Synthesis of Maximum-Efficiency Beam Arrays via Convex Programming and Compressive Sensing

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***Abstract*—The power synthesis of maximally-sparse arrays such to maximize the beam efficiency over a target region Ω is addressed. In particular, Ω is the unique input parameter of the proposed design procedure and, once it has been fixed, the approach allows identifying the minimum number of elements required to achieve inside it a beam efficiency close to the theoretical maximum. The problem is cast as a Convex Programming one which exploits at best the Compressive Sensing theory. Comparisons with state-of-the-art methods are provided.**

***Index Terms*-Beam efficiency, power synthesis, sparse arrays.**

I. Introduction

The synthesis of a radiating system is commonly referred as "optimal" [1]-[19] if, and only if, it allows satisfying one of the two following conditions:

a) for fixed antenna's spatial or electrical resources, the radiation performances are maximized;

b) for guaranteed radiation characteristics, the antennas' spatial or electrical required resources are minimized.

In the following, focusing on array antennas, two different instances respectively referring to conditions *a)* and *b)* above are considered.

The first instance, which belongs to problems wherein condition *a)* is pursued, is the maximization of the Beam Transmission Efficiency (BTE), i.e., the ratio between the power deposited over a given 'target' area and the overall radiated power [2]-[5]. This problem, which will be referred in the following as *Problem 1*, is very important in those applications where the transfer of energy (rather than of information) is of interest, e.g., wireless power transmissions [3] and microwave hyperthermia [6].

The second instance, which belongs to problems wherein condition *b)* is pursued, is the synthesis of 'Maximally-Sparse' Arrays (MSAs) [9]-[17], i.e., arrays able to achieve given radiation performances by exploiting the minimum number of radiating elements. This problem, which will be referred in the following as *Problem 2*, is crucial in several applications, including multiple-input-multiple-output systems and radar or satellite communications [10].

As far as *Problem 1* is concerned, it is usually solved either as an eigenvalue problem [2], or by exploiting the theory of prolate spheroidal functions [5], or global optimization [3]. In particular, the approach in [2] turned out able to outperform all previous methods. Sometime later, by formulating the problem as a Convex Programming (CP) one, the technique in [4] provided BTE values very close to the ones shown in [2] by using a lower number of array elements.

As far as *Problem 2* is concerned, it is usually solved by using either global optimization [9], or density taper [10], or the matrix pencil method [11]-[13], or Linear Programming (LP) procedures exploitable as long as the radiated field is real [14]. Recently, several approaches exploiting the Compressive Sensing (CS) theory also appeared [15]-[17], and the one in [17] allowed lowering the minimum number of array elements achievable by all methods in [13]-[16].

Notably, although the results 'separately' accomplished for *Problem 1* [2]-[5] and *Problem 2* [9]-[17] are indeed remarkable, there seems being plenty of room for improvement when jointly considering both of them. In fact, *Problem 2* has never been approached by choosing the BTE as the parameter which is kept constant while minimizing the number of array elements. This can obviously lead to MSAs exhibiting poor BTE performances. On the other side, *Problem 1* has never been approached by using MSAs. This circumstance is a relevant limitation, given the advantages offered by MSAs with respect to Uniformly Spaced Arrays (USAs) in terms of size, weight, bandwidth, and sidelobe level [10]. The only contribution addressing the BTE optimization through sparse arrays is [3] wherein, however, the elements' excitations are not exploited as a degree of freedom of the synthesis and hence (as it will be shown in Section III) the achieved elements' number can be considerably lowered.

In the attempt of filling such a gap, this paper presents a new strategy for the synthesis of arrays being at the same time maximally sparse and able to achieve BTE values very close to 100%. The technique is conceived as the unification of the two design procedures respectively devised in [4] (to solve *Problem 1*) and [17] (to solve *Problem 2*) in such a way to keep the advantages of both of them.

In the following, the proposed method is described in Section II and assessed in Section III. Conclusions follow.

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II. THE SYNTHESIS PROCEDURE

The approach can be used in both cases of linear and planar MSAs. However, by the sake of simplicity, in the following it will be described only for the case of 1-D arrays. The only input parameter is the 'target' area over which the BTE must be maximized, say Ω. The beam efficiency inside this region (which is supposed being in the far-field zone of the source) will be expressed as in [2]-[4], i.e., $BTE_\Omega = P_\Omega / P$ (P_Ω and P representing, respectively, the power transmitted over Ω and the overall radiated power). The unknowns of the problem are both the array elements locations and excitations, which represents a relevant novelty as all previous approaches aimed at maximizing the BTE exploited *only one* of these two degrees of freedom (see for instance [2]-[4]).

The design procedure consists of two consecutive steps, i.e., a *feasibility criterion* allowing the dimensioning of the source followed by an *actual synthesis* algorithm, which are separately discussed in the two following Subsections.

II.A Step 1: Feasibility criterion and source dimensioning

Once Ω has been fixed, this step is aimed at identifying:

- the minimum number of elements required to an USA in order to achieve a beam efficiency equal to 99.99%;
- an optimal power pattern to be exploited as 'reference' by step 2 in order to finalize the MSA design.

This can be done by exploiting (as auxiliary 'virtual' array) a linear USA composed by N isotropic elements and having an inter-element spacing equal to d. Note that, by virtue of the theory in [7], its power pattern can be expressed as a linear function of $2N+1$ complex coefficients $\{D_p\}$ as follows:

$$P(u) = \sum_{p=-N+1}^{N-1} D_p e^{jp\beta du} \quad (1)$$

with $\beta = 2\pi/\lambda$ (λ denoting the wavelength) and $u = \sin\theta$ (θ being the angle between the boresight and observation directions). Let us also recall the optimization problem cast in [4], i.e.:

$$\min_{D_{-N+1},\, \dots,\, D_{N-1}} \; -\int_{\Omega} \sum_{p=-N+1}^{N-1} D_p e^{jpu\beta d} du \quad (2.a)$$

subject to:

$$0 \le \sum_{p=-N+1}^{N-1} D_p e^{jpu\beta d} \le UB(u) \quad (2.b)$$

$$D_p = D_{-p}^{*} \quad p = 1, \dots, N-1 \quad (2.c)$$

$$\frac{\int_{\Omega} \sum_{p=-N+1}^{N-1} D_p e^{jpu\beta d} du}{\int_{-1}^{1} \sum_{p=-N+1}^{N-1} D_p e^{jpu\beta d} du} \ge A \quad (2.d)$$

wherein A is a real constant (such that $0<A<1$) and constraints (2.b),(2.c) (* denoting complex conjugation) guarantee that $P(u)$ is a real non-negative function lower than UB (which can be used as a 'protection' bound), while the minimization of the objective function (2.a) under constraint (2.d) entails that the power deposited inside Ω is maximum and that $BTE_\Omega \ge A$.

By taking into account the fact that (2) is a LP problem (and hence that one can find its unique optimal solution in a very fast and effective fashion), once Ω has been fixed step 1 of the proposed synthesis procedure is performed as follows:

Set {d= λ/2, UB(θ)=1 ∀θ, A=0.9999} and repeatedly solve problem (2) for decreasing values of N until constraints become so strict that no solution exists anymore.

A number of comments are in order about the array coming out from such a procedure, denoting with $P_1(u)$ and N_1, respectively, its power pattern and overall elements number.

First, $P_1(u)$ will be factorable as follows:

$$P_1(u) = F_1(u) F_1(u)^* \quad with \quad F_1(u) = \sum_{n=1}^{N} a_n e^{jnu\beta d} \quad (3)$$

$F_1(u)$ being the far field radiated by the 'virtual' array at hand and $a_1, \dots, a_N$ being the corresponding elements excitations. It is also worth noting that factorization (3) is not unique, as $2^{R/2}$ different fields satisfying it can be identified (R being the number of zeroes of P_1 not lying on the unit circle) [7],[18].

Second, by virtue of the adopted A value, the array will provide over the region Ω a beam efficiency at least equal to 99.99%, i.e., very close to the maximum theoretical value achievable by any antenna of whatever size.

Third, by virtue of both the adopted decreasing N values and $\lambda/2$ spacing [7], it will be guaranteed that, excluding 'superdirective' solutions [8], no USAs composed by less than N_1 elements can achieve the same BTE_Ω value. Therefore, the above 'virtual' array represents the *minimally-redundant* USA associated to the specific problem at hand.

II.B Step 2: Actual Synthesis of the MSA

This step is aimed at synthesizing a MSA able to achieve radiation characteristics equivalent to the ones of the 'virtual' array coming out from step 1 by exploiting less then N_1 elements. This can be done through the following procedure:

i. factorize P_1 and select, amongst all equivalent fields, the one having the minimum ℓ_1 norm of excitations, say F_{1REF};

ii. given the length of the USA coming out from step 1, i.e., $L_1 = (N_1 - 1)\lambda/2$, sample with a rate in the order of $\lambda/10$ the domain $[0, L_1]$ into M equispaced points, say $x_1, \dots, x_M$. Then, define a further USA having locations $x_1, \dots, x_M$ and find its excitations $b_1, \dots, b_M$ by solving the following CP problem:

$$\min_{b_1, \dots, b_M} \sum_{m=1}^{M} |b_m| \quad (4.a)$$

subject to:

$$\begin{cases} |F_2(\theta)|^2 \le g(\theta) \;\; \forall \theta \in \tau_1 & (4.b) \\ \dfrac{\|F_2(\theta) - F_{1REF}(\theta)\|_2}{\|F_{1REF}(\theta)\|_2} \le \varepsilon \;\; \forall \theta \in \tau_2 & (4.c) \end{cases}$$

with:

$$F_2(\theta) = \sum_{m=1}^{M} b_m e^{j\beta x_m \sin\theta} \quad (4.d)$$

iii. find the MSA elements locations, say $y_1, \dots, y_S$, by performing two operations on the USA coming out from step **ii**: discarding the antennas having a negligible excitation amplitude and then substituting each couple of elements whose distance is lower than a threshold σ with one element placed in the middle point between them;

iv. determine slight shifts $\Delta y_1,\ldots,\Delta y_S$ on the above elements locations (and the associated optimal excitations $c_1,\ldots,c_S$) by solving the following optimization problem:

$$\min_{c_1,\ldots,c_S,\Delta y_1,\ldots,\Delta y_S} \frac{\left\| F_3(\theta) - F_{1REF}(\theta) \right\|_2}{\left\| F_{1REF}(\theta) \right\|_2} \quad \forall \theta \in \tau_2 \qquad (5.a)$$

subject to:

$$\left| F_3(\theta) \right|^2 \leq g(\theta) \;\; \forall \theta \in \tau_1 \qquad (5.b)$$

wherein $F_3(u)$ represents the far field radiated by the MSA at the end of the overall synthesis procedure, i.e.:

$$F_3(\theta) = \sum_{s=1}^{S} c_s e^{j\beta(y_s + \Delta y_s)\sin\theta} \qquad (5.c)$$

The rationale of the above steps is described in the following.

Step **i** is aimed at identifying the far-field distribution to be exploited as reference in the subsequent steps. The selected solution, i.e., $F_{1REF}(u)$, has the following properties:

- by coming out from (3), it still provides $\mathrm{BTE}_\Omega \geq 0.9999$;
- by corresponding to the USA's excitations set having the minimum ℓ_1 norm, it is the field which more easily lends itself to a CS-based 'sparsification' process [15]-[17].

Step **ii** is aimed at 'sparsifying' the array layout coming out from step **i**. In fact, the ℓ_1 norm in (4.a) is the 'common alternative' to ℓ_0 norm when minimizing the number of elements of an USA through the CS theory [16]. As far as constraints (4.b) and (4.c) are concerned, the former allows controlling the power-pattern sidelobes in the angular region denoted with τ_1 (where g is an upper-bound function chosen by the user) while the latter ensures that, in the angular region denoted by τ_2, the sought array factor fits F_{1REF} with a precision ε. As a crucial feature which belongs also to step **iv**, it is worth noting that this step pursues a field fitting just in the main-beam zone, while upper bounds are used in the sidelobes region. Such a choice allows recovering a significant number of degrees of freedom with respect to the cases wherein a field fitting is pursued over the whole spectral domain [16],[17].

Step **iii** allows a suitable 'sparsification' of the USA layout coming out from step **ii**. Obviously, while severely reducing the number of array elements, the performed operations may induce a worsening of radiation performances. This possible worsening is the reason underlying execution of step **iv**.

Step **iv** is aimed at refining the elements locations (and at identifying the optimal excitations associated to them) so as to recover, without increasing the number of elements, from possible radiation losses induced by step **iii**. In fact, solving problem (5) leads to radiation characteristics equivalent to those grant by (4) while saving M-S elements.

It is worth noting that problem (5) may be cast as a 'nearly CP' one by adding to it the following constraint:

$$-\eta \leq \beta \Delta y_s \leq \eta \;\; \forall s \qquad (6)$$

with $0<\eta<<1$. In fact, this linear constraint entails that:

$$F_3(\theta) \approx \sum_{s=1}^{S} c_s e^{j\beta y_s \sin\theta}\left(1 + j\beta\Delta y_s \sin\theta\right) \qquad (7)$$

which is a linear function of the locations' shifts. Under these conditions, one can also add a linear constraint counteracting mutual coupling by avoiding too-small element spacings, i.e.:

$$\left[(y_{S+1} + \Delta y_{S+1}) - (y_S + \Delta y_S) \right] \geq \psi \qquad s = 1,\ldots,S-1 \qquad (8)$$

where ψ is an user-defined constant.

III. Numerical Assessment

The approach has been assessed by identifying the savings in terms of array elements number (for equivalent or better BTE performances) with respect to the state-of-the-art methods. In particular, in the first two test cases the synthesis of linear arrays is dealt with and the outcomes are compared with the ones achievable through the approach in [4] which, in turn, allowed a saving of elements with respect to the method in [2]. Then, in the third test case, the synthesis a planar array is considered and the results are compared with the ones in [2]-[4]. In all cases, an isotropic element pattern is considered.

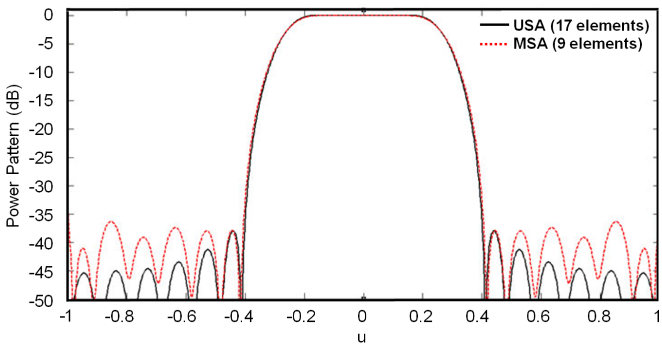

Fig. 1. First test case: $\Omega=\{u$: $-0.4\leq u\leq 0.4\}$. Power patterns radiated by the minimally-redundant equispaced array (black curve, BTE_Ω=99.99%) and the synthesized maximally-sparse array (red curve, BTE_Ω=99.98%).

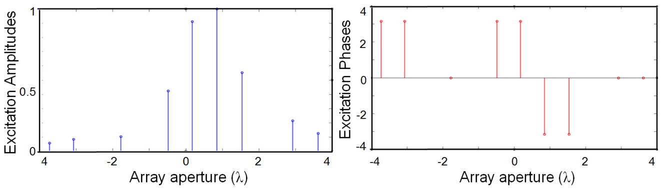

Fig. 2. Excitation amplitudes (on the left) and phases (on the right) corresponding to the power pattern depicted in red color in Fig. 1.

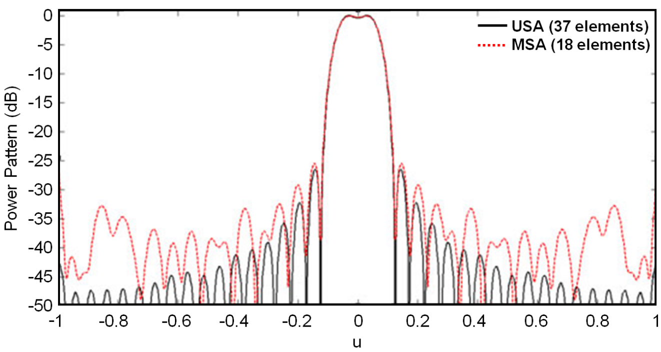

Fig. 3. Second test case: $\Omega=\{u$: $-0.1\leq u\leq 0.1\}$. Power patterns radiated by the minimally-redundant equispaced array (black curve, BTE_Ω=99.99%) and the synthesized maximally-sparse array (red curve, BTE_Ω=99.91%).

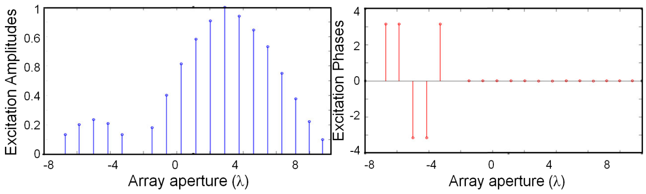

Fig. 4. Excitation amplitudes (on the left) and phases (on the right) corresponding to the power pattern depicted in red color in Fig. 3.

In the first test case, the target region has been set as $\Omega=\{u: -0.4\leq u\leq 0.4\}$. By performing step 1 of the procedure, $N_1=17$ resulted the minimum number of elements required to get a $BTE_\Omega=0.9999$ through an USA. The achieved power pattern is shown in Fig. 1. Then, by executing step 2, a MSA composed by 9 elements located over an aperture of length 7.34λ with a minimum inter-element spacing of 0.66λ has been synthesized. The achieved excitations and locations are depicted in Fig. 2. The corresponding power pattern, which is shown in Fig. 1, gives $BTE_\Omega=0.9998$. Therefore, by losing just the 0.01% of the beam efficiency, the synthesized MSA allowed saving 47% of elements with respect to the minimally-redundant USA achievable through method in [4].

In the second test case, the target region has been set as $\Omega=\{u: -0.1\leq u\leq 0.1\}$. Execution of step 1 of the procedure allowed ascertaining that the minimum number of elements required to achieve a $BTE_\Omega=0.9999$ through an USA is $N_1=37$. The power pattern coming out from this step is reported in Fig. 3. Then, by means of step 2, a MSA composed by 18 elements located over an aperture of length 13.43λ with a minimum inter-element spacing of 0.79λ has been synthesized. The achieved MSA elements excitations and locations are depicted in Fig. 4. The corresponding power pattern, which is shown in Fig. 3, provides a $BTE_\Omega=0.9991$. Therefore, by losing just the 0.08% of the beam efficiency, the MSA allowed saving the 51% of elements with respect to the minimally-redundant USA achievable by the method in [4].

As a third test case, the problem dealt with both in [2] and [3], i.e., the synthesis of a square array such to optimize the BTE inside the region $\Omega=\{(u,v): -0.2\leq u\leq 0.2, -0.2\leq v\leq 0.2\}$ has been considered. The maximum BTE_Ω achieved in [2] and [3] by exploiting a 100-elements array is reported in Table I. To assess the present approach, the problem has been first solved by exploiting the technique in [4]. By so doing, it has been possible to get (for an equal number of array elements) a BTE_Ω increase of 6.8% and 1.4% with respect to [3] and [2], respcetively. Then, the problem has been solved through the present approach. In particular, we applied step 1 of the procedure with $\Omega=\{u: -0.2\leq u\leq 0.2\}$ and then applied step 2 to a square array having factorable excitations and generating, along both the u and v cuts, a far-field equal to the F_{1REF} distribution coming out from step 1. The achieved power pattern and array layout (wherein the minimum spacing is 0.47λ) are shown in Fig. 5. Notably, the present method allowed at the same time saving 46% of elements and increasing of 8.6%, 3.2%, and 1.7% the BTE_Ω with respect to [3], [2], and [4], respectively. These results, which are summarized in Tab. I, confirm the efficacy of the approach.

IV. Conclusions

A new approach to the optimal synthesis of maximum-efficiency beams through sparse arrays has been presented. The technique allows minimizing the elements number of arrays able to achieve a beam efficiency very close to 100%, and hence it is expected to keep competitive even in case of slight manufacturing errors.

The joint use of both a Compressive-Sensing engine and a Convex-Programming formulation allowed outperforming the state-of-the-art methods.

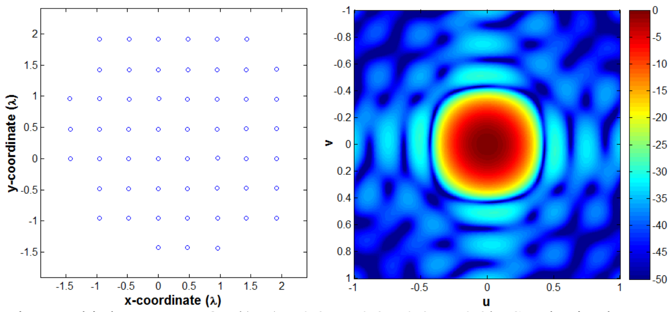

Fig. 5. Third test case: $\Omega=\{(u,v): -0.2\leq u\leq 0.2, -0.2\leq v\leq 0.2\}$. Synthesized array layout (on the left) and power pattern [dB] (on the right). BTE_Ω=99.61%.

TABLE I
COMPARISON WITH PREVIOUS TECHNIQUES (PLANAR ARRAYS CASE)

	Ref. [3]	Ref. [2]	Ref. [4]	This method
BTE_Ω (%)	91.06	96.45	97.89	99.61
Number of elements	100	100	100	54

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