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On a Linear Differential Game in the Hilbert Space ℓ^2

Marks Ruziboev [1,2,†], Gafurjan Ibragimov [3,4,†], Khudoyor Mamayusupov [5,†], Adkham Khaitmetov [6,†] and Bruno Antonio Pansera [7,*,†]

1 School of Engineering, Central Asian University, Milliy Bog St 264, Tashkent 111221, Uzbekistan; m.ruziboyev@centralasian.uz
2 Faculty of Mathematics, University of Vienna, Oskar-Morgenstern Platz 1, 1090 Vienna, Austria
3 V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent 100174, Uzbekistan; gofurjon.ibragimov@tsue.uz
4 Department of Applied Mathematics, Tashkent State University of Economics, Tashkent 100006, Uzbekistan
5 Department of Mathematics, New Uzbekistan University, Mustaqillik ave.54, Tashkent 100007, Uzbekistan; mamayusupov@mathinst.uz
6 Department of Mathematics and Information Technologies, Fiscal Institute under the Tax Committee, Tashkent 100173, Uzbekistan; adhamhaitmetov@gmail.com
7 Department of Law, Economics and Human Sciences, University Mediterranea of Reggio Calabria, 89124 Reggio Calabria, Italy
* Correspondence: bruno.pansera@unirc.it
† These authors contributed equally to this work.

Abstract: Two player pursuit evasion differential game and time optimal zero control problem in ℓ^2 are considered. Optimal control for the corresponding zero control problem is found. A strategy for the pursuer that guarantees the solution for the pursuit problem is constructed.

Keywords: infinite system; control function; pursuer; evader; strategy

MSC: 49N05; 49N75; 91A23; 91A24; 93C15

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1. Introduction and Statement of the Problem

For every $i \in N$, $d_i \times d_i$ matrices Ai are given with $2 \le d_i \le d$, where $d \ge 2$ is a fixed integer. We consider a differential game described by the following countable system of differential equations

$$\dot{x}_i = A_i x_i + u_i - v_i, x_i(0) = x_{i0} \in \mathbb{R}^{d_i}, i = 1, 2, \dots, \tag{1}$$

where u_i is a control parameter of the pursuer, v_i is a control parameter of the evader, both assumed to be locally integrable functions with values in $\mathbb{R}^{d_i}$ for all $i \in \mathbb{N}$ satisfying certain constraints (see Definition 2). For convenience, we form column vector $x = (x_1^*, x_2^*, \dots)^*$, where x_i^* is the transpose of $x_i \in \mathbb{R}^{d_i}$, for $i = 1, 2, \dots$ (Actually, we are just concatenating vectors $x_1, x_2, \dots$ one below another to obtain an infinite vector. Below we adopt this point of view, which simplifies the notation considerably). Assume that $x_0 = (x_{10}, x_{20}, \dots,) \in \ell^2$, i.e., $\|x\|^2 = \sum_{i=0}^{\infty} \|x_{i0}\|^2 < +\infty$, where $\|x_{i0}\|$ is the Euclidean norm of $x_{i0} \in \mathbb{R}^{d_i}$. Let $A = \mathrm{diag}(A_1, A_2, \dots)$ be an operator on $\mathbb{R}^\infty$ whose action is defined as $Ax = \mathrm{diag}([A_1x_1]^*, [A_2x_2]^*, \dots)$ for $x \in \mathbb{R}^\infty$. We would like to define e^{tA} for suitable classes of matrices A_i. The problem here is that A is not necessarily defined on ℓ^2 or even on ℓ^∞ (i.e., Ax is not necessarily in ℓ^∞ for $x \in \ell^\infty$ since $|A_i x_i|$ may go to infinity as $n \to \infty$). Therefore, we have to justify the existence of solutions to the above Cauchy problem for initial points in ℓ^2.

We consider a pursuit-evasion differential game which consists of two separate problems as usual. The pursuit game can be completed in time $T > 0$ provided there exists a control function of the pursuer $u : \mathbb{R} \to \ell^2$ such that for any control of $v : \mathbb{R} \to \ell^2$ the solution of $x : \mathbb{R} \to \ell^2$ of (2) for any x_0 satisfies $x(T) = 0$. In this case, T is called the guaranteed pursuit time. Below we state precise conditions that are imposed on u, v.

Motivation for the setup comes from control problems for evolutionary PDEs, where using suitable decomposition of the control problem (see, for example, [1–5]) x_i would be a Fourier coefficient of an unknown function, while u_i and v_i would be that of control parameters. Also, the setup is of independent interest as a controlled system in a Banach space (for works in this spirit see for example [6–13]). Differential games for infinite dimensional systems are also well studied, for example, when the evolution of the system is governed by parabolic equations pursuit-evasion problems are considered in [14–16], where the problem for the partial differential equations is reduced to an infinite system of ordinary differential equations. Pursuit and evasion games with many players considered in [17–20].

For us, the system (1) is a toy model of a system consisting of countably many point masses moving in $\mathbb{R}^n$ with simple motions which are not interacting with each other. It is the first step in understanding the system of weakly interacting controllable particles in a more natural setting, e.g., for considering control problems for systems considered in [21].

2. Main Results

As we pointed out earlier, we have to justify the existence of solutions of the following Cauchy problem

$$\dot{x}_i = A_i x_i + w_i, x_i(0) = x_{i0} \in \mathbb{R}^{d_i}, i = 1, 2, \dots, \tag{2}$$

with $w_i : \mathbb{R} \to \mathbb{R}^{d_i}$ locally integrable. We look for solutions of (2) from the space of continuous functions $C([0, T]; \ell^2)$ for some $T > 0$, such that the coordinates $x_i(\cdot)$ of $x : [0, T] \to \ell^2$ are almost everywhere differentiable.

Definition 1. *We say that a family of matrices $\{A_i\}_{i\in\mathbb{N}}$ is uniformly normalizable if there exists a family $\{P_i\}_{i\in\mathbb{N}}$ of non-singular matrices and a constant $C \geq 1$ such that $\|P_i\| \cdot \|P_i^{-1}\| \leq C$ and $P_i A_i P_i^{-1}$ is a matrix in the Jordan normal form for all $i \in \mathbb{N}$.*

Notice that there exist uniformly normalizable families of matrices, i.e., if all elements are already in Jordan's normal form, then we may take $P_i = \mathrm{Id}$ for all $i \in \mathbb{N}$. On the other hand, one can construct families, which aren't uniformly normalizable. In this work we will assume that the family of matrices in (2) and (1) are uniformly normalizable and control parameters of the players satisfy the following constraint.

Definition 2. *Fix $\theta > 0$ and let $B(\theta)$ be the set of all functions $w(\cdot) = (w_1(\cdot), w_2(\cdot), \dots)$, $w : [0, T] \to \ell^2$, with measurable coordinates $w_i(\cdot) \in \mathbb{R}^{d_i}$, $0 \leq t \leq T$, $i = 1, 2, \dots$, that satisfy the constraint*

$$\sum_{i=1}^{\infty} \int_0^T \|w_i(s)\|^2 ds \leq \theta^2. \tag{3}$$

$B(\theta)$ is called the set of admissible control functions.

We have the following

Theorem 1. *Let $\{A_i\}$ be a family of uniformly normalizable matrices. If the real parts of eigenvalues of matrices $A_1, A_2, ..$ are negative, then for any $w \in B(\theta)$, $\theta > 0$ system (2) has a unique solution for any $z_0 \in \ell_2$. Moreover, the corresponding components of the solution $x(t) = (x_1(t), x_2(t), \dots)$ are given by*

$$x_i(t) = e^{tA_i} x_{i0} + \int_0^t e^{(t-s)A_i} w_i(s) ds, \quad i \in \mathbb{N}. \tag{4}$$

Definition 3. *System* (2) *is called* ***globally asymptotically stable*** *if* $\lim_{t\to+\infty} x(t) = 0$ *for a solution* $x(t)$ *of* (2) *with any initial condition* $x_0 \in \ell^2$ *and* $w_i \equiv 0$ *for all* $i \in \mathbb{N}$. *Further, system* (2) *is* ***null-controllable*** *from* $x_0 \in \ell^2$ *if there exists an admissible control* $u \in B(\theta)$ *and* $T = T(u) \in \mathbb{R}_+$ *such that the solution of* (2) *starting from* x_0 *satisfies* $x(T) = 0$. *We say that system* (2) *is null-controllable in large if it is null-controllable from any* $x_0 \in \ell^2$. *Also,* $\inf_{u\in B(\theta)} T(u)$ *is called* ***optimal time of translation*** *and* $u \in B(\theta)$ *realizing the minimum is called* ***time optimal control***.

Theorem 2. *Under the assumptions of Theorem 1 system* (2) *is globally asymptotically stable and null controllable in large. Time optimal control exists and can be constructed explicitly.*

Notice that the explicit form of the time optimal control requires some preliminary contraction and it is given in Section 4.

Further, we consider a pursuit-evasion differential game (1). Fix $\rho, \sigma > 0$. A function $u(\cdot) \in B(\rho)$ ($v(\cdot) \in B(\rho)$) is called an admissible control of the pursuer (evader).

Definition 4. *A function* $u : [0, T] \times \ell^2 \to \ell^2$ *with coordinates* $u_k(t) = v_k(t) + \omega_k(t)$, $\omega \in B(\rho - \sigma)$, *which is an admissible control of the pursuer for every* $v \in \ell^2$ *is called a strategy of the pursuer.*

Theorem 3. *Suppose that* $\rho > \sigma$ *and the assumptions of Theorem 1 are satisfied. Then, for any admissible control of the evader* v *there exists a strategy of the pursuer* u *and* $\vartheta_1 > 0$ *such that the solution of* (1) *satisfies* $z(\tau) = 0$ *for some* $0 \le \tau \le \vartheta_1$, *i.e., the game* (1) *can be completed within time* ϑ_1.

3. Existence and Uniqueness

Notice that if we define $x(t) = (x_1(t), x_2(t), \dots)$ by setting every component x_i as in (4), then $x(t)$ satisfies the equation and initial conditions in (2). This also implies the uniqueness of a solution. Thus it is sufficient to prove that $x(\cdot) \in C([0, T], \ell^2)$ for any $T > 0$. Now we will show that $x(t) \in \ell^2$ for all $t \ge 0$.

3.1. Estimate for $\|e^{tA}\|$

Since $A = \{A_1, A_2, \}$we have $e^{tA} = \{e^{tA_1}, e^{tA_2}, \dots\}$. Recall that for every i there exists a non-singular transformation $P_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$ such that $A_i = P_i J_i P_i^{-1}$, where J_i is the Jordan normal form of A_i.

Thus,
$$\|e^{tA_i}\| \le \|P_i\| \cdot \|e^{tJ_i}\| \cdot \|P_i^{-1}\| \le C\|e^{tJ_i}\|.$$

By the assumption all eigenvalues $\lambda_{i1}, \dots, \lambda_{id_i}$ of matrices have negative real part, letting $2\alpha_i = -\max_{1\le j\le d_i} Re(\lambda_{jd_i}) > 0$, we can find a polynomial $Q_i(t)$ of degree at most $d_i \le d$ (see §13 in [22]) such that

$$\|e^{tA_i}\| \le C|Q_i(t)|e^{-2t\alpha_i} \le \bar{C}e^{-t\alpha_i}. \tag{5}$$

Thus, for any $x \in \ell^2$ and $t \in [0, +\infty)$ we have

$$\|e^{tA}x\|^2 = \sum_{i=1}^{\infty} \|e^{tA_i}x_i\|^2 \le \sum_{i=1}^{\infty} \|e^{tA_i}\|^2\|x_i\|^2 \le \bar{C}^2\|x\|^2. \tag{6}$$

This implies that $e^{tA} : \ell^2 \to \ell^2$ is a bounded linear operator for every $t \in [0, +\infty)$. Also, it is standard to check that e^{tA} is a semigroup, i.e., $e^{(t+s)A} = e^{tA}e^{sA}$.

3.2. *Proof of Theorem 1*

We start by showing that $x(t) \in \ell^2$ for all $x_0 \in \ell^2$ and for all $t \in [0, T]$. Indeed,

$$\begin{aligned} \|x(t)\|^2 &\leq \sum_{i=1}^{\infty} \|e^{tA_i} x_{i0} + \int_0^t e^{(t-s)A_i} w_i(s) ds\|^2 \\ &\leq 2\|e^{tA} x_0\|^2 + 2\sum_{i=1}^{\infty} \| \int_0^t e^{(t-s)A_i} w_i(s) ds\|^2. \end{aligned} \tag{7}$$

Let us estimate the last term of the above inequality. We have

$$\| \int_0^t e^{(t-s)A_i} w_i(s) ds\|^2 \leq \bar{C}^2 \left(\int_0^T \|w_i(s)\| ds \right)^2 \leq \bar{C}^2 T \left(\int_0^T \|w_i(s)\|^2 ds \right) \tag{8}$$

where in the last step we have used the Cauchy–Schwartz inequality for 1 against $\|w_i(s)\|$. First substituting (8) into (7) and then using (6), (5) and constraint (3) we obtain

$$\|x(t)\|^2 \leq 2\bar{C}^2 \|x_0\|^2 + 2\bar{C}^2 T\theta^2,$$

which proves the claim.

We are now ready to prove that $x(t) = (x_1(t), x_2(t), \dots) \in \mathcal{C}([0, T], \ell^2)$ for any $T > 0$. Since $\|x(t)\|^2$ is bounded by a constant independent of t, for any $\varepsilon > 0$ there exists $N = N(\varepsilon, t, t_0) \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} \|x_i(t) - x_i(t_0)\|^2 \leq \frac{\varepsilon}{2}. \tag{9}$$

For any $t, t_0 \in [0, T]$ with $t_0 \leq t$ we have

$$\begin{aligned} &\sum_{i=1}^{N} \|x_i(t) - x_i(t_0)\|^2 \leq \sum_{i=1}^{N} \|e^{tA_i} - e^{t_0 A_i}\|^2 \cdot \|x_{i0}\|^2 + \\ &+ \sum_{i=1}^{N} \left\| e^{tA_i} \int_0^t e^{-sA_i} w_i(s) ds - e^{t_0 A_i} \int_0^{t_0} e^{-sA_i} w_i(s) ds \right\|^2 = (I) + (II). \end{aligned}$$

We start by estimating (I). Notice that

$$\|e^{tA_i} - e^{t_0 A_i}\| \leq \|P_i\| \cdot \|P_i^{-1}\| \cdot \|e^{tJ_i} - e^{t_0 J_i}\| \leq \bar{C} \|e^{t_0 J_i}\| \cdot |Q_i(t - t_0)| e^{-|t-t_0|\alpha_i}. \tag{10}$$

Recall that $Q_i(t - t_0)$ is a polynomial of degree at most d with coefficients depending only on the dimension d_i of J_i. Thus, we can find δ independent of i such that

$$\bar{C}^3 |Q_i(t - t_0)| \cdot \|x_0\|^2 < \frac{\varepsilon}{4} \text{ and } \bar{C}^3 |Q_i(t - t_0)| < \frac{\varepsilon}{8T} \tag{11}$$

for all $t \in (t_0 - \delta, t_0 + \delta)$. Thus, by (5) and (11) we have the following estimate for (I):

$$\begin{aligned} \sum_{i=0}^{N} \|e^{tA_i} - e^{t_0 A_i}\|^2 \cdot \|x_0\|^2 &\leq \sum_{i=0}^{N} \|e^{tA_i}\|^2 \cdot \|e^{(t-t_0)A_i} - \|^2 \cdot \|x_0\|^2 \\ &\leq \sum_{i=0}^{N} \bar{C}^3 |Q_i(t - t_0)| e^{-|t-t_0|\alpha_i} \cdot \|x_0\|^2 < \frac{\varepsilon}{4}. \end{aligned} \tag{12}$$

For (II) we write

$$\sum_{i=1}^{N}\left\|(e^{tA_i}-e^{t_0A_i})\int_0^t e^{-sA_i}w_i(s)ds - e^{t_0A_i}\left(\int_{t_0}^t e^{-sA_i}w_i(s)ds\right)\right\|^2.$$

Thus, for every i every sum and of the above sum is bound by

$$\left(\int_0^t \|e^{(t-s)A_i}-e^{(t_0-s)A_i}\|\cdot\|w_i(s)\|ds+\int_{t_0}^t \|e^{(t_0-s)A_i}\|\cdot\|w_i(s)\|ds\right)^2.$$

Applying inequality (10) and (11) to the first summand and inequality (5) to the second we obtain

$$\sum_{i=1}^{N}\left(\frac{\varepsilon}{8T}\int_0^T 1_{[0,t]}\cdot\|w_i(s)\|ds+\int_0^T 1_{[t_0,t]}\cdot\bar{C}e^{-t\alpha_i}\cdot\|w_i(s)\|ds\right)^2,$$

which is bound by

$$\sum_{i=1}^{N}\left(\int_0^T\left(\frac{\varepsilon}{8T}1_{[0,t]}+1_{[t_0,t]}\cdot\bar{C}\right)\|w_i(s)\|ds\right)^2.$$

Now, using Cauchy–Schwartz inequality we obtain

$$\left(\frac{\varepsilon}{8}+|t_0-t|\cdot\bar{C}\right)^2\sum_{i=1}^{N}\int_0^T\|w_i(s)\|^2ds.$$

Since $|t-t_0|<\delta$ choosing $\delta>0$ sufficiently small and using (3) we bound the latter expression by $\frac{\varepsilon}{4}$ for all $\varepsilon<4\theta^2$. Therefore, we conclude $(II)<\frac{\varepsilon}{4}$. Combining this, estimate (12) and (9) implies that we obtain that $x(\cdot)\in\mathcal{C}([0,1],1)$, hence finishes the proof.

4. Proof of Theorem 2

4.1. Asymptotic Stability

We will show that $\|x(t)\|\to 0$ as $t\to\infty$. Since $x_0\in\ell^2$ for any ε there exists $N=N(\varepsilon)$ such that $\sum_{i=N+1}^{\infty}\|x_{i0}\|^2<\frac{\varepsilon}{2\bar{C}}$. By (4) and (5) we have

$$\begin{aligned}\|x(t)\|^2 &= \sum_{i=1}^{\infty}\|x_i(t)\|^2=\sum_{i=1}^{\infty}\|e^{tA_i}x_{i0}\|^2\\ &\le \sum_{i=1}^{N}\bar{C}e^{-t\alpha_i}\|x_{i0}\|^2+\bar{C}\sum_{i=N+1}^{\infty}\|x_{i0}\|^2.\end{aligned}$$

Letting $\alpha_{\min}=\min_{1\le i\le N}\alpha_i>0$, from the above inequality we obtain

$$\begin{aligned}\|x(t)\|^2 &\le \sum_{i=1}^{N}\bar{C}e^{-t\alpha_i}\|x_{i0}\|^2+\bar{C}\sum_{i=N}^{\infty}\|x_{i0}\|^2\\ &\le \bar{C}e^{-t\alpha_{\min}}\|x_0\|+\frac{\varepsilon}{2}.\end{aligned}$$

There exists t_ε such that $\bar{C}e^{-t\alpha_{\min}}\|x_0\|\le\frac{\varepsilon}{2}$ for all $t<t_\varepsilon$. This finishes the proof.

Notice that if $\alpha_{\inf}=\inf_{i\ge 1}\alpha_i>0$, then the system is exponentially stable. Since in this case we do not have to cut at N and can write

$$\|x(t)\|^2\le\sum_{i=1}^{\infty}\bar{C}e^{-t\alpha_i}\|x_{i0}\|^2\le\bar{C}e^{-t\alpha_{\inf}}\sum_{i=1}^{\infty}\|x_{i0}\|^2\le\bar{C}e^{-t\alpha_{\inf}}\|x_0\|^2.$$

4.2. Gramians

In this subsection, we prove null controllability of (1) and hence the proof of Theorem 2. Our approach relies on Gramian operators and observability inequalities. Set

$$W(\tau) = \int_0^\tau e^{-sA} \cdot e^{-sA^*} ds, \quad \tau \in \mathbb{R}$$

where A^* is the adjoint of A in ℓ^2. The definition implies

$$W(\tau) = \{W_1(\tau), W_2(\tau), W_3(\tau) \dots\}, \text{ with } W_i(\tau) = \int_0^\tau e^{-sA_i} \cdot e^{-sA_i^*} ds, i \geq 1.$$

Since A_i is a finite matrix for every i we have that $W_i(\tau)$ is a positive definite, symmetric and invertible operator (Notice, that $W(\tau)$ is not necessarily a bounded operator for fixed $\tau \in \mathbb{R}$.) for every i and $\tau \in \mathbb{R}$, i.e., $W_i^{-1}(\tau)$ exists and bounded. Define

$$W^{-1}(\tau) = \{W_1^{-1}(\tau), W_2^{-1}(\tau), W_3^{-1}(\tau) \dots\}$$

It is clear that $W_1^{-1}(\tau)$ is inverse to $W_1(\tau)$ for each $\tau \in \mathbb{R}$. We will show that $W^{-1}(\tau) : \ell^2 \to \ell^2$ is a bounded linear operator. For every $s \in \mathbb{R}$, $i \in \mathbb{N}$ and $x_i \in \mathbb{R}^{d_i}$ we have

$$\begin{aligned} \langle W_i(\tau)x_i, x_i \rangle &= \int_0^\tau \|e^{-sA_i^*} x_i\|^2 ds \geq \int_0^\tau (m(P_i) e^{\beta_i s} m(P_i^{-1}) \|x_i\|)^2 ds \\ &= \int_0^\tau \frac{e^{2\beta_i s} \|x_i\|^2}{\|P_i\|^2 \cdot \|P_i^{-1}\|^2} ds. \end{aligned}$$

where $m(P_i)$ is the minimum seminorm of P_i and $\beta_i = -\min_{1 \leq j \leq d_i} Re(\lambda_j)$. Since the eigenvalues of A_i assumed to have strictly negative real part bounded away from zero, we have $\beta = \inf_i \beta_i > 0$. Therefore, we have

$$\|W_i(\tau)\| \geq \frac{\langle W_i(\tau)x_i, x_i \rangle}{\|x\|^2} \geq \frac{1}{C^2} \int_0^\tau e^{2\beta_i s} ds \geq \frac{1}{C^2 \beta_i} \left(e^{2\beta_i \tau} - 1 \right),$$

which implies

$$\|W_i^{-1}(\tau)\| \leq 2C^2 \beta_i \left(e^{2\beta_i \tau} - 1 \right)^{-1} \leq C^2/\tau. \tag{13}$$

Further, for any $x = (x_1, x_2, \dots) \in \ell^2$ with $\|x\| = 1$ we have

$$\|W^{-1}(\tau)x\|^2 = \sum_{i=1}^\infty \|W_i^{-1}(\tau)x_i\|^2 \leq \sum_{i=1}^\infty \|W_i^{-1}(\tau)\|^2 \cdot \|x_i\|^2 \leq C^2/\tau. \tag{14}$$

4.3. Null Controllability in Large

Below we assume that $\theta > 0$ and the set of admissible control is defined as in Section 2. Recall that $x(t) = e^{tA}x_0 + e^{tA}\int_0^t e^{-sA} w(s) ds$ is the unique solution of system (2) with an initial state $x(0) = x_0$. It is standard to check that the function

$$u^0(t) = -e^{-tA^*} \cdot W^{-1}(\tau)x_0 \quad \text{for every} \quad x_0 \in \ell^2, \tau \in \mathbb{R}^+ \tag{15}$$

solves the control problem if it is admissible, i.e., $\int_0^\tau e^{-sA} u^0(s) ds = -x_0$ for every fixed $\tau \in \mathbb{R}^+$. Indeed, by (15) we have

$$-\int_0^\tau e^{-tA_i} u^0 dt = \int_0^\tau e^{-tA_i} e^{-tA_i^*} dt \cdot W_i^{-1}(\tau) x_{i0} = x_{i0}, \text{ for all } i \in \mathbb{N}. \tag{16}$$

Therefore, it remains to show that u^0 is admissible, i.e., there exists $\tau > 0$ such that $\|u^0\|^2 = \sum\limits_{i=1}^\infty \int\limits_0^\tau \|u_i^0(s)\|^2 ds \leq \theta^2$, $u_i^0(s) \in \mathbb{R}^{d_i}$.

By definition of $W(\tau)$ and Chauchy–Schwarz inequality we have

$$\begin{aligned}\int_0^\tau \|u^0(t)\|^2 dt &= \int_0^\tau \|e^{-tA^*}W^{-1}(\tau)u^0\|^2 dt \\ &= \int_0^\tau \left\langle e^{-tA} \cdot e^{-tA^*}W^{-1}(\tau)x_0, W^{-1}(\tau)x_0 \right\rangle dt \\ &= \langle x_0, W^{-1}(\tau)x_0\rangle \le \|x_0\|^2 \cdot \|W^{-1}(\tau)\|.\end{aligned} \tag{17}$$

This together with inequality (14) and (15) implies that u^0 is admissible if

$$C\|x_0\|^2/\sqrt{\tau} \le \theta^2. \tag{18}$$

This finishes the proof, since the left hand side of (18) decays as τ grows.

4.4. Time Optimal Control

Equation (14) shows that $\langle x_0, W^{-1}(\tau)x_0\rangle$ is decreasing as τ for every $x_0 \in \ell^2$. Thus, for every $x_0 \in \ell^2$ there exists a unique $\vartheta \in \mathbb{R}_+$ such that

$$\langle x_0, W^{-1}(\tau)x_0\rangle > \theta^2, \text{ for } \tau > \vartheta, \text{ and } \langle x_0, W^{-1}(\vartheta)x_0\rangle = \theta^2. \tag{19}$$

We claim that ϑ is the optimal time. We use the following result from [23].

Proposition 1. *Let $B(t)$, $t \in [0, \vartheta_0]$ be a continuous matrix-function of the order d with a determinant not identically* 0 *on* $[0, \vartheta_0]$. *Then among the measurable functions $w : [0, \vartheta_0] \to \mathbb{R}^d$, satisfying the condition $\int_0^{\vartheta_0} B(s)w(s)ds = w_0 \in \mathbb{R}^d$ the function defined almost everywhere on $[0, \vartheta]$ by the formula $w(s) = B^*F^{-1}(\vartheta_0)x_0$, $F(\vartheta_0) = \int_0^{\vartheta_0} B(s)B^*(s)ds$ gives a minimum to the functional $\int_0^{\vartheta_0} |w(s)|^2 ds$.*

Assume that there is an admissible control $u(\cdot)$ defined on $[0, \vartheta)$ such that $x(\tau) = 0$ for some $\tau < \vartheta$. By definition we have

$$e^{\tau A_i}x_{0i} + \int_0^\tau e^{(\tau-s)A_i}u_i(s)ds = 0 \text{ for all } i \in \mathbb{N}.$$

Since $e^{(\tau-s)A_i}$ is a continuous matrix function we can apply the above proposition for every $i \in \mathbb{N}$ and conclude that the functional $J(u) = \int_0^\tau \sum_{k=1}^\infty \|u_i(s)\|^2 ds$ is minimized by u^0 defined in (15). Thus we have

$$J(u) \ge J(u^0) = \int_0^\tau \sum_{k=1}^\infty \|u_i^0(s)\| ds = \langle x_0, W^{-1}(\tau)x_0\rangle > \langle x_0, W^{-1}(\vartheta)x_0\rangle = \theta^2.$$

This shows that $u(\cdot)$ is not admissible. This contradiction implies that ϑ is the optimal time of translation to the origin and $u^0(t) = -e^{-tA^*} \cdot W^{-1}(\vartheta)x_0$ is the time optimal control.

5. Differential Game Problem: Proof of Theorem 3

We now consider the game problem (1). Recall that the equation

$$\langle x_0, W^{-1}(\tau)x_0\rangle = (\rho - \theta)^2$$

has a unique solution ϑ_1. Fix $T > \vartheta$. We define

$$u(t, v) = v - e^{-tA^*} \cdot W^{-1}(\vartheta_1)x_0 \tag{20}$$

Let $v(\cdot)$ be any admissible control of the evader. We show that (20) is admissible.

$$\|u(t, v)\| = \|v\| + \|e^{-tA^*}W^{-1}(\vartheta_1)x_0\| \le \sigma + \langle x_0, W^{-1}(\vartheta_1)x_0\rangle^{1/2} = \rho.$$

Also, it is easy to show that $x(\vartheta_1) = 0$. This completes the proof.

6. Conclusions

In this paper, we studied an infinite controllable system consisting of independent finite dimensional blocks. We solved the optimal zero control problem and constructed a guaranteed strategy for pursuer to complete the pursuit game. We use Gramians in order to construct optimal control. It would be more desirable to consider a more general equation than (1), But we left this for further investigation. Since, our results don't generalize to this setting, and also one needs to find an analog of the Kalmann condition on controllability.

We proved that the pursuit game can be completed if $\rho < \sigma$. Since in our setting the system is globally asymptotically stable and we expect that it is possible to complete the pursuit game for any $\rho, \sigma > 0$. However, it turned out to be a challenging problem to define the strategy for $0 < \rho < \sigma$. Also, we didn't attempt here evasion problem. We think that in the interval $(0, \vartheta_1)$ evasion is possible. However, we leave this for future work.

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