\} = \sum_{i=1}^{p} \sigma_{0,\chi_{i,j}}^{2} \;; \qquad \mathrm{dev}\left\{\sigma_{U_j}^{2I}\right\} = \sum_{i=1}^{p} \Delta\sigma_{\chi_{i,j}}^{2}\, \hat{e}_i^I . \tag{25a,b}$$

Taking into account Eq. (24), the *LB* and *UB* of the variance of the $j-$th random displacement can be evaluated as follows:

$$\underline{\sigma}_{U_j}^{2} = \sum_{i=1}^{p} \sigma_{0,\chi_{i,j}}^{2} - \sum_{i=1}^{p} \Delta\sigma_{\chi_{i,j}}^{2} \;; \qquad \overline{\sigma}_{U_j}^{2} = \sum_{i=1}^{p} \sigma_{0,\chi_{i,j}}^{2} + \sum_{i=1}^{p} \Delta\sigma_{\chi_{i,j}}^{2} . \tag{26a,b}$$

3.3 Imprecise uniform *PDF*

Let us assume that the fluctuations of the uncertain parameters around the nominal value are represented by random variables X_i with uniform *imprecise PDF* [28], i.e.:

$$p_{X_i}(x_i; a_i^I) = \begin{cases} \dfrac{1}{2a_i^I}, & \text{for } -a_i^I \le x_i \le a_i^I \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

where $a_i^I = \left[\underline{a}_i, \overline{a}_i\right] = a_{0,i}(1+\Delta\alpha_i \hat{e}_i^I)$ with $a_{0,i} > 0$, $\Delta\alpha_i < 1$ and $\hat{e}_i^I = [-1,+1]$, ($i = 1,2,\ldots,p$). Figure 1 displays three typical realizations of the uniform *imprecise PDF* of the random variables X_i obtained setting the basic parameter a_i^I in Eq. (27) equal to the nominal value $a_{0,i}$, the *LB*, $\underline{a}_i$, and the *UB*, $\overline{a}_i$.

The *imprecise PDF* of the auxiliary random variables $\chi_{i,j}(X_i)$ in Eq. (13) can be derived from the knowledge of the uniform *imprecise PDF* $p_{X_i}(x_i;a_i^I)$ (see Eq. (27)) as [1],[2]:

$$p_{\chi_{i,j}}(y_i,a_i^I)=\begin{cases}\dfrac{1}{2a_i^I}\dfrac{\left|A_{i,j}\right|}{\left(1-B_{i,j}y_i\right)^2} & \text{for } y_{i,1}^I\le y_i\le y_{i,2}^I\\ 0, & \text{otherwise}\end{cases} \tag{28}$$

where

$$y_{i,1}^I=\frac{a_i^I}{-\left|A_{i,j}\right|+a_i^I B_{i,j}};\ \ y_{i,2}^I=\frac{a_i^I}{\left|A_{i,j}\right|+a_i^I B_{i,j}}. \tag{29a,b}$$

Based on the *imprecise PDF* in Eq. (28), the interval mean-value and variance of the auxiliary random variables $\chi_{i,j}$ can be evaluated analytically as follows [2]:

$$\begin{aligned}\mu_{\chi_{i,j}}^I&=\mathrm{E}^I\left\langle\chi_{i,j}\right\rangle=\frac{A_{i,j}}{2B_{i,j}^2a_i^I}\left[\frac{2B_{i,j}a_i^I}{A_{i,j}}+\ln\left(\frac{A_{i,j}-B_{i,j}a_i^I}{A_{i,j}+B_{i,j}a_i^I}\right)\right];\\ \sigma_{\chi_{i,j}}^{2I}&=E^I\left\langle\chi_{i,j}^2\right\rangle-\left(\mu_{\chi_{i,j}}^I\right)^2=\frac{A_{i,j}^2}{4B_{i,j}^4\left(a_i^I\right)^2\left[B_{i,j}^2\left(a_i^I\right)^2-A_{i,j}^2\right]}\\ &\times\left\{\left[A_{i,j}^2-B_{i,j}^2\left(a_i^I\right)^2\right]\left[\ln\left(\frac{A_{i,j}-B_{i,j}a_i^I}{A_{i,j}+B_{i,j}a_i^I}\right)\right]^2-4B_{i,j}^2\left(a_i^I\right)^2\right\}.\end{aligned} \tag{30a,b}$$

In the previous equations, $A_{i,j}$ and $B_{i,j}$ are the coefficients appearing in the assumed ratio of polynomial *response surface* defined in Eq.(5). Such coefficients can be efficiently evaluated fitting the exact solution to the approximate one at $2p$ sampling points selected as follows: $X_i=X_i^{(1)}=-a_{0,i}$, $X_j=0$, $i\neq j=1,2,\ldots,p$; $X_i=X_i^{(2)}=a_{0,i}$, $X_j=0$, $i\neq j=1,2,\ldots,p$. Once the $2p+1$ sets of linear algebraic equations in Eqs. (8a-c) are solved, Eqs. (7a,b) yield the following explicit expressions of the coefficients $A_{i,j}$ and $B_{i,j}$:

$$A_{i,j} = \frac{a_{0,i}\left(U_j^{(i,2)} - U_j^{(i,1)}\right)}{2\left(U_{0,j} - U_j^{(i,1)}\right)\left(U_{0,j} - U_j^{(i,2)}\right)};$$
$$B_{i,j} = -\frac{2U_{0,j} - U_j^{(i,1)} - U_j^{(i,2)}}{2\left(U_{0,j} - U_j^{(i,1)}\right)\left(U_{0,j} - U_j^{(i,2)}\right)}. \qquad \text{(31a,b)}$$

Equations (30a,b) allow us to derive the bounds of the interval mean-value and variance of the auxiliary random variables, $\mu^I_{\chi_{i,j}}$ and $\sigma^{2I}_{\chi_{i,j}}$ (see Eqs. (14a,b)), and the associated midpoint and deviation amplitude, in explicit form. Then, by applying Eqs. (23a,b) and Eqs. (26a,b), approximate explicit expressions of the *LB* and *UB* of the interval mean-value and variance of the displacements can be obtained as well.

4. STRUCTURAL PERFORMANCE ASSESSMENT IN THE CONTEXT OF IMPRECISE PROBABILITY

4.1 Basic concepts and definitions

The primary objective of engineering design is to ensure that a structural system satisfies various criteria of performance, safety, serviceability during its service life. To this aim, for a selected *performance criterion*, system's *reliability* needs to be evaluated. The basic idea of reliability analysis relies on the comparison between the *demand*, *S*, represented by the most relevant structural response caused by external loads, and the *capacity*, *R*, namely the corresponding resistance of the structure.

In reliability analysis, a measure of the risk is the *probability of failure*, $\mathcal{P}_{\mathcal{F}}$, while a measure of the success is the *probability of success* or *survival probability*, $\mathcal{P}_{\mathcal{S}} = 1 - \mathcal{P}_{\mathcal{F}}$. These probabilities can be calculated in the following alternative ways:

$$\mathcal{P}_{\mathcal{F}} = \Pr\langle R \le S\rangle = \Pr\langle R/S \le 1\rangle = \Pr\langle \ln R - \ln S \le 0\rangle;$$
$$\mathcal{P}_{\mathcal{S}} = \Pr\langle S < R\rangle = \Pr\langle R/S > 1\rangle = \Pr\langle \ln R - \ln S > 0\rangle. \qquad \text{(32a,b)}$$

Both the response, *S*, and the resistance, *R*, are herein modeled as statistically independent random variables with lognormal distribution so that they cannot take unrealistic negative values. Under this assumption, the *probability of failure* reads as follows [42]:

$$\mathcal{P}_{\mathcal{F}} = 1 - \Phi(\beta) \tag{33}$$

where $\Phi(\bullet)$ denotes the *cumulative distribution function* (*CDF*) of the *standard normal variable*; $\beta = \Phi^{-1}\left(1 - \mathcal{P}_{\mathcal{F}}\right)$ is the *safety index* or *reliability index*, which, using the well-known relationships for lognormal distributions, can be expressed as:

$$\beta = \frac{\ln\left[\dfrac{\mu_R}{\mu_S}\sqrt{\dfrac{1+\delta_S^2}{1+\delta_R^2}}\right]}{\sqrt{\ln\left[(1+\delta_R^2)\left(1+\delta_S^2\right)\right]}}. \tag{34}$$

In the previous equation, δ_R and δ_S are the *coefficients of variation* of the random variables *R* and *S*, defined, respectively, as:

$$\delta_R = \frac{\sigma_R}{\mu_R}; \quad \delta_S = \frac{\sigma_S}{\mu_S} \tag{35a,b}$$

where μ_R and σ_R are the mean-value and standard deviation of the random variable *R*, while μ_S and σ_S are the mean-value and standard deviation of the random variable *S*.

4.2 Interval reliability index and failure probability

When the structure possesses uncertain parameters modeled as random variables with *imprecise PDF*, suitable definitions of the *reliability index* as well as of the *probability of failure* need to be introduced. In 1995, Elishakoff [43] deemed that, if the basic parameters of the input *PDF* are modelled as intervals, the reliability belongs to an interval and the *reliability index* is also an interval quantity. Afterwards, by combining the classical reliability theory and the interval analysis [9],[10], Elishakoff and co-workers [44], showed that a small error in the estimation of the basic parameters may lead to a remarkable error in the resulting *probability of failure*; the same result was achieved by introducing the concept of *interval reliability index* [45]. In this section, relying on the pioneering studies by Elishakoff and co-workers [43]-[45], an efficient procedure to perform reliability assessments under *imprecise probabilities* is developed.

The response, *S*, and the resistance, *R*, are modelled as statistically independent random variables having lognormal distributions. In particular, the resistance, *R*, is assumed to be characterized by a *precise* lognormal *PDF*, $p_R(r;\mu_R,\sigma_R)$, with mean-value μ_R and standard deviation σ_R. On account of the imprecise character of the random structural parameters, the response, *S*, is assumed to have a lognormal *imprecise PDF*, $p_S(s;\mu_S^I,\sigma_S^I)$, with interval mean-value and standard deviation μ_S^I and σ_S^I. The latter can be efficiently computed by applying the proposed *response surface* approach.

Under this assumption, by interval extension of Eq.(34), the *interval reliability index* takes the following form:

$$\beta^I = \left[\underline{\beta}, \overline{\beta}\right] = \ln\left[\frac{\mu_R}{\mu_S^I}\sqrt{\frac{1+\delta_S^{2I}}{1+\delta_R^2}}\right]\frac{1}{\sqrt{\ln\left[(1+\delta_R^2)\left(1+\delta_S^{2I}\right)\right]}} \tag{36}$$

where δ_R and δ_S^I are the *coefficients of variation* of the resistance and response, respectively:

$$\delta_R = \frac{\sigma_R}{\mu_R}; \quad \delta_S^I = \frac{\sigma_S^I}{\mu_S^I}. \tag{37a,b}$$

The *probability of failure* has an interval nature too and, by interval extension of Eq. (33), is defined as:

$$\mathcal{P}_{\mathcal{F}}^I = \left[\underline{\mathcal{P}}_{\mathcal{F}}, \overline{\mathcal{P}}_{\mathcal{F}}\right] = 1-\Phi\left(\beta^I\right). \tag{38}$$

By inspection of Eq. (36), the *interval reliability index* β^I may be viewed as a function of two interval parameters, say the interval mean-value and standard deviation of the response *S*, μ_S^I and σ_S^I. Thus, the *LB* and *UB* of the *interval reliability index* can be evaluated as the minimum and maximum, respectively, among all possible values of $\beta \in \beta^I = \left[\underline{\beta}, \overline{\beta}\right]$ obtained as the interval mean-value and standard deviation of the response *S* range over their intervals, $\mu_S^I = [\underline{\mu}_S, \overline{\mu}_S]$ and $\sigma_S^I = [\underline{\sigma}_S, \overline{\sigma}_S]$, i.e.:

$$\underline{\beta} = \min_{\mu_S \in \mu_S^I,\ \sigma_S \in \sigma_S^I} \left\{ \ln\left[\frac{\mu_R}{\mu_S} \sqrt{\frac{1+\delta_S^2}{1+\delta_R^2}} \right] \frac{1}{\sqrt{\ln\left[(1+\delta_R^2)\left(1+\delta_S^2\right)\right]}} \right\};$$
$$\overline{\beta} = \max_{\mu_S \in \mu_S^I,\ \sigma_S \in \sigma_S^I} \left\{ \ln\left[\frac{\mu_R}{\mu_S} \sqrt{\frac{1+\delta_S^2}{1+\delta_R^2}} \right] \frac{1}{\sqrt{\ln\left[(1+\delta_R^2)\left(1+\delta_S^2\right)\right]}} \right\}. \qquad (39a,b)$$

In order to identify the values of the interval mean-value and standard deviation of the response *S* which give the *LB* and *UB* of the *interval reliability index*, sensitivities of β are evaluated by direct differentiation of Eq. (36) with respect to $\mu_S \in \mu_S^I = [\underline{\mu}_S, \overline{\mu}_S]$ and $\sigma_S \in \sigma_S^I = [\underline{\sigma}_S, \overline{\sigma}_S]$, obtaining:

$$S_{\beta,\mu_S} = \left.\frac{\partial \beta}{\partial \mu_S}\right|_{\substack{\mu_S=\mu_{0,S}\\ \sigma_S=\sigma_{0,S}}} = -\frac{(\mu_{S,0}^2 + 2\sigma_{S,0}^2)\ln\left[(1+\delta_R^2)(1+\delta_{S,0}^2)\right] + \sigma_{S,0}^2 \ln\left[\frac{\mu_R}{\mu_{S,0}}\sqrt{(1+\delta_R^2)(1+\delta_{S,0}^2)}\right]}{\mu_{S,0}(\mu_{S,0}^2 + 2\sigma_{S,0}^2)\left\{\ln\left[(1+\delta_R^2)(1+\delta_{S,0}^2)\right]\right\}^{3/2}} \qquad (40)$$

and

$$S_{\beta,\sigma_S} = \left.\frac{\partial \beta}{\partial \sigma_S}\right|_{\substack{\mu_S=\mu_{0,S}\\ \sigma_S=\sigma_{0,S}}} = \frac{\sigma_{S,0}\ln\left[\frac{\mu_{S,0}}{\mu_R}(1+\delta_R^2)\sqrt{(1+\delta_R^2)(1+\delta_{S,0}^2)}\right]}{(\mu_{S,0}^2 + 2\sigma_{S,0}^2)\left\{\ln\left[(1+\delta_R^2)(1+\delta_{S,0}^2)\right]\right\}^{3/2}}. \qquad (41)$$

In the previous equations, $\mu_{0,S}$ and $\sigma_{0,S}$ are the nominal mean-value and standard deviation of the response *S* pertaining to the structure with *precise PDF* of the uncertain parameters; $\delta_{S,0} = \sigma_{S,0} / \mu_{S,0}$ is the corresponding *coefficient of variation*. Eqs. (40) and (41) provide information about the change of the *reliability index* due to small changes of the mean-value and standard deviation of the response *S* around the nominal values $\mu_{0,S}$ and $\sigma_{0,S}$. Taking into account that $\mu_R > \mu_{S,0}$, by inspection of Eq. (40), it is found that $S_{\beta,\mu_S} < 0$, that is the *interval reliability index* is a monotonic decreasing function of the mean-value $\mu_S \in \mu_S^I = [\underline{\mu}_S, \bar{\mu}_S]$. Furthermore, by examining Eq. (41), it can be readily inferred that $S_{\beta,\sigma_S} < 0$ if

$$\ln\left[\frac{\mu_{S,0}}{\mu_R}\right] < \frac{1}{2}\left\{3\ln(1+\delta_R^2) + \ln(1+\delta_{S,0}^2)\right\}. \tag{42}$$

The previous condition, is certainly satisfied if the *coefficients of variations* δ_R and $\delta_{S,0}$ are not very large. Under this assumption, therefore, $S_{\beta,\sigma_S} < 0$ and the *interval reliability index* is a monotonic decreasing function of the standard deviation $\sigma_S \in \sigma_S^I = [\underline{\sigma}_S, \bar{\sigma}_S]$ of the response *S* as well. This implies that the *LB* and *UB* of the *interval reliability index* can be evaluated analytically setting in Eq. (36) both $\mu_S^I = [\underline{\mu}_S, \bar{\mu}_S]$ and $\sigma_S^I = [\underline{\sigma}_S, \bar{\sigma}_S]$ equal to their *UB* and *LB*, respectively, obtaining:

$$\begin{aligned} \underline{\beta} &= \beta(\bar{\mu}_S, \bar{\sigma}_S) = \ln\left[\frac{\mu_R}{\bar{\mu}_S}\sqrt{\frac{1+(\bar{\sigma}_S/\bar{\mu}_S)^2}{1+\delta_R^2}}\right]\frac{1}{\sqrt{\ln\left[(1+\delta_R^2)\left(1+(\bar{\sigma}_S/\bar{\mu}_S)^2\right)\right]}}; \\ \bar{\beta} &= \beta(\underline{\mu}_S, \underline{\sigma}_S) = \ln\left[\frac{\mu_R}{\underline{\mu}_S}\sqrt{\frac{1+(\underline{\sigma}_S/\underline{\mu}_S)^2}{1+\delta_R^2}}\right]\frac{1}{\sqrt{\ln\left[(1+\delta_R^2)\left(1+(\underline{\sigma}_S/\underline{\mu}_S)^2\right)\right]}}. \end{aligned} \tag{43a,b}$$

Then, according to Eq.(38), the best possible value (or *LB*) and the worst possible value (or *UB*) of the *probability of failure* can be evaluated, respectively, as:

$$\underline{\mathcal{P}}_{\mathcal{F}} = 1 - \Phi\left(\overline{\beta}\right); \quad \overline{\mathcal{P}}_{\mathcal{F}} = 1 - \Phi\left(\underline{\beta}\right). \tag{44a,b}$$

Obviously, the *LB* and *UB* of the interval *survival probability* are given, respectively, as:

$$\underline{\mathcal{P}}_{\mathcal{S}} = 1 - \overline{\mathcal{P}}_{\mathcal{F}} = \Phi\left(\underline{\beta}\right); \quad \overline{\mathcal{P}}_{\mathcal{S}} = 1 - \underline{\mathcal{P}}_{\mathcal{F}} = \Phi\left(\overline{\beta}\right). \tag{45a,b}$$

The previous bounds allow us to compute the highest expected *failure probability,* $\overline{\mathcal{P}}_{\mathcal{F}}$, (see Eq. (44b)) which corresponds to the *LB* of the *survival probability* (see Eq. (45a)).

5. NUMERICAL APPLICATIONS

5.1 Frame structure with uncertain Young's moduli

The first application concerns the two-bay three-storey frame with uncertain Young's moduli shown in Figure 2. The following geometrical and mechanical properties are assumed: nominal Young's modulus of the material $E_0 = 31.5\ \text{GPa}$ equal for all the elements; rectangular cross-section of beams and columns with width and thickness $b = 0.30\ \text{m}$ and $h = 0.50\ \text{m}$, respectively; lengths of columns and beams $H = 3\ \text{m}$ and $L = 4\ \text{m}$, respectively. The frame is subjected at each floor to deterministic concentrated and distributed static loads of intensity $f = 100\ \text{kN}$ and $p = 100\ \text{kN/m}$ (see Figure 2), respectively. Young's moduli, $E_i = E_0\left(1 + X_i\right)$, ($i = 1,2,3$), of beams and columns at each floor are assumed to be uncertain (see Figure 2) with fluctuations X_i around the nominal value modeled as zero-mean independent random variables with *imprecise PDF*. Two different probabilistic models are assumed for the random fluctuations: *i*) uniform *imprecise PDF*; *ii*) truncated Gaussian *imprecise PDF*. In both cases, the horizontal displacement of node 10, U_{10} , is selected as response quantity of interest. The accuracy of the proposed method is demonstrated by performing appropriate comparisons with the results obtained by applying a procedure resulting from the combination of *MCS* and the *Vertex Method* (*VM*) [46], labeled as *MCS-VM* [28].

5.1.1 Uniform *imprecise PDF*

Let us assume that the random fluctuations X_i of Young's moduli $E_i = E_0\left(1+X_i\right)$, $(i=1,2,3)$ are described by a uniform *imprecise PDF* (see Eq. (27)) with basic parameters defined by the intervals $a_i^I = \left[\underline{a}_i, \overline{a}_i\right] = a_0(1+\Delta\alpha\hat{e}_i^I)$, characterized by the same midpoint value and deviation amplitude.

For validation purposes, first the statistics of the response are evaluated in the framework of the classical probability theory, that is assuming that the random variables X_i have a *precise* uniform distribution in the interval $[-a_0, a_0]$. Figure 3 shows the proposed mean-value and standard deviation of the displacement U_{10} along with the ones obtained by performing *MCS*, with $N_\mathrm{g} = 3\times10^4$ samples, versus the parameter a_0. A very good agreement between the proposed closed-form expressions of response statistics (Eqs. (30a,b) for $a_i^I = a_0$) and *MCS* data is observed even for $a_0 = 0.4$.

In the case of random fluctuations X_i described by a uniform *imprecise PDF*, the results provided by the *MCS-VM* [28] are assumed as reference solutions. In the present case, the *MCS-VM* requires $N_\mathrm{c}\times N_\mathrm{g}$ deterministic structural analyses, where $N_\mathrm{c} = 2^3$ is the number of all possible combinations of the endpoints of the interval basic parameters a_i^I, $(i=1,2,3)$, of the uniform *imprecise PDF* in Eq. (27), while $N_\mathrm{g} = 3\times10^4$ is the number of considered samples. The deviation amplitude of the interval basic parameters $a_i^I = a_0(1+\Delta\alpha\hat{e}_i^I)$ is set equal to $\Delta\alpha = 0.2$ for all the random fluctuations X_i, $(i=1,2,3)$, when not otherwise specified.

Figure 4 displays the bounds of the interval mean-value and standard deviation of the displacement U_{10} versus a_0. Notice that the proposed bounds, defined in closed-form (see Eqs. (22) and (26)), are very close to the ones obtained by applying the *MCS-VM*. Furthermore, it is observed that the region of interval statistics of the response becomes wider when larger values of the parameter a_0 are considered, which imply higher degrees of uncertainty.

In order to scrutinize the influence of imprecision on the interval statistics of the response, in Figure 5 the bounds of the interval mean-value and standard deviation of the displacement U_{10} versus the deviation amplitude $\Delta\alpha$ of the interval basic parameters $a_i^I = a_0(1+\Delta\alpha\hat{e}_i^I)$ of the uniform *imprecise PDF* ($a_0 = 0.2$) are plotted. The comparison with the results provided by the *MCS-VM* demonstrates the accuracy of the proposed bounds even for large values of $\Delta\alpha$, which imply high imprecision of the *PDF* of the uncertain parameters. As expected, the region enclosed by the *LB* and *UB* becomes wider as the deviation amplitude $\Delta\alpha$ of the interval basic parameters increases.

To assess the performance of the frame structure with *imprecise* random Young's moduli, the formulation outlined in Section 4 is applied. To this aim, the displacement U_{10} is selected as critical response quantity, that is $S \equiv U_{10}$, and is assumed to have a lognormal *imprecise PDF*, $p_{U_{10}}(u_{10};\mu_{U_{10}}^I,\sigma_{U_{10}}^I)$, with interval mean-value and standard deviation, $\mu_{U_{10}}^I$ and $\sigma_{U_{10}}^I$, evaluated by the proposed method (see Figure 4). The corresponding resistance, R, is modelled as a lognormally distributed random variable with mean-value and standard deviation equal to $\mu_R = 0.0223\,\mathrm{m}$ and $\sigma_R = 0.0036\,\mathrm{m}$, respectively.

Figure 6 shows the *PDF* of the resistance R, $p_R(r;\mu_R,\sigma_R)$, along with three realizations of the *imprecise PDF* of the displacement U_{10}, $p_{U_{10}}(u_{10};\mu_{U_{10}}^I,\sigma_{U_{10}}^I)$, obtained setting the interval mean-value and standard deviation equal to their *LB* $(\underline{\mu}_{U_{10}},\underline{\sigma}_{U_{10}})$ and *UB* $(\overline{\mu}_{U_{10}},\overline{\sigma}_{U_{10}})$, as well as to the values $(\mu_{0,U_{10}}, \sigma_{0,U_{10}})$ pertaining to the uniform *PDF* (28) of the random variables X_i with nominal basic parameter a_0. Notice that the largest area of overlap between the *PDF*s of R and U_{10}, which gives a qualitative measure of the *probability of failure*, is obtained when the interval mean-value and standard deviation of U_{10} are set equal to their *UB*, $\bar{\mu}_{U_{10}}$ and $\bar{\sigma}_{U_{10}}$. This result is physically consistent and is in agreement with Eqs. (43a) and (44 b), which estimate the *LB* of the *interval reliability index* and the associated *UB* of the *interval failure probability* as the values corresponding to the *UB* of both the mean-value and standard deviation of the critical response S. Furthermore, by inspection of Figure 6, it is observed that classical probability theory would underestimate the actual *failure probability*.

In Figure 7, the bounds of the *interval reliability index* and *failure probability* for the displacement U_{10} versus the nominal basic parameter a_0 of the *PDF* of the uncertain Young's moduli are plotted. The nominal values pertaining to a uniform *precise PDF* with basic parameter a_0 are also reported. A very good match between the proposed bounds and those predicted by the *MCS-VM* is observed. In Figure 7, $\beta_{\text{Target}} = 3.8$ is the target value for the *reliability index* indicated by Eurocode EN 1990 [47] for reliability class RC2 structural members, for reference period 50 years and ultimate limit state. As expected, the *reliability index* and the *failure probability* are significantly affected by the imprecision of the *PDF* of the random Young's moduli. Numerical results reported in Figure 7 demonstrate that, assuming the nominal value a_0 of the basic parameters of the *PDF* of the random variables X_i, may lead to serious underestimation of the *failure probability* and thus to unsafe design. In particular, by inspection of Figure 7c, it is inferred that the classical probabilistic analysis would consider the structure safe for values of the parameter a_0 for which the *UB* of the *failure probability* exceeds the target value prescribed by Eurocode EN 1990 [47].

5.1.2 Truncated Gaussian *imprecise PDF*

In order to demonstrate the versatility of the proposed *response surface* approach, the random fluctuations X_i of the uncertain Young's moduli $E_i = E_0\left(1+X_i\right)$, ($i=1,2,3$), of the frame structure under consideration (see Figure 2) are now assumed to have a zero-mean truncated Gaussian *imprecise PDF*, defined as follows:

$$p_{X_i}\left(x_i;\sigma_i^I,a_i^I\right)=\begin{cases}\dfrac{\phi\left(\dfrac{x_i}{\sigma_i^I}\right)}{\sigma_i^I\left[\Phi\left(\dfrac{a_i^I}{\sigma_i^I}\right)-\Phi\left(-\dfrac{a_i^I}{\sigma_i^I}\right)\right]}, & \text{for } -a_i^I\le x_i\le a_i^I\\ 0, & \text{otherwise}\end{cases} \tag{46}$$

where $\phi(\bullet)$ is the standard Gaussian *PDF*; $a_i^I=\left[\underline{a}_i,\overline{a}_i\right]=a_{0,i}(1+\Delta\alpha_i\hat{e}_i^I)$, $\sigma_i^I=\left[\underline{\sigma}_i,\overline{\sigma}_i\right]=\sigma_{0,i}(1+\Delta\alpha_i\hat{e}_i^I)$ with $a_{0,i}>0$, $\sigma_{0,i}>0$, $\Delta\alpha_i<1$ and $\hat{e}_i^I=[-1,+1]$, ($i=1,2,3$). Figure 8a shows three typical realizations of the truncated Gaussian *imprecise PDF* of the random variables X_i obtained setting the interval parameter a_i^I equal to the nominal value $a_{0,i}$ and the standard deviation of the associated normal distribution σ_i^I in Eq. (46) equal to three different values, namely the nominal value $\sigma_{0,i}$, the *LB*, $\underline{\sigma}_i$, and the *UB*, $\overline{\sigma}_i$. Similarly, Figure 8b displays three typical realizations of the truncated Gaussian *imprecise PDF* obtained assuming the interval standard deviation σ_i^I equal to the nominal value $\sigma_{0,i}$ and the interval basic parameter a_i^I equal to three different values, namely the nominal value $a_{0,i}$, the *LB*, $\underline{a}_i$, and the *UB*, $\overline{a}_i$. As expected, when the parameters $\sigma_i\in\sigma_i^I$ and $a_i\in a_i^I$ increase, the truncated *Gaussian PDF* tends to the uniform distribution. Numerical results presented in this section are obtained assuming that the truncated Gaussian *imprecise PDF*s of the random fluctuations of all Young's moduli $E_i = E_0\left(1+X_i\right)$, ($i=1,2,3$), are characterized by the same nominal values of the basic parameters and the same deviation amplitudes, i.e. $a_i^I = a_0(1+\Delta\alpha\hat{e}_i^I)$ and $\sigma_i^I = \sigma_0(1+\Delta\alpha\hat{e}_i^I)$, ($i=1,2,3$), with $\Delta\alpha = 0.2$.

Figure 9 and 10 show the bounds of the interval mean-value and standard deviation of the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the truncated Gaussian *imprecise PDF* of the random variables X_i for three different values of the nominal standard deviation σ_0, say $\sigma_0 = 0.1, 0.2, 0.3$. The proposed estimates are in very good agreement with the bounds provided by the *MCS-VM*, which, in the present case, requires $N_{\mathrm{c}} \times N_{\mathrm{g}}$ deterministic structural analyses, where $N_{\mathrm{c}} = 2^6$ is the number of all possible combinations of the endpoints of the interval basic parameters a_i^I and σ_i^I $(i=1,2,3)$ of the truncated Gaussian *imprecise PDF* in Eq. (46), while $N_{\mathrm{g}} = 3\times 10^4$ is the number of considered samples. Notice that, as the nominal standard deviation σ_0 increases, the pattern of the bounds of response statistics versus the nominal basic parameter a_0 consistently approaches the one pertaining to the case of random variables X_i with uniform *imprecise PDF* (see Figure 4). Furthermore, also in this case, the region of response statistics becomes wider as the basic parameter a_0 increases.

As in the previous example, reliability assessment of the frame structure is performed assuming the displacement U_{10} as critical response quantity, that is $S \equiv U_{10}$, having a lognormal *imprecise PDF*, $p_{U_{10}}(u_{10}; \mu_{U_{10}}^I, \sigma_{U_{10}}^I)$, with interval mean-value and standard deviation, $\mu_{U_{10}}^I$ and $\sigma_{U_{10}}^I$, evaluated by the proposed method (see Figures 9 and 10). The corresponding resistance, R, is modelled again as a lognormally distributed random variable with mean-value and standard deviation equal to $\mu_R = 0.0223\,\mathrm{m}$ and $\sigma_R = 0.0036\,\mathrm{m}$, respectively.

Figure 11 shows the *PDF* of the resistance R, $p_R(r;\mu_R,\sigma_R)$, along with three realizations of the *imprecise PDF* of the displacement U_{10}, $p_{U_{10}}(u_{10};\mu^I_{U_{10}},\sigma^I_{U_{10}})$, obtained setting the interval mean-value and standard deviation equal to their *LB* $(\underline{\mu}_{U_{10}},\underline{\sigma}_{U_{10}})$ and *UB* $(\overline{\mu}_{U_{10}},\overline{\sigma}_{U_{10}})$, as well as to the values $(\mu_{0,U_{10}},\sigma_{0,U_{10}})$ pertaining to the truncated Gaussian *PDF* (46) of the random variables X_i with nominal basic parameters a_0 and σ_0. In agreement with Eqs. (43a) and (44b), the largest area of overlap between the *PDF*s of R and U_{10}, which gives a qualitative measure of the *probability of failure*, also in this case, is obtained when the interval mean-value and standard deviation of U_{10} are set equal to their *UB*, $\overline{\mu}_{U_{10}}$ and $\overline{\sigma}_{U_{10}}$.

In Figures 12 and 13, the bounds of the *interval reliability index* and *failure probability* for the displacement U_{10} versus the nominal basic parameter a_0 of the *PDF* of the random fluctuations X_i are plotted for three different values of the nominal standard deviation σ_0, say $\sigma_0 = 0.1, 0.2, 0.3$. The nominal values pertaining to the truncated Gaussian *precise PDF* with nominal basic parameters a_0 and σ_0 are also reported. As already mentioned, $\beta_{\text{Target}} = 3.8$ is the target value indicated by Eurocode EN 1990 [47] for the *reliability index*. Notice that the proposed bounds are very close to those predicted by the *MCS-VM*. As expected, the *reliability index* decreases while the *probability of failure* increases as larger values of the nominal standard deviation σ_0 are considered. Also in this case, imprecision of the *PDF* of the random Young's moduli significantly affects structural safety. As can be inferred from Figure 13, the classical probabilistic analysis may seriously underestimate the *failure probability*, especially for large degrees of uncertainty of Young's moduli. By inspection of the enlargement in Figure 13d, it is observed that, with reference to the target *failure probability* indicated by Eurocode EN 1990 [47], in the context of *imprecise probability*, the structure fails for a value of the parameter a_0 smaller than the one predicted assuming a truncated Gaussian *precise PDF*.

5.2 Grid structure with uncertain Young's moduli

As second application, the grid structure shown in Figure 14, which represents part of the roof of an existing building located in Italy, is considered. The grid structure is made of concrete with the following mechanical properties: nominal Young's modulus $E_0 = 3.15 \times 10^7$ kN/m^2, Poisson's ratio $\nu = 0.2$ and unit weight $\gamma = 25\,\mathrm{kN/m}^3$. The finite element model of the grid structure consists of 142 frame finite elements with rectangular cross-section having width $0.80\ \mathrm{m}$ and thickness $0.20\ \mathrm{m}$. The total number of degrees-of-freedom is $n = 474$. The grid structure is subjected to downward transversal forces of intensity $F = 14$ kN applied to each node. Young's moduli of $p = 8$ elements are assumed to be uncertain, $E_i = E_0(1 + X_i)$, ($i = 1, 2, \ldots, 8$), (see element numbering in Figure 14), with fluctuations around the nominal value modeled as zero-mean random variables X_i described by uniform *imprecise PDF* (see Eq. (27)) with interval basic parameters $a_i^I = [\underline{a}_i, \overline{a}_i] = a_0(1 + \Delta\alpha \hat{e}_i^I)$ whose deviation amplitude is set to $\Delta\alpha = 0.2$ when not otherwise specified. The random displacement of node A along the z axis, U_{Az}, is selected as response quantity of interest.

First, the accuracy of the assumed ratio of polynomial *response surface* (see Eq. (5)) is assessed in the context of the classical probability theory assuming that the random fluctuations X_i have a uniform *precise PDF* with nominal value a_0 of the basic parameters. Figure 15 displays the mean-value and standard deviation of the displacement U_{Az} versus the nominal basic parameter a_0. The comparison with *MSC* data demonstrates the high accuracy of the proposed closed-form expressions (see Eqs. (30a,b) for $a_i^I = a_0$) of response statistics even for increasing degrees of uncertainty.

In the context of *imprecise probability*, the proposed bounds of response statistics are contrasted with those obtained by applying the *MCS-VM*. The latter requires $N_{\mathrm{c}} \times N_{\mathrm{g}}$ deterministic structural analyses, where $N_{\mathrm{c}} = 2^8$ is the number of all possible combinations of the endpoints of the interval basic parameters a_i^I, ($i = 1, 2, \ldots, 8$), of the uniform *imprecise PDF* in Eq. (28), while $N_{\mathrm{g}} = 3 \times 10^4$ is the number of considered samples.

In Figure 16, the bounds of the interval mean-value and standard deviation of the displacement U_{Az} versus the nominal basic parameter a_0 are plotted. Also in this case, the proposed analytical expressions of the bounds of response statistics (see Eqs. (23) and (26)) are in very good agreement with *MCS-VM* data.

The proposed method yields very accurate results even for increasing degrees of imprecision of the input *PDF*, as shown in Figure 17 where the bounds of the interval mean-value and standard deviation of the displacement U_{Az} versus the deviation amplitude $\Delta\alpha$ of the interval basic parameters a_i^I are plotted. By inspection of this figure, it is inferred that the region of response statistics widens as the degree of imprecision of the *PDF* increases. This implies that classical probabilistic analysis underestimates the statistics of the response as much as higher is the deviation amplitude of the interval basic parameters of the input *PDF*.

In Figure 18, the bounds of the *interval reliability index* and *failure probability* for the displacement U_{Az} versus the nominal basic parameter a_0 of the uniform *PDF* of the random fluctuations X_i are plotted. The nominal values pertaining to the uniform *precise PDF* with nominal basic parameter a_0 are also reported. As pointed out in the previous sections, $\beta_{\mathrm{Target}} = 3.8$ is the target value indicated by Eurocode EN 1990 [47] for the *reliability index*. A very good match between the proposed bounds and the ones provided by the *MCS-VM* is observed. Also in this case, the *reliability index* and the *probability of failure* are significantly affected by the imprecision of the *PDF* of the random Young's moduli. By inspection of Figure 13, it is observed that, according to the classical probabilistic analysis, the grid structure would be safe for any value of the parameter a_0 within the considered range since the *probability of failure* is always smaller than the target value indicated by Eurocode EN 1990 [47]. Conversely, the *UB* of the *interval failure probability* predicted in the context of the *imprecise probability* exceeds the target value as the parameter a_0 increases.

6. CONCLUSIONS

The analysis of discretized structures under deterministic static loads with random parameters described by *imprecise Probability Density Functions* (*PDF*s) has been addressed. Imprecision, due to vague or incomplete information on the uncertain properties, has been taken into account by modelling the basic parameters of the underlying *PDF* as interval variables. As a result, the statistics of the response turn out to be described by intervals. In this context, the main concern to ensure safe design is to predict the influence of imprecision on the *probability of failure* which has an interval nature as well. The bounds of interval statistics and *failure probability* can be predicted by *Monte Carlo Simulation* (*MCS*) nested into the classical combinatorial procedure loops, which enables to process simultaneously the random and interval character of uncertainties. Unfortunately, this approach requires tremendous computational burden and becomes prohibitive for real engineering problems. To overcome this limitation, a novel procedure which does not rely neither on *MCS* nor on combinatorial strategies has been developed.

The key idea behind the proposed method is to combine the well-established *response surface* approach with both classical probability theory and the so-called *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*) [3]. In particular, using a ratio of polynomial *response surface* [1],[2], the statistics of the response have been derived as approximate explicit functions of the interval basic parameters of the *PDF* of the uncertain properties. The *IIA* via *EUI* allows us to handle the interval character of uncertainties in a such a way that conservatism affecting computations carried out in the context of the *classical interval analysis* is counteracted. Furthermore, computational times are drastically reduced compared to the joint application of *MCS* and the classical combinatorial procedure. Indeed, only a few deterministic analyses at selected sampling points are needed to define the ratio of polynomial *response surface*. Furthermore, the bounds of interval statistics are provided by handy formulas. In particular, in the case of uncertainties with uniform *imprecise* distribution, approximate explicit expressions of the bounds of the interval mean-value and variance of the displacements have been obtained. Based on this result, the bounds of the *interval reliability index* and the associated *interval failure probability* have been evaluated analytically as well.

Numerical results concerning a two-bay three-storey frame and a grid structure with uncertain Young’s moduli have demonstrated the accuracy and efficiency of the proposed procedure. The versatility of method has been shown by considering both a uniform and Gaussian truncated *imprecise PDF* of the uncertain parameters. In the context of reliability assessment, it has been shown that the *failure probability* is highly sensitive to the variation of the basic parameters of the *PDF* of the uncertain material properties. This implies that classical probabilistic analysis may led to unsafe design when the actual distribution deviates from the assumed one.

The proposed approach provides an efficient tool to predict the worst-case scenario (the maximum *failure probability*) when only imprecise information on the uncertain parameters is available and the application of the classical probabilistic analysis may lead to unreliable estimates of the *failure probability*.

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Figure captions

Figure 1. Uniform *imprecise PDF* of the random variables X_i.

Figure 2. Frame structure with uncertain Young's moduli described by *imprecise PDF*.

Figure 3. a) Mean-value and b) standard deviation of the displacement U_{10} of the frame versus the basic parameter a_0 of the uniform *precise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS* data.

Figure 4. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS-VM* data.

Figure 5. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{10} of the frame versus the deviation amplitude of the interval basic parameters $a_i^I = a_0(1+\Delta\alpha\hat{e}_i^I)$ of the uniform *imprecise PDF* of the random variables X_i ($a_0 = 0.2$): comparison between the proposed estimates and *MCS-VM* data.

Figure 6. Realizations of the lognormal *imprecise PDF* of the displacement U_{10} of the frame with random fluctuations of Young's moduli described by a uniform *imprecise PDF* and lognormal *precise PDF* of the corresponding resistance.

Figure 7. Bounds and nominal value of the a) *interval reliability index* and b) *interval failure probability* for the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i; c) enlargement for $a_0 < 0.25$: comparison between the proposed estimates and *MCS-VM* data.

Figure 8. Truncated Gaussian *imprecise PDF* of the random variables X_i: realizations for three different values of a) the interval standard deviation σ_i^I ($a_{0,i} = 0.4$) and b) of the interval parameter a_i^I ($\sigma_{0,i} = 0.3$).

Figure 9. Bounds of the interval mean-value of the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the truncated Gaussian *imprecise PDF* of the random variables X_i for a) $\sigma_0 = 0.1$, b) $\sigma_0 = 0.2$, c) $\sigma_0 = 0.3$: comparison between the proposed estimates and *MCS-VM* data.

Figure 10. Bounds of the interval standard deviation of the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the truncated Gaussian *imprecise PDF* of the random variables X_i for a) $\sigma_0 = 0.1$, b) $\sigma_0 = 0.2$, c) $\sigma_0 = 0.3$: comparison between the proposed estimates and *MCS-VM* data.

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Figure 14. Grid structure with uncertain Young's moduli of $p = 8$ elements described by *imprecise PDF*: a) 3D model; b) planar view.

Figure 15. a) Mean-value and b) standard deviation of the displacement U_{Az} of the grid structure versus the nominal basic parameter a_0 of the uniform *precise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS* data.

Figure 16. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{Az} of the grid structure versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS-VM* data.

Figure 17. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{Az} of the grid structure versus the deviation amplitude of the interval basic parameters $a_i^I = a_0(1+\Delta\alpha\hat{e}_i^I)$ of the uniform *imprecise PDF* of the random variables X_i ($a_0 = 0.2$): comparison between the proposed estimates and *MCS-VM* data.

Figure 18. Bounds and nominal value of the a) *interval reliability index* and b) *interval failure probability* for the displacement U_{Az} of the grid structure versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS-VM* data.

Figure 1

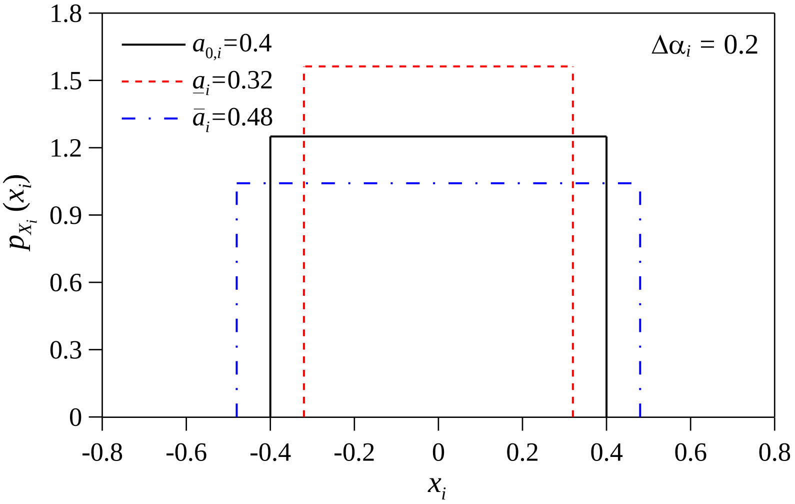


Figure 1. Uniform *imprecise PDF* of the random variables X_i .

Figure 2

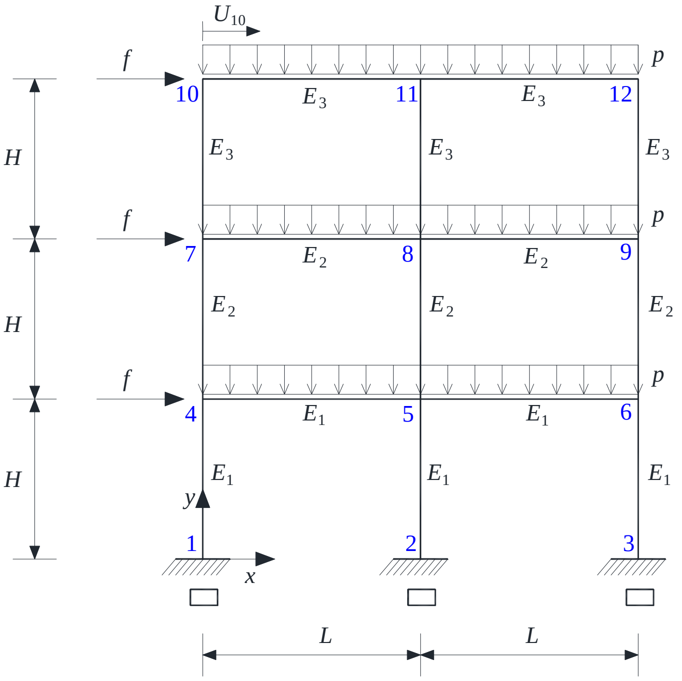


Figure 2. Frame structure with uncertain Young's moduli described by *imprecise PDF*.

Figure 3

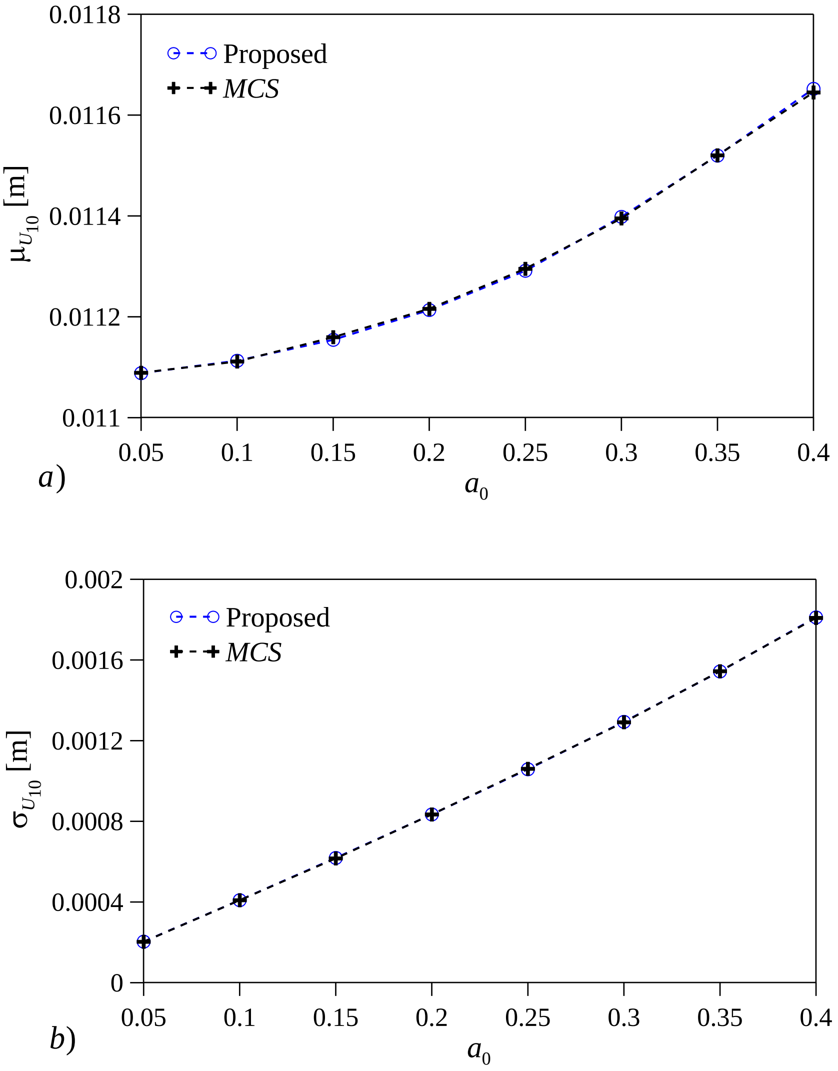


Figure 3. a) Mean-value and b) standard deviation of the displacement U_{10} of the frame versus the basic parameter a_0 of the uniform *precise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS* data.

Figure 4

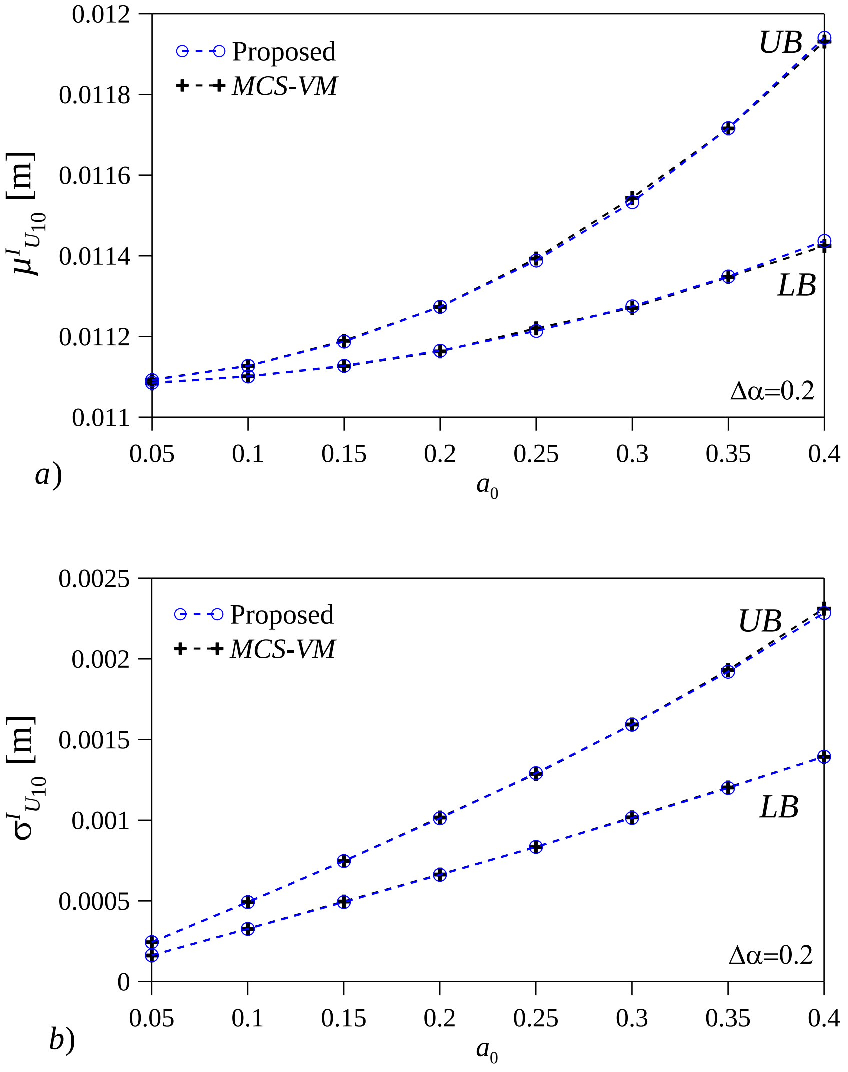


Figure 4. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS-VM* data.

Figure 5

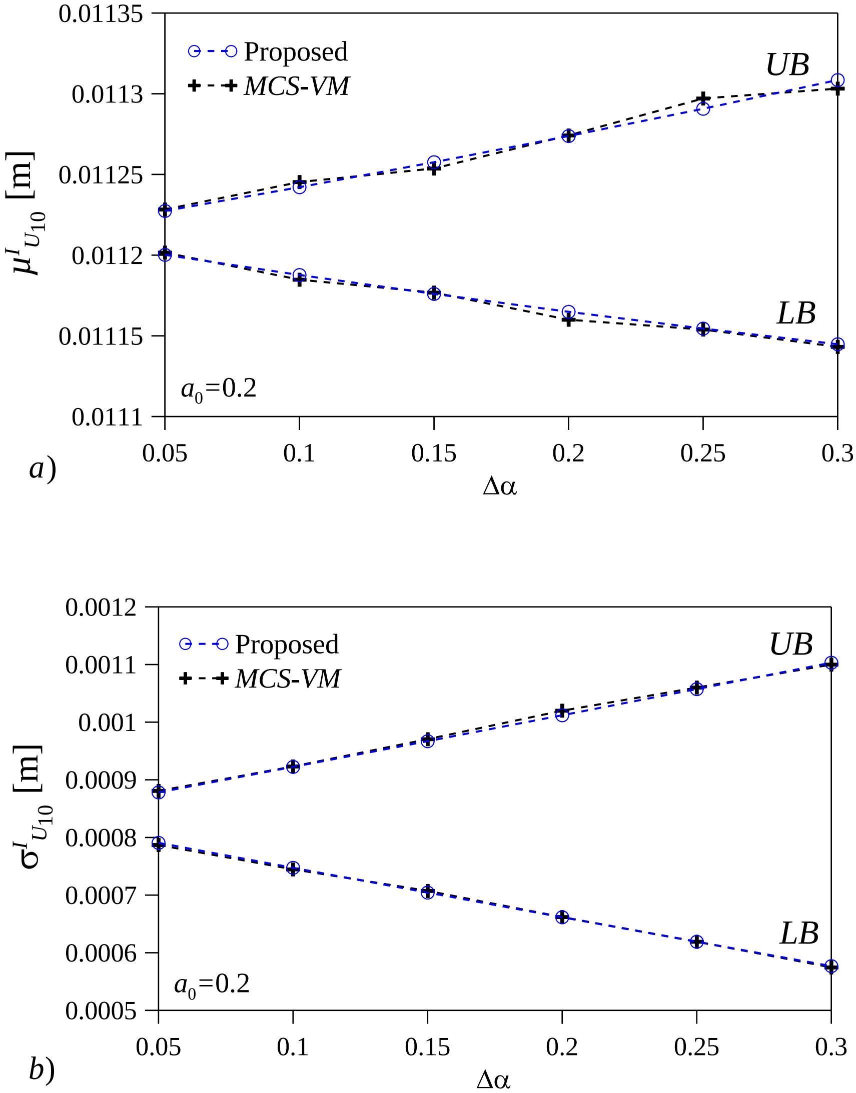


Figure 5. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{10} of the frame versus the deviation amplitude of the interval basic parameters $a_i^I = a_0(1+\Delta\alpha\hat{e}_i^I)$ of the uniform *imprecise PDF* of the random variables X_i ($a_0 = 0.2$): comparison between the proposed estimates and *MCS-VM* data.

Figure 6

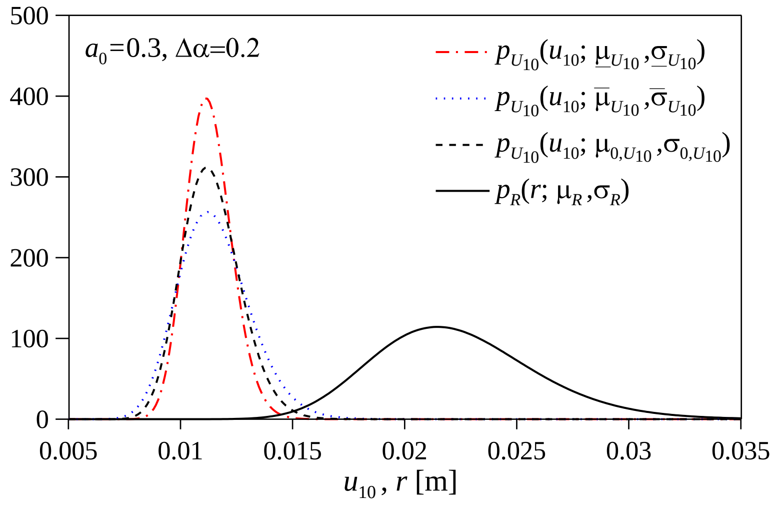


Figure 6. Realizations of the lognormal *imprecise PDF* of the displacement U_{10} of the frame with random fluctuations of Young's moduli described by a uniform *imprecise PDF* and lognormal *precise PDF* of the corresponding resistance.

Figure 7

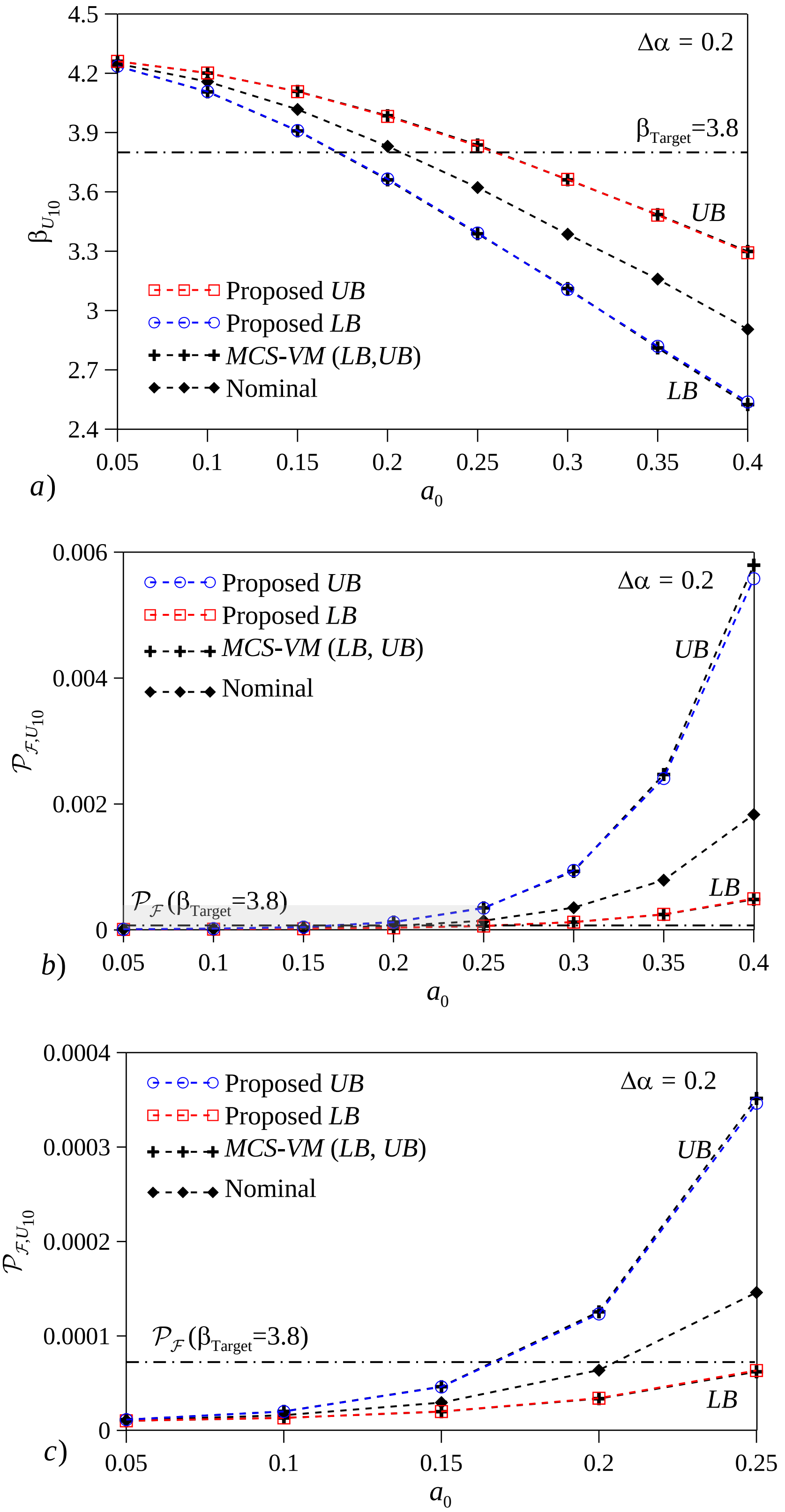


Figure 7. Bounds and nominal value of the a) *interval reliability index* and b) *interval failure probability* for the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i; c) enlargement for $a_0 < 0.25$: comparison between the proposed estimates and *MCS-VM* data.

Figure 8

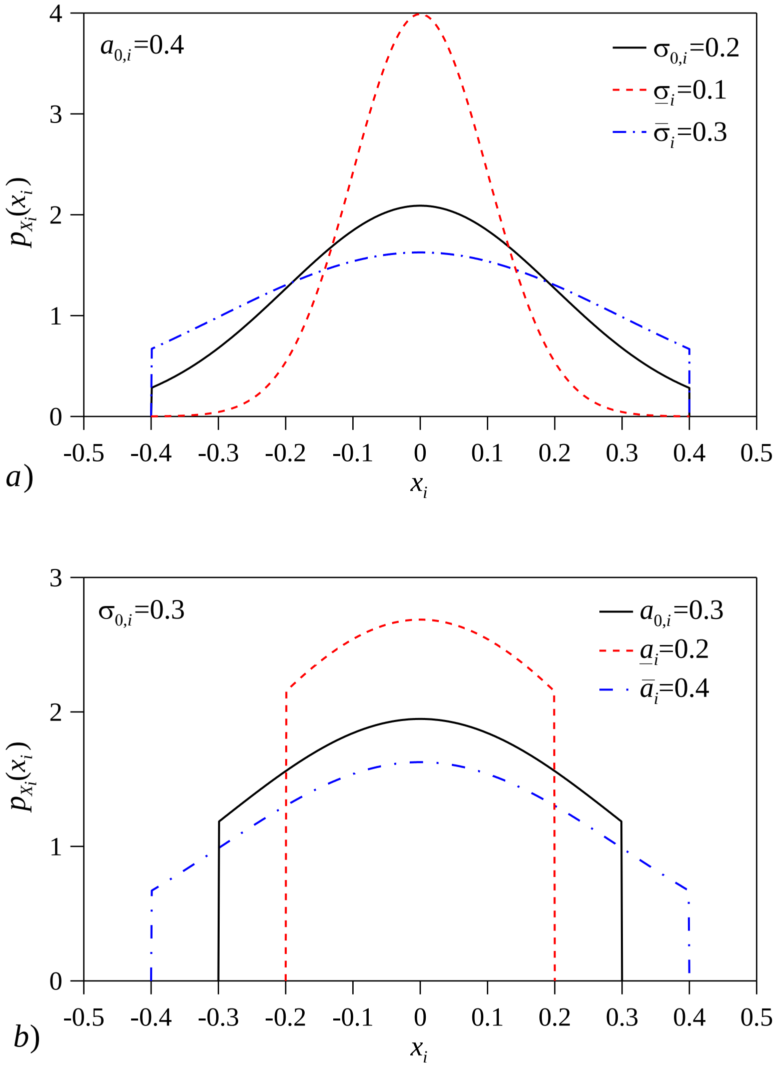


Figure 8. Truncated Gaussian *imprecise PDF* of the random variables X_i: realizations for three different values of a) the interval standard deviation $\sigma_i^I = \sigma_{0,i}(1+\Delta\alpha_i \hat{e}_i^I)$, ($a_{0,i} = 0.4$, $\sigma_{0,i} = 0.2$, $\Delta\alpha_i = 0.5$) and b) of the interval parameter $a_i^I = a_{0,i}(1+\Delta\alpha_i \hat{e}_i^I)$ ($\sigma_{0,i} = 0.3$, $a_{0,i} = 0.3$, $\Delta\alpha_i = 0.33$).

Figure 9

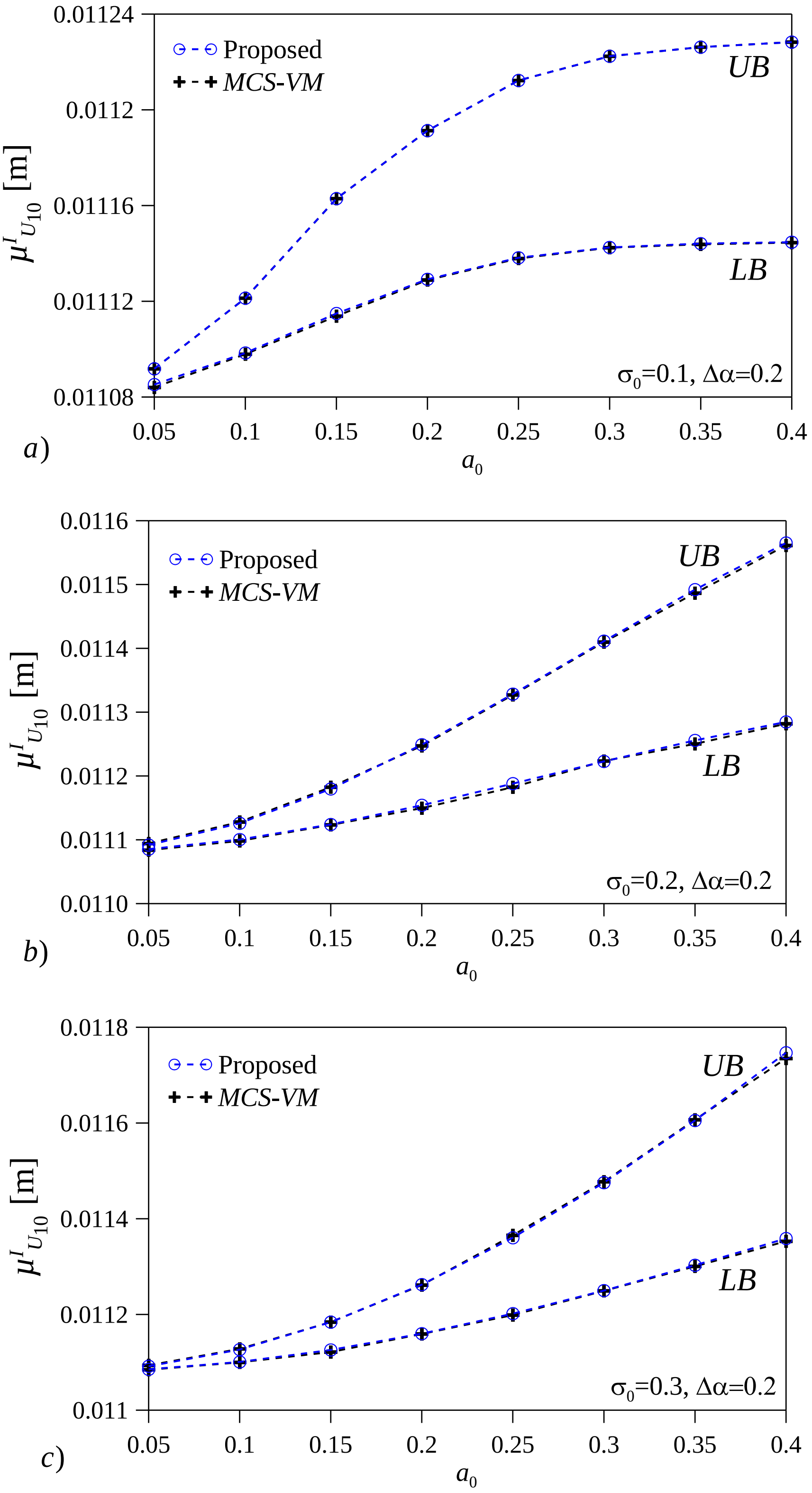


Figure 9. Bounds of the interval mean-value of the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the truncated Gaussian *imprecise PDF* of the random variables X_i for a) $\sigma_0 = 0.1$, b) $\sigma_0 = 0.2$, c) $\sigma_0 = 0.3$: comparison between the proposed estimates and *MCS-VM* data.

Figure 10

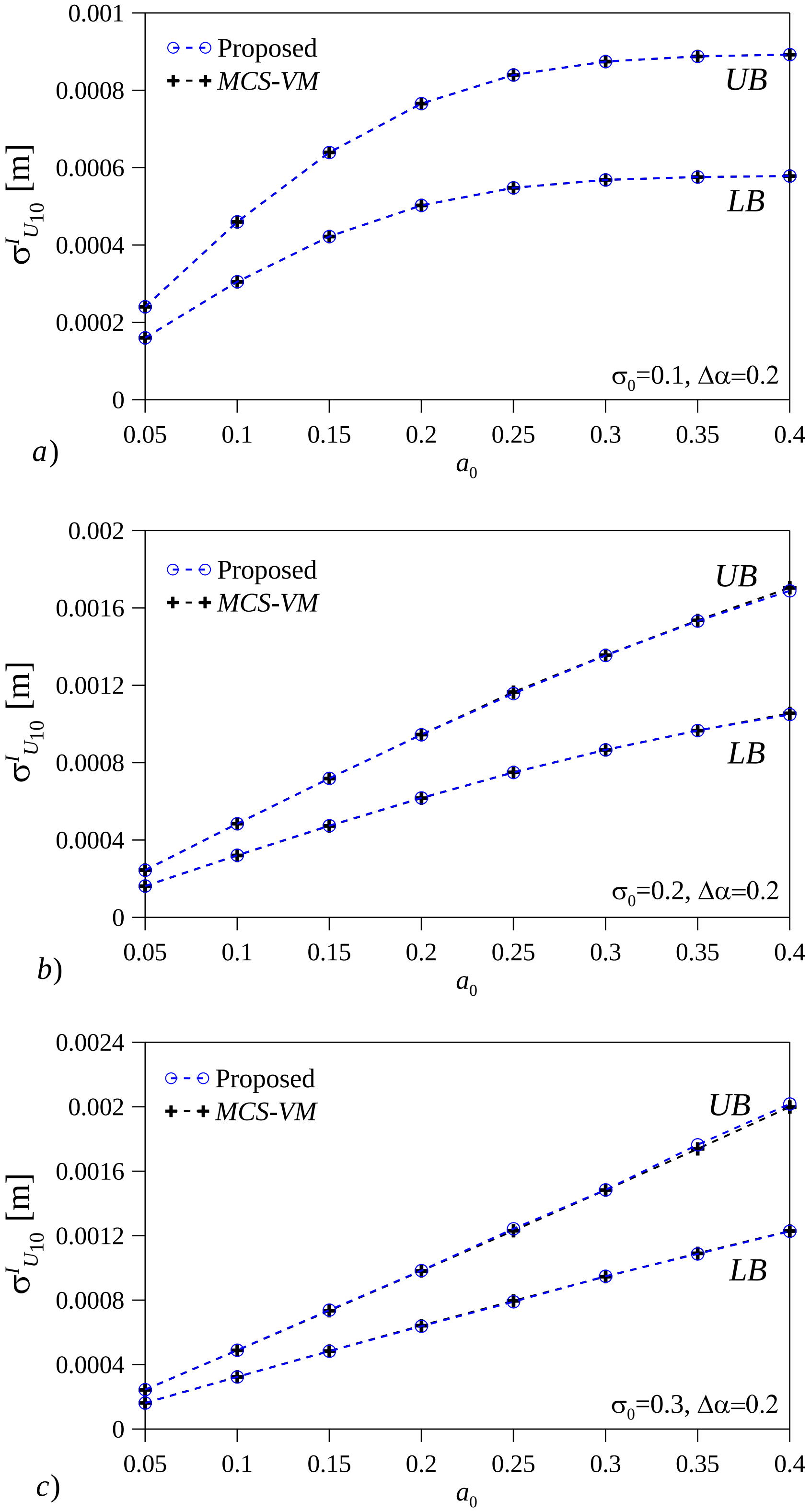


Figure 10. Bounds of the interval standard deviation of the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the truncated Gaussian *imprecise PDF* of the random variables X_i for a) $\sigma_0 = 0.1$, b) $\sigma_0 = 0.2$, c) $\sigma_0 = 0.3$: comparison between the proposed estimates and *MCS-VM* data.

Figure 11

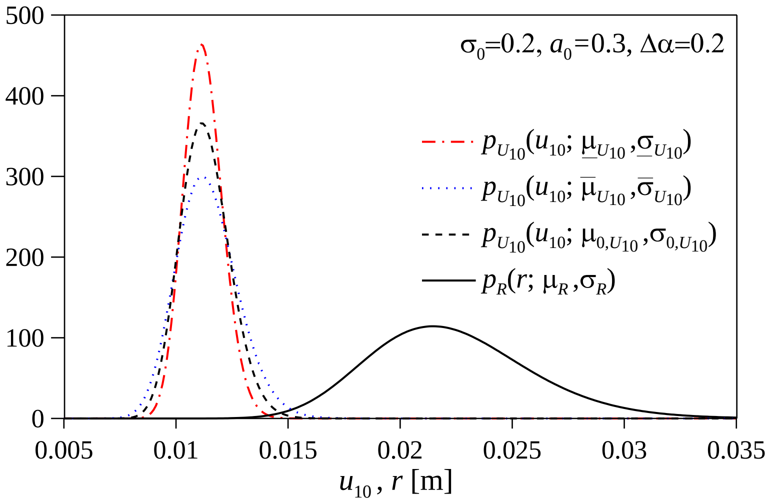


Figure 11. Realizations of the lognormal *imprecise PDF* of the displacement U_{10} of the frame with random fluctuations of Young's moduli described by a truncated Gaussian *imprecise PDF* and lognormal *precise PDF* of the corresponding resistance.

Figure 12

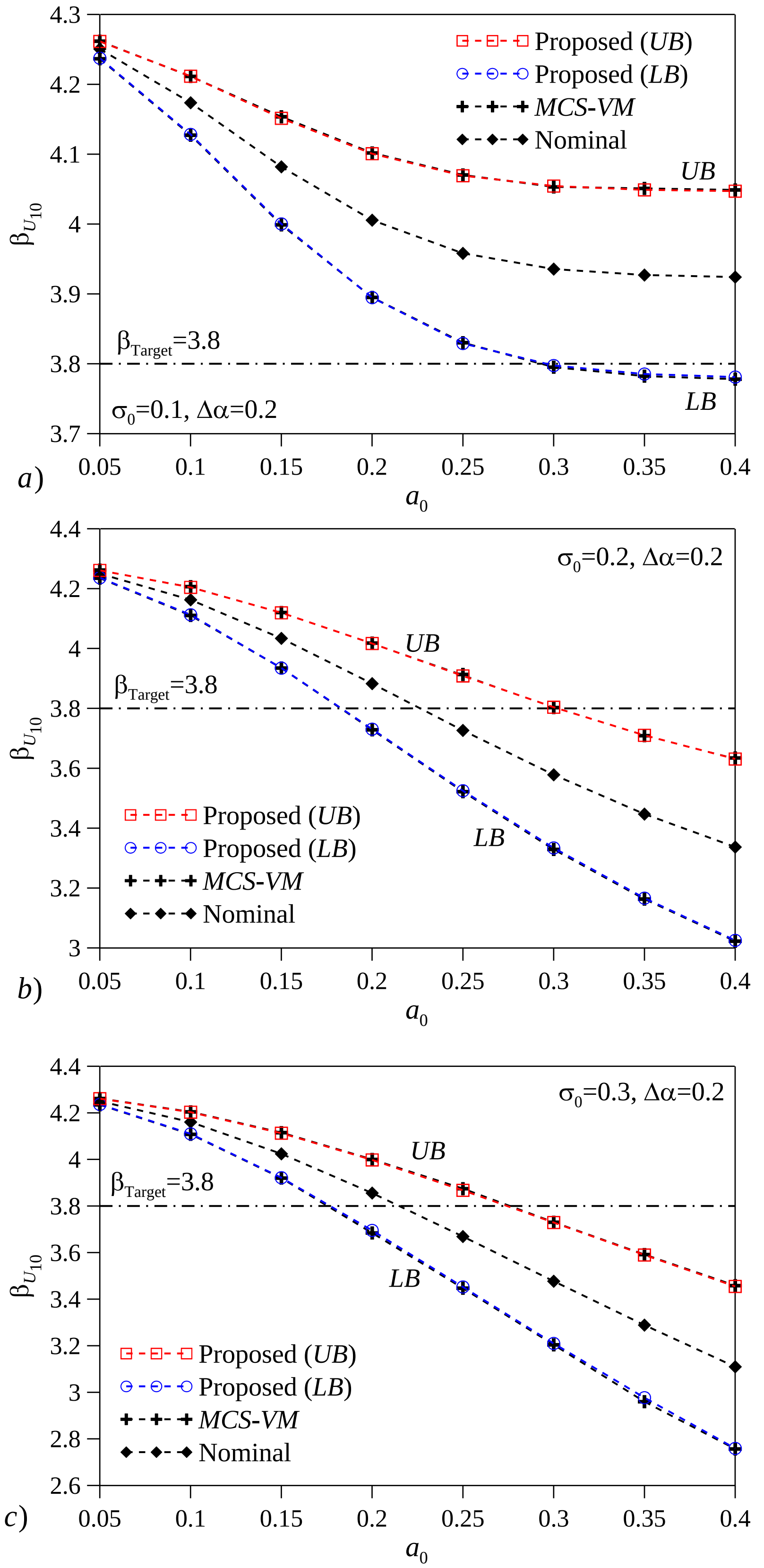
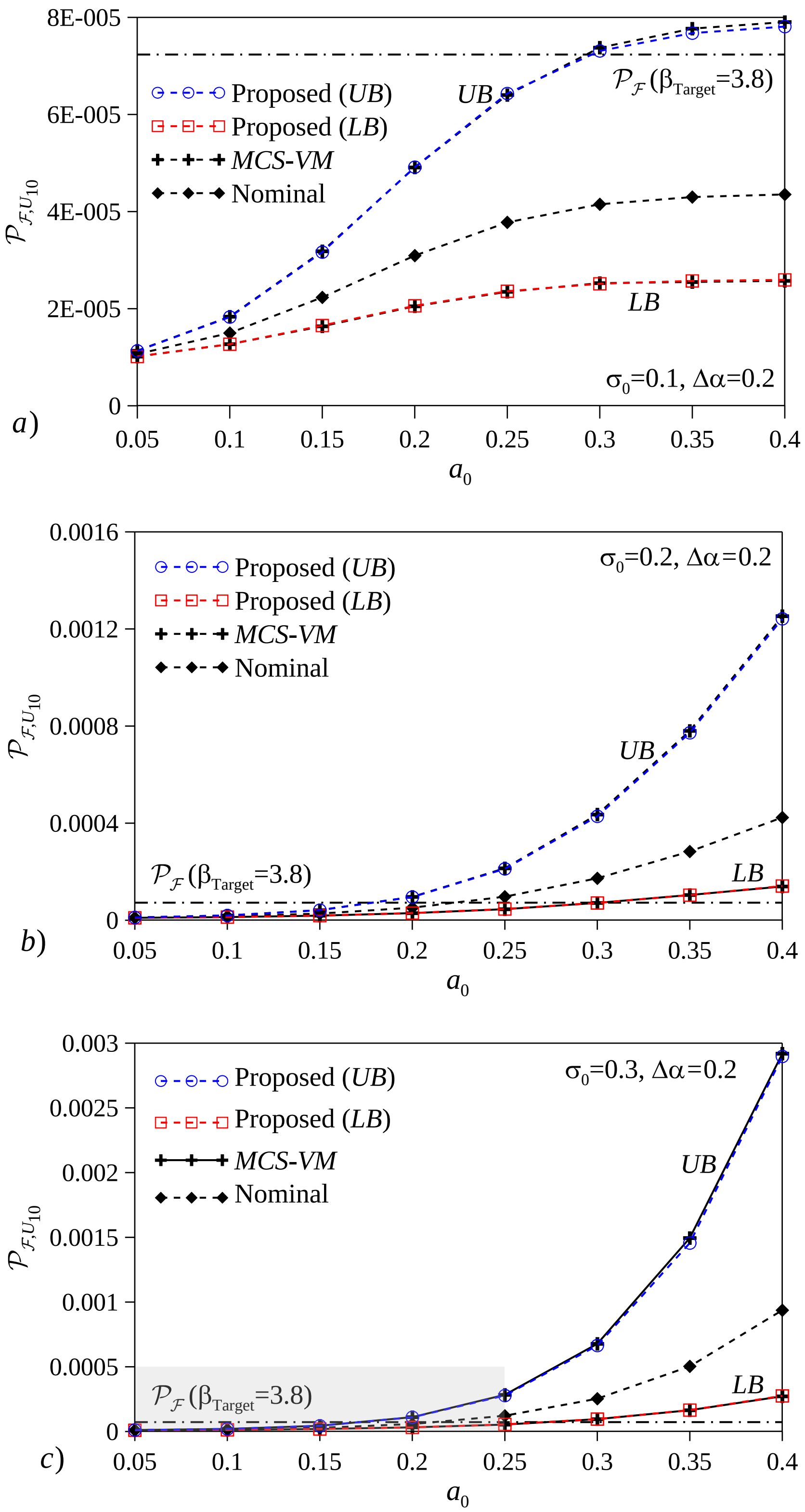


Figure 12. Bounds and nominal value of the *interval reliability index* for the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the truncated Gaussian *imprecise PDF* of the

random variables X_i for a) $\sigma_0 = 0.1$, b) $\sigma_0 = 0.2$ and c) $\sigma_0 = 0.3$: comparison between the proposed estimates and *MCS-VM* data.

Figure 13


8E-005
6E-005
4E-005
2E-005
0
$\mathcal{P}_{\mathcal{F},U_{10}}$
Proposed (*UB*)
Proposed (*LB*)
MCS-VM
Nominal
UB
LB
$\mathcal{P}_{\mathcal{F}}$ (β_{Target}=3.8)
σ_0=0.1, $\Delta\alpha$=0.2
a)
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
a_0
0.0016
0.0012
0.0008
0.0004
0
$\mathcal{P}_{\mathcal{F},U_{10}}$
Proposed (*UB*)
Proposed (*LB*)
MCS-VM
Nominal
σ_0=0.2, $\Delta\alpha$=0.2
UB
LB
$\mathcal{P}_{\mathcal{F}}$ (β_{Target}=3.8)
b)
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
a_0
0.003
0.0025
0.002
0.0015
0.001
0.0005
0
$\mathcal{P}_{\mathcal{F},U_{10}}$
Proposed (*UB*)
Proposed (*LB*)
MCS-VM
Nominal
σ_0=0.3, $\Delta\alpha$=0.2
UB
LB
$\mathcal{P}_{\mathcal{F}}$ (β_{Target}=3.8)
c)
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
a_0

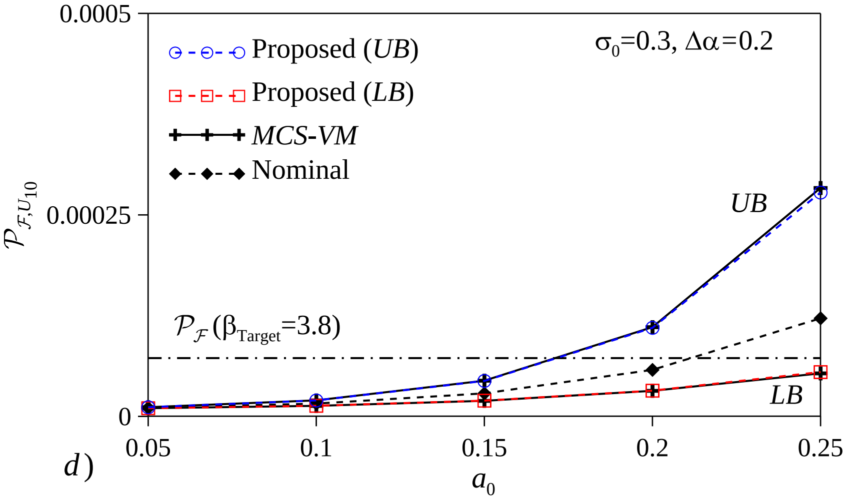


Figure 13. Bounds and nominal value of the *interval failure probability* for the displacement U_{10} of the frame versus the nominal basic parameter a_0 of the truncated Gaussian *imprecise PDF* of the random variables X_i for a) $\sigma_0 = 0.1$, b) $\sigma_0 = 0.2$ and c) $\sigma_0 = 0.3$; d) enlargement for $a_0 < 0.25$: comparison between the proposed estimates and *MCS-VM* data.

Figure 14

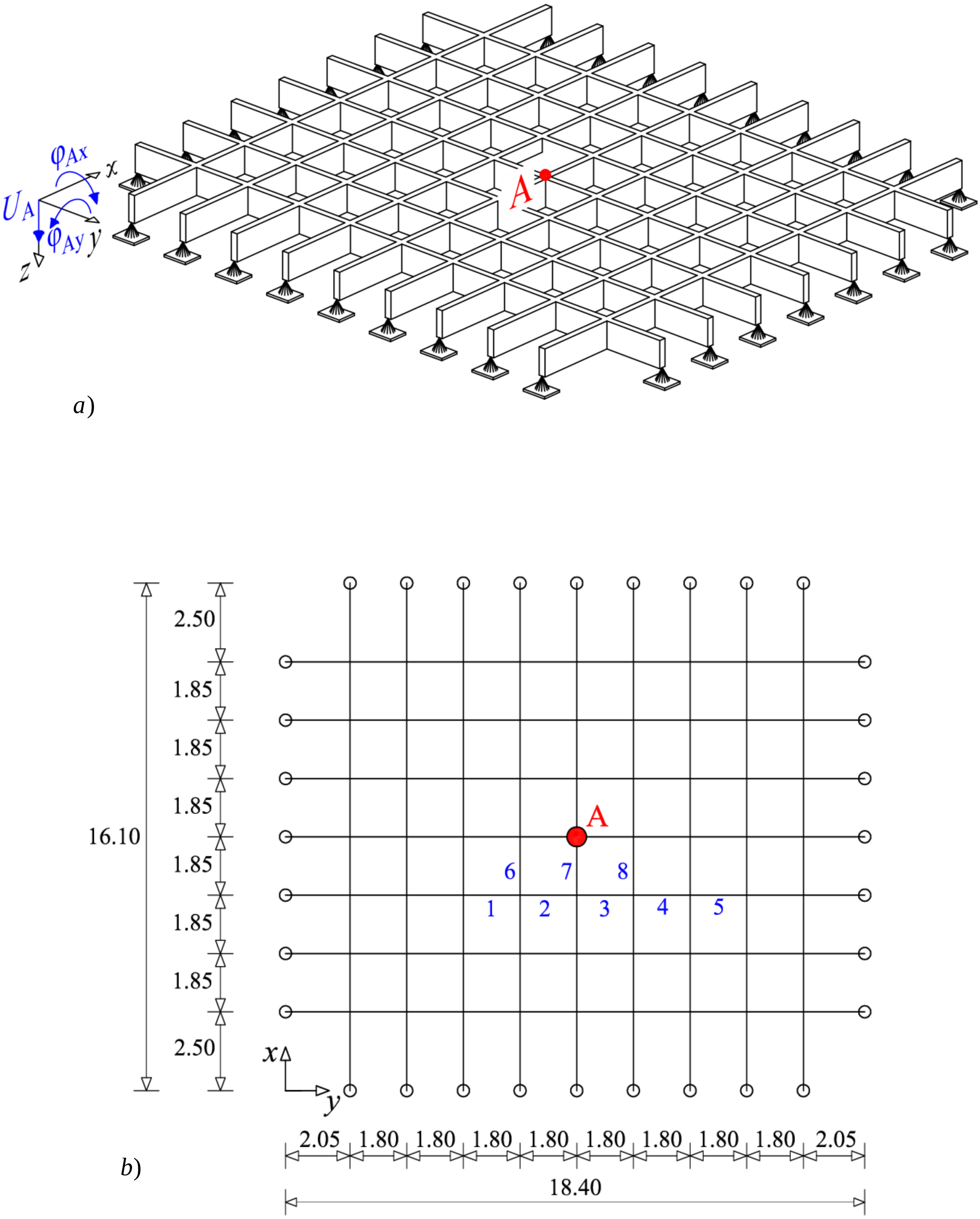


Figure 14. Grid structure with uncertain Young's moduli of $p = 8$ elements described by *imprecise PDF*: a) 3D model; b) planar view.

Figure 15

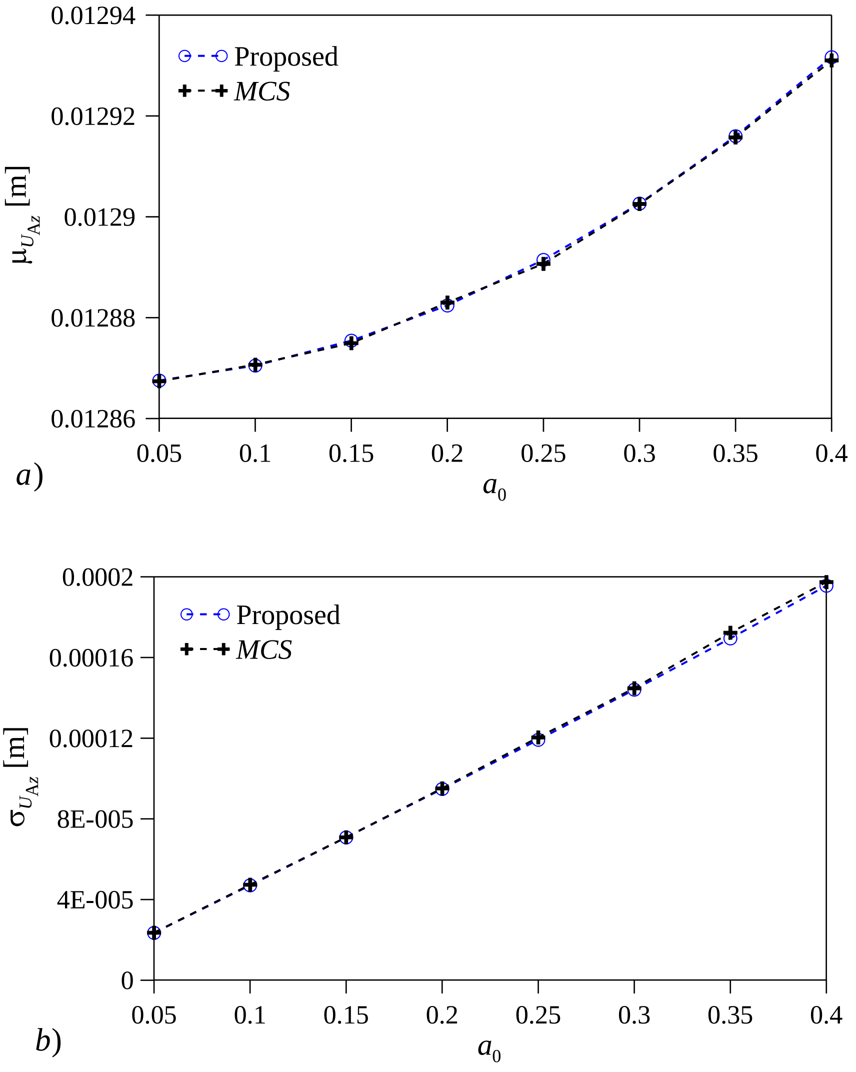


Figure 15. a) Mean-value and b) standard deviation of the displacement U_{Az} of the grid structure versus the nominal basic parameter a_0 of the uniform *precise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS* data.

Figure 16

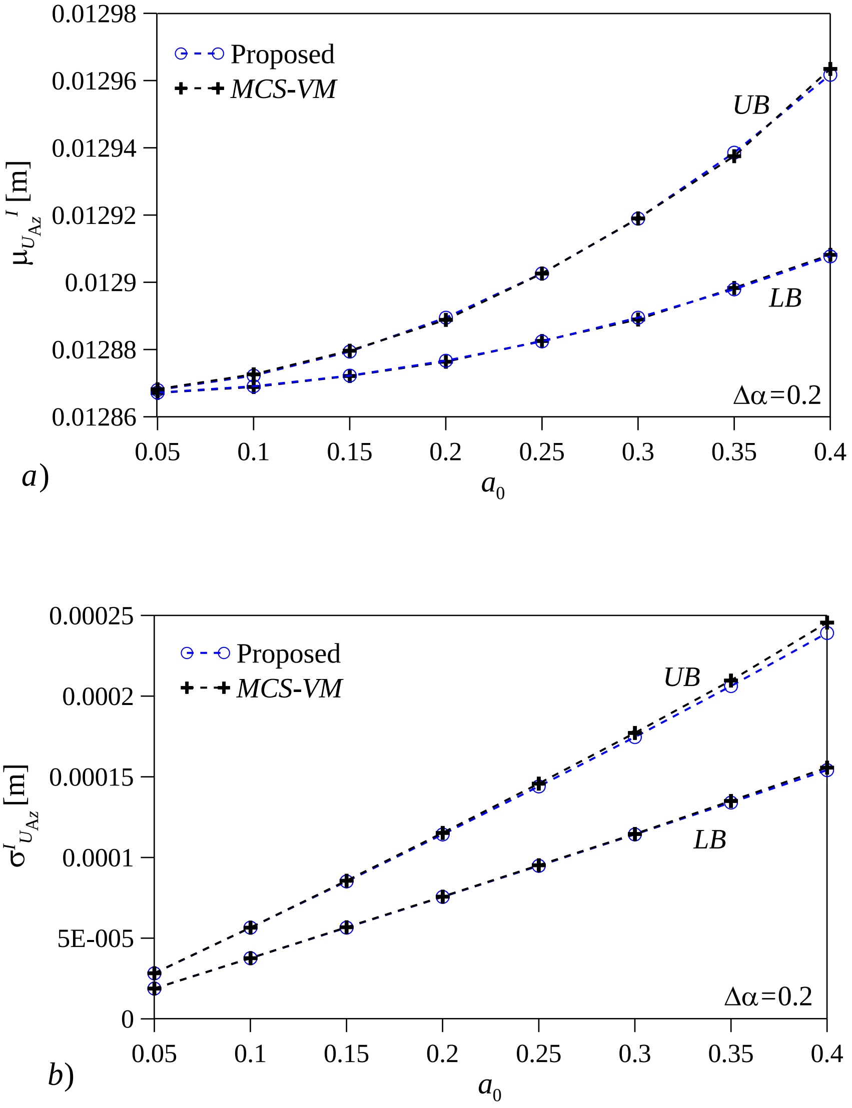


Figure 16. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{Az} of the grid structure versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS-VM* data.

Figure 17

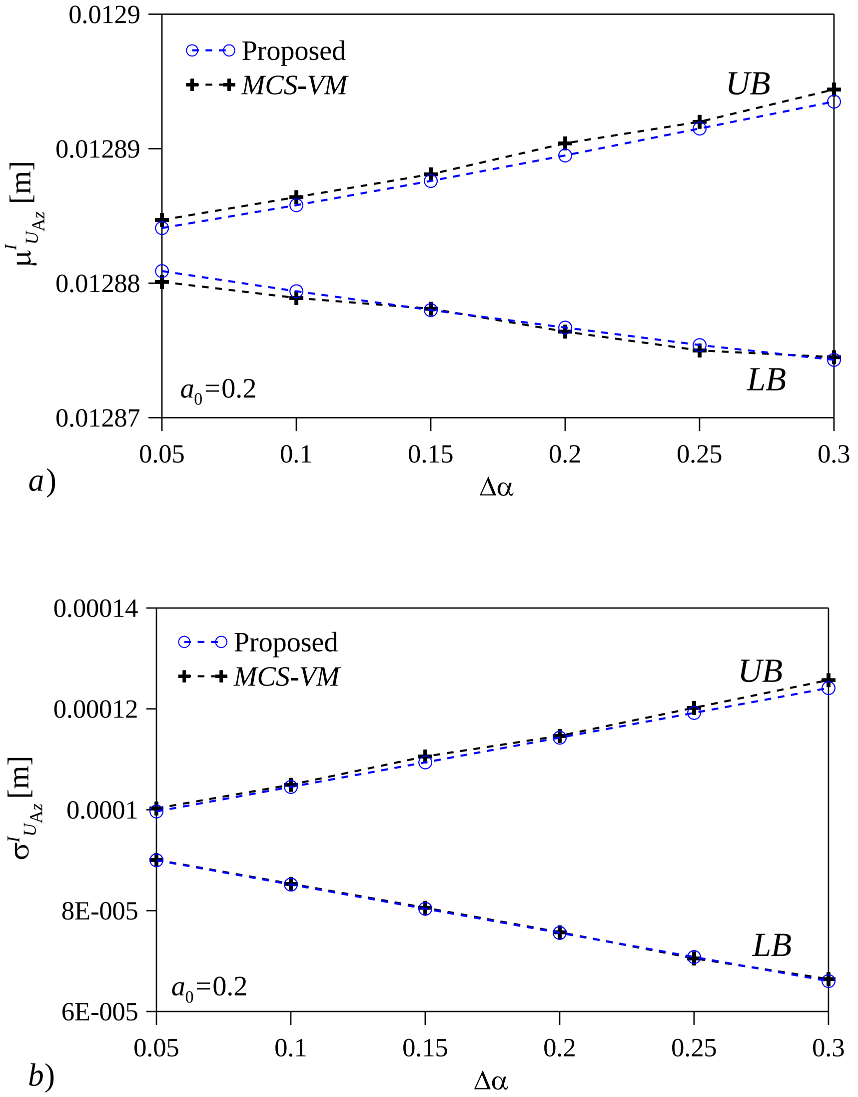


Figure 17. Bounds of the interval a) mean-value and b) standard deviation of the displacement U_{Az} of the grid structure versus the deviation amplitude of the interval basic parameters $a_i^I = a_0(1+\Delta\alpha\hat{e}_i^I)$ of the uniform *imprecise PDF* of the random variables X_i $(a_0 = 0.2)$: comparison between the proposed estimates and *MCS-VM* data.

Figure 18

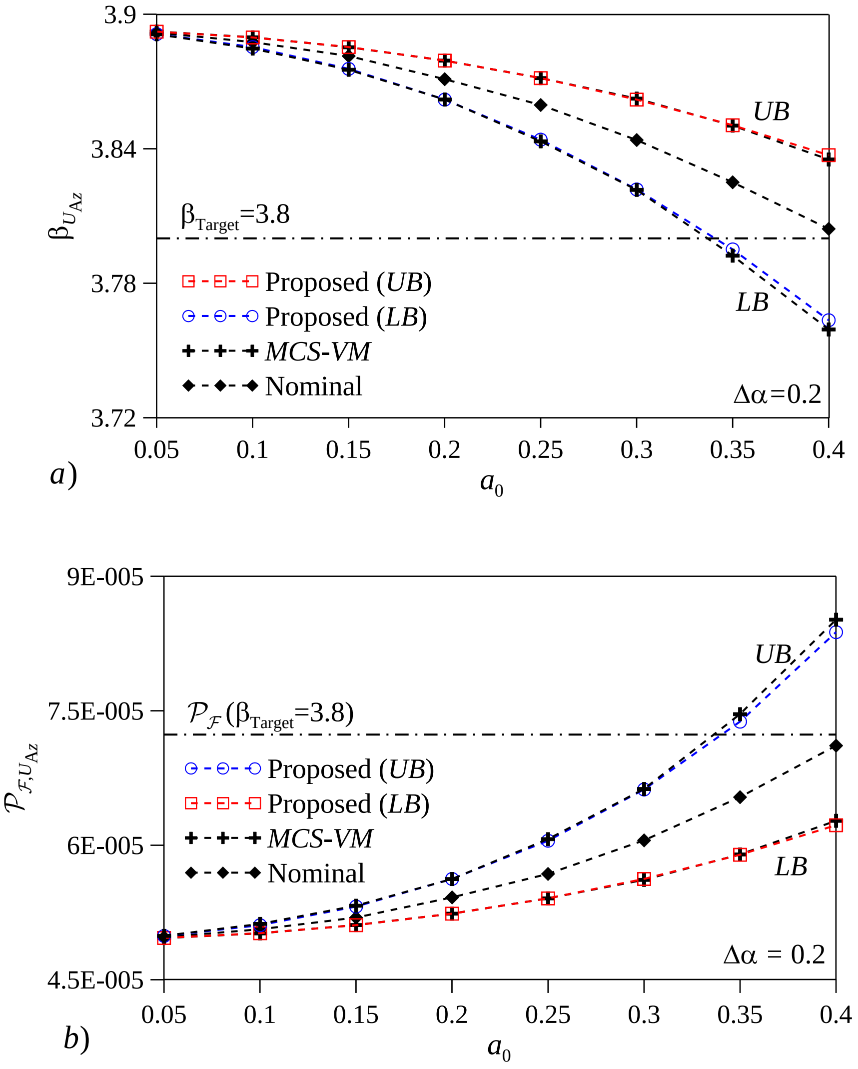


Figure 18. Bounds and nominal value of the a) *interval reliability index* and b) *interval failure probability* for the displacement U_{Az} of the grid structure versus the nominal basic parameter a_0 of the uniform *imprecise PDF* of the random variables X_i: comparison between the proposed estimates and *MCS-VM* data.