Optimal Pursuit Time in Linear Differential Game

BASHIR MAI UMAR[1], AHMAD YAHAYA HARUNA[2], JEWAIDU RILWAN[3],
BRUNO ANTONIO PANSERA[4]

[1]Department of Mathematics Federal University, Gashua,
P.M.B 1005, Yobe,
NIGERIA

[2]Mathematics Unit, Department of General Studies,
Federal Polytechnic Kabo, Kano,
NIGERIA

[3]Department of Mathematical Sciences, Bayero University, Kano,
P.M.B., 3011, Kano,
NIGERIA

[4]Department of Law, Economics and Human Sciences & Decisions_lab,
University Mediterranea of Reggio Calabria,
ITALY

Abstract: We consider a differential game of many pursuers and one evader. The motion of the players is described by a linear differential equations. Control functions of the players are subject to generalized integral constraints. The position of the pursuit and evader at some time t is described by $x(t)$ and $y(t)$ respectively, $z(t)$ represents the state of the game express as $z(t) = y(t) - x(t)$. Game is said to be completed if $z(t) = 0$, that is if the position of pursuer and evader is the same. Pursuer tries to complete the game and evader pursues the opposite goal. We construct a formula for guaranteed pursuit time and prove that it an optimal pursuit time. To this end, we construct the optimal strategies of the players. Lastly, we demonstrate our results with a numerical example.

Key-Words: Optimal pursuit time; integral generalized constraint; players control functions.

Received: March 6, 2025. Revised: April 22, 2025. Accepted: June 16, 2025. Published: July 28, 2025.

1 Introduction

The study of differential games has its foundations in [1]; the approach proposed by the author did not involve the use of the classics variational techniques, in fact, resembled the dynamic programming used for optimization problems. Following the work of [1], [2], applied variational calculus techniques in solving a simple differential game problem.

The need for such a theoretical framework is dictated by the fact that differential games are defined in a specific time interval where all players adopt their strategies continuously.

The strategy adopted by the players is described by the control function u_i, where u_i derives from a certain set of possible choices U_i, and depends on the player's discretion, i.e. on the choices he makes based on his knowledge of the strategies of the other participants and the state of the system (see, for instance, [3], [4], [5], [6], [7], [8]).

Differential game is a mathematical approach of investigating conflict which can be modeled as system of differential equation. Given the notable applications that these games have in the fields of war, cybersecurity and robotics, interest in this type of game has intensified in recent decades (see, for example, [9], [10], [11], [12]).

Research problems usually considered in differential game are (but not limited to) pursuit differential game, evasion differential game, pursuit-evasion differential game problems, guaranteed pursuit time and optimal pursuit time. The origin of the study of this topic lies in the observation of natural phenomena such as the daily struggle between lion and gazelle, between those who chase and those who are forced to flee. This suggests to us that the chase game presents two classes of players who have different goals: on the one hand there are the pursuers who define strategies to capture the evader, on the other there are the evaders who implement escape strategies.

This class of games can be further differentiated by considering a different number of pursuers and evaders, games of capture or escape, simple games or linear differential games. In fact, from the initial game with a pursuer and an evader, the scholars moved on to considering those with many pursuers and an evader and then studying those with m pursuers and n evaders. See, [13], [14], [15], for the case of the evasion from many pursuers in the simple motion case, or, [16], for the case on multiplayer pursuit on a closed convex set with integral constraints. The article demonstrate a theorem that lays the foundations for an accurate study of evasion and capture strategies in terms of the total resources of the players involved. Following this idea, capture, i.e. that instant of time in which the positions of those who escape and those who chase coincide, will be obtained when the resolutions of the pursuers are greater than those of the evaders.

Subsequently, in [17], they demonstrate that if the total resources of the evaders exceed those of the pursuers then the evasion is complete.

In particular, this paper is devoted to study a problem in the class of linear differential games. A linear differential game is a game with the players'dynamic equations of the form:

$$\begin{aligned} \dot{x}\ (t) &= \lambda(t)x(t) + u(t), \quad x(0) = x_0 \\ \dot{y}\ (t) &= \lambda(t)y(t) + v(t), \quad y(0) = y_0\ , \quad t > 0, \end{aligned}$$

where x, y are state-variable of pursuer and evader, respectively, $0 \leq t < \infty$, $\lambda(t)$ is a function, $u(t)$ and $v(t)$ are the control functions of the pursuer and evader respectively satisfying certain constraints. In some cases, the dynamics of the players are giving in a closed form as:

$$\dot{z}\ (t) = \lambda(t)z(t) + v(t) - u(t),$$

where $z(t) = y(t) - x(t)$ (usually called the state of the game at time $t > 0$).

The players'controls are measurable functions that determine the form or nature of the strategy the player uses. It serves as an input by the players to achieve their goals. For this input to be acceptable, it has to satisfies some restrictions which hinder player's potential from achieving its goal, called the constraints, which are of different types (integral, geometric, and mixed constraints) see, for example, [18], [19]. Integral and geometric constraints are described by:

$$\int_0^\infty |u(t)|^2 dt \leq \rho^2, \ \ and\ \ |u(t)| \leq \rho, \quad t \in [0, \infty),$$

respectively, where u is a control function, ρ is a positive number. A mixed constraint comprises both. The formula which measures the performance of a player at any given time in the game is known as the payoff. Moreover, the value of payoff when the players uses their best strategies is the game value. Many results were published on differential game with integral constraint (see, [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]). In some of these research works, players dynamics are described by ordinary differential equations.

Optimal pursuit time game problem, is an interesting game problem because a formula for a certain time that will guarantee pursuit and evasion at the same time must be constructed. In [26], the study considered a pursuit differential game of one pursuer and one evader described by infinite system of first order differential equations. A sufficient condition of completion of pursuit is obtained, strategy for the pursuer is constructed and an explicit formula for the guaranteed pursuit time is give by

$$T' = \sup T_i, \ \ T_i = \frac{1}{2\lambda_i}\left[\ln\left(1 + 2\lambda_i\left(\frac{|z_{i0}|}{\rho_i - \sigma_i}\right)^2\right)\right]. \qquad (1)$$

Later on, the result was extended in [33] and the guaranteed pursuit time found in [26], was proved to be an optimal pursuit time. The study, [32], examined guaranteed pursuit time of a differential game described by an infinite system of first order differential equations

$$\dot{z}_k + \beta_k z_k = u_k - v_k, \ \ z_k(0) = z_k^0, \ \ k = 1, 2, \cdots, \qquad (2)$$

where $z_k, u_k, v_k \in \mathbb{R}^n$ and $\beta_k > 0$. In the paper, pursuer tries to transfer the state of the system from an initial state to another state (i.e., $z(t) = z^1, \ \ t \in [0, \theta]$), in contrast to what is considered, [26], where the case $z(t) = 0, \ \ t \in [0, \theta]$. Sufficient conditions were obtained that ensure completion of the game at the guaranteed pursuit time;

$$\theta = \sup\theta_k, \theta_k = \frac{1}{2\beta_k}\left[\ln\left(2\beta_k\left(\frac{\|z_k^0 - z_k^1 e^{\beta_k\theta_k}\|}{\rho_k - \sigma_k}\right)^2 + 1\right)\right]. \qquad (3)$$

In the present paper a differential game with many pursuers and one evader is considered and the control functions of the players is subject to generalized integral constraints. Players' optimal strategies and the formula of an optimal pursuit time are constructed.

The article is organized as follows. In Section 2 the linear differential game with countably infinite pursuers is presented; we introduce the optimal pursuit time as the time T' in which pursuit can be completed in $[0, T']$ and evasion is possible in $[0, T')$. In Section 3 we prove the main result, introduce a formula for the guaranteed pursuit time and prove that it is an optimal pursuit time, and build the evasion strategy. Application of differential game arising in finance, specifically in the theory of option pricing with an "interval model", [34].

The primary innovation of this paper lies in the formulation and analysis of a linear differential game involving multiple pursuers and a single evader, where all players' dynamics are described by linear differential equations under generalized integral control constraints. Unlike existing works that often focus on pairwise interactions or impose simpler control bounds, we extend the framework to a multi-agent pursuit-evasion scenario with more realistic and flexible energy-like control limitations. Key contributions include:

1. State Reduction and Game Reformulation: By introducing the relative state variable $z(t) = y(t) - x(t)$, we reduce the high-dimensional dynamics into a simplified yet analytically tractable form that captures the essence of the pursuit-evasion interaction.
2. Guaranteed Pursuit Time: We derive a closed-form expression for the guaranteed pursuit time, offering an explicit criterion for when at least one pursuer is guaranteed to intercept the evader under optimal conditions.
3. Optimal Strategies Construction: We provide explicit constructions of optimal strategies for both the pursuers and the evader that ensure either successful capture or evasion depending on the configuration of the constraints and initial conditions.
4. Proof of Optimality: We rigorously prove that the guaranteed pursuit time obtained is indeed optimal, thereby validating the theoretical soundness and practical relevance of our solution.
5. Numerical Demonstration: A comprehensive numerical example is included to illustrate the applicability of our results and demonstrate the efficiency of the proposed strategies in a concrete setting.

Overall, this work significantly broadens the scope of classical pursuit-evasion games by addressing the challenges of multiple-agent interactions under general integral constraints and contributes both to the theory of differential games and its applications in robotics, defense, and surveillance systems.

While differential games with linear dynamics and classical control constraints have been extensively studied, most works focus on two-player models or assume simple control bounds. The problem of guaranteed pursuit involving multiple pursuers against a single evader under generalized integral constraints remains relatively under explored. Furthermore, explicit formulas for optimal pursuit time and construction of optimal strategies in such settings are rare in the literature. This paper addresses this gap by formulating and solving a linear differential game with multiple pursuers, deriving the guaranteed pursuit time, and validating the results numerically.

2 Statement of the Problem

This section aims to explore the definition of the linear differential game. Before entering the specific problem, we first present in detail the system of differential equations that characterize the game and the definition of the concept of admissible strategy and control. We consider a linear differential game

with infinitely many pursuers $P_j, j \in J = \{1, 2, \cdots\}$ and one evader E in the Euclidean space $\mathbb{R}^n$

Let the dynamic equations of the players be given by the following linear differential equations;

$$P_j: \quad \dot{x}_j = -\lambda x_j + u_j, \quad x_j(0) = x_j^0, \tag{4}$$

$$E: \qquad \dot{y} = -\lambda y + v, \quad y(0) = y^0, \tag{5}$$

where $x_j, x_j^0, u_j, y, y^0, v \in \mathbb{R}^n$ and λ is a given positive number, $u_j = (u_{j1}, u_{j2}, \cdots, u_{jn})$ and $v = (v_1, v_2, \cdots, v_n)$ are control parameters of pursuers and evader respectively.
Combining (4) and (5) by $z_j = y - x_j$, we have;

$$\dot{z}_j + \lambda z_j = v - u_j, \quad z_j(0) = z_j^0, \ j \in J. \tag{6}$$

where $z_j, z_j^0, u_j, v \in \mathbb{R}^n$ and the initial state $z_j^0 = y^0 - x_j^0$.

Obviously the evolution of the positions occupied by the players (4) and (5) depends on time and therefore become functions of time. Let $[0, T']$ be an interval of $\mathbb{R}$ and $L_p(0, T')$ be the collection of Lebesgue measurable functions defined on $(0, T')$, where $\int_0^{T'} \|f(t)\|^p \, dt < \infty, \ 1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Let

$$\|f\|_{L_p(0,T')} = \left(\int_0^{T'} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} < \infty$$

for any $f \in L_p(0, T')$.

Lemma 1. *[35] Suppose (X, S, μ) is a measure space and $1 \le p < \infty$. Then,*

$$\|f + g\|^p \le 2^p \left(\|f\|^p + \|g\|^p \right). \tag{7}$$

Definition 1. *A measurable function $u_j : [0, T'] \longrightarrow \mathbb{R}^n$, ($v : [0, T'] \longrightarrow \mathbb{R}^n$) is called an admissible control of the j^{th}, $j \in J$, pursuer (evader) if it satisfies the following*

$$\int_0^{T'} \|u_j(s)\|^p \, ds \le \rho_j^p, \quad \left(\int_0^{T'} \|v(t)\|^p \, dt \le \sigma^p \right) \tag{8}$$

respectively, where ρ_j and σ are given positive number.

As [33], reports, if we replace the parameters u_j and v in the equation (6) with some admissible controls $u_j(t)$ and $v(t)$, $0 \leq t \leq T'$, then the problem at the initial value (6) has a unique solution on the time interval $[0, T']$. The solution $z(t) = (z_1(t), z_2(t), ...), 0 \leq t \leq T'$, of infinite system of differential equations (6), is considered in the space $L_p(0, T')$.

Definition 2. *Let $\psi_j(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ be measurable. A function $U_j : [0, T'] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of the form,*

$$\begin{aligned} U_j(t, v(t)) &= v(t) + \psi_j(t) \\ &= (v_1(t) + \psi_{j1}(t), \cdots, v_n(t) + \psi_{jn}(t)), \end{aligned} \tag{9}$$

where $U_j(t, v(t)) = v(t) + \psi_j(t) \in \mathbb{R}^n$, $j \in J$, is called strategy of the j^{th} pursuer, if for any admissible control of the evader $v(\cdot)$, the inequality,

$$\int_0^{T'} \left\|U_j(t, v(t))\right\|^p \, dt \leq \rho_j^p,$$

holds.

Before the strategy of the evader is defined, we extend the system (6)-(8) by introducing the following state variables p_j and q by the equations

$$\dot{p}_j\,(t) = -\left\|u_j(t)\right\|^p, \;\; p_j(0) = \rho_j^p \tag{10}$$

and

$$\dot{q}\,(t) = -\left\|v(t)\right\|^p, \;\; q(0) = \sigma^p. \tag{11}$$

Such an extension is a typical technique applied in studying games with integral constraints. If t is a current time than

$$p_j(t) = \rho_j^p - \int_0^t \left\|u_j(s)\right\|^p ds \tag{12}$$

and

$$q(t) = \sigma^p - \int_0^t \left\|v(s)\right\|^p ds. \tag{13}$$

The functions $p_j(t)$ and $q(t)$ are called the control resources of the jth pursuer and evader, respectively. The quantity $\int_0^t \left\|u_j(s)\right\|^p ds$ expresses the amount of energy spent by the pursuer of control u_j. The quantity $p_j(t)$ is the amount of energy remained for the control u_j which the jth pursuer can use starting from time t. The quantities $\int_0^t \left\|v(s)\right\|^p ds$ and $q(t)$ also have similar meanings for the evader. That is The quantity $\int_0^t \left\|v(s)\right\|^p ds$ expresses the amount of energy spent by the evader of control v. The quantity $q(t)$ is the amount of energy remained for the control v which the evader can use starting from time t

Definition 3. *A function $V(t) : \mathbb{R}^n \to \mathbb{R}^n$; $t > 0$, defined by*

$$V(t) = \begin{cases} V_0(t), & 0 \leq t \leq \tau, \; q_0(\tau) \geq 0, \\ 0, & \tau < t \leq \tau + \epsilon, \\ u_j(t - \epsilon) & t > \tau + \epsilon \end{cases} \tag{14}$$

is called strategy of evader, where $V_0(t), t > 0$, is a measurable function such that

$$q_0(t) = \sigma^p - \int_0^t \left\|V_0(s)\right\|^p ds, \tag{15}$$

$t = \tau$ is the first time for which $p_j(t) > q_0(t), u_j(t) = (u_{j1}(t), u_{j2}(t), \cdots, u_{jn}(t))$ is any admissible control of pursuer, ϵ is a positive number.

Definition 4 (Guaranteed Pursuit Time, T'). *Pursuit is said to be completed at time $T' > 0$, from the initial positions $z_j^0 = (z_{j1}^0, z_{j2}^0, \cdots, z_{jn}^0)$, if there exist strategies of the pursuers $U_j(t, v(t))$, such that for any admissible control of the evader $v(t)$, $0 \leq t \leq T'$, we have $z_j(t) = 0$, $j \in J$, $0 \leq t \leq T'$, at some τ, $0 \leq \tau \leq T'$, i.e., $z_j(\tau) = 0$, at some τ, $0 \leq \tau \leq T'$. In the sequel, the number T' is called guaranteed pursuit time.*

Definition 5. *A guaranteed pursuit time T' is called optimal pursuit time if there exists a strategy of the evader V such that for any admissible control of the pursuer $z_j(t) \neq 0$ for all $t \in [0, T')$.*

Problem 1. *Find optimal pursuit time in the game (6)-(8), and construct the strategy for the evader.*

3 Main Result

In this section we introduce the formula for the optimal pursuit time in the game (6)-(8).

Lemma 2.

$$\left\|\int_0^t e^{\lambda s} u_j(s) ds\right\| \leq \rho_j \varphi_\lambda(0, t)$$

Proof. Taking the left hand side we have

$$\begin{aligned} \left\|\int_0^t e^{\lambda s} u_j(s) ds\right\| &\leq \int_0^t \left\|e^{\lambda s} u_j(s)\right\| ds \\ &\leq \left(\int_0^t e^{\lambda q s} ds\right)^{\frac{1}{q}} \left(\int_0^t \left\|u_j(s)\right\|^p ds\right)^{\frac{1}{p}} \\ &\leq \rho_j \varphi_\lambda(0, t) \end{aligned}$$

□

Theorem 1. *Let* $\rho_j > 2\sigma,\ \ j \in J$ *and* $\sup_j \frac{\|z_{j0}\|}{\frac{\rho_j}{2}-\sigma} < \infty$. *Then*

$$T' = \sup_j T_j, \ \ T_j := \frac{1}{\lambda q}\ln\left(1 + \lambda q\left(\frac{\|z_{j0}\|}{\frac{\rho_j}{2}-\sigma}\right)^q\right) \tag{16}$$

is an optimal pursuit time.

Proof. Theorem 1 above is a two-step proof, first we show that T' is a guaranteed pursuit and then we show that it is also an optimal pursuit time, proving that on the interval $[0, T')$ evasion is possible.

3.1 Completion of Pursuit

The hypothesis of the Theorem 1, $\sup_j \frac{\|z_{j0}\|}{\frac{\rho_j}{2}-\sigma} < \infty$ implies that T' is finite.

Let $v(t), 0 \le t \le T'$, be an arbitrary control of the evader. Construct the pursuers' strategy as follows

$$U_j(t, v(t)) = \begin{cases} v(t) + \frac{z_{j0}e^{\lambda t(q-1)}}{\varphi_\lambda^q(0,T_j)}, & 0 \le t \le T_j, \\ v(t), & T_j < t \le T', \end{cases} \tag{17}$$

where

$$\varphi_\lambda(0,t) = \left(\int_0^t e^{\lambda q s}\,ds\right)^{\frac{1}{q}}, \quad t \ge 0. \tag{18}$$

Claim 1: Admissibility of the pursuers'strategy.

To verify admissibility of the pursuers' strategy (17) using Lemma 1. We have

$$\begin{aligned}\int_0^{T'} \|U_j(t,v(t))\|^p\,dt &= \int_0^{T_j}\left\|v(t) + \frac{z_{j0}e^{\lambda t(q-1)}}{\varphi_\lambda^q(0,T_j)}\right\|^p dt + \int_{T_j}^{T'}\|v(t)\|^p\,dt \\ &\le \int_0^{T_j} 2^p\left(\|v(t)\|^p + \left\|\frac{z_{j0}e^{\lambda t(q-1)}}{\varphi_\lambda^q(0,T_j)}\right\|^p\right)dt + \int_{T_j}^{T'}\|v(t)\|^p\,dt \end{aligned} \tag{19}$$

$$\begin{aligned} &< 2^p\int_0^{T_j}\|v(t)\|^p\,dt + 2^p\int_0^{T_j}\left\|\frac{z_{j0}e^{\lambda t(q-1)}}{\varphi_\lambda^q(0,T_j)}\right\|^p dt + 2^p\int_{T_j}^{T'}\|v(t)\|^p\,dt \\ &= 2^p\left(\int_0^{T_j}\|v(t)\|^p\,dt + \int_{T_j}^{T'}\|v(t)\|^p\,dt\right) + 2^p\int_0^{T_j}\left\|\frac{z_{j0}e^{\lambda t(q-1)}}{\varphi_\lambda^q(0,T_j)}\right\|^p dt \\ &= 2^p\int_0^{T'}\|v(t)\|^p\,dt + 2^p\frac{\|z_{j0}\|^p}{\varphi_\lambda^{pq}(0,T_j)}\int_0^{T_j}e^{\lambda t(q-1)p}\,dt \\ &\le 2^p\sigma^p + 2^p\frac{\|z_{j0}\|^p}{\varphi_\lambda^{pq}(0,T_j)}\int_0^{T_j}e^{\lambda t(q-1)p}\,dt \\ &= 2^p\sigma^p + 2^p\frac{\|z_{j0}\|^p}{\varphi_\lambda^{pq}(0,T_j)}\int_0^{T_j}e^{\lambda(pq-p)t}\,dt \\ &= 2^p\sigma^p + 2^p\frac{\|z_{j0}\|^p}{\varphi_\lambda^{pq}(0,T_j)}\varphi_\lambda^{pq-p}(0,T_j) \\ &= 2^p\sigma^p + 2^p\|z_{j0}\|^p\varphi_\lambda^{pq-p-pq}(0,T_j) \\ &= 2^p\sigma^p + 2^p\|z_{j0}\|^p\varphi_\lambda^{-p}(0,T_j). \end{aligned} \tag{20}$$

The time T_j, defined by (16) indeed satisfies

$$\varphi_\lambda(0,T_j) = \left(\int_0^{T_j} e^{\lambda q s}\,ds\right)^{\frac{1}{q}} = \left(\frac{e^{\lambda q T_j} - 1}{\lambda q}\right)^{\frac{1}{q}} = \frac{\|z_{j0}\|}{\frac{\rho_j}{2}-\sigma},$$

and

$$\varphi_\lambda^{-p}(0,T_j) = \frac{\left(\frac{\rho_j}{2}-\sigma\right)^p}{\|z_{j0}\|^p}.$$

Hence

$$\varphi_\lambda^{-p}(0,T_j) \le \frac{(\frac{\rho_j}{2})^P - (\sigma)^p}{\|z_{j0}\|^p}.$$

Therefore, inequality (20) consequently becomes

$$\begin{aligned}\int_0^{T'}\|U_j(t,v(t))\|^p\,dt &\le 2^p\sigma^p + 2^p\|z_{j0}\|^p\left(\frac{(\frac{\rho_j}{2})^p - (\sigma)^p}{\|z_{j0}\|^p}\right) \\ &\le 2^p\sigma^p + 2^p\left(\frac{\rho_j}{2}\right)^p - 2^p\sigma^p \\ &= \rho_j^p.\end{aligned}$$

Thus, the strategy (17) is indeed admissible.

Claim2: T' is a guaranteed pursuit time.

We prove now that T' is a guaranteed pursuit time, i.e. $z_j(T') = 0$.

Let $v(t), 0 \le t \le T'$, be a admissible control of the evader. Replacing $u_j(t)$ in the equation

$$\begin{aligned}&\dot z_j(t) + \lambda z_j(t) = v(t) - u_j(t), \\ &z_j(0) = z_{j0}, \ \ j \in J,\end{aligned}$$

by the strategy $U_j(t, v(t))$ constructed in (17), we obtain

$$\begin{aligned} z_j(T') &= e^{-\lambda T'}z_{j0} + \int_0^{T'} e^{-\lambda(T'-s)}(v(s) - U_j(s,v(s)))\,ds \\ &= e^{-\lambda T'}\left(z_{j0} + \int_0^{T'} e^{\lambda s}(v(s) - U_j(s,v(s)))\,ds\right) \\ &= e^{-\lambda T'}\left(z_{j0} + \int_0^{T'} e^{\lambda s}v(s)\,ds - \left(\int_0^{T_j} e^{\lambda s}\left(v(s) + \frac{z_{j0}e^{\lambda s(q-1)}}{\varphi_\lambda^q(0,T_j)}\right)ds + \int_{T_j}^{T'} e^{\lambda s}v(s)\,ds\right)\right) \\ &= e^{-\lambda T'}\left(z_{j0} + \int_0^{T'} e^{\lambda s}v(s)\,ds - \left(\int_0^{T'} e^{\lambda s}v(s)\,ds + \int_0^{T_j} e^{\lambda s}\left(\frac{z_{j0}e^{\lambda s(q-1)}}{\varphi_\lambda^q(0,T_j)}\right)ds\right)\right) \\ &= e^{-\lambda T'}\left(z_{j0} - \int_0^{T_j} e^{\lambda s}\left(\frac{z_{j0}e^{\lambda s(q-1)}}{\varphi_\lambda^q(0,T_j)}\right)ds\right) \\ &= e^{-\lambda T'}\left(z_{j0} - \frac{z_{j0}}{\varphi_\lambda^q(0,T_j)}\int_0^{T_j} e^{\lambda s}e^{\lambda s(q-1)}\,ds\right) \\ &= e^{-\lambda T'}\left(z_{j0} - \frac{z_{j0}}{\varphi_\lambda^q(0,T_j)}\int_0^{T_j} e^{\lambda q s}\,ds\right) \\ &= e^{-\lambda T'}\left(z_{j0} - \frac{z_{j0}}{\varphi_\lambda^q(0,T_j)}\varphi_\lambda^q(0,T_j)\right) \\ &= e^{-\lambda T'}(z_{j0} - z_{j0}) \\ &= 0. \end{aligned} \tag{21}$$

Therefore, the equality $z_j(T') = 0$ holds. Thus, the time T' is a guaranteed pursuit time. □

Once the existence of a guaranteed pursuit time has been demonstrated, we move on to the second step, i.e. the demonstration that evasion is possible at time $t \in [0, T')$.

3.1.1 Constructing the Strategy of the Evader

To prove that T' is an optimal pursuit time, we need to define the evasion strategy that guarantees the evasion on the time interval $[0, T')$.

We construct evader's strategy in two parts. The first part of it is as follows:

let

$$v(s) = \frac{z_{j0} e^{\lambda s} \sigma}{\|z_{j0}\| \varphi_\lambda(0, T_j)}, \quad 0 \le s \le \tau, \;\; j = 1, 2, \cdots, n. \quad (22)$$

where τ, $\tau < T_j$, is some time at which $q(\tau) = p_j(\tau)$, and

$$\varphi_\lambda(0, t) = \left(\int_0^t e^{\lambda q s} \, ds \right)^{\frac{1}{q}}, \quad t \ge 0. \quad (23)$$

When $t = 0$, $p_j(0) = \rho_j^p > \sigma^p = q(0)$.

The evader uses (22) unless $q(\tau) = p_j(\tau)$ at some time τ. Evader uses the second part of the strategy when starting from the time τ as follows:

$$v(s) = \begin{cases} 0, & \tau \le s \le \tau + \epsilon, \\ u_j(t - \epsilon) & t > \tau + \epsilon \end{cases} \quad (24)$$

where ϵ is a positive number.

Claim 3: The evasion is guaranteed on the time interval $[0, T')$ At this point, we show that evasion is guaranteed on the time interval $[0, T')$ using the constructed strategy (22)-(24). It suffices to show that $z_j(t) \ne 0$ as long as $q(t) \ne p_j(t)$ that is $q(t) < p_j(t)$ precisely. We proof this by contradiction. Let there exist a control of the pursuer $u_j(t)$ such that $z_j(\tau') = 0$ at some $0 \le \tau' \le \tau$, where $q(\tau') \le p_j(\tau')$.

The solution of the dynamic equation (6) in τ, by definition of τ, is equal to zero.

Therefore we have:

$$z_j(\tau) = e^{-\lambda \tau} \left(z_{j0} + \int_0^\tau e^{\lambda s} (v(s) - u_j(s)) \, ds \right) = 0,$$

hence

$$z_{j0} + \int_0^\tau e^{\lambda s} v(s) ds = \int_0^\tau e^{\lambda s} u_j(s) \, ds. \quad (25)$$

We consider now the

$$\left\| z_{j0} + \int_0^\tau e^{\lambda s} v(s) ds \right\| = \left\| \int_0^\tau e^{\lambda s} u_j(s) \, ds \right\|. \quad (26)$$

Use Hölder's inequality to estimate the right hand side of (26)

$$\left\| \int_0^\tau e^{\lambda s} u_j(s) \, ds \right\| \le \rho_j \varphi_\lambda(0, \tau) \quad (27)$$

Assertion 1. *[11] Among the measurable functions* $u(\cdot), u : [0, T) \to \mathbb{R}^n$ *satisfying* $\int_0^T e^{\lambda s} u_j(s) \, ds = \gamma$, *the control defined by the formula*

$$u(s) = \varphi_\lambda^{-1}(0, T)\gamma \;\; a.e \;\; on \;\; [0, T], \;\; \varphi_\lambda(0, T) = \int_0^T e^{\lambda s} ds \quad (28)$$

gives the minimum to the functional $\int_0^T \|u_j(s)\|^p \, ds$.

If γ equal to the left hand side of (25) and $T = \tau$, then according to Assertion 1 the control

$$u(s) = \varphi_\lambda^{-1}(0, \tau) \left(z_{j0} + \int_0^\tau e^{\lambda s} v(s) ds \right) \quad (29)$$

gives the minimum to the functional $\int_0^T \|u_j(s)\|^p \, ds$.

Then the function $\varphi_\lambda^p(0, t)$ is increasing in $[0, T')$. Combining (22),(25) and (29) we have

$$\begin{aligned} \int_0^\tau \|u_j(s)\|^p \, ds - \int_0^\tau \|v(s)\|^p \, ds =& \rho_j^p \tau - \frac{\sigma^p . \varphi_\lambda^p(0, \tau)}{\varphi_\lambda^p(0, T_j)} \\ >& \rho_j^p - \frac{\sigma^p . \varphi_\lambda^p(0, \tau)}{\varphi_\lambda^p(0, T_j)} \\ >& \rho_j^p - \frac{\sigma^p . \varphi_\lambda^p(0, T_j)}{\varphi_\lambda^p(0, T_j)} \\ =& \rho_j^p - \sigma^p. \end{aligned} \quad (30)$$

This implies that

$$\sigma^p - \int_0^\tau \|v(s)\|^p \, ds > \rho_j^p - \int_0^\tau \|u_j(s)\|^p \, ds. \quad (31)$$

That is $q(t) > p_j(t)$ which is a contradiction.

Thus for time t on the interval $[0,\tau]$, $z_j(t) \ne 0$, in particular $z_j(\tau) \ne 0$. Starting from the time τ the evader use the strategy (24). It is easy to see that

$$z_j(t) = e^{-\lambda t} \left(z_{j0} + \int_0^t e^{\lambda s} (v(s) - u_j(s)) \, ds \right) = 0$$

if and only if

$$y_j(t) = z_{j0} + \int_0^t e^{\lambda s} (v(s) - u_j(s)) ds = 0. \quad (32)$$

Therefore it is enough to show the inequality $y_j(t) \ne 0$, $t \in [\tau, \infty)$. Let $\tau \le t \le \tau + \epsilon$.

Then

$$\begin{aligned} y_j(t) =& z_{j0} + \int_0^t e^{\lambda s} (v(s) - u_j(s)) ds \\ =& y_j(\tau) + \int_\tau^t e^{\lambda s} (v(s) - u_j(s)) ds \\ =& y_j(\tau) + \int_\tau^t e^{\lambda s} (-u_j(s)) ds \end{aligned} \quad (33)$$

we have

$$\begin{aligned}\left\|-\int_{\tau}^{t} e^{\lambda s} u_j(s) ds\right\| &\leq \\ &\leq \varphi_{\lambda}(\tau, t)\left(\int_{\tau}^{t}\left\|u_j(s)\right\|^p ds\right)^p \qquad (34) \\ &\leq \varphi_{\lambda}(\tau, \tau+\epsilon)\rho_j.\end{aligned}$$

If $b-a=\epsilon, \;\; a \in [0, T]$, then

$$\begin{aligned}\varphi_{\lambda}(a, b) &= \left(\int_{a}^{b} e^{q\lambda s} ds\right)^{\frac{1}{q}} \\ &= \left(\frac{1}{q\lambda}\left(e^{q\lambda b}-e^{q\lambda a}\right)\right)^{\frac{1}{q}} \\ &= e^{\lambda a}\left(\frac{1}{q\lambda}\left(e^{q\lambda(b-a)}-1\right)\right)^{\frac{1}{q}} \\ &\leq e^{\lambda T}\left(\frac{1}{q\lambda}\left(e^{q\lambda\epsilon}-1\right)\right)^{\frac{1}{q}}\end{aligned}$$

Let ϵ be chosen in such a way that

$$\rho_j e^{\lambda T}\left(\frac{1}{q\lambda}\left(e^{q\lambda\epsilon}-1\right)\right)^{\frac{1}{q}} \leq \frac{\left\|y_j(\tau)\right\|}{q}.$$

Then

$$\varphi_{\lambda}(\tau, \tau+\epsilon)\rho_j \leq \frac{\left\|y_j(\tau)\right\|}{q} \qquad (35)$$

and so

$$\|y(t)\| = \left\|y_j(\tau)\right\| - \int_{\tau}^{t} e^{\lambda s}\left\|u_j(s)\right\| ds \geq \frac{\left\|y_j(\tau)\right\|}{q}, \quad \tau \leq t \leq \tau+\epsilon.$$

Let now $t > \tau+\epsilon$. Then

$$\begin{aligned}y(t) &= y_j(\tau) - \int_{\tau}^{\tau+\epsilon} e^{\lambda s} u_j(s) ds + \int_{\tau+\epsilon}^{t} e^{\lambda s}(v(s)-u_j(s)) ds \\ &= y_j(\tau) - \int_{\tau}^{\tau+\epsilon} e^{\lambda s} u_j(s) ds + \int_{\tau+\epsilon}^{t} e^{\lambda s} v(s) ds - \int_{\tau+\epsilon}^{t} e^{\lambda s} u_j(s) ds \\ &= y_j(\tau) - \int_{\tau}^{t} e^{\lambda s} u_j(s) ds + \int_{\tau+\epsilon}^{t} e^{\lambda s} u_j(t-\epsilon) ds. \qquad (36)\end{aligned}$$

Since

$$\int_{\tau+\epsilon}^{t} e^{\lambda s} u_j(s-\epsilon) ds = \int_{\tau}^{t-\epsilon} e^{\lambda s} u_j(s) ds \qquad (37)$$

therefore (36) implies

$$\begin{aligned}y_j(t) &= y_j(\tau) - \int_{\tau}^{t} e^{\lambda s} u_j(s) ds + \int_{\tau}^{t-\epsilon} e^{\lambda s} u_j(s) ds \\ &= y_j(\tau) - \int_{\tau}^{t} e^{\lambda s} u_j(s) ds - \int_{t-\epsilon}^{\tau} e^{\lambda s} u_j(s) ds \\ &= y_j(\tau) - \int_{t-\epsilon}^{\tau} e^{\lambda s} u_j(s) ds.\end{aligned}$$

Hence,

$$\begin{aligned}\left\|y_j(t)\right\| &\geq \left\|y_j(\tau)\right\| - \int_{t-\epsilon}^{t} e^{\lambda s}\left\|u_j(s)\right\| ds \\ &\geq \left\|y_j(\tau)\right\| - \rho_j \varphi_{\lambda}(t-\epsilon, t) \geq \frac{\left\|y_j(\tau)\right\|}{q},\end{aligned}$$

for all $t \geq \tau+\epsilon$.

Thus $\left\|y_j(t)\right\| \geq \frac{\|y_j(\tau)\|}{q} > 0$ for all $t \geq \tau$. In other words, evasion is possible on the interval $[0, T_j)$.

Show that evasion is possible on the interval $[0, T')$, $T' = sup_{j=1,2,...} T_j$.

Let $t \in [0, T')$ be any time. Then by definition of supremum $t \in [0, T_j)$ at some $j \in \mathbb{N}$. As proved above $y_j(t) \neq 0$ for $t \in [0, T_j)$, and hence $z_j(t) \neq 0$. In its turn this inequality implies that $z_j(t) \neq 0$ meaning that evasion is possible on the interval $[0, T')$. Thus, T' defined by above is optimal pursuit time. Proof is complete.

4 Numerical Example

Consider a differential game described by (6) with the first pursuer ($j=1$) in $\mathbb{R}^3$. The initial positions of the players are given by $x_1^0 = (0,0,0)$ and $y^0 = (1,0,0)$ respectively. Let $\lambda = 3$, $\rho_1 = 4$, $\sigma = 1$, $p = 4$ and $q = \frac{4}{3}$, then $z_1^0 = y^0 - x_1^0 = (1,0,0)$. $\|z_1^0\| = 1$.

From the initial positions,this first pursuer can catch the evader at time $t = 0.4$, since from Theorem 1 the time is given by

$$T_j = \frac{1}{\lambda q} \ln\left(1 + \lambda q\left(\frac{\|z_{j0}\|}{\frac{\rho_j}{2} - \sigma}\right)^q\right). \qquad (38)$$

By substituting all the values given above we have

$$\begin{aligned}T_1 &= \frac{1}{3.\frac{4}{3}} \ln\left(1 + 3.\frac{4}{3}\left(\frac{1}{\frac{4}{2}-1}\right)^{\frac{4}{3}}\right) \qquad (39) \\ &= \frac{1}{4}\ln(5) = 0.4, \qquad (40)\end{aligned}$$

hence from Theorem 1 pursuit will occur at time $t = 0.4$.

Let the pursuer use the strategy below

$$U_1(t, v(t)) = \begin{cases} v(t) + \frac{z_1^0 e^{3t(\frac{1}{3})}}{\varphi_3^{\frac{4}{3}}(0, 0.4)}, & 0 \leq t \leq 0.4, \\ v(t), & 0.4 < t \leq T', \end{cases} \qquad (41)$$

Pursuit is said to occur when ever the geometric position of any of the pursuers with that of evader remain the same, that is $z_j = 0$ and from (21)

$$z_j(T') = e^{-\lambda T'}\left(z_{j0} - \int_{0}^{T_j} e^{\lambda s}\left(\frac{z_{j0} e^{\lambda s(q-1)}}{\varphi_{\lambda}^{q}(0, T_j)}\right) ds\right),$$

therefore using the first pursuer we have

$$\begin{aligned} z_1(0.4) =& e^{-3.(0.4)}\left((1,0,0)-\int_0^{0.4} e^{3s}\left(\frac{(1,0,0)e^{3.s(\frac{1}{3})}}{\varphi_3^{\frac{4}{3}}(0,0.4)}\right)ds\right) \\ =& e^{-1.2}\left((1,0,0)-\frac{(1,0,0)}{\varphi_3^{\frac{4}{3}}(0,0.4)}\int_0^{0.4} e^{3s}e^{s}\,ds\right) \\ =& e^{-1.2}\left((1,0,0)-\frac{(1,0,0)}{\varphi_3^{\frac{4}{3}}(0,0.4)}\int_0^{0.4} e^{4s}\,ds\right). \end{aligned}$$

Recall that

$$\varphi_\lambda(0,T_j)=\left(\int_0^{T_j} e^{\lambda q s}\,ds\right)^{\frac{1}{q}}=\left(\frac{e^{\lambda q T_j}-1}{\lambda q}\right)^{\frac{1}{q}}=\frac{\|z_{j0}\|}{\frac{\rho_j}{2}-\sigma}. \tag{42}$$

This implies that

$$\varphi_3^{\frac{4}{3}}(0,0.4)=\int_0^{0.4} e^{3\cdot\frac{4}{3}s}\,ds=\frac{e^{4(0.4)}-1}{4}=\frac{5-1}{4}=1 \tag{43}$$

Therefore

$$\begin{aligned} z_1(0.4) =& e^{-1.2}\left((1,0,0)-\frac{(1,0,0)}{\varphi_3^{\frac{4}{3}}(0,0.4)}\int_0^{0.4} e^{4s}\,ds\right) \\ =& e^{-1.2}\left((1,0,0)-\frac{(1,0,0)}{1}.1\right) \\ =& e^{-1.2}\left((1,0,0)-(1,0,0)\right) \\ =& e^{-1.2}\left((0,0,0)\right)=\mathbf{0}. \end{aligned}$$

Thus pursuit is completed.

For evasion, we use the strategy

$$v_{(s)=\frac{z_{j0}e^{\lambda s}\sigma}{\|z_{j0}\|\varphi_\lambda(0,T_j)}+u_j(s),} \tag{44}$$

with $\lambda=3$, $j=1$, $\rho_1=4$, $\sigma=1$, $p=4$ and $q=\frac{4}{3}$, $z_1^0=y^0-x_1^0=(1,0,0)$ and $\|z_1^0\|=1$, then the strategy became

$$\begin{aligned} v(s) =& \frac{(1,0,0)e^{3s}(1)}{\|(1,0,0)\|\,\varphi_3(0,0.4)}+u_1(s) \\ =& \frac{(1,0,0)e^{3s}}{1(1)}+u_1(s) \end{aligned}$$

since

$$\begin{aligned} \varphi_3(0,0.4) =& \left(\int_0^{0.4} e^{3\cdot\frac{4}{3}s}\right)^{\frac{4}{3}} ds \\ =& \left(\frac{e^{4(0.4)}-1}{4}\right)^{\frac{4}{3}} \\ =& \left(\frac{5-1}{4}\right)^{\frac{4}{3}} \\ =& 1. \end{aligned}$$

Therefore the strategy is

$$v(s) = (1,0,0)e^{3s}+u_1(s)$$

Solution of players dynamic is described by

$$z_j(\tau) = e^{-\lambda\tau}\left(z_{j0}+\int_0^{\tau} e^{\lambda s}(v(s)-u_j(s))\,ds\right)=0,$$

At this junction let substitute the above strategy (44) i.e

$$v(s) = (1,0,0)e^{3s}+u_1(s).$$

Therefore we have

$$\begin{aligned} z_1(0.4) =& e^{-3(0.4)}\left((1,0,0)+\int_0^{0.4} e^{3s}(v(s)-u_j(s))\,ds\right)=0 \\ =& e^{-3(0.4)}\left((1,0,0)+\int_0^{0.4} e^{3s}(v(s)-u_j(s))\,ds\right)=0. \end{aligned}$$

Let

$$y(0.4) = (1,0,0)+\int_0^{0.4} e^{3s}(v(s)-u_1(s))\,ds.$$

Using the strategy we have

$$\begin{aligned} y(0.4) =& (1,0,0)+\int_0^{0.4} e^{3s}(v(s)-u_1(s))\,ds \\ =& (1,0,0)+\int_0^{0.4} e^{3s}((1,0,0)e^{3s}+u_1(s)-u_1(s))\,ds \\ =& (1,0,0)+\int_0^{0.4} e^{3s}((1,0,0)e^{3s})\,ds \\ =& (1,0,0)+\int_0^{0.4} e^{6s}((1,0,0))\,ds \\ =& (1,0,0)+(1,0,0)\int_0^{0.4} e^{6s}\,ds \\ =& (1,0,0)+(1,0,0)(1.7) \\ =& (1,0,0)+(1.7,0,0) \\ =& (2.7,0,0)\neq\mathbf{0}. \end{aligned}$$

This show that the position of the players is not the same , therefore evasion is possible.

5 Conclusion

In this paper we studied a differential game of many pursuers and one evader described by an infinite system of first-order differential equations. We have responded to an open issue indicated in [33], where control functions of pursuers and evader are subject to generalized integral constraints.

We introduced the optima pursuer time defined by formula 16 and proved that it is indeed an optimal pursuit time.

Moreover, we have constructed strategies of players.

6 Suggestion for Further Research

One may consider a Leader-Follower Model in Group Pursuit.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

The authors wrote, reviewed and edited the content as needed and verifies that none utilised artificial intelligence (AI) tools were used. The authors take full responsibility for the content of the publication.

References:

[1] Isaacs, R.”Differentialgames I, 11, 111, IV”, FUND. Corporation Research Memorandum RM-1391, 1399,1411, 1468,(1954- 1956).

[2] Berkowitz, L. D. and Fleming, I. H. ”On differential games with integral payoff”, Annals of Math. Study. No. 39, Princeton, N.J.: Princeton University Press,1957, pp. 413- 435.

[3] Petrosyan, L. A. ”Differential games of pursuit.” World Scientific,Singapore, 1993.

[4] Pontryagin, L.S. ”Selected works.”Nauka, Moscow, 1988.

[5] Pshenichnii, B.N.”Simple pursuit by several objects.” Cybern. Syst. Anal.12(3):145–146, (1976).

[6] Pshenichnii, B.N., Chikrii, A.A., Rappoport,J.S. ”An efficient method of solving differential games with many pursuers.” Dokl. Akad. Nauk. SSSR 256: 530–535, (1981) (in Russian).

[7] Satimov, N.Y., Rikhsiev, B.B., Khamdamov, A.A.”On a pursuit problem for n person linear differential and discrete games with integral constraints.” Math. USSR Sbornik. 46(4):456–469, (1983).

[8] Satimov, N.Y., Fazilov, A.Z., Hamdamov, A.A.”On pursuit and evasion problems for multi-person linear differential and discrete games with integral constraints.” Diff. Uravneniya 20(8):1388–1396, (1984).

[9] Wang, C., Chen, S., Wang, Y., Pansera, B.A. and Luckraz, S. ”On the Welfare and Policy Implications of a Two-Period Real Option Game”, Socio-Economic Planning Sciences, Volume 90, December 2023, 101717.

..

[10] Rilwan, J., Ferrara, M., Badakaya, A. J. and Pansera, B. A. ”On pursuit-evasion game problems with Gronwall-type Constraints”, Quality and Quantity, 1-12, (2023).

[11] Ibragimov, G. I., Abdulla A. A., and Marzieh K. ”Solution of a linear pursuit-evasion game with integral constraints.” ANZIAM Journal 52 (2010): E59-E75.

[12] Rilwan, J., Kumam, P., Ibragimov, G. I., Badakaya, A. J. and Ahmed, I. ”A differential game problem of many pursuers and one evader in the Hilbert Space ℓ_2”, Differential Equations and Dynamical Systems, (2020).

[13] Chernous’ko, F.L. ”A problem of evasion of several pursuers.”Prikl. Mat. Mekh. 40(1):14–24, (1976).

[14] Chodun, W. ”Differential games of evasion with many pursuers.” J. Math. Anal. Appl., 142(2):370–389, (1989).

[15] Borowko, P., Rzymowski, W., Stachura, A. ”Evasion from many pursuers in the simple motion case.” J. Math. Anal. Appl. 135(1):75–80, (1988).

[16] Ibragimov, G., Satimov, N. ”A multiplayer pursuit differential game on a closed convex set with integral constraints.” In: Abstract and applied analysis. Article ID 460171: 12 p. (2012).

[17] Ibragimov, G. I., Ferrara, M., Kuchkarov, A. & Pansera, B.A., ”Simple motion evasion differential game of many pursuers and evaders with integral constraints.” Dynamic Games and Applications, 8 (2), 352–378, (2018).

[18] Croft, H.T.”Lion and man: a post script.” J. Lond. Math. Soc. 39:385–390, (1964).

[19] Ivanov, R. P. ”Simple pursuit-evasion on a compact convex set.” Doklady Akademii Nauk SSSR 254(6):1318–1321, (1980).

[20] Ahmed, I., Kumam, W., Ibragimov, G. I. and Rilwan, J. ”Pursuit differential game problem with multiple players on a closed convex set with more general integral constraints”, Thai Journal of Mathematics, 18(2), 551-561, (2020).

[21] Alias, I. A., Ibragimov, G. I., Kuchkarov, A. S. and Sotvoldiyev, A. ”Differential game with many pursuers when controls are subjected to coordinate-wise integral constraints”, Malaysian Journal of Mathematical Sciences, 10(2), 195-207, (2016).

[22] Ferrara, M., Ibragimov, G. I. and Salimi, M. ”Pursuit-evasion game of many players with coordinate-wise integral constraints on a convex set in the plane”, Atti della Accademia Peloritana dei Pericolanti, Classe di Scienze Fisiche, Matematiche e Naturali, 95(2), (2017).

[23] Ibragimov, G. I. ”On a Multi-person Pursuit problem with integral constraints on the controls of the players”, Mathematical Notes, 70(2), 181-191, (2001).

[24] Ibragimov, G. I. ”A game problem on closed convex set”, Siberian Advances in Mathematics, 12(3), 16-31, (2002).

[25] Ibragimov, G. I., Khakestari, M. and Kuchkarov, A. S. ”Solution of a linear pursuit-evasion differential game with closed and convex terminal set”, ITB Journal of Science, 44(1), 1-12, (2012).

[26] Ibragimov, G. I., Tukhtasinov, M., Hasim, R. and Alias, I. ”A Pursuit Problem Described by Infinite Systems of Differential Equations with Coordinate-Wise Integral Constraints on Controls”, Malaysian Journal of Mathematical Sciences, 9(1), 67-76, (2015).

[27] Ibragimov, G. I., Ferrara, M., Salimi, M. and Ismail, N. ”Pursuit and evasion games for an infinite system of differential equations”, Bulletin of the Malaysian Mathematical Sciences Society, 45(1), 69-81, (2022).

[28] Karapanan, P., Ibragimov, G. I., Alias, I. A. and Hasim, R. M. ”A guaranteed result for the differential game of two Pursuers and one Evader on a Cylinder”, Malaysian Journal of Mathematical Science, 15(2), 293-302, (2021).

[29] Kuchkarov, A. S. ”Solution of simple pursuit-evasion problem when evader moves on a given curve”, International Game Theory Review, 12(03), 223-238, (2010).

[30] Leong W. J, and Ibragimov, G. I. ” A Multiperson Pursuit Problem on a Closed Convex Set in Hilbert Spaces”, Far East Journal of Applied Mathematics, 33(2), 205-214 (2008).

[31] Rakhmanov, A., Ibragimov, G. I. and Ferrara, M. ”Linear pursuit differential game under phase constraint on the state of evader”, Discrete Dynamics in Nature and Society, Article ID 1289456, 6, (2016).

[32] Waziri, U., and Ibragimov, G. I. ”Guaranteed pursuit time in a differential game with coordinate-wise integral constraints” AIP Conference Proceedings, 2184(1), (2019).

[33] Ibragimov, G., Hasim, R.M., Rakhmanov, A., and Alias, I.A. ”Optimal Pursuit Time in Differential Game for an Infinite System of Differential Equations.” Malaysian Journal of Mathematical Sciences 10(S): 267–277, (2016).

[34] Bernhard, P., El Farouq, N. & Thiery, S. An impulsive differential game arising in finance with interesting singularities. *Advances In Dynamic Games: Applications To Economics, Management Science, Engineering, And Environmental Management*. pp. 335-363, (2006).

[35] Axler, S. ”Lp Space In: Measure, Integration and Real Analysis.” Springer, Cham. 282, (2020).

Contribution of Individual Authors to the Creation of the Scientific Article

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in the Scientific Article

This research was funded by the National Science, Research and Innovation Fund (NSRF), and King Mongkut’s University of Technology North Bangkok with Contract no. KMUTNB-FF-67-B-02.

Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Creative Commons Attribution License 4.0 (Attribution 4.0 International , CC BY 4.0)