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Triangular Bernstein moment-based identification of algebraic curves

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Abstract

Extending our previous results, in this paper we present a theoretical improvement of a strategy for the identification of binary images with algebraic boundaries. Such identification is obtained from few samples and it is based on a representation of the image shape in terms of non-separable bivariate Bernstein polynomials piecewisely defined over triangular domains.

Keywords: Image shape recovery; Algebraic curves; Bernstein polynomials; Image moments; Refinable sampling kernels

1. Introduction

This paper discusses a theoretical improvement of the strategy presented in [2] where Bernstein polynomials are used to derive a stable procedure for the reconstruction of a class of binary images (i.e. characteristic functions of a given bounded domain) whose shape is modelled as an algebraic curve. Indeed, since the numerical stability of algebraic curves defined by implicit equations can be enhanced by selecting appropriate polynomial basis representations, the use of non-separable bivariate Bernstein polynomials turns out to be very effective to identify an algebraic domain from noisy image samples.

To be more precise, let $\Omega = [0, L_1] \times [0, L_2]$ be a rectangular region containing a given bounded open domain D, whose boundary is an algebraic curve described by

$$\partial D = \left\{ x \in \mathbb{R}^2 : \ p(x) = 0 \right\},$$

where p is a two-variable polynomial of degree $n \geq 1$. Such boundary ∂D identifies the *shape* of the image $I = \chi_D$.

It is known that an algebraic curve of degree n as above can be uniquely determined from the two-dimensional moments

$$M_{i,j} = \int_{\Omega} x_1^i x_2^j \, dx_1 \, dx_2, \tag{1}$$

of order less than or equal to n. More details on such a result can be found in the seminal books on moment problems [12] and, in particular, [13, Chapter IX], while a more precise statement is given in [20, Section 2].

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The problem of recovering a binary image with algebraic boundary is known to be very sensitive to noise even in case a high number of moments is used (see [11,15,18]). It has been faced by Fatemi et al. in [9] where the authors, to overcome instability and to reduce the impact of errors on the recovery algorithm, assumed to have as input data a discrete set of uniform samples of the image I, that is the quantities

$$d_k = \frac{1}{\tau}\int_{\Omega} I(x)\,\varphi\left(\frac{x}{\tau} - k\right)dx, \quad k \in \mathbb{Z}^2, \tag{2}$$

where $\tau \geq 1$ is a sampling parameter and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a specific *sampling kernel*.

It is additionally assumed that such kernel satisfies the *polynomial generation* (or reproduction, according to other definition in literature) property up to a certain degree m, i.e. there exist coefficient sequences $(c_k^{(\alpha)} : k \in \mathbb{Z}^2)$ such that

$$\sum_{k\in\mathbb{Z}^2} c_k^{(\alpha)} \varphi(x-k) = x^{\alpha}, \quad |\alpha| = 0, \ldots, m. \tag{3}$$

Under these assumptions, the *image moments* (1) can be expressed as linear combinations of the *image samples* (2). The coefficients of such linear combinations obviously depend on the sampling kernel φ and are solution to a system of linear equations, called the *image moment equation*, whose coefficient matrix consists of the computed moments (1).

Nevertheless, if the bivariate polynomial identifying the boundary is simply expressed in terms of monomials as

$$p(x_1, x_2) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}\, x_1^i\, x_2^j, \quad x_1, x_2 \in \Omega, \tag{4}$$

the process of turning image samples into image moments can be very sensitive to even a modest amount of additive noise corrupting the image samples.

Fatemi et al. [9] developed a modified formulation based on *generalized moments* to address the instability of the recovery. This formulation leads to a non trivial constrained optimization problem which is solved by quadratic programming.

Such instability problems are connected to the fact that, in the reconstruction process, an important role is played by the coefficient involved in the reproduction of the polynomial basis by means of the sampling kernels. When the monomial basis is used, such coefficients (say c_k, $k \in \mathbb{Z}$) have the same growth rate as the monomials (i.e. they behave like $|k|^j$), so their amplitude increases too much depending on the degree of the polynomial and the size of the domain. As we will see, the moments are computed as sums of the image samples weighted by the reproduction coefficients, and this implies that the image samples close to the borders of the image have a higher impact than the central samples. Since our binary image model may contain noise at the borders, such noise affects the moments and the effect becomes more severe as the order of moments grows.

In order to control such instability issues, a representation in a basis different from the monomial one is proposed in [2]. Since representations in terms of Bernstein polynomials are known to enhance stability in several numerical contexts (cf. [8]), the idea is to express p in terms of Bernstein polynomials as:

$$p(x_1, x_2) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \gamma_{i,j}\, B_{i,j}^n(x_1, x_2), \quad x_1, x_2 \in T \supseteq \Omega, \tag{5}$$

and to derive a novel formulation of the image moment equation that can be solved to recover an algebraic domain from image samples affected by noise avoiding instabilities. Note that the triangular region T in (5) is any triangle embedding the image domain Ω, but its choice has an impact on the representation coefficients.

It is well known that (see [14]) over an arbitrary triangular region T defined by the three points $\boldsymbol{S}, \boldsymbol{Q}, \boldsymbol{R}$ expressing an arbitrary point as

$$\boldsymbol{P} = w\boldsymbol{S} + u\boldsymbol{Q} + v\boldsymbol{R}, \qquad w, u, v \in [0, 1], \quad u + v + w = 1,$$

the bivariate polynomials

$$B_{i,j}^{n,T}(u, v) = \binom{n}{i}\binom{n-i}{j} u^i\, v^j\, w^{n-i-j}, \;\; i = 0, \ldots, n, \;\; j = 0, \ldots, n-i, \tag{6}$$

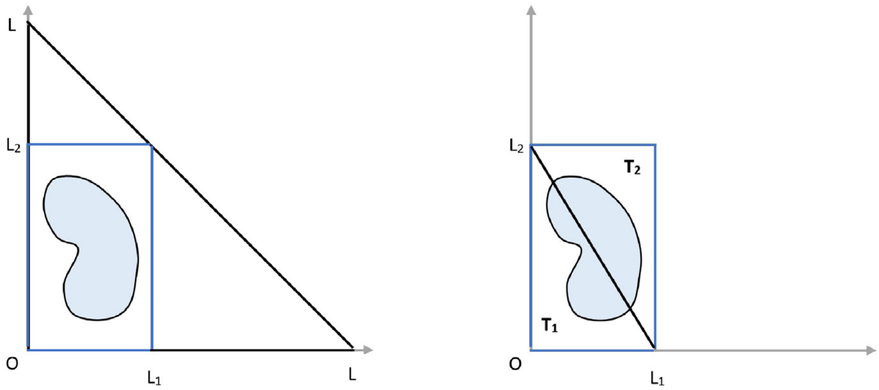

Fig. 1. The domain D of the algebraic curve is contained inside the rectangular region $\Omega = [0, L_1] \times [0, L_2]$, which we embed inside a triangle with vertices $(0, 0)$, $(L, 0)$, $(0, L)$ with $L = L_1 + L_2$ (left) or which we split into two triangles (right).

are a basis of the polynomial space Π_n. Hence, in [2] the authors adopted the simple solution of embedding Ω into a triangle with vertices $(0, 0)$, $(L_1 + L_2, 0)$, $(0, L_1 + L_2)$, as illustrated in Fig. 1 (left).

Since the size of the triangle affects the Bernstein coefficients and consequently the polynomial representation, increasing the overall computational cost of the procedure as well as its stability, here we theoretically explore a different approach: We consider the splitting of the domain Ω into two adjacent triangles: T_1 having vertices $(0, 0)$, $(L_1, 0)$, $(0, L_2)$ and T_2 having vertices (L_1, L_2), $(0, L_2)$, $(L_1, 0)$ (the situation is represented in Fig. 1 (right)), so that a piecewise Bernstein representation of p can be considered over the whole rectangle (i.e. the minimal embedding domain).

For shortness and without loss of generality, we can assume $L_1 = L_2 = L$ since typically any binary rectangular image can be padded with zero entries so to model the image plane Ω as a square domain, with the advantage of simplifying several computations.

Therefore, the focus of this theoretical paper is to derive and investigate the image shape identification problem when the shape model is an algebraic polynomial expressed in terms of bivariate Bernstein polynomials over two isosceles triangles. Such a solution turns out to be mathematical more elegant, more stable, and less computationally expensive, hence more effective in the actual reconstruction of the image shapes. In addition, in this paper we also better underline that the Bernstein representation is the most reasonable one, not only with respect to monomials, but also to other polynomial bases, like Chebyshev polynomials.

The rest of the paper is organized as follows. In Section 2, we discuss the behaviour of the reproduction coefficients involved in the reproduction property (3) for monomials, Bernstein, and Chebyshev polynomials. In Section 3, we discuss some properties of Bernstein polynomials represented on a triangular partition of a quadrilateral region. Then, for this type of polynomial representation, in Section 4 we show how to recover an algebraic curve written in Bernstein form through the solution of a system of moment equations. Finally, in Section 5 we describe how such moments can be obtained from the image samples.

2. Bernstein versus monomial and Chebyshev representation

The aim of this section is to better underline that the Bernstein representation is the most reasonable one, not only with respect to monomials, but also to other polynomial bases, like Chebyshev polynomials. Therefore, we compare the behaviour of the reproduction coefficients involved in the reproduction property (3) for monomials, Bernstein, and Chebyshev polynomials. To illustrate the situation, w.l.o.g. we consider the univariate case. In fact, as better described in Section 5, choosing 2D sampling kernels which are separable (i.e. product of univariate refinable functions), the computation is reduced to the univariate case, even when non separable triangular Bernstein polynomials are used.

We thus recall from [2] the procedure for computing the coefficients $c_k^p, k \in \mathbb{Z}$, involved in the expansion of a given polynomial $p \in \Pi_m$ on an interval $[a, b]$ in terms of translates of the sampling kernel φ

$$p = \sum_{k=a+1-N}^{b-1} c_k^p \varphi(\cdot - k), \tag{7}$$

assuming that the support of φ is $[0, N]$.

The kernel φ can be the basic limit function of a convergent subdivision scheme, with the assumption that it generates polynomials up to the degree m. It can be, for example, a B-spline or a pseudo-spline [1,3–5].

Based on results in [17], it has been shown in [2] that such coefficients depend on the derivatives of the reproduced polynomial p evaluated at the integers.

In fact we have

$$c_k^p = \sum_{s=0}^{m} \frac{A_s}{s!} D^s p(k), \quad k = -N + a + 1, \dots, b - 1, \tag{8}$$

where the quantities A_s, $s = 0, \dots, m$, are found as solution to the upper triangular system:

$$\mu_0 A_0 = 1, \qquad \sum_{s=0}^{k} \frac{\mu_{k-s} A_s}{(k-s)!s!} = 0, \quad k = 1, \dots, m.$$

In the above system, the coefficients μ_j are the discrete moments of φ:

$$\mu_j = \sum_{\ell=-N}^{0} \varphi(-\ell)\ell^j,$$

which are easily computed even without knowing the analytical expression of the kernel φ, since, as limit function of a subdivision scheme, it is *refinable*, i.e. there exists a coefficient sequence $(p_k : k = 0, \dots, N)$ such that:

$$\varphi = \sum_{k=0}^{N} p_k \varphi(2 \cdot -k). \tag{9}$$

In such a case, the values at the integers can be computed by solving an eigenvector problem derived from the refinement Eq. (9).

In the case of Bernstein polynomials B_j^n on the interval $[a, b]$:

$$B_j^n(t) = \frac{1}{(b-a)^n} \binom{n}{j} (t-a)^j (b-t)^{n-j}, \ j = 0, \dots, n,$$

by using the recurrence relation for the first order derivatives [7], the sth derivative can be explicitly obtained as:

$$D^s B_j^n(t) = \frac{1}{(b-a)^s} \frac{n!}{(n-s)!} \sum_{l=\max\{0,j+s-n\}}^{\min\{j,s\}} (-1)^{l+s} \binom{s}{l} B_{j-l}^{n-s}(t), \quad j = 0, \dots, n.$$

In the case of a jth degree Chebyshev polynomial of the first kind on the interval $[a, b]$:

$$T_j(t) = \sum_{k=0}^{j/2} (-1)^k \frac{j}{2(j-k)} \binom{j-k}{k} \left(\frac{2}{b-a}\right)^{j-2k} (2t - a - b)^{j-2k},$$

one can use the formula for the derivatives in [19]:

$$\begin{aligned} D^s T_j(t) = \left(\frac{2}{b-a}\right)^s \sum_{j=0}^{(j-s)/2} j(j-1-j)^{\underline{s-1}} \binom{s+j-1}{s-1} T_{j-s-2j}(t) \\ - \frac{(-1)^{j-s}+1}{2^{2-s}} j((j+s)/2-1)^{\underline{s-1}} \binom{(j+s)/2-1}{s-1}, \end{aligned}$$

where $x^{\underline{m}}$ denotes the falling factorial $x(x-1)\dots(x-m+1)$.

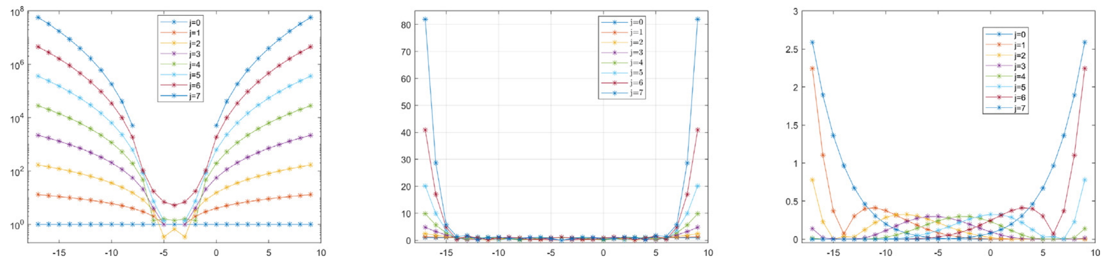

Fig. 2. Behaviour of the coefficients generating the monomials x^j (left), the Chebyshev polynomials T_j (centre), the Bernstein polynomials B_j^7 (right), for $j = 0, \dots, 7$. The kernel φ is the B-spline of order 8.

From a comparison with the derivative expressions of the monomials, it turns out that the growth of the coefficients involved in (7) is much more controlled both in the Chebyshev and in the Bernstein case than in the monomial case, where we have:

$$D^s t^\ell = \frac{\ell!}{(\ell-s)!} t^{\ell-s}, \quad \ell = 0, \dots, m, \ s = 0, \dots, \ell.$$

The different behaviour of the coefficients is also evidenced in Fig. 2.

Nevertheless, the amplitude of the reproduction coefficients associated to the Bernstein polynomials $B_j^n, 0 \le j \le n$, involved in the representation of a polynomial of degree n is much smaller than the amplitude of the reproduction coefficients of the Chebyshev polynomials T_j, $0 \le j \le n$ involved in the same representation.

This allows to confirm that Bernstein representation is to be preferred in particular in our context, where the reproduction coefficients play a role for the reconstruction of the image shapes from moments, as mentioned in the Introduction and better explained in the next sections.

We conclude by underlying the fact that the choice of the refinable function (e.g B-spline or Daubechies or pseudo-splines) does not have significant effects on the final results, as also pointed out in [2].

3. Piecewise–Bernstein representation of bivariate polynomials on square domains

Our aim is to obtain an expression of the bivariate polynomial $p(x_1, x_2)$ over the domain $\Omega = [0, L]^2$ in terms of piecewise Bernstein polynomials over two triangles.

The representation of p over the triangle T_1 is based on the Bernstein polynomials

$$B_{i,j}^{n,1}(x_1, x_2) = \frac{1}{L^n}\binom{n}{i}\binom{n-i}{j}(L - x_1 - x_2)^i x_1^j x_2^{n-i-j}, \quad i = 0, \dots, n, \ \ j = 0, \dots, n-i, \tag{10}$$

which means

$$p(x_1, x_2) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} b_{i,j,n-i-j}^1 \, B_{i,j}^{n,1}(x_1, x_2), \quad x_1, x_2 \in T_1.$$

Note that the previous expression is sometimes given in equivalent forms (see [14]), for example

$$p(x_1, x_2) = \sum_{i+j+k=n} b_{i,j,k}^1 \, B_{i,j,k}^{n,1}(x_1, x_2), \quad x_1, x_2 \in T_1,$$

where

$$B_{i,j,k}^{n,1}(x_1, x_2) = \frac{1}{L^n}\binom{n}{i}\binom{n-i}{j}(L - x_1 - x_2)^i x_1^j x_2^k \quad i + j + k = n.$$

The representation of p over the triangle T_2 reads as

$$p(x_1, x_2) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \tilde{b}_{i,j,n-i-j}^1 \, B_{i,j}^{n,2}(x_1, x_2), \quad x_1, x_2 \in T_2,$$

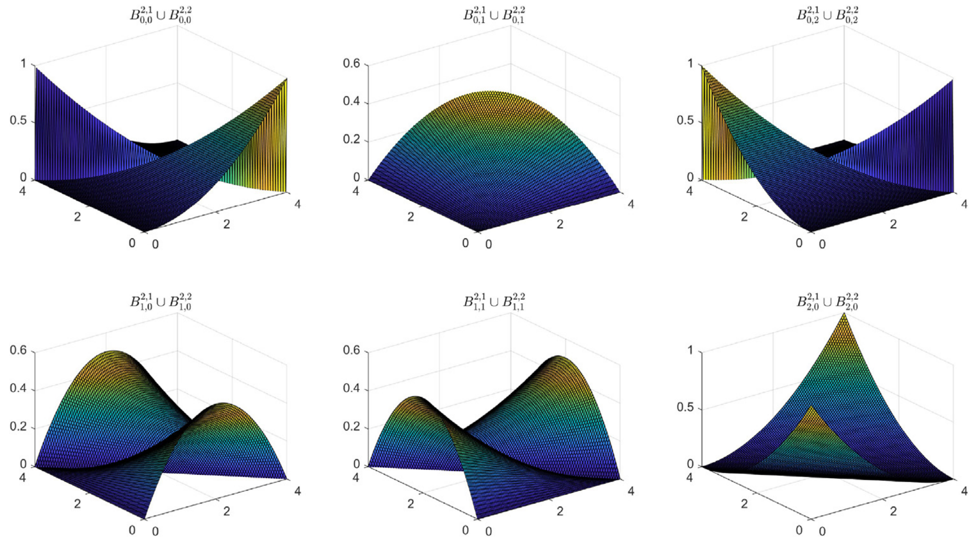

Fig. 3. Case $n = 2$, $L_1 = L_2 = 4$. Plots of the union of the two Bernstein polynomials $B^{n,1}_{i,j}$, $B^{n,2}_{i,j}$ defined over the two triangles T_1, T_2, for $i = 0, \ldots, n,\ j = 0, \ldots, n - i$.

with the Bernstein polynomials

$$B^{n,2}_{i,j}(x_1, x_2) = \frac{1}{L^n}\binom{n}{i}\binom{n-i}{j}(x_1 + x_2 - L)^i (L - x_1)^j\,(L - x_2)^{n-i-j}, \tag{11}$$

for $i = 0, \ldots, n,\ \ j = 0, \ldots, n - i$, satisfying

$$B^{n,2}_{i,j}(x_1, x_2) = B^{n,1}_{i,j}(L - x_1, L - x_2), \quad x_1, x_2 \in T_2. \tag{12}$$

Again, we can also use the equivalent alternative representation

$$p(x_1, x_2) = \sum_{i+j+k=n} \tilde{b}^1_{i,j,k}\, B^{n,2}_{i,j,k}(x_1, x_2), \quad x_1, x_2 \in T_2,$$

where the Bernstein polynomials are

$$B^{n,2}_{i,j,k}(x_1, x_2) = \frac{1}{L^n}\binom{n}{i}\binom{n-i}{j}(x_1 + x_2 - L)^i (L - x_1)^j\,(L - x_2)^k \quad i + j + k = n. \tag{13}$$

The plots of the union of the two Bernstein polynomials $B^{n,1}_{i,j}$, $B^{n,2}_{i,j}$ defined over $\Omega = T_1 \cup T_2$, for each pair (i, j), $i = 0, \ldots, n,\ j = 0, \ldots, n - i$, are shown in Fig. 3. Though the interesting case, considered in [2], corresponds to $n = 4$, and thus to a total of 15 plots, for page spacing reasons here we fix $n = 2$.

On the whole domain Ω, the polynomial p has then the piecewise representation

$$p(x_1, x_2) = \begin{cases} \displaystyle\sum_{i+j+k=n} b^1_{i,j,k}\, B^{n,1}_{i,j,k}(x_1, x_2), & (x_1, x_2) \in T_1, \\[2ex] \displaystyle\sum_{i+j+k=n} b^2_{i,j,k}\, B^{n,2}_{i,j,k}(x_1, x_2), & (x_1, x_2) \in T_2. \end{cases} \tag{14}$$

The crucial aspect is that this representation is unique if we impose the regularity conditions of the polynomial p along the common edge of the triangles, that is the diagonal of the square.

The link between the coefficients $b^1_{i,j,k}$ and $b^2_{i,j,k}$ is given by [14]:

$$b^2_{i,j,k} = \sum_{\nu+\mu+s=i} b^1_{\nu,k+\mu,j+s} B^{i,1}_{\nu,\mu,s}(L,L), \quad j+k=n-i \quad i=0,\ldots,n, \tag{15}$$

and therefore, p in (14) depends on $\binom{n+2}{2}$ coefficients only, as expected.

The relation connecting the two sequences of coefficients $b^1_{i,j,k}$, $b^2_{i,j,k}$, $k=n-i-j$, can be written in a matrix form as follows. Let us arrange the sequences as vectors following a lexicographical order:

$$\mathbf{b}^1 = [b^1_{0,0}, b^1_{0,1}, \ldots b^1_{0,n}, b^1_{1,0}, \ldots, b^1_{1,n-1}, \ldots, b^1_{n-1,0}, \ldots, b^1_{n-1,1}, b^1_{n,0}]^T,$$

$$\mathbf{b}^2 = [b^2_{0,0}, b^2_{0,1}, \ldots b^2_{0,n}, b^2_{1,0}, \ldots, b^2_{1,n-1}, \ldots, b^2_{n-1,0}, \ldots, b^2_{n-1,1}, b^2_{n,0}]^T.$$

Then,

$$\mathbf{b}^2 = \boldsymbol{A}\mathbf{b}^1$$

where $\boldsymbol{A}$ is the matrix with elements given by the values of the Bernstein polynomials $B^{i,1}_{\nu,\mu,s}$ at the point (L,L).

The structure of the matrix $\boldsymbol{A}$ for $n=4$ (the case considered in [2]) is

$$\boldsymbol{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 2 & 1 & 0 & 0 & -2 & -2 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 2 & 1 & 0 & 0 & -2 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 2 & 1 & 0 & 0 & -2 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 3 & 3 & 1 & 0 & -3 & -6 & -3 & 0 & 3 & 3 & 0 & -1 & 0\\
1 & 3 & 3 & 1 & 0 & -3 & -6 & -3 & 0 & 3 & 3 & 0 & -1 & 0 & 0\\
1 & 4 & 6 & 4 & 1 & -4 & -12 & -12 & -4 & 6 & 12 & 6 & -4 & -4 & 1
\end{pmatrix}.$$

The piecewise representation in (14) can then be rewritten in a more compact form as:

$$p(x) = \begin{cases} \boldsymbol{B}^{n,1}(x)\,\mathbf{b}^1, & x \in T_1, \\ \boldsymbol{B}^{n,2}(x)\boldsymbol{A}\mathbf{b}^1, & x \in T_2. \end{cases} \tag{16}$$

where $\boldsymbol{B}^{n,1}(x)$ and $\boldsymbol{B}^{n,2}(x)$ are the row vectors constructed from the functions in (10), (11) by using the same ordering as for $\mathbf{b}^1$ and $\mathbf{b}^2$.

The Bernstein representation of p in Ω thus depends on $\binom{n+2}{2}$ free parameters that can be obtained by considering the binary image $I = \chi_D$ and generalizing the arguments from [15]. This is what we detail in the next section.

4. Identification of algebraic boundaries from piecewise Bernstein moments

In this section, we show how to formulate the problem of recovering an algebraic curve in Bernstein form from the solution of a system of moment equations working over the two triangles as explained in Section 3. We follow the arguments used in [15] to formulate such a system of linear equation based on the B-moments. The existence and uniqueness of its solution, providing the coefficients of the polynomial describing the bounding curve, is a direct consequence of Theorem 2.2 in [15] combined with a change of basis argument.

We begin by reviewing the Stokes' Theorem based on Whitney's generalization [21], from which we deduce the formulation over two triangles rather than one triangle as done in [2].

Theorem 4.1. *Let $D \subset \mathbb{R}^2$ be bounded and open, with boundary ∂D that is smooth up to a set of measure zero in $\mathbb{R}$. Let $n(x)$ be the outward pointing normal to D at $x \in \partial D$. Then, given a vector field X on $\mathbb{R}^2$ and a differentiable function f, we have*

$$\int_D div(X)\, f(x)\, dx + \int_D X \cdot \nabla f(x)\, dx = \int_{\partial D} X \cdot n(x) f(x)\, ds.$$

We next take

$$f(x) = (x_1 + x_2 - L)^k p(x),$$

where $k \in \mathbb{N}$ and p is a polynomial of degree $n \geq 1$ vanishing on ∂D.

Letting $X = (x_1 + x_2 - L, x_1 + x_2 - L)$, after few manipulations, Stokes' theorem gives the following:

$$2(1+k) \int_D (x_1 + x_2 - L)^k p(x)\, dx + \int_D (x_1 + x_2 - L)^k \left(X \cdot \nabla p(x)\right)\, dx = 0. \tag{17}$$

We now consider that p is given as in (14) and apply some properties of the Bernstein polynomials stated in the next propositions.

The first property concerns the multiplication of a Bernstein polynomial with a monomial factor.

Proposition 4.1. *For a fixed $k \in \mathbb{N}$, the following equalities hold:*

$$(x_1 + x_2 - L)^k\, B^{n,1}_{i,j}(x) = q^{n,k}_i\, B^{n+k,1}_{i+k,j}(x), \quad x \in T_1, \tag{18}$$

$$(x_1 + x_2 - L)^k\, B^{n,2}_{i,j}(x) = \tilde{q}^{n,k}_i\, B^{n+k,2}_{i+k,j}(x), \quad x \in T_2, \tag{19}$$

where

$$q^{n,k}_i = (-1)^k \tilde{q}^{n,k}_i \quad \tilde{q}^{n,k}_i = L^k \frac{n!\,(i+k)!}{i!\,(n+k)!} = \frac{L^k (i+k)\cdots(i+1)}{(n+k)\cdots(n+1)}. \tag{20}$$

Proof. Multiplication of (10) and of (11) by $(x_1 + x_2 - L)^k$ produces

$$(L - x_1 - x_2)^k\, B^{n,1}_{i,j}(x_1, x_2) = \frac{\binom{n}{i}\binom{n-i}{j}}{L^n} (L - x_1 - x_2)^{i+k}\, x_1^j\, x_2^{n+k-(i+k)-j},$$

and

$$(x_1 + x_2 - L)^k B^{n,2}_{i,j}(x_1, x_2) = \frac{\binom{n}{i}\binom{n-i}{j}}{L^n} (x_1 + x_2 - L)^{i+k} (L - x_1)^j\, (L - x_2)^{n+k-(i+k)-j},$$

and therefore the claim. □

The second property is related to a classical recurrence result concerning the partial derivatives of Bernstein polynomials on both T_1 and T_2 (see [6] for details).

Proposition 4.2. *The partial derivatives of the polynomials $B^{n,1}_{i,j}$ and $B^{n,2}_{i,j}$ with respect to the variables x_1, x_2 are respectively given by*

$$\frac{\partial}{\partial x_1} B^{n,1}_{i,j}(x) = \frac{n}{L}\left(B^{n-1,1}_{i,j-1}(x) - B^{n-1,1}_{i-1,j}(x)\right),\ i = 1, \ldots, n,\ \ j = 0, \ldots, n-i,$$

$$\frac{\partial}{\partial x_2} B^{n,1}_{i,j}(x) = \frac{n}{L}\left(B^{n-1,1}_{i,j}(x) - B^{n-1,1}_{i-1,j}(x)\right),\ i = 0, \ldots, n,\ \ j = 0, \ldots, n-i,$$

$$\frac{\partial}{\partial x_1} B^{n,2}_{i,j}(x) = \frac{n}{L}\left(B^{n-1,2}_{i-1,j}(x) - B^{n-1,2}_{i,j-1}(x)\right),\ i = 1, \ldots, n,\ \ j = 0, \ldots, n-i,$$

$$\frac{\partial}{\partial x_2} B^{n,2}_{i,j}(x) = \frac{n}{L}\left(B^{n-1,2}_{i-1,j}(x) - B^{n-1,2}_{i,j}(x)\right),\ i = 0, \ldots, n,\ \ j = 0, \ldots, n-i,$$

with the convention that $B^{n,\ell}_{i,j}(x) = 0,\ \ \ell = 1, 2$ whenever $i < 0,\ \ i > n,\ \ j < 0, j > n$.

In particular from the above expressions we deduce:

$$(x_1+x_2-L)\,\frac{\partial}{\partial x_1}\,B^{n,1}_{i,j}(x) = -(i+1)\,B^{n,1}_{i+1,j-1}(x) + i\,B^{n,1}_{i,j}(x),$$

$$(x_1+x_2-L)\,\frac{\partial}{\partial x_2}\,B^{n,1}_{i,j}(x) = -(i+1)\,B^{n,1}_{i+1,j}(x) + i\,B^{n,1}_{i,j}(x),$$

and similarly

$$(x_1+x_2-L)\,\frac{\partial}{\partial x_1}\,B^{n,2}_{i,j}(x) = i\,B^{n,2}_{i,j}(x) - (i+1)\,B^{n,2}_{i+1,j-1}(x),$$

$$(x_1+x_2-L)\,\frac{\partial}{\partial x_2}\,B^{n,2}_{i,j}(x) = i\,B^{n,2}_{i,j}(x) - (i+1)\,B^{n,2}_{i+1,j}(x).$$

Hence for $X = (x_1+x_2-L, x_1+x_2-L)$

$$X\cdot\nabla B^{n,1}_{i,j}(x) = -(i+1)\,B^{n,1}_{i+1,j-1}(x) + 2i\,B^{n,1}_{i,j}(x) - (i+1)\,B^{n,1}_{i+1,j}(x),$$

and

$$X\cdot\nabla B^{n,2}_{i,j}(x) = -(i+1)\,B^{n,2}_{i+1,j-1}(x) + 2i\,B^{n,2}_{i,j}(x) - (i+1)\,B^{n,2}_{i+1,j}(x).$$

Thus, (17) becomes

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{j=0}^{n-i} b^1_{i,j}\int_{D\cap T_1}(x_1+x_2-L)^k\Big[-(i+1)\,B^{n,1}_{i+1,j-1}(x)\\
&\quad +(2+2i+2k)\,B^{n,1}_{i,j}(x) - (i+1)\,B^{n,1}_{i+1,j}(x)\Big]\,dx\\
&+\sum_{i=0}^{n}\sum_{j=0}^{n-i} b^2_{i,j}\int_{D\cap T_2}(x_1+x_2-L)^k\Big[-(i+1)\,\tilde{B}^{n,2}_{i+1,j-1}(x)\\
&\quad +(2+2i+2k)\,\tilde{B}^{n,2}_{i,j}(x) - (i+1)\,\tilde{B}^{n,2}_{i+1,j}(x)\Big]\,dx = 0\end{aligned}$$

which, in virtue of (18) and (19), takes the form

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{j=0}^{n-i}\Big(b^1_{i,j}\Big[(i+1)q^{n,k}_{i+1}\Big(2\,m^{n+k,1}_{i+k,j} - m^{n+k,1}_{i+1+k,j-1} - m^{n+k,1}_{i+1+k,j}\Big)\Big]\\
&\quad + b^2_{i,j}\Big[(i+1)\tilde{q}^{n,k}_{i+1}\Big(2\,m^{n+k,2}_{i+k,j} - m^{n+k,2}_{i+1+k,j-1} - m^{n+k,2}_{i+1+k,j}\Big)\Big]\Big) = 0,\end{aligned} \tag{21}$$

where we have used the equalities:

$$(i+1+k)q^{n,k}_i = (i+1)q^{n,k}_{i+1}, \quad (i+1+k)\tilde{q}^{n,k}_i = (i+1)\tilde{q}^{n,k}_{i+1}$$

and where we have denoted with

$$m^{\ell,\epsilon}_{i,j} = \int_{D\cap T_\epsilon} B^{\ell,\epsilon}_{i,j}(x)dx, \quad \ell\in\mathbb{N}_0,\ \epsilon\in\{1,2\} \tag{22}$$

the *triangular Bernstein moments* (*B-moments*). We then define the matrices $\mathbf{G}^1$, $\mathbf{G}^2$, respectively with elements

$$\begin{aligned}g^1(k,\ell(i,j)) = &\quad (i+1)q^{n,\alpha}_{i+1}(2\,m^{n+k,1}_{i+k,j} - m^{n+k,1}_{i+1+k,j-1} - m^{n+k,1}_{i+1+k,j}),\\
g^2(k,\ell(i,j)) = &\quad (i+1)\tilde{q}^{n,\alpha}_{i+1}(2\,m^{n+k,2}_{i+k,j} - m^{n+k,2}_{i+1+k,j-1} + m^{n+k,2}_{i+1+k,j}),\end{aligned} \tag{23}$$

where the column index

$$\ell(i,j) = (n+1)i - \frac{(i-1)i}{2} + j$$

ranges from 0 to $\binom{n+2}{2} - 1$. Then, fixing $K \in \mathbb{N}_0$, from (21) and (23) we obtain the homogeneous equations

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i} g^1(k, \ell(i,j))\, b^1_{i,j} + g^2(k, \ell(i,j))\, b^2_{i,j} = 0, \quad 0 \le k \le K \tag{24}$$

that can also be written as

$$\mathbf{G}^1\mathbf{b}^1 + \mathbf{G}^2\mathbf{b}^2 = 0, \quad \text{where} \quad \mathbf{G}^\epsilon \in \mathbb{R}^{K\times\binom{n+2}{2}},\ \mathbf{b}^\epsilon \in \mathbb{R}^{\binom{n+2}{2}}, \quad \epsilon \in \{1,2\},$$

or, in virtue of (15), $\mathbf{b}^2 = \mathbf{A}\mathbf{b}^1$,

$$\left(\mathbf{G}^1 + \mathbf{G}^2\mathbf{A}\right)\mathbf{b}^1 = 0. \tag{25}$$

It is observed in [15] that, though the minimal number of equations in (24) required to exactly recover the desired polynomial is $\binom{2n+2}{2}$, in practice one needs a much less number of equations, in most cases corresponding to a square matrix in (25).

Nevertheless a more stable approach can be obtained if the common solution of two systems

$$\begin{aligned} &\mathbf{G}^1\mathbf{b}^1 = 0,\\ &\mathbf{G}^2\mathbf{A}\mathbf{b}^1 = 0. \end{aligned} \tag{26}$$

is found

A similar approach has been used in the experimentation carried out in [2,9]. In the latter, in particular, it was shown that the strategy leading to an overdetermined system of $2(\lceil n/2\rceil + 1)^2$ produces the non trivial solution to (24), as a direct consequence of Theorem 2.2 in [15], unique under the constraint $b^1_{0,0} = 1$, thus yielding the searched polynomial coefficients.

5. Computation of the triangular Bernstein moments

This section aims at discussing how the polynomial generation property of the kernel φ can help to compute the triangular Bernstein moments. Indeed, from the previous section it is clear that, in order to set up Eqs. (21), what is needed is a tool for the computation of the B-moments (22), and this can be achieved from the image samples.

From our assumption on I, we observe that

$$m^{\ell,\epsilon}_{i,j} = \int_{D\cap T_\epsilon} B^{\ell,\epsilon}_{i,j}(x)dx = \int_{T_\epsilon} B^{\ell,\epsilon}_{i,j}(x)I(x)dx \quad \ell \in \mathbb{N}_0,\ \epsilon \in \{1,2\}.$$

We assume that the kernel φ is refinable and possesses the polynomial generation property up to degree $n+k$. This means that there exists a set of coefficients $\{C^{\ell,\epsilon}_k,\ \epsilon \in \{1,2\}, k \in \mathbb{Z}^2\}$ such that locally, on the triangles T_1 and T_2, it satisfies,

$$B^{\ell,1}_{i,j}(x) = \sum_{k\in Z_1} C^{\ell,1}_k \varphi(x-k), \quad \ell = 0,\ldots,n+k, \quad x \in T_1, \tag{27}$$

$$B^{\ell,2}_{i,j}(x) = \sum_{k\in Z_2} C^{\ell,2}_k \varphi(x-k), \quad \ell = 0,\ldots,n+k, \quad x \in T_2, \tag{28}$$

where the coefficient indexes vary in the sets Z_1, Z_2 given by:

$$Z_1 = \{k = (k_1,k_2) \in \mathbb{Z}^2 : -N+1 \le k_1, k_2 \le L-1, k_1+k_2 \le L-1\},$$

and

$$Z_2 = \{k = (k_1,k_2) \in \mathbb{Z}^2 : -N+1 \le k_1, k_2 \le L-1, k_1+k_2 \ge L+1-2N\},$$

where $[0,N]^2$ is the support of φ.

Based on (27) and (28), the B-moments can be written as

$$m^{\ell,\epsilon}_{i,j} = \int_{T_\epsilon} B^{\ell,\epsilon}_{i,j}(x)I(x)\,dx = \sum_{k\in Z_\epsilon} C^{\ell,\epsilon}_k \int_{T_\epsilon} \varphi(x-k)I(x)\,dx, \quad \epsilon \in \{1,2\}. \tag{29}$$

Hence, if a discrete set of triangle samples

$$d_k^\epsilon = \int_{T_\epsilon} \varphi(x-k) I(x)\, dx, \quad \epsilon \in \{1,2\},$$

from the binary image $I = \chi_D$ is available, the B-moments reduce to

$$m_{i,j}^{\ell,\epsilon} = \sum_{k \in Z_\epsilon} C_k^{\ell,\epsilon} d_k^\epsilon \quad \epsilon \in \{1,2\}, \tag{30}$$

and so, in order to obtain the parameters associated to the polynomial representing the desired curve, only the reproduction coefficients are needed to compute the moments and, consequently, to solve the system (24).

If we additionally assume that the refinable function φ is separable, i.e. $\varphi(x) = \phi(x_1)\phi(x_2)$, then a further simplification is possible for finding the reproduction coefficients in (30).

In fact, based on simple manipulations on the bivariate Bernstein polynomials (10) and (11), it can be seen that they can be written in a separable way in terms of product of two univariate Bernstein polynomials, as follows:

$$B_{i,j}^{\ell,1}(x_1,x_2) = i! \sum_{m=j}^{i+j} \binom{\ell}{m} b_j^m(x_1) b_{\ell-i-j}^{\ell-m}(x_2), \quad (x_1,x_2) \in T_1, \tag{31}$$

$$B_{i,j}^{\ell,2}(x_1,x_2) = i! \sum_{m=j}^{i+j} \binom{\ell}{m} b_j^m(L-x_1) b_{\ell-i-j}^{\ell-m}(L-x_2), \quad (x_1,x_2) \in T_2, \tag{32}$$

where b_i^ℓ is the ℓth degree univariate Bernstein polynomial

$$b_i^\ell(t) = \frac{1}{L^\ell}\binom{\ell}{i} t^i (L-t)^{\ell-i}, \; i = 0,\dots,\ell, \quad t \in [0,L].$$

So, assuming the polynomial reproduction property of ϕ is satisfied over $[0, L]$, namely

$$b_i^\ell = \sum_{k=1-N}^{L-1} c_k^{\ell,i} \phi(\cdot - k)$$

we have

$$B_{i,j}^{\ell,1}(x_1,x_2) = \frac{i!}{2^n} \sum_{k_1,k_2=1-N}^{L-1} \sum_{m=j}^{i+j} \binom{\ell}{m} c_{k_1}^{m,j} c_{k_2}^{\ell-m,n-i-j} \phi(x_1-k_1)\phi(x_2-k_2), \quad (x_1,x_2) \in T_1, \tag{33}$$

which, compared to (27), gives

$$C_{k_1,k_2}^{\ell,1} = \frac{i!}{2^n} \sum_{m=j}^{i+j} \binom{\ell}{m} c_{k_1}^{m,j} c_{k_2}^{\ell-m,n-i-j}, \quad k_1,k_2 \in Z_1.$$

Analogously

$$C_{k_1,k_2}^{\ell,2} = \frac{i!}{2^n} \sum_{m=j}^{i+j} \binom{\ell}{m} c_{L-N-k_1}^{m,j} c_{L-N-k_2}^{\ell-m,n-i-j}, \quad k_1,k_2 \in Z_2.$$

We conclude mentioning that such approach needs the computation of the triangular samples of the binary image $I = \chi_D$ given in (29). This can be done through quadrature rules for integration with respect to refinable functions on assigned nodes as discussed [10] or through Gaussian quadrature for refinable weight functions as discussed in [16].

6. Conclusions

In this paper, we have proposed a strategy for the reconstruction of the algebraic boundary of a binary image expressed in terms of non-separable bivariate Bernstein polynomials. Such a strategy does not suffer of the instabilities connected to the reconstruction from moments, which typically arise when the algebraic domain is expressed in the monomial basis. Our procedure differs from what proposed in [2] in that the Bernstein polynomials are defined piecewisely over the whole rectangular region defining the image domain. Thus, it avoids the embedding of such domain in a larger triangle, which has the unpleasant consequence of increasing the size of the problem and the computational cost. The experimentation conducted so far shows that the results in terms of reconstruction errors are comparable to [2], but the overall algorithm/code can be rendered in more elegant and time efficient way. The details of the implementation and the illustration of the experimental results are beyond the scope of this paper, whose aim is purely a theoretical description of the procedure. Nevertheless an extensive campaign of tests over images of different nature (for example biological images) is the focus of an ongoing research and the results will be the core of a future paper.

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