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Generating Special Curves for Cubic Polynomials

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Abstract: An algorithmic method is proposed to generate all cubic polynomials with a critical orbit relation. We generate curves (polynomials of parameters) that correspond to those functions with critical orbit relations. The irreducibility of the polynomials obtained is left as an open problem. Our approach also works to generate critical orbit relations in all families of rational functions with active critical points.

Keywords: critical orbit relations; resultant; recurrence relation

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1. Introduction and Statement of the Problem

In their influential paper, Baker and DeMarco [1] explored the distribution of postcritically finite (PCF) polynomials through the perspective of the Zariski topology in algebraic geometry. They defined special algebraic subvarieties as subvarieties within the space of degree-d polynomials that contain a Zariski-dense set of postcritically finite polynomials. They then posed the problem of classifying all such subvarieties.

Baker and DeMarco called an algebraic variety in the space of degree d polynomials (or rational maps) "special" if it contains infinitely many postcritically finite polynomials (rational maps) and asked to classify such special varieties [1]. Recently, this problem was solved for the cubic polynomials in [2] and for all degrees in [3] by Favre and Gauthier. They showed that the special curves can be characterized by the orbits of critical points in three cases: one of the critical points is persistently preperiodic, the two critical points have a persistent critical orbit relation, or there is a symmetry via $z \mapsto -z$, in which case the curve is $b = 0$, ([2] Theorem A). The special curves where the critical point is periodic were studied by Milnor [4]. The works of Milnor were generalized for rational maps with a preperiodic critical point by Buff, Epstein, and Koch and they showed that prefixed subvarieties are irreducible [5].

This research builds on significant advances in holomorphic dynamics, particularly the classification of special algebraic varieties containing postcritically finite polynomials, as established by Baker, DeMarco, Favre, and Gauthier [1,3] . By addressing critical orbit relations in cubic polynomials, our study extends the computational and theoretical techniques used to explore stability and bifurcation in parameter spaces. This work introduces an iterative framework for generating dynamical systems with specific critical orbit relations, offering new tools for understanding the structure and irreducibility of COR varieties. These contributions are pivotal for advancing computational approaches in dynamical systems and analyzing the parameter spaces of complex polynomials.

The approach is applicable for generating critical orbit relations in all families of rational functions with active critical points. We address the irreducibility of the polynomials in the space of cubic polynomials for some low-degree cases. In this regard, we would like to remark that experiments with other cases show that the irreducibility problem is very hard and for each case one needs to use different tools.

Every cubic polynomial has the Branner–Hubbard form $p(z) = z^3 - 3a^2z + b$, where a, b are complex numbers [6]. The critical points are at $\pm a$. Cubic polynomials as a dynamical system and the parameter plane of cubic polynomials were studied intensively by many in [4,6–9]. Define the iterates of $p(z)$ by letting $p^0(z) = z$ and $p^{n+1}(z) = p(p^n(z))$, for $z \in \mathbb{C}$. In this paper, we solve the following problem.

Problem 1. *For each pair of (n, m), find all polynomials $p(z) = z^3 - 3a^2z + b$ such that*

$$p^n(a) = p^m(-a). \tag{1}$$

Here, we also require that $p^{n-1}(a) \neq p^{m-1}(-a)$ if $n \geq 1$ and $m \geq 1$. In fact, we give an answer to this problem as an algebraic curve on complex variables a, b, which is the Zariski closure of the set of points that solve the Problem 1. The irreducibility of the obtained curves is of great importance and is left as an open problem (see Proposition 2).

The orbit of a point z is $\{z, p(z), \ldots, p^{\circ n}(z), \ldots\}$. A point $z_0 \in \mathbb{C}$ is called a critical point of a polynomial p if $p'(z_0) = 0$. A polynomial p is called postcritically finite if the orbit of all critical points of p is finite as a set. A point $z_0 \in \mathbb{C}$ is called a fixed point of a polynomial p if $p(z_0) = z_0$. The derivative $p'(z_0)$ at the fixed point is called its multiplier. Two polynomials p and q are called affine conjugate if and only if there exists an affine map $\phi(z) = k_1 z + k_2$ with $k_1 \neq 0, k_2 \in \mathbb{C}$ (conjugacy) such that $\phi(p(z)) = q(\phi(z))$ for all $z \in \mathbb{C}$. Note that here ϕ plays the role of affine change of coordinates in z-plane and in $w = p(z)$ planes at the same time. In the study of dynamical systems such a dynamical change in coordinates produces the same dynamics. By scaling ($z \mapsto kz$), we can make the leading coefficient of any polynomial 1. In general, a cubic polynomial has two critical points so by a change of coordinates (translation ($z \mapsto z + k$) we can put them at some $\pm a$ and obtain the Branner–Hubbard form.

We also consider cubic polynomials written in the form

$$q(z) = z^3 + \mu z^2 + \lambda z \tag{2}$$

with a fixed point at the origin with the multiplier λ. The family in this form is denoted by $\mathrm{Per}_1(\lambda)$. Critical points of q solve the quadratic equation

$$3z^2 + 2\mu z + \lambda = 0.$$

Denote the critical points by c_1 and c_2. If $\lambda \neq 0$ then it is not possible to label (mark) the critical points without taking the square root function in the complex plane.

Problem 2. *For each pair of (n, m), find all polynomials $q(z) = z^3 + \mu z^2 + \lambda z$ such that*

$$q^n(c_1) = q^m(c_2). \tag{3}$$

These two problems are equivalent to each other but it is easy to work with the first one as the critical points are marked at $\pm a$. Indeed, let z_0 be any fixed point of $p(z) = z^3 - 3a^2z + b$; by the change of coordinates $z \mapsto z + z_0$ we can put its fixed point to the origin to obtain a cubic polynomial of the form (2). Now we start with a cubic polynomial of the from (2) and let $c_0 = (c_1 + c_2)/2$ be the midpoint of the critical points c_1 and c_0 of $q(z) = z^3 + \mu z^2 + \lambda z$ (the midpoint of the line segment joining the two critical points). By the change in coordinates $z \mapsto z + c_0$ we can put its critical points to two points symmetric with respect to the origin. The resulting cubic polynomial is of the Branner–Hubbard form. Now for $p(z) = z^3 - 3a^2z + b$ and $q(z) = z^3 + \mu z^2 + \lambda z$, for convenience, denote the change in the coordinate by ϕ with $\phi(a) = c_1$ and $\phi(-a) = c_2$ then $p(z) = \phi^{-1}(q(\phi(z)))$ and for all $k \in \mathbb{N}$ we obtain $p^k(z) = \phi^{-1}(q^k(\phi(z))$. Finally, we assume that p satisfies the Problem 1 for (n, m), i.e., $p^n(a) = p^m(-a)$. Then, $\phi^{-1}(q^n(\phi(a))) = \phi^{-1}(q^m(\phi(-a)))$. Applying ϕ to both sides of the latter we obtain $q^n(\phi(a)) = q^m(\phi(-a))$ or $q^n(c_1) = q^k(c_2)$ which solves Problem 2 with the same (n, m). Going backward, we can show that every solution to Problem 2 is also a solution to Problem 1.

One can also consider the following subproblem if we restrict to a slice $\lambda = 1$ and work in $\mathrm{Per}_1(1)$.

Problem 3. *For each pair of (n, m), find all polynomials $q(z) = z^3 + \mu z^2 + z$ such that*

$$q^n(c_1) = q^m(c_2). \tag{4}$$

The third problem has the same difficulty as the second one but both can be derived from the first.

Two distinct cubics $z^3 - 3a^2z + b$ and $z^3 - 3a'^2z + b'$ are affine conjugate if and only if $a' = -a$ and $b' = -b$; the conjugacy is $z \mapsto -z$. Indeed, for $a \neq 0$ assume that there is an affine conjugacy: $p_1(k_1z + k_2) = k_1p_2(z) + k_2$. Take the derivative from both sides and obtain $p_1'(k_1z + k_2) = p_2'(z)$. It yields that the conjugacy $\phi(z) = k_1z + k_2$ sends the critical points of p_2 to the critical points of p_1. If we substitute $z = \pm a_2$ we must have $\pm k_1a_2 + k_2 = \pm a_1$ or $\pm k_1a_2 + k_2 = \mp a_1$. In both cases, we obtain $k_2 = 0$. Then, we obtain $p_1(k_1z) = k_1p_2(z)$ and directly obtain $k_1 = \pm 1$. Thus, the conjugacy is the identity or it is $z \mapsto -z$. If $a = 0$ then the conjugacy is $(k_1z + k_2)^3 + b_1 = k_1(z^3 + b_2) + k_2$. By collecting common terms we obtain $k_1 = \pm 1$ and $k_2 = 0$, which finishes the claim. The conjugacy $z \mapsto -z$ interchanges the markings (labeling) of critical points $\pm a$. Thus, the moduli space, consisting of all affine conjugacy classes of cubics with marked critical points, can be identified with coordinates $(a^2, b^2) \in \mathbb{C}^2$. It means that all four pairs $(\pm a, \pm b)$ correspond to the same equivalence class.

Definition 1. *For a polynomial p with critical points c_1 and c_2 a* **critical orbit relation** *is a quadruple (n, m, c_1, c_2) (in short, a pair (n, m)) with nonnegative integers n and m such that*

$$p^n(c_1) = p^m(c_2). \tag{5}$$

If $c_1 = c_2$ then we require $n \neq m$. If $c_1 = c_2$ then the critical orbit relation is of the first type and for $c_1 \neq c_2$ it is of the second type. In this paper, we only consider critical orbit relations of the second type.

We do not require such n, m to be exact but only ask if (5) is true then $f^{n-1}(c_1) \neq f^{m-1}(c_2)$ and we call it a **minimal relation**. Every critical orbit relation of the form $(n, 0)$ satisfies our requirement so it is minimal. As the Equation (5) is symmetric with respect to n and m, it suffices to consider only the cases of $n \geq m$.

For a polynomial $p(z) = z^3 - 3a^2z + b$, a **critical orbit relation** is a pair (n, m) with non-negative integers n and m such that for the critical points a and $-a$ we have (1).

Let $p_t(z)$ for $t \in \chi$ be a holomorphic family (χ is a complex manifold) of cubic polynomials and mark critical points $c_1(t), c_2(t)$ of p_t. A point $t = t_0$ belongs to the stability locus [10] if the Julia sets $J(p_t)$ move holomorphically in a neighborhood of t_0. Alternatively, a point $t = t_0$ belongs to the stability locus if the sequence

$$\{t \mapsto p_t^n(c_i(t))\}$$

forms a normal family on some neighborhood of t_0 for both $i \in \{1, 2\}$. A point $t = t_0$ belongs to the **bifurcation locus** if stability fails at t_0. In complex dynamics, the bifurcation locus (also known as the activity locus [10,11]) of a parameterized family of one-variable holomorphic functions refers to the set of parameter points where small changes in the parameter cause significant changes in the dynamical behavior. This locus is often seen as a counterpart to the Julia set but in the context of parameter space. The most well-known example of a bifurcation locus is the boundary of the Mandelbrot set [10].

The following lemma motivates us to study functions with interacting distinct critical points. It is analogous to Lemma 2.3 in [11].

Lemma 1. *Assume that $\{t \mapsto p_t^n(c_1(t))\}$ is not normal at t_0 for $c_1(t)$ (the bifurcation locus is not empty) and $\#\{\text{orbit of } c_2(t)\} \geq 3$ persists throughout χ (or in every sufficiently small neighborhood of t_0). Then, there are infinitely many parameters $t \in \chi$ such that $c_1(t)$ and $c_2(t)$ are in critical orbit relations. In particular, there are infinitely many parameters $t \in \chi$ such that $p_t^n(c_1(t)) = c_2(t)$, where $n \in \mathbb{N}$ depends on t.*

Proof. Consider t_0 from the bifurcation locus. In a small neighborhood of t_0 let $c_2^0(t) \neq c_2^1(t)$ be two preimages of $c_2(t)$. An application of Montel's theorem with the triple $c_2^0(t), c_2^1(t), c_2(t)$, which is persistent, finishes the proof. If there are no two preimages, then they must coincide, creating a critical point, which must be $c_1(t)$ as there are only two critical points, so that the image of it is the critical point $c_2(t)$ so that the relation is of the form $p_t(c_1(t)) = c_2(t)$. □

The Problem 1 may be reduced to computing the result of two polynomials $p^n(z) - p^m(-z)$ and $z^2 - a^2$, with respect to variable z [12] (see Proposition 1). The result (denoted by $Res_z(p^n(z) - p^m(-z), z^2 - a^2)$) is a polynomial on the parameters a, b.

We propose a procedure for constructing a sequence of auxiliary polynomials (see Lemma 2) from which we directly arrive at the polynomial equation in the parameters a, b, a zero set of which corresponds to a critical orbit relation (1) for each pair n, m.

The zero set of the obtained polynomials can also be viewed as a family of a critical orbit relation, known as a COR variety (Critical orbit relations), in the parameter space of cubic maps, as described in [13]. These COR varieties are fundamental in the analysis of holomorphic and arithmetic dynamical systems, and they have been extensively studied [13–16]. With this paper, we proposed a new class of COR curves. The problem of the irreducibility of these new curves can be studied by the use of the techniques developed in [5,15–18].

The notion of poscritical minimality in families of rational functions was introduced in [19,20]. The condition on functions we consider in this paper is weaker than the condition for poscritical minimality. Moreover, the functions with a critical orbit relation are not necessarily postcritically finite. They can well be postcritically infinite. The above-mentioned result of Favre and Gauthier [2] shows that each variety defining the critical orbit relation contains infinite postcritically finite cubic polynomials. If we consider a slice in the cubic family consisting of polynomials with a persistent attracting or parabolic periodic point (e.g., $\mathrm{Per}_1(1)$), all such maps will be postcritically infinite. But in these slices, there always exist those with a critical orbit relation. On the other hand, if the slice consists of entirely cubics with a persistent superattracting period point then all cubics with a critical orbit relation are necessarily postcritically finite.

2. From Critical Orbit Relations to Recurrence Relations

This section is a preliminary to a quantitative answer to the main Lemma 1, which is the main theme of the paper. Below we reduce the orbit of the critical point to a simple looking form with universal polynomials that are recursively obtained. In what follows, the powers in the polynomials p represent iterations while the powers of newly introduced polynomials and the variables or in the algebraic expressions represent the algebraic power.

Lemma 2. *There exist sequences $\{A_n(a,b)\}_{n\geq 0}$ and $\{B_n(a,b)\}_{n\geq 0}$ of polynomials of parameters a,b such that for all $n \geq 0$ the equality $p^n(\pm a) = \pm a A_n(a,b) + B_n(a,b)$ holds.*

Proof. As $p^0(z) = z$, set $A_0(a,b) = 1$ and $B_0(a,b) = 0$. Recurrently define polynomials $A_n(a,b)$ and $B_n(a,b)$ such that $A_{n+1}(a,b)z + B_{n+1}(a,b) = p(A_n(a,b)z + B_n(a,b))$ for $z = \pm a$. It follows that the recurrence relation is the following.

$$A_{n+1}(a,b) = A_n(a,b)(a^2 A_n^2(a,b) + 3B_n^2(a,b) - 3a^2), \tag{6}$$

$$B_{n+1}(a,b) = B_n^3(a,b) + 3a^2 B_n(a,b)(A_n^2(a,b) - 1) + b. \tag{7}$$

The above formulas are obtained by substituting $z^2 = a^2$, $z^3 = a^2 z$ for $z = \pm a$ into the expansion of $(A_n(a,b)z + B_n(a,b))^3 - 3a^2(A_n(a,b)z + B_n(a,b)) + b$ and combining the common terms. □

Denote by $\deg_a Q(a,b)$ the degree of a two-variable polynomial $Q(a,b)$ with respect to a variable a and by $\deg Q$ the degree of a two-variable polynomial $Q(a,b)$, which is the sum of degrees of variables a and b in the highest monomial of Q. For instance, $\deg_a(a^2 - ab^4) = 2$ and $\deg(a^2 - ab^4) = 5$. It is easy to see from the recurrence relations that $\deg_a A_n(a,b) = \deg A_n(a,b) = 3^n - 1$ for $n \geq 1$, $\deg_a B_n(a,b) = 3^n - 3$ and $\deg B_n(a,b) = 3^n - 2$ for $n \geq 1$ (to ease the notations in some places of this paper we drop the arguments in writing functions).

The following lemma gives even more structure to the introduced polynomials $\{A_n(a,b)\}_{n\geq 0}$ and $\{B_n(a,b)\}_{n\geq 0}$. It also helps to reduce by half the degrees of the later obtained algebraic curves that represent each critical orbit relation.

Lemma 3. *There exist sequences $\{\tilde{A}_n(x,y)\}_{n\geq 0}$ and $\{\tilde{B}_n(x,y)\}_{n\geq 0}$ of polynomials such that for every $n \geq 0$ one has $A_n(a,b) = \tilde{A}_n(a^2,b^2)$ and $B_n(a,b) = b\tilde{B}_n(a^2,b^2)$.*

Proof. We proceed by induction on n. For $n = 0$ the statement is true as $A_0 = 1$ and $B_0 = 0$, set $\tilde{A}_0 = 1$ and $\tilde{B}_0 = 0$. Suppose that the statement holds for n:

$$A_n(a,b) = \tilde{A}_n(a^2,b^2) \text{ and } B_n(a,b) = b\tilde{B}_n(a^2,b^2).$$

By the inductive hypothesis,

$$\begin{aligned} A_{n+1}(a,b) &= A_n(a,b)(a^2 A_n^2(a,b) + 3B_n^2(a,b) - 3a^2) \\ &= \tilde{A}_n(a^2,b^2)(a^2 \tilde{A}_n^2(a^2,b^2) + 3b^2 \tilde{B}_n^2(a^2,b^2) - 3a^2). \end{aligned}$$

Set

$$\tilde{A}_{n+1}(x,y) = \tilde{A}_n(x,y)(x\tilde{A}_n^2(x,y) + 3y\tilde{B}_n^2(x,y) - 3x), \tag{8}$$

so that $A_{n+1}(a,b) = \tilde{A}_{n+1}(a^2,b^2)$. Similarly,

$$\begin{aligned} B_{n+1}(a,b) &= B_n^3(a,b) + 3a^2 B_n(a,b)(\tilde{A}_n^2(a^2,b^2) - 1) + b \\ &= b^3\tilde{B}_n^3(a^2,b^2) + 3a^2 b\tilde{B}_n(a,b)(\tilde{A}_n^2(a^2,b)^2 - 1) + b \\ &= b\big(b^2\tilde{B}_n^3(a^2,b^2) + 3a^2\tilde{B}_n(a^2,b^2)(\tilde{A}_n^2(a^2,b^2) - 1) + 1\big). \end{aligned}$$

Set

$$\tilde{B}_{n+1}(x,y) = y\tilde{B}_n^3(x,y) + 3x\tilde{B}_n(x,y)(\tilde{A}_n^2(x,y) - 1) + 1, \tag{9}$$

so that $B_{n+1}(a,b) = b\tilde{B}_{n+1}(a^2,b^2)$. This finishes the proof. □

In fact, by the above we have

$$p^n(\pm a) = \pm a\tilde{A}_n(a^2,b^2) + b\tilde{B}_n(a^2,b^2)$$

for all $n \geq 0$.

The recurrence formulas that define A_n and B_n allow us to easily construct a linear equation in z from a critical orbit relation. If $n > m$ the critical relation $p^n(z) - p^m(-z) = 0$ becomes a linear equation in z, and for $n = m$ the critical relation reduces to $A_n(a,b) = 0$.

3. Main Technical Results

In this section, the critical orbit relations are reduced to polynomial relations. We consider them case by case.

Case of (n,n). By Lemma 2 we have

$$p^n(a) - p^n(-a) = A_n(a,b)a + B_n(a,b) - (-A_n(a,b)a + B_n(a,b)) = 2A_n(a,b)a, \quad n \geq 1.$$

This implies that the critical orbit relation reduces to $A_n(a,b) = 0$.

If $A_1 = -2a^2$ vanishes then $a = 0$. In this case, both critical points collide so the critical orbit relation is $(0,0)$. This means that there is no cubic polynomial with an exact critical orbit relation $(1,1)$. There is also a topological reason why the case $(1,1)$ is not realized in the cubic polynomial family.

Denote $P_{n,n}(a,b) = A_n(a,b)/A_1(a,b)$. Then,

$$p^n(a) - p^n(-a) = -4a^3 P_{n,n}(a,b), \tag{10}$$

and (6) yields

$$P_{n,n}(a,b) = A_n(a,b)/A_1(a,b) = A_{n-1}(a,b)/A_1(a,b)(a^2A_{n-1}^2(a,b) + 3B_{n-1}^2(a,b) - 3a^2).$$

Set for $n \geq 2$

$$\tilde{P}_{n,n}(a,b) = a^2A_{n-1}^2(a,b) + 3B_{n-1}^2(a,b) - 3a^2,$$

or by Lemma 3 we can write

$$\tilde{P}_{n,n}(a,b) = a^2\tilde{A}_{n-1}^2(a^2,b^2) + 3b^2\tilde{B}_{n-1}^2(a^2,b^2) - 3a^2. \tag{11}$$

This implies that, for $n \geq 2$, we can write the following

$$P_{n,n}(a,b) = P_{n-1,n-1}(a,b) \cdot \tilde{P}_{n,n}(a,b). \tag{12}$$

For $n \geq 1$, $\tilde{P}_{n,n}(x,y) = Q_{n,n}(x^2,y^2)$, where

$$Q_{n,n}(x,y) = x\tilde{A}^2_{n-1}(x,y) + 3y\tilde{B}^2_{n-1}(x,y) - 3x. \tag{13}$$

If there is a minimal critical orbit relation (n,n), then $\tilde{P}_{n,n}(a,b) = 0$.

Moreover, $\deg_a P_{n,n}(a,b) = \deg P_{n,n}(a,b) = 3^n - 3$ and $\deg_a \tilde{P}_{n,n}(a,b) = \deg \tilde{P}_{n,n}(a,b) = 2 \cdot 3^{n-1}$ for $n \geq 1$.

From (12) we obtain that if $P_{n,n}(a,b) = 0$ and the relation is minimal then $P_{n-1,n-1}(a,b) \neq 0$ it yields that $\tilde{P}_{n,n}(a,b) = 0$. By definition

$$\deg P_{n,n}(a,b) = \deg A_n - 2 = 3^n - 1 - 2 = 3^n - 3$$

and

$$\deg \tilde{P}_{n,n}(a,b) = 2 + 2\deg A_{n-1} = 2 + 2(3^{n-1} - 1) = 2 \cdot 3^{n-1}.$$

Moreover, $\deg_a P_{n,n}(a,b) = \deg P_{n,n}(a,b)$ and $\deg_a \tilde{P}_{n,n}(a,b) = \deg \tilde{P}_{n,n}(a,b)$.

Case of $n > m$ for $m = 0$ and $m = 1$. Note that if $n \neq m$ we have

$$p^n(a) - p^m(-a) = aA_n + B_n - (-aA_m + B_m) = a(A_n + A_m) + (B_n - B_m)$$

and

$$p^m(a) - p^n(-a) = aA_m + B_m - (-aA_n + B_n) = a(A_n + A_m) - (B_n - B_m).$$

We denote

$$P_{n,m}(a,b) := (p^n(a) - p^m(-a))(p^m(a) - p^n(-a)), \tag{14}$$

then $P_{n,m}(a,b) = a^2(A_n + A_m)^2 - (B_n - B_m)^2$. Recall that $A_0 = 1$, $B_0 = 0$ and $A_1 = -2a^2$, $B_1 = b$. For $n \geq 1$ we have that $P_{n,0} = a^2(A_n(a,b) + 1)^2 - B^2_n(a,b)$. By Lemma 3 set

$$\tilde{P}_{n,0}(a,b) = P_{n,0}(a,b) = a^2(\tilde{A}_n(a^2,b^2) + 1)^2 - b^2\tilde{B}^2_n(a^2,b^2). \tag{15}$$

Note that the critical orbit relation $(n,0)$ is minimal. An easy calculation shows that

$$P_{n,1} = \left(a^2(A_{n-1} + 1)^2 - B^2_{n-1}\right)^2 \cdot \left(a^2(A_{n-1} - 2)^2 - B^2_{n-1}\right). \tag{16}$$

For $n \geq 1$ set

$$\tilde{P}_{n,1}(a,b) = a^2(A_{n-1}(a,b) - 2)^2 - B^2_{n-1}(a,b),$$

or by Lemma 3 we can write it as

$$\tilde{P}_{n,1}(a,b) = a^2(\tilde{A}_{n-1}(a^2,b^2) - 2)^2 - b^2\tilde{B}^2_{n-1}(a^2,b^2) \tag{17}$$

then the above implies that

$$P_{n,1}(a,b) = P^2_{n-1,0}(a,b) \cdot \tilde{P}_{n,1}(a,b). \tag{18}$$

The following trivially follows by (15)–(18) and from the degrees of A_n and B_n.

For $n \geq 1$ set

$$Q_{n,0}(x,y) = x(\tilde{A}_n(x,y) + 1)^2 - y\tilde{B}^2_n(x,y), \tag{19}$$
$$Q_{n,1}(x,y) = x(\tilde{A}_{n-1}(x,y) - 2)^2 - y\tilde{B}^2_{n-1}(x,y) \tag{20}$$

then $\tilde{P}_{n,0}(x,y) = Q_{n,0}(x^2,y^2)$ and $\tilde{P}_{n,1}(x,y) = Q_{n,1}(x^2,y^2)$. If there is a minimal critical orbit relation $(n,1)$ then $\tilde{P}_{n,1}(a,b) = 0$. Moreover, $\deg_a \tilde{P}_{n,0}(a,b) = \deg \tilde{P}_{n,0}(a,b) = 2\cdot 3^n$ and $\deg_a \tilde{P}_{n,1} = \deg \tilde{P}_{n,1} = 2\cdot 3^{n-1}$ for $n \geq 1$.

Case of $n > m > 1$. By following Taylor's formula for cubic polynomials

$$p(z) = p(w) + p'(w)(z-w) + p''(w)(z-w)^2/2 + p'''(w)(z-w)^3/6$$

and since $p'(w) = 3w^2 - 3a^2$, $p''(w) = 6w$, and $p'''(w) = 6$ we obtain $p(z) - p(w) = (z-w)(3w^2 - 3a^2 + 3w(z-w) + (z-w)^2)$. After simplification it becomes

$$p(z) - p(w) = (z-w)(z^2 + zw + w^2 - 3a^2).$$

By substituting $p^{n-1}(a)$ and $p^{m-1}(-a)$ instead of z and w, respectively, we obtain

$$\begin{aligned} p^n(a) - p^m(-a) =& (p^{n-1}(a) - p^{m-1}(-a))(p^{n-1}(a)^2 + p^{n-1}(a)p^{m-1}(-a) \\ &+ p^{m-1}(-a)^2 - 3a^2) \end{aligned}$$

Similarly,

$$\begin{aligned} p^m(a) - p^n(-a) =& (p^{m-1}(a) - p^{n-1}(-a))(p^{m-1}(a)^2 + p^{m-1}(a)p^{n-1}(-a) \\ &+ p^{n-1}(-a)^2 - 3a^2) \end{aligned}$$

Now, if we multiply the left-hand sides (which is $P_{n,m}(a,b)$ as was defined in (14)) and the right-hand sides of the latter two identities (product of the first items is $P_{n-1,m-1}(a,b)$) and substituting the iterates in the second items of the products on the right-hand sides by Lemma 2 correspondingly and denote by $\tilde{P}_{n,m}(a,b)$ then

$$\begin{aligned} \tilde{P}_{n,m}(a,b) =& \big(a^2(A_{n-1}^2 - A_{n-1}A_{m-1} + A_{m-1}^2) + B_{n-1}^2 + B_{n-1}B_{m-1} + B_{m-1}^2 - 3a^2\big)^2 \\ &- a^2\big((2A_{n-1} - A_{m-1})B_{n-1} - (2A_{m-1} - A_{n-1})B_{m-1}\big)^2. \end{aligned} \tag{21}$$

and we obtain the factorization

$$P_{n,m}(a,b) = P_{n-1,m-1}(a,b)\cdot \tilde{P}_{n,m}(a,b).$$

Let $n > m > 1$ and set

$$\begin{aligned} Q_{n,m}(x,y) =& \Big(x(\tilde{A}_{n-1}^2(x,y) - \tilde{A}_{n-1}(x,y)\tilde{A}_{m-1}(x,y) + \tilde{A}_{m-1}^2(x,y)) \\ &+ y(\tilde{B}_{n-1}^2(x,y) + \tilde{B}_{n-1}(x,y)\tilde{B}_{m-1}(x,y) + \tilde{B}_{m-1}^2(x,y)) - 3x\Big)^2 \\ &- xy\Big((2\tilde{A}_{n-1}(x,y) - \tilde{A}_{m-1}(x,y))\tilde{B}_{n-1}(x,y) \\ &+ (\tilde{A}_{n-1}(x,y) - 2\tilde{A}_{m-1}(x,y))\tilde{B}_{m-1}(x,y)\Big)^2, \end{aligned} \tag{22}$$

then $\tilde{P}_{n,m}(x,y) = Q_{n,m}(x^2,y^2)$. We have $P_{n,m}(a,b) = P_{n-1,m-1}(a,b)\cdot\tilde{P}_{n,m}(a,b)$ and if there is a minimal critical orbit relation (n,m) then $\tilde{P}_{n,m}(a,b) = 0$.

By definition $\deg P_{n,n}(a,b) = \deg_a P_{n,m}(a,b) = 2\deg_a A_n + 2 = 2\cdot 3^n$ and $\deg \tilde{P}_{n,m}(a,b) = \deg_a \tilde{P}_{n,m}(a,b) = 4\deg_a A_{n-1} + 4 = 4\cdot 3^{n-1}$.

Here is an application of the above computations. The problem of the exclusion of the variable z from the system $p^n(z) - p^m(-z) = 0$ and $z^2 - a^2 = 0$ is equivalent to finding the resultant of the two polynomials, with respect to variable z [12]. We obtain the following as a direct corollary of the above computations.

Proposition 1. *For all pairs of n, m, we have if $n \neq m$ then*

$$Res_z\left(p^n(z) - p^m(-z), z^2 - a^2\right) = -P_{n,m}(a,b),$$

and

$$Res_z\left(p^n(z) - p^n(-z), z^2 - a^2\right) = -16a^6 P_{n,n}^2(a,b).$$

Proof. For the case of $n \neq m$ by ([21] Theorem 1.3.1) and by (14) we obtain

$$Res_z\left(p^n(z) - p^m(-z), z^2 - a^2\right) = (p^n(a) - p^m(-a))(p^n(-a) - p^m(a)) = -P_{n,m}(a,b).$$

For the other case, by ([21] Theorem 1.3.1) and by (10) we obtain

$$Res_z\left(p^n(z) - p^n(-z), z^2 - a^2\right) = (p^n(a) - p^n(-a))(p^n(-a) - p^n(a)) = -16a^6 P_{n,n}^2(a,b).$$

□

4. Main Result and Its Proof

Our main result is the following:

Theorem 1. *Every minimal critical orbit relation $(n,m) \neq (1,1)$ (1) reduces to an algebraic equation $\tilde{P}_{n,m}(a,b) = 0$ on the parameters a, b defined in Section 3. In particular, there are infinitely many cubic polynomials for each minimal critical orbit relation (n,m), except the relation $(1,1)$, which does not exist.*

Let us remark that if the critical orbit relation $(1,1)$ exists then any nearby point to the critical value must have four preimages but our polynomial is cubic so it is not realized for this family. The main result concludes that all other critical orbit relations are realized.

Proof of Theorem 1. The first part of the theorem has been considered in the previous section for all three cases ((n,n), (n,m) for $n > m$ and $m = 0$ and $m = 1$, (n,m) for $n > m > 1$). For each case, the zero level of polynomials $\tilde{P}_{n,m}(a,b)$ corresponds to minimal (n,m) critical orbit relation. The degree counts show that all but $(1,1)$ critical orbit relations are realized so that there are infinitely many cubic polynomials for every minimal critical orbit relation $(n,m) \neq (1,1)$. It finishes the proof and answers to Problems 1 and 2 as these two are the same. □

We would like to remark that for the minimal critical orbit relation (n,m) the curves defined by $\tilde{P}_{n,m}(a,b) = 0$ contain points (a_0, b_0) satisfying some other minimal critical orbit relation $(n', m') \neq (n,m)$. So the required minimality is not equivalent to exactness. Our definition of minimality was introduced to factor those $P_{n,m}(a,b)$ as much as possible. We conjecture that there are no further factorizations than those we found in this paper. If one needs to find the exact critical orbit relation it is necessary to study the intersection of these distinct algebraic curves defined by all pairs of $\tilde{P}_{n,m}(a,b) = 0$ and $\tilde{P}_{n',m'}(a,b) = 0$. Then, the exact critical orbit relations for particular (n,m) are obtained from the curve $\tilde{P}_{n,m}(a,b) = 0$ by removing infinitely many discrete points that belong also to other such curves. It is clear that an algebraic curve defined by $\tilde{P}_{n,m}(a,b) = 0$ on complex variables a, b is the Zariski closure of the set of points that solve the Problem 1 for the critical orbit relation (n,m).

If we consider a slice $\mathrm{Per}_1(\lambda)$, for a given λ, then if $|\lambda| <= 1$ this slice does not contain postcritically finite maps. One can show that in this slice the above defined minimality is equivalent to exactness.

The moduli space can be described by the coordinates (a^2, b^2), let $x = a^2$ and $y = b^2$ then instead of $\tilde{P}_{n,m}(a^2, b^2)$ we can work well with $Q_{n,m}(x, y)$. We obtained the following.

Corollary 1. *In the moduli space of cubic polynomials of the form $z^3 - 3a^2z + b$ with coordinates $x = a^2$ and $y = b^2$ the minimal critical orbit relation (n, m) corresponds to the set $\{(x, y) \in \mathbb{C}^2 : Q_{n,m}(x, y) = 0\}$, where $Q_{n,m}(x, y)$ is defined by (13), (19), (20), (22), respectively. It is never empty, except for the relation $(1, 1)$. The degree of the curve $\mathcal{S}_{n,m}$ is half of the degree of the polynomial $\tilde{P}_{(n,m)}(a, b)$.*

Denote $\mathcal{S}_{n,m} = \{(x, y) \in \mathbb{C}^2 : Q_{n,m}(x, y) = 0\}$ the affine algebraic curve in $\mathbb{C}^2$. It seems that each curve $\mathcal{S}_{n,m}$, except $\mathcal{S}_{1,1}$ (which is an empty set), is irreducible. These curves are analogous to those defined by Milnor [4].

Denote $\mathrm{Crit}(n, m) = \{(a, b) : \tilde{P}_{n,m}(a, b) = 0\}$. In the parameter space except (n, n) for all cases the algebraic curves $\mathrm{Crit}(n, m)$ in $\mathbb{C}^2$ have two components as $\tilde{P}_{n,m}(a, b)$ which are the difference of two squares. But it seems that each factor of $\tilde{P}_{n,m}(a, b)$ is irreducible over $\mathbb{C}$ in this trivial factorization, which we leave as an open problem together with the irreducibility of $\tilde{P}_{n,n}(a, b)$. Moreover, the coefficients of each of $\tilde{P}_{n,m}(a, b)$ are integers.

We end this section with the following.

Conjecture. Every curve $\mathcal{S}_{n,m}$ is irreducible in $\mathbb{C}[x, y]$.

If this conjecture holds true, it implies that the structure of the parameter space remains consistent for points with critical orbit relations. If the curves were reducible, their reduced components would behave differently, causing the parameter space to split into non-uniform regions. However, visualizations and experimental results show no indication of such asymmetrical regions within the parameter space.

5. Examples of Special Curves in $\mathbb{C}^2$

Here are some examples of these special curves in $\mathbb{C}^2$.

$$\mathcal{S}_{0,0} = \{x = 0\}, \;\; \mathcal{S}_{1,0} = \{x(2x - 1)^2 - y = 0\},$$

$$\mathcal{S}_{2,0} = \{x(8x^4 - 6x^2 + 6xy - 1)^2 - y(12x^3 - 3x + y + 1)^2 = 0\},$$
$$\mathcal{S}_{2,1} = \{4x(x + 1)^2 - y = 0\}, \;\; \mathcal{S}_{2,2} = \{4x^3 - 3x + 3y = 0\},$$
$$\mathcal{S}_{3,1} = \{x(8x^4 - 6x^2 + 6xy + 2)^2 - y(12x^3 - 3x + y + 1)^2 = 0\},$$

and

$$\mathcal{S}_{3,3} = \{64x^9 - 96x^7 + 528x^6y + 36x^5 - 288x^4y + 108x^3y^2 + 72x^3y + 27x^2y - 18xy^2 - 18xy + 3y^3 + 6y^2 + 3y - 3x = 0\}.$$

Proposition 2. *The curves $\mathcal{S}_{0,0}$, $\mathcal{S}_{1,0}$, $\mathcal{S}_{2,0}$, $\mathcal{S}_{2,1}$, and $\mathcal{S}_{2,2}$ are irreducible in $\mathbb{C}[x, y]$.*

Some examples of these curves are obtained in [22].

Proof. The curves $\mathcal{S}_{0,0}$, $\mathcal{S}_{1,0}$, $\mathcal{S}_{2,1}$, and $\mathcal{S}_{2,2}$ are irreducible as they can be identified with the complex plain $\mathbb{C}$, graphs of polynomials (Figure 1). Let us show that the curve $\mathcal{S}_{2,0}$ is irreducible. We need to show that the polynomial $Q_{2,0}(x, y) = x(8x^4 - 6x^2 + 6xy - 1)^2 - y(12x^3 - 3x + y + 1)^2$ is irreducible in $\mathbb{Q}[x, y]$. Since its linear part does not vanish, it is irreducible in $\mathbb{C}[x, y]$ as well (see ([5] Lemma 5) or ([17] Theorem 4.6)). First, we show that the polynomial $Q_{2,0}(x, y)$ does not have a factor which is a polynomial of only x. On the contrary, assume $Q_{2,0}(x, y) = f(x)g(x, y)$. As $Q_{2,0}(x, 0) = x(8x^4 - 6x^2 - 1)^2$ and $Q_{2,0}(0, y) = -y(y + 1)^2$ we obtain $f(0) = \pm 1$. Since $Q_{2,0}(x, 0) = f(x)g(x, 0)$ we obtain, without loss of generality, that $f(x) = 8x^4 - 6x^2 - 1$. Now divide $Q_{2,0}(x, y)$ by $f(x)$ and obtain a nonzero reminder, which is a contradiction.

Note that in $\mathbb{Z}_{13}$ we have $Q_{2,0}(-2, y) = 12(y^3 + 6y^2 + 2)$ now apply the Eisenstein criterion for prime number 2, which proves the claim. □

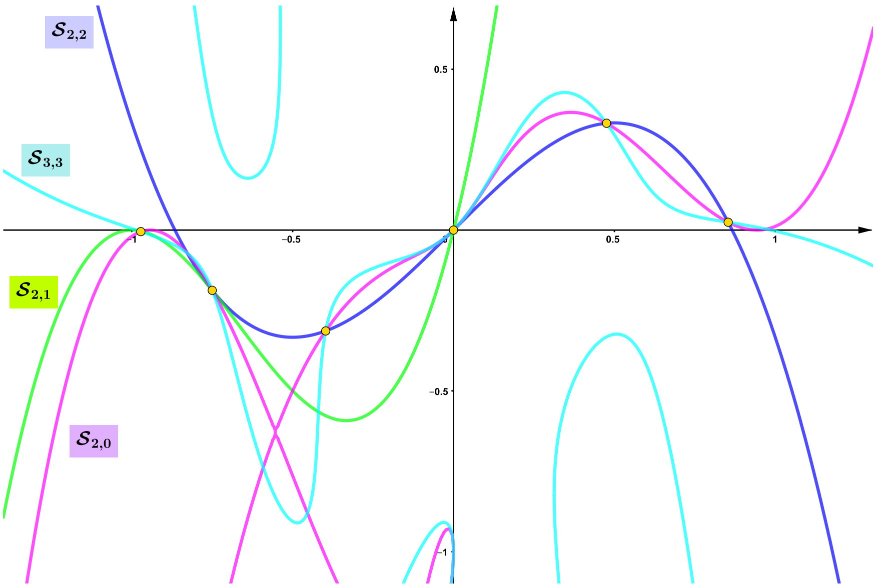

Figure 1. The graphs of the curves $\mathcal{S}_{2,0}, \mathcal{S}_{2,1}, \mathcal{S}_{2,2}, \mathcal{S}_{3,3}$.

In Table 1 the degrees of $\mathcal{S}_{n,m}$ for $n \geq 2$ have been listed in a row.

Table 1. The degree row of $\mathcal{S}_{n,m}$ for $n \geq 2$.

	m						
	0	**1**	**2**	**.**	**.**	$\boldsymbol{n-1}$	$\boldsymbol{n}$
n	3^n	3^{n-1}	$2 \cdot 3^{n-1}$	.	.	$2 \cdot 3^{n-1}$	3^{n-1}

6. Two Methods to Solve Problem 3

To solve Problem 3 one can try to use the method developed for Problem 1 by obtaining analogous results of Sections 2 and 3. The factorizations of Section 3 for $\tilde{P}_{n,m}$ are no longer needed. We can still reduce each critical orbit relation into a polynomial equation. In this section by briefly go through the main parts of the former and also develop a new method for the latter. The main Lemma 2 will become the following.

Lemma 4. *Let $q(z) = z^3 + \mu z^2 + z$ be a cubic polynomial with critical points c_1 and c_2 that solve the equation $3z^2 + 2\mu z + 1 = 0$. Then, for all $n \geq 0$ the equality $q^n(c) = A_n(\mu)c + B_n(\mu)$ holds for $c = c_1$ and $c = c_2$ where polynomials $A_n(\mu)$ and $B_n(\mu)$ are recurrently defined as*

$$A_{n+1}(\mu) = A_n(\mu)\left(\frac{4\mu^2 - 3}{9}A_n^2(\mu) - \frac{2\mu(3B_n(\mu) + \mu)}{3}A_n(\mu) + 3B_n^2(\mu) + 2\mu B_n(\mu) + 1\right), \tag{23}$$

$$B_{n+1}(\mu) = B_n^3(\mu) + \mu B^2(\mu) - \frac{3B_n(\mu) + \mu}{3}A_n^2(\mu) + B_n(\mu) + \frac{2}{9}\mu A_n(\mu) \tag{24}$$

with $A_0(\mu) = 1$, $B_0(\mu) = 0$.

Proof of this lemma goes alongside of the proof of Lemma 2 where in the expansion of $(A_n(\mu)z + B_n(\mu))^3 + \mu(A_n(\mu)z + B_n(\mu))^2 + (A_n(\mu)z + B_n(\mu))$ we substitute $z^2 = -\frac{2}{3}z - \frac{1}{3}$ and $z^3 = \frac{4\mu^2-3}{9}z + \frac{2}{9}\mu$.

Now we deal with the critical orbit relations and reduce them to polynomial relations. Since both critical points c_1 and c_2 are the roots of $3z^2 + 2\mu z + 1 = 0$, Vieta's formula yields that $c_2 = -2\mu/3 - c_1$, which can be used to exclude the unknown c_2 from the critical orbit relations.

Case of (n, n). By Lemma 4 we have

$$q^n(c_1) - q^n(c_2) = A_n(\mu)c_1 + B_n(\mu) - (A_n(\mu)c_2 + B_n(\mu)) = A_n(\mu)(c_1 - c_2), \ \ n \geq 1.$$

If $c_1 = c_2$ then the discriminate of $3z^2 + 2\mu z + 1 = 0$ vanishes: $4(\mu^2 - 3) = 0$. So that if $\mu = \pm\sqrt{3}$ then the critical orbit relation is $(0, 0)$. For all other values of μ the critical points c_1 and c_2 are distinct and the (n, n) critical relation reduces to $A_n(\mu) = 0$. Direct calculation yields $A_1(\mu) = \frac{2(3-\mu^2)}{9}$ so that $A_1(\mu) = 0$ does not produce a $(1, 1)$ critical orbit relation, so this case is empty.

Denote $P_{n,n}(\mu) = A_n(\mu)/A_1(\mu)$. From (23) we obtain

$$P_{n,n}(\mu) = A_n(\mu)/A_1(\mu) = A_{n-1}(\mu)/A_1(\mu)\Big(\frac{4\mu^2 - 3}{9}A_{n-1}^2(\mu) - \frac{2\mu(3B_{n-1}(\mu) + \mu)}{3}A_{n-1}(\mu) + 3B_{n-1}^2(\mu) + 2\mu B_{n-1}(\mu) + 1\Big).$$

Set

$$\tilde{P}_{n,n}(\mu) = \frac{4\mu^2 - 3}{9}A_{n-1}^2(\mu) - \frac{2\mu(3B_{n-1}(\mu) + \mu)}{3}A_n(\mu) + 3B_{n-1}^2(\mu) + 2\mu B_{n-1}(\mu) + 1,$$

This implies that for $n \geq 2$, we have the following factorization

$$P_{n,n}(\mu) = P_{n-1,n-1}(\mu) \cdot \tilde{P}_{n,n}(\mu). \tag{25}$$

Case of $n > m$. For $m = 0$ we have

$$q^n(c_1) - c_2 = A_n(\mu)c_1 + B_n(\mu) - c_2 = 0. \tag{26}$$

Substituting $c_2 = -2\mu/3 - c_1$, into (22) yields

$$q^n(c_1) - c_2 = A_n(\mu)c_1 + B_n(\mu) + 2\mu/3 + c_1 = (A_n(\mu) + 1)c_1 + B_n(\mu) + \frac{2\mu}{3} = 0$$

Now the problem, the $(n, 0)$ critical orbit relation, reduces to the following system of equations from which we need to exclude $c = c_1$.

$$(A_n(\mu) + 1)c + B_n(\mu) + \frac{2\mu}{3} = 0$$
$$3c^2 + 2\mu c + 1 = 0.$$

This system reduces to

$$3\big(B_n(\mu) + \frac{2\mu}{3}\big)^2 - 2\mu\big(B_n(\mu) + \frac{2\mu}{3}\big)(A_n(\mu) + 1) + (A_n(\mu) + 1)^2 = 0.$$

For $m \geq 1$, using the identity

$$q(z) - q(w) = (z - w)(z^2 + zw + w^2 + \mu(z + w) + 1),$$

we obtain

$$q^n(c_1) - q^m(c_2) = (q^{n-1}(c_1) - q^{m-1}(c_2))\big(q^{n-1}(c_1)^2 + q^{n-1}(c_1)q^{m-1}(c_2) + q^{m-1}(c_2)^2 + \mu(q^{n-1}(c_1) + q^{m-1}(c_2)) + 1\big) = 0. \quad (27)$$

If $m = 1$ then the latter simplifies to

$$\begin{aligned} q^n(c_1) - q(c_2) &= (q^{n-1}(c_1) - c_2)(q^{n-1}(c_1)^2 + q^{n-1}(c_1)c_2 + c_2^2 + \mu(q^{n-1}(c_1) + c_2) + 1) \\ &= (q^{n-1}(c_1) - c_2)^2(q^{n-1}(c_1) + 2c_2 + \mu) \\ &= (q^{n-1}(c_1) - c_2)^2(A_{n-1}(\mu)c_1 + B_{n-1}(\mu) + 2(-2\mu/3 - c_1) + \mu) \\ &= (q^{n-1}(c_1) - c_2)^2\big((A_{n-1}(\mu) - 2)c_1 + B_{n-1}(\mu) - \mu/3\big) = 0. \end{aligned}$$

Now the problem, the $(n, 1)$ critical orbit relation, reduces to the following system of equations from which we need to exclude $c = c_1$.

$$\begin{aligned} (A_{n-1}(\mu) - 2)c + B_{n-1}(\mu) - \mu/3 = 0 \\ 3c^2 + 2\mu c + 1 = 0. \end{aligned}$$

This system reduces to

$$3\big(B_{n-1}(\mu) - \mu/3\big)^2 - 2\mu\big(B_{n-1}(\mu) - \mu/3\big)\big(A_{n-1}(\mu) - 2\big) + \big(A_{n-1}(\mu) - 2\big)^2 = 0.$$

It remains to consider the case of $n > m \geq 2$. By (27) this case reduces to solve the following system.

$$\begin{aligned} q^{n-1}(c_1)^2 + q^{n-1}(c_1)q^{m-1}(c_2) + q^{m-1}(c_2)^2 + \mu(q^{n-1}(c_1) + q^{m-1}(c_2)) + 1 = 0 \\ 3c_1^2 + 2\mu c_1 + 1 = 0. \end{aligned}$$

By Lemma 4 and substituting $c_2 = -2\mu/3 - c_1$ into the first equation of the latter system we obtain the following.

To obtain the latter we substituted identities $c_1^2 = -2\mu c_1/3 - 1/3$, $c_2^2 = 2\mu c_1/3 + 4\mu^2/9 - 1/3$, and $c_1 c_2 = 1/3$.

Finally, this system reduces to

$$3\big(B_{n-1}(\mu) - \mu/3\big)^2 - 2\mu\big(B_{n-1}(\mu) - \mu/3\big)\big(A_{n-1}(\mu) - 2\big) + \big(A_{n-1}(\mu) - 2\big)^2 = 0.$$

Instead of this, we propose an alternative method to solve Problem 3. In fact, it is not much alternative as it uses all the solutions of Problem 1 developed in Sections 2 and 3. The idea is to work with a different parametrization of $\mathrm{Per}_1(1)$ other than $q(z) = z^3 + \mu z^2 + z$.

We start with the Branner–Hubbard form of a cubic polynomial $p(z) = z^3 - 3a^2 z + b$ and require that it has a multiple fixed point, not necessarily at the origin. It means that the fixed point equation $z^3 - 3a^2 z + b = z$ has a multiple solution, which in turn, is equivalent to the vanishing of the discriminant of $z^3 - (3a^2 + 1)z + b$, which is $4(3a^2 + 1)^3 - 27b^2 = 0$. Now solve the latter for b^2 and substitute it in $\tilde{P}_{n,m}(a, b)$ that is defined in Section 3 (by Formulas (10), (15), (16) and (21)) as these all are polynomials of a^2 and b^2 and thus obtain polynomials of a^2 only. For Problem 3 the corresponding result becomes stronger and, in particular, answers affirmatively on the infiniteness of cubic polynomials with critical orbit relations. Let us state it as a theorem without giving a proof.

Theorem 2. *The slice of* $\mathbb{C}^2$ *defined by* $4(3a^2 + 1)^3 - 27b^2 = 0$ *represents* $\mathrm{Per}_1(1)$ *and cubic polynomials* $q(z) = z^3 + \mu z^2 + z$ *with exact critical orbit relations* (n, m) *corresponding to level sets*

$\tilde{P}_{n,m}(a,b) = 0$, where $\tilde{P}_{n,m}(a,b)$ are defined in Formulas (11), (15), (16), (21) with a substitution $b^2 = 4(3a^2+1)^3/27$. Moreover, the obtained critical orbit relations $(n,m) \neq (1,1)$ are all realized and are exact and there are infinitely many cubic polynomials $q(z) = z^3 + \mu z^2 + z$ with a critical orbit relation except the relation $(1,1)$.

The idea of its proof is as follows. Let us note that every map $q(z) = z^3 + \mu z^2 + z$ has a parabolic fixed point at the origin and such a fixed point attracts the infinite orbit of a critical point [4]. If both critical points are in a relation then this relation is exact. By substituting $A_n(a,b) = \tilde{A}_n(a^2,b^2)$ and $B_n(a,b) = b\tilde{B}_n(a^2,b^2)$ from Lemma 3 with (8), (9) and $b^2 = 4(3a^2+1)^3/27$ into (11) we obtain $\tilde{P}_{n,n}(a,b) = a^2\tilde{A}^2_{n-1}(a^2,b^2) + 3b^2\tilde{B}^2_{n-1}(a^2,b^2) - 3a^2 = a^2\tilde{A}^2_{n-1}(a^2,b^2) + \frac{4(3a^2+1)^3}{9}\tilde{B}^2_{n-1}(a^2,b^2) - 3a^2$, which is a polynomial of variable a. Similarly, we can show that after substitutions the polynomials defined in Formulas (15), (16), (21) are polynomials of variable a. Now introduce notations $x = a^2$ and $y = b^2$ then

$$y = 4(3a^2+1)^3/27 = 4(3x+1)^3/27 = 4(x+1/3)^3.$$

By abusing the notation, (8) and (9) become the following after the substitutions.

$$\tilde{A}_{n+1}(x) = \tilde{A}_n(x)(x\tilde{A}^2_n(x) + 12(x+1/3)^3\tilde{B}^2_n(x) - 3x), \tag{28}$$
$$\tilde{B}_{n+1}(x) = 4(x+1/3)^3\tilde{B}^3_n(x) + 3x\tilde{B}_n(x)(\tilde{A}^2_n(x) - 1) + 1, \tag{29}$$

Denote the leading term of a polynomial f by $LT(f)$ and the leading coefficient by $LC(f)$. We obtain the following recurrence relations for the leading terms and coefficients.

$$LT(\tilde{A}_{n+1}(x)) = xLT(\tilde{A}_n(x))^3 + 12x^3LT(\tilde{B}_n(x))^2LT(\tilde{A}_n(x)), \tag{30}$$
$$LT(\tilde{B}_{n+1}(x)) = 4x^3LT(\tilde{B}_n(x))^3 + 3xLT(\tilde{B}_n(x))LT(\tilde{A}_n(x))^2, \tag{31}$$

with $LT(\tilde{A}_1(x)) = -2x$ and $LT(\tilde{B}_1(x)) = 1$.

$$LC(\tilde{A}_{n+1}(x)) = LC(\tilde{A}_n(x))^3 + 12LC(\tilde{B}_n(x))^2LC(\tilde{A}_n(x)), \tag{32}$$
$$LC(\tilde{B}_{n+1}(x)) = 4LC(\tilde{B}_n(x))^3 + 3LC(\tilde{B}_n(x))LC(\tilde{A}_n(x))^2, \tag{33}$$

with $LC(\tilde{A}_1(x)) = -2$ and $LC(\tilde{B}_1(x)) = 1$. To solve the recurrence relation, introduce $t_n = LC(\tilde{A}_n(x))/LC(\tilde{B}_n(x))$ and divide the first equation by the second and obtain

$$t_{n+1} = \frac{t_n(t_n^2+12)}{3t_n^2+4},$$

with $t_1 = -2$. This recurrence relation produces a constant sequence $t_n = -2$ as -2 is a fixed point of the rational function $t \mapsto \frac{t(t^2+12)}{3t^2+4}$. Now it is easy to see that $LC(\tilde{A}_n(x)) = -2^{2\cdot 3^{n-1}-1}$ and $LC(\tilde{B}_n(x)) = 2^{2\cdot 3^{n-1}-2}$. Corresponding degrees are as follows:

$$\deg(\tilde{A}_{n+1}(x)) = 1 + 3\deg(\tilde{A}_n(x)), \tag{34}$$
$$\deg(\tilde{B}_{n+1}(x)) = 3 + 3\deg(\tilde{B}_n(x)), \tag{35}$$

with $\deg\tilde{A}_1 = 1$ and $\deg\tilde{B}_1 = 0$. Solving these recurrence relations one obtains

$$\deg\tilde{A}_n = (3^n - 1)/2$$

and

$$\deg\tilde{B}_n = (3^n - 3)/2.$$

Finally, $LT(\tilde{A}_n(x)) = -2^{2\cdot 3^{n-1}-1}x^{(3^n-1)/2}$ and $LT(\tilde{B}_n(x)) = 2^{2\cdot 3^{n-1}-2}x^{(3^n-3)/2}$. Note that

$$\tilde{P}_{n,n}(x) = x\tilde{A}_{n-1}^2(x) + 12(x+1/3)^3\tilde{B}_{n-1}^2(x) - 3x,$$
$$\tilde{P}_{n,0}(x) = x(\tilde{A}_n(x)+1)^2 - 4(x+1/3)^3\tilde{B}_n^2(x),$$
$$\tilde{P}_{n,1} = x(\tilde{A}_{n-1}(x)-2)^2 - 4(x+1/3)^3\tilde{B}_{n-1}^2(x),$$

and for $n > m \geq 2$

$$\begin{aligned}\tilde{P}_{n,m}(x) = &\Big(x(\tilde{A}_{n-1}^2(x) - \tilde{A}_{n-1}(x)\tilde{A}_{m-1}(x) + \tilde{A}_{m-1}^2(x))\\ &+ 4(x+1/3)^3(\tilde{B}_{n-1}^2(x) + \tilde{B}_{n-1}(x)\tilde{B}_{m-1}(x) + \tilde{B}_{m-1}^2(x)) - 3x\Big)^2\\ &- 4x(x+1/3)^3\Big((2\tilde{A}_{n-1}(x) - \tilde{A}_{m-1}(x))\tilde{B}_{n-1}(x) + (\tilde{A}_{n-1}(x) - 2\tilde{A}_{m-1}(x))\tilde{B}_{m-1}(x)\Big)^2.\end{aligned}$$

We have that

$$LT(\tilde{P}_{n,n}) = xLT(A_{n-1})^2 + 12x^3LT(B_{n-1})^2 = 16^{3^{n-2}}x^{3^{n-1}}.$$

For the rest of the polynomials there are resonances. The term $x\tilde{A}_n^2(x)$ resonates with the term $4(x+1/3)^3\tilde{B}_{n-1}^2(x)$ so that for $\tilde{P}_{n,0}(x)$ and $\tilde{P}_{n,1}(x)$ half of their degrees are dropped. For the polynomial $\tilde{P}_{n,m}(x)$ the resonance happens for $x^2(\tilde{A}_{n-1}^2(x)^2$ with $16x(x+1/3)^3\tilde{A}_{n-1}(x)^2$ resulting in a drop of the degree. But one can show that

$$\deg\tilde{P}_{n,0}(x) = (3^n+1)/2,$$

$$\deg\tilde{P}_{n,1}(x) = (3^n+3)/6,$$

and for $n > m \geq 2$ we have that

$$\deg\tilde{P}_{n,m}(x) = 2\cdot 3^{n-1}.$$

Here are some examples: $\tilde{P}_{2,0}(x) = -128x^5 - 308x^4/3 - 340x^3/27 - 904x^2/243 - 139x/729 - 3844/19683$, $\tilde{P}_{2,1}(x) = 4x^2 + 8x/3 - 4/27$, $\tilde{P}_{2,2}(x) = 16x^3 + 12x^2 + x + 4/9$, $\tilde{P}_{3,1}(x) = 64x^5 + 124x^4/3 - 16x^3/27 + 392x^2/243 + 2048x/729 - 3844/19683$, $\tilde{P}_{3,3}(x) = 4096x^9 + 6144x^8 + 2816x^7 + 5504x^6/9 + 1136x^5/3 + 1604x^4/9 + 2344x^3/81 + 472x^2/81 + 139x/243 + 3844/6561$.

7. Summary and Further Discussions

This paper considered the problem of generating cubic polynomials with all possible critical orbit relations. We explicitly found that for polynomials, the level sets correspond to specific critical orbit relations. We also considered the sub-problems where we considered some slices in the parameter plane. The slices are cubic polynomials with fixed points with constant multipliers, Problem 2, and the multiplier 1, where the cubic polynomials have parabolic fixed points, and Problem 3. For the latter case, we gave two solutions to the problem. The statement of the main Lemma 2 is true for all other families of polynomials or rational functions as long as we have two interacting critical points that solve a quadratic equation of the parameters of the family. The idea is to reduce high powers in iterates of the critical point to a linear map (affine) of the critical point and then to reduce the critical relation into a linear equation and solve it explicitly. For some low degrees, the irreducibility is studied in Proposition 2. We propose a conjecture that all obtained polynomials in this paper are irreducible but this problem could be very hard to solve in full generality. To continue this line of research the other problems are as follows: One can also consider critical orbit relation for Blashke products. In this case, maps depend only on real analysis of the parameters so that the obtained equations will not be complex polynomials but polynomials involving parameters and their complex conjugates. The family considered in [23] is also a perfect example of the methods proposed in this paper.

One of the applications of the results in this paper is to study the number of stable components in the parameter plane of cubic polynomials as they contain unique centers where maps have critical orbit relations [19,20,24]. We can label them by the number of iterates for a critical orbit to reach the immediate basins of some fixed points. The difficulty one needs to overcome this problem is that the polynomials that we obtain do not determine the exact location of the critical point. The critical point could be in any Fatou component or both critical points could be at the same component and have a relation with each other. So the solution for this could be to use the Blaschke products that are mentioned above to exclude the latter case and obtain correct numbers on the number of stable components.

Another open problem is to study the interaction of more than two critical points. Now the critical points solve cubic equations or higher-order equations. In this case, we can modify the main Lemma 2 such that the iterates are now quadratic or one degree lower than the critical equation. The findings in this paper have significant implications for computational dynamics and related fields. By providing a systematic method for generating dynamical systems with specified orbit relations, our approach offers tools to study parameter spaces and the stability of holomorphic families. These insights can be utilized in modeling phenomena in physics, biology, and other sciences where dynamical systems are prevalent.

In computational dynamics, the iterative techniques described here can improve algorithms for detecting critical orbit relations, which are pivotal in visualizing fractals, such as the Mandelbrot set. Furthermore, understanding the structure and irreducibility of COR varieties aids in optimizing parameter exploration in high-dimensional spaces.

Future research could extend this framework to higher-degree polynomials or other families of rational functions. Additionally, exploring connections between critical orbit relations and arithmetic dynamics might yield new methods for solving Diophantine equations. The interdisciplinary nature of this work invites collaboration across mathematics, physics, and computer science to tackle complex systems and advance computational methods for analyzing their behavior.

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