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Optimal Pursuit Game of Two Pursuers and One Evader with the Grönwall-Type Constraints on Controls

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Abstract: We studied a simple motion differential game of two pursuers and one evader in $\mathbb{R}^2$. The control functions of players are subjected to the Grönwall-type constraints. If the state of the evader coincides with the state of a pursuer, then the game is considered complete. The pursuers attempt to complete the game as earlier as possible. The evader attempts to avoid being captured or delays the capture time. We found an equation for the optimal pursuit time and construct the optimal strategies of players.

Keywords: differential game; Grönwall constraints; pursuers; evader; optimal pursuit time; optimal strategy

MSC: 91A23; 49N75

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1. Introduction

Differential games with many players have received a great deal of attention in the literature (see, for more, [1–11]). Most of the literature considers differential games when the pursuers move faster than the evaders to complete the game. The paper [2] is devoted to the simultaneous multiple capturing of rigidly coordinated evaders by several pursuers. Moreover, there has been increasing interest in studying differential games with faster evaders (see, for example [12,13]).

Differential games of optimal approaches are difficult branches of differential games. The main problems for such differential games involve finding the value functions, to construct the optimal strategies of players. In the case of one pursuer and one evader, Isaacs [14] successfully applied the main equation of differential games—the Hamilton–Jacobi–Isaacs equation—to concrete differential games (to obtain the value function), although the existence and uniqueness of the solutions of the equation were not theoretically established yet.

Subbotin [15] established that the main equation of a differential game may not have a solution or may have infinitely many solutions. In this regard, Subbotin [15] obtained necessary and sufficient conditions for a function (to be the value function of the game) by introducing the notions of u-stability and v-stability. However, this condition contains two nonlinear partial derivative inequalities, and solving these inequalities is very difficult. Therefore, various methods were used by the researchers to find the value function for a differential game.

For example, in [16], the value function was guessed, and then it was shown that it satisfied the conditions of u-stability and v-stability [15]. Petrosyan [17] effectively used the method of the pursuit center to solve optimal pursuit problems. To prove the optimality of strategies, in the paper by Ibragimov [18], the solution of a differential game in half space was used. Another was proposed by Jang and Tomlin [19].

The algorithm proposed in [8] for the multiplayer differential game, which is based on the Apollonius circle, allows the pursuers to optimize the allocation of resources and ensures the capture of the evader for the minimum time.

The paper by Samatov et al. [20] deals with a differential game of optimal pursuit of one pursuer and one evader with the Grönwall-type constraints on the controls for players. In that paper, the optimal strategies of players were constructed and an optimal pursuit time was found.

In the present paper, we consider a differential game of the optimal pursuit of two pursuers and one evader when the control functions of players were subjected to Grönwall-type constraints. We found the optimal pursuit time in terms of reachability sets and constructed optimal strategies of players. To prove the main theorem, we considered an auxiliary differential game in a half-plane.

2. Statement of Problem

Let the dynamics of two pursuers (x_1, x_2) and one evader (y) be described in $\mathbb{R}^2$ by the following differential equations:

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(0) = x_{i0},\ i = 1,2, \\ &\dot{y} = v, \quad\ \ y(0) = y_0, \end{aligned} \tag{1}$$

where $x_i, y, x_{i0}, y_0 \in \mathbb{R}^2$, u_i, and v stand for the control parameters of the i-th pursuer x_i, $i = 1,2$, and the evader y, respectively.

Definition 1. *Measurable functions $u_i(t) = (u_{i1}(t), u_{i2}(t))$ and $v(t) = (v_1(t), v_2(t))$, $t \geq 0$ that satisfy the following constraints*

$$|u_i(t)|^2 \leq {\rho_i}^2 + 2k\int_0^t |u_i(s)|^2 ds,\ \ t \geq 0, \tag{2}$$

$$|v(t)|^2 \leq \sigma^2 + 2k\int_0^t |v(s)|^2 ds,\ \ t \geq 0, \tag{3}$$

are called admissible controls of the pursuers (x_i, $i = 1,2$) and evader (y), respectively, where ρ_1, ρ_2, σ, and k are the given positive numbers.

We let $\mathbb{U}_1$, $\mathbb{U}_2$, and $\mathbb{V}$ denote the set of all admissible controls of the pursuers (x_1, x_2) and evader (y), respectively.

The trajectories of the pursuers and the evader corresponding to admissible controls $u_i(\cdot) \in \mathbb{U}_i$ and $v(\cdot) \in \mathbb{V}$ are defined by the following equations

$$x_i(t) = x_{i0} + \int_0^t u_i(s)ds,\ i = 1,2,\ \ y(t) = y_0 + \int_0^t v(s)ds,$$

respectively. We need the following statement.

Lemma 1 ([21])**.** *If, for the positive numbers ρ and k,*

$$|\omega(t)|^2 \leq \rho^2 + 2k\int_0^t |\omega(s)|^2 ds,$$

then $|\omega(t)| \le \rho e^{kt}$, *where* $\omega(t)$, $t \ge 0$, *are measurable functions.*

By Lemma 1, for the admissible controls $u_i(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, we have

$$|u_i(t)| \le \rho_i e^{kt}, \;\; |v(t)| \le \sigma e^{kt}, \;\; t \ge 0. \tag{4}$$

It should be noted that (4) does not imply (2) and (3). It is not difficult to verify that if

$$|u_i(t)| = \rho_i e^{kt}, \;\; |v(t)| = \sigma e^{kt}, \;\; t \ge 0, \tag{5}$$

then Equations (2) and (3) are satisfied, respectively.

Next, we give definitions for the optimal strategies of players and optimal pursuit time.

2.1. Guaranteed Pursuit Time

Let $H(x,r)$ (respectively, $S(x,r)$) denote the ball (sphere) of radius r, centered at x, and let O be the origin.

Definition 2. *We call the function*

$$U_i(x_{i0}, y_0, t, v), \;\; U_i : \mathbb{R}^2 \times \mathbb{R}^2 \times [0,\infty) \times H(O, \sigma e^{kt}) \to H(O, \rho_i e^{kt}), \;\; i \in \{1,2\},$$

strategy of the pursuer x_i, *if for any* $v(\cdot) \in \mathbb{V}$ *and for* $u_i = U_i(x_{i0}, y_0, t, v(t))$, *the initial value problem* (1) *has a unique solution* $(x_i(t), y(t))$, *and*

$$|U_i(x_{i0}, y_0, t, v(t))|^2 \le \rho_i^2 + 2k \int_0^t |U_i(x_{i0}, y_0, s, v(s))|^2 ds, \; t \ge 0,$$

In other words, the pursuer x_i uses information about the initial states x_{i0}, y_0, and the value of the control parameter $v(t)$ at the current time t.

Definition 3. *We say that the strategies* $U_i = U_i(x_{i0}, y_0, t, v(t))$, $i = 1,2$, *ensure the completion of the game for the time* $T(U_1, U_2)$ *if, for any* $v(\cdot) \in \mathbb{V}$, *we have* $x_i(\tau) = y(\tau)$ *for some* $i \in \{1,2\}$ *and* $\tau \in [0, T(U_1, U_2)]$, *where* $(x_1(t), x_2(t), y(t))$ *is the solution of the initial value problem* (1) *with* $u_i = U_i(x_{i0}, y_0, t, v(t))$, $i = 1,2$.

We call the number $T(U_1, U_2)$ a guaranteed pursuit time corresponding to the strategies U_1, U_2. It should be noted that any time T', $T' \ge T(U_1, U_2)$ is also a guaranteed pursuit time corresponding to the same strategies U_1, U_2. Let $T^*(U_1, U_2)$ denote the greatest lower bound of the numbers $T(U_1, U_2)$ corresponding to the strategies U_1, U_2.

The pursuers attempt to minimize $T^*(U_1, U_2)$ by choosing their strategies U_1, U_2, and the evader attempts to maximize $T^*(U_1, U_2)$ by choosing $v(\cdot) \in \mathbb{V}$. If, for some strategies U_{10}, U_{20} of pursuers, $\inf\limits_{U_1, U_2} T^*(U_1, U_2) = T^*(U_{10}, U_{20})$, then U_{10}, U_{20} are called optimal strategies of pursuers and the number $T^*(U_{10}, U_{20})$ is called a guaranteed pursuit time in the game.

2.2. Guaranteed Evasion Time

Definition 4. *A continuous function*

$$V(x_{10}, x_{20}, y_0, t, x_1, x_2, y), \;\; V : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times [0,\infty) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \; H(O, \sigma e^{kt})$$

is called a strategy of the evader if, for any $u_i(\cdot) \in \mathbb{U}_i$, $i = 1,2$, *and for* $v = V(x_{10}, x_{20}, y_0, t, x_1, x_2, y)$, *the initial value problem* (1) *has a unique solution* $(x_1(t), x_2(t), y(t))$ *and along this solution*

$$|V(x_{10}, x_{20}, y_0, t, x_1(t), x_2(t), y(t))|^2 \le \sigma^2 + 2k \int_0^t |V(x_{10}, x_{20}, y_0, s, x_1(s), x_2(s), y(s))|^2 ds, \; t \ge 0.$$

Definition 5. *We say that strategy V guarantees the evasion on the interval of time $[0, T(V))$ if, for any $u_i(\cdot) \in \mathbb{U}_i$, $i = 1, 2$, we have $x_i(t) \neq y(t)$, for all $i = 1, 2$ and $t \in [0, T(V))$. We let $T_*(V)$ denote the least upper bound of the numbers $T(V)$ corresponding to strategy V. Moreover, we call the number $T_*(V)$ a guaranteed evasion time corresponding to the strategy V.*

The evader attempts to maximize the number $T_*(V)$ by choosing the strategy V, and the pursuers attempt to minimize the number $T_*(V)$ by choosing the controls $u_i(\cdot) \in \mathbb{U}_i$, $i = 1, 2$.

Definition 6. *If for some strategy V_0 of the evader $\sup\limits_V T_*(V) = T_*(V_0)$, then V_0 is called the optimal strategy of the evader, and the number $T_*(V_0)$ is called a guaranteed evasion time in the game. If $T^*(U_{10}, U_{20}) = T_*(V_0)$, then this number is called the optimal pursuit time in the game.*

Problem 1. *Construct the optimal strategies of the pursuers U_{10}, U_{20}, and evader V_0, and find the optimal pursuit time in game (1).*

This is an example of a quote.

3. Main Result

Let

$$R_i(t) = \rho_i \int\limits_0^t e^{ks} ds,\ i = 1, 2,\ r(t) = \sigma \int\limits_0^t e^{ks} ds.$$

It is not difficult to verify that the set of all points reachable by the pursuer x_i (respectively, by the evader y) from the point x_{i0} (y_0) at $t = 0$ to the time t is the ball $H(x_{i0}, R_i(t))$ (respectively, $H(y_0, r(t))$).

In this section, we prove the following main result of the paper.

Theorem 1. *The number*

$$\theta = \min\{t \geq 0 \mid H(y_0, r(t)) \subset H(x_{10}, R_1(t)) \cup H(x_{20}, R_2(t))\} \tag{6}$$

is the optimal pursuit time in game (1).

A Differential Game in the Half-Plane

To prove the theorem, we consider an auxiliary differential game of one pursuer x and one evader y whose dynamics are described by the following equations:

$$\begin{aligned} &\dot{x} = u, \quad x(0) = x_0 = (x_{10}, x_{20}), \quad |u(t)|^2 \leq \rho^2 + 2k\int\limits_0^t |u(s)|^2 ds,\ \ t \geq 0, \\ &\dot{y} = v, \quad y(0) = y_0 = (y_{10}, y_{20}), \quad |v(t)|^2 \leq \sigma^2 + 2k\int\limits_0^t |v(s)|^2 ds,\ \ t \geq 0. \end{aligned} \tag{7}$$

It is assumed that $\rho > \sigma$. Let $R(t) = \rho \int\limits_0^t e^{ks} ds$ and let the circumferences $S(x_0, R(\theta_0))$ and $S(y_0, r(\theta_0))$ intersect for some $\theta_0 > 0$. We pass a straight line Γ perpendicular to the vector $y_0 - x_0$ through the intersection points of these circumferences (see Figure 1). We denote the half-plane bounded by Γ and containing the point x_0 by X. Note that the half-plane X may not contain the point y_0. The evader must be in the half-plane X at the time θ_0 and the pursuer attempts to realize the equation $x(t) = y(t)$ as early as possible.

Lemma 2. *If the position of the evader is $y(\theta_0) \in X$, then the strategy*

$$u = v - (v, e)e + e\sqrt{(\rho^2 - \sigma^2)e^{2kt} + (v, e)^2}, \;\; e = \frac{y_0 - x_0}{|y_0 - x_0|}. \tag{8}$$

of the pursuer guarantees the equation $x(\tau) = y(\tau)$ at $0 < \tau \leq \theta_0$.

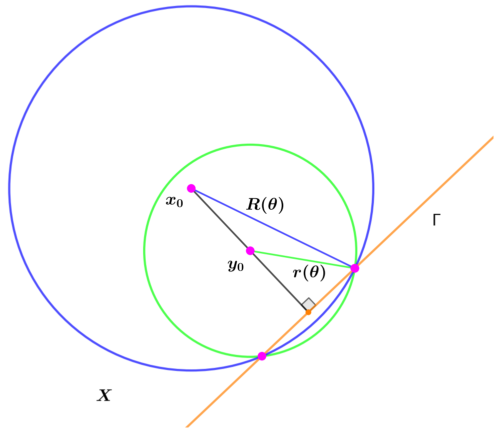

Figure 1. Game mechanism design in the half-plane X.

Proof. Without any loss of generality, we assume that X is the upper-half-plane bounded by the x-axis. Then, clearly, $x_{10} = y_{10}$. It is not difficult to show that

$$\theta_0 = \frac{1}{k}\ln\left(1 + k\sqrt{\frac{x_{20}^2 - y_{20}^2}{\rho^2 - \sigma^2}}\right).$$

Moreover, for strategy (8), it can be easily verified that $|u(t)|^2 = \left(\rho^2 - \sigma^2\right)e^{2kt} + |v(t)|^2$, and so

$$|u(t)|^2 \leq \rho^2 + 2k\int_0^t |u(s)|^2 ds, \; t \geq 0,$$

meaning that strategy (8) is admissible.

Next, since $y_0 - x_0$ is perpendicular to the x-axis, $e = \frac{y_0 - x_0}{|y_0 - x_0|} = (0, -1)$, and so the strategy (8) takes the form

$$u_1 = v_1, \;\; u_2 = -\sqrt{(\rho^2 - \sigma^2)e^{2kt} + v_2^2}. \tag{9}$$

The condition $y(\theta_0) \in X$ can be written as follows

$$\int_0^{\theta_0} v_2(s)ds \geq -y_{20}. \tag{10}$$

By (9) $u_1(t) = v_1(t)$, $t \geq 0$ and, hence, $x_1(t) = y_1(t)$, $t \geq 0$. Therefore, it suffices to show that $x_2(\tau) = y_2(\tau)$ at some $0 < \tau \leq \theta_0$. To this end, we consider the following vector function $f(t) = \left(\sqrt{\rho^2 - \sigma^2}e^{kt}, v_2(t)\right)$, $t \geq 0$. Then

$$\begin{aligned} x_2(\theta_0) - y_2(\theta_0) &= x_{20} - y_{20} - \int\limits_0^{\theta_0} \sqrt{(\rho^2 - \sigma^2)e^{2ks} + v_2^2(s)}ds - \int\limits_0^{\theta_0} v_2(s)ds \\ &= x_{20} - y_{20} - \int\limits_0^{\theta_0} |f(s)|ds - \int\limits_0^{\theta_0} v_2(s)ds \end{aligned}$$

Since $\int\limits_0^{\theta_0} |f(s)|ds \geq \left|\int\limits_0^{\theta_0} f(s)ds\right|$, then

$$\begin{aligned} x_2(\theta_0) - y_2(\theta_0) &\leq x_{20} - y_{20} - \left|\int\limits_0^{\theta_0} f(s)ds\right| - \int\limits_0^{\theta_0} v_2(s)ds \\ &= x_{20} - y_{20} - \left|\left(\frac{\sqrt{\rho^2 - \sigma^2}}{k}(e^{k\theta_0} - 1), \int\limits_0^{\theta_0} v_2(s)ds\right)\right| - \int\limits_0^{\theta_0} v_2(s)ds \\ &= x_{20} - y_{20} - \left(\frac{\rho^2 - \sigma^2}{k^2}(e^{k\theta_0} - 1)^2 + \left(\int\limits_0^{\theta_0} v_2(s)ds\right)^2\right)^{1/2} - \int\limits_0^{\theta_0} v_2(s)ds. \end{aligned} \quad (11)$$

Letting $\int_0^{\theta_0} v_2(s)ds = \xi$ on the right-hand side of the last equation we consider the following function:

$$f(\xi) = x_{20} - y_{20} - \sqrt{\frac{\rho^2 - \sigma^2}{k^2}(e^{k\theta_0} - 1)^2 + \xi^2} - \xi,$$

whereby (10) $\xi \geq -y_{20}$. For the derivative of $f(\xi)$ we have

$$f'(\xi) = -\frac{\xi}{\sqrt{\frac{\rho^2-\sigma^2}{k^2}(e^{k\theta_0} - 1)^2 + \xi^2}} - 1 < 0.$$

Hence, the function $f(\xi)$ is decreasing and so it takes its greatest value at $\xi = \int\limits_0^{\theta_0} v_2(s)ds = -y_{20}$. Therefore, using this, we obtain from (11)

$$\begin{aligned} x_2(\theta_0) - y_2(\theta_0) &\leq x_{20} - y_{20} - \left(\frac{\rho^2 - \sigma^2}{k^2}\left(e^{k\theta_0} - 1\right)^2 + y_{20}^2\right)^{1/2} + y_{20} \\ &= x_{20} - \left(\frac{\rho^2 - \sigma^2}{k^2}\left(e^{k\theta_0} - 1\right)^2 + y_{20}^2\right)^{1/2} = 0 \end{aligned}$$

Combining this inequality with $x_2(0) - y_2(0) > 0$ and the fact that the $x_2(t) - y_2(t)$ is continuous, we conclude that $x_2(\tau) - y_2(\tau) = 0$ at some $0 < \tau < \theta_0$.

Recalling that $x_1(t) = y_1(t)$, $t \geq 0$, which implies in particular $x_1(\tau) = y_1(\tau)$, we obtain $x(\tau) = y(\tau)$. Consequently, θ_0 is a guaranteed pursuit time in game (8). The proof of Lemma 2 is complete. □

Next, we prove Theorem 1.

Proof. We show first that θ is a guaranteed pursuit time in game (1). To this end, we let the pursuers apply the following strategies:

$$u_i = v - (v, e_i)e_i + e_i\sqrt{(\rho_i^2 - \sigma^2)e^{2kt} + (v, e_i)^2}, \ e_i = \frac{y_0 - x_{i0}}{|y_0 - x_{i0}|}, \ i = 1, 2. \quad (12)$$

It can be easily verified that

$$|u_i(t)|^2 \leq \rho_i^2 + 2k\int_0^t |u_i(s)|^2 ds, \ t \geq 0, \ i = 1, 2,$$

and so strategies (9) are admissible.

For the time θ, we consider the following two cases:

Case 1. $H(y_0, r(\theta)) \subset H(x_{i_0 0}, R_{i_0}(\theta))$ for some $i_0 \in \{1, 2\}$ (see Figure 2).

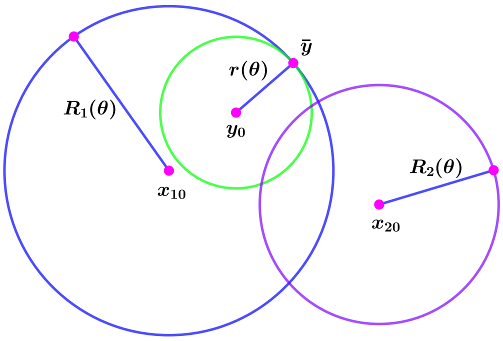

Figure 2. Case 1: $H(y_0, r(\theta)) \subset H(x_{10}, R_1(\theta))$.

In this case, strategies (12) guarantee the completion of the pursuit for the time

$$\theta = \frac{1}{k}\ln\left(1 + k\frac{|x_{i_0 0} - y_0|}{\rho - \sigma}\right).$$

More precisely, only the pursuer x_{i_0} can complete the game for the time θ. Figure 2 illustrates Case 1, where $i_0 = 1$ and only the first pursuer x_1 can complete the game by the time θ [20].

Case 2. (see Figure 3).

$$H(y_0, r(\theta)) \not\subseteq H(x_{i0}, R_i(\theta)), \ i = 1, 2, \ H(y_0, r(\theta)) \subset H(x_{10}, R_1(\theta) \cup H(x_{20}, R_2(\theta)). \quad (13)$$

In Case 2, by the definition of θ, we have the following relation

$$H(y_0, r(t)) \not\subset H(x_{10}, R_1(t)) \cup H(x_{20}, R_2(t)), \ 0 < t < \theta. \quad (14)$$

We show that θ is a guaranteed pursuit time in game (1). It can be shown that, for some $\bar{y} \in S(y_0, r(\theta))$, we have $\bar{y} \in S(x_{10}, R_1(\theta)) \cap S(x_{20}, R_2(\theta))$ and $\bar{y} \notin H(x_{10}, R_1(t)) \cup H(x_{20}, R_2(t)), 0 < t < \theta$. We pass straight lines Γ_i from the point $\bar{y}$ perpendicular to vectors $y_0 - x_{i0}$, $i = 1, 2$. We denote the half-plane bounded by the straight line Γ_i that contains the point x_{i0} by X_i, $i = 1, 2$.

One can show similar to Assertion 4 (Appendix, [22]) that $H(y_0, r(\theta)) \subset X_1 \cup X_2$. Combining this inclusion with the inclusion $y(\theta) \in H(y_0, r(\theta))$, we have either $y(\theta) \in X_1$ or $y(\theta) \in X_2$. If $y(\theta) \in X_1$, then by Lemma 2, we obtain $x_1(\tau_1) = y(\tau_1)$ at some $0 \leq \tau_1 \leq \theta$; if $y(\theta) \in X_2$, then by Lemma 2 $x_2(\tau_2) = y(\tau_2)$ at some $0 \leq \tau_2 \leq \theta$. This observation shows that θ is a guaranteed pursuit time in game (1).

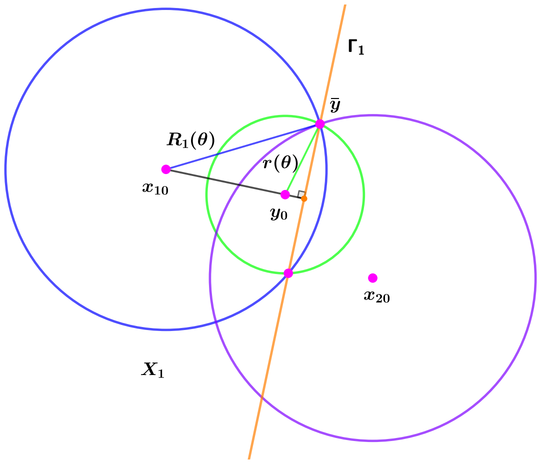

Figure 3. Half space X_1 bounded by Γ_1.

Next, we show that θ is a guaranteed evasion time in game (1) in both Cases 1 and 2. We let the evader apply the following strategy

$$v(t) = \frac{\bar{y} - y_0}{|\bar{y} - y_0|}\sigma e^{kt}, \quad t \geq 0, \tag{15}$$

where $\bar{y}$ is defined as above in Case 2, and $\bar{y} \in S(y_0, r(\theta)) \cap S(x_{10}, R_1(\theta))$ in Case 1. Strategy (15) is admissible. Indeed, since

$$|v(t)| = \left|\frac{\bar{y} - y_0}{|\bar{y} - y_0|}\sigma e^{kt}\right| = \sigma e^{kt},$$

and so it satisfies condition (3). Moreover, since $|\bar{y} - y_0| = \int_0^\theta \sigma e^{ks} ds$, we have

$$y(\theta) = y_0 + \int_0^\theta v(s)ds = y_0 + \int_0^\theta \frac{\bar{y} - y_0}{|\bar{y} - y_0|}\sigma e^{ks} ds = y_0 + \bar{y} - y_0 = \bar{y},$$

where the evader reaches the point $\bar{y}$ at the time θ.

What is left is to show that $x_i(t) \neq y(t)$ for all $0 \leq t < \theta$ and $i = 1, 2$. The following reasoning works for the definitions of $\bar{y}$ in both Cases 1 and 2. We assume the contrary, let $x_{i_0}(\tau) = y(\tau)$ at $\tau < \theta$ and $i_0 \in \{1, 2\}$. For definiteness, we assume that $i_0 = 1$; that is, $x_1(\tau) = y(\tau)$. Then using the equation $y(\theta) = \bar{y}$, we have

$$\begin{aligned} |\bar{y} - x_{10}| &\leq |\bar{y} - x_1(\tau)| + |x_1(\tau) - x_{10}| = |\bar{y} - y(\tau)| + \rho_1 \int_0^\tau e^{ks} ds \\ &= \sigma \int_\tau^\theta e^{ks} ds + \rho_1 \int_0^\tau e^{ks} ds < \rho_1 \int_0^\theta e^{ks} ds = R_1(\theta). \end{aligned}$$

This means $\bar{y}$ belongs to the interior of the ball $H(x_{10}, R_1(\theta))$, and so $\bar{y} \in H(x_{10}, R_1(t_1))$ for some $t_1 < \theta$. This contradicts condition (14). Hence, $x_i(t) \neq y(t)$ for all $0 \leq t < \theta$ and $i = 1, 2$, meaning that θ is a guaranteed evasion time. Thus, θ is the optimal pursuit time. The proof of the theorem is complete. □

4. Conclusions

We studied a simple motion differential game of two pursuers and one evader in $\mathbb{R}^2$. The control functions of players are subjected to the Grönwall-type constraints. We found an equation for the optimal pursuit time and constructed the optimal strategies of players. The optimal strategies of pursuers are defined by Equation (12) and the optimal strategy of the evader is defined by (15). The optimal strategy of the evader (15) satisfies the equation $|v(t)| = \sigma e^{kt}$. Moreover, according to the Grönwall-type constraint (3), we have $|v(t)| \leq \sigma e^{kt}$. Therefore, we can say that the evader moves at its maximal speed. The equation $|v(t)| = \sigma e^{kt}$ and (12) imply that $|u_i(t)| = \rho_i e^{kt}$, $i = 1, 2$, meaning that the pursuers move with maximal speed as well.

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