Quality & Quantity
https://doi.org/10.1007/s11135-023-01616-9

On pursuit and evasion game problems with Grönwall-type constraints

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Accepted: 17 January 2023

Abstract

We study a fixed duration pursuit-evasion differential game problem of one pursuer and one evader with Grönwall-type constraints (recently introduced in the work of Samatov et al. (Ural Math J 6:95–107, 2020b)) imposed on all players' control functions. The players' dynamics are governed by a generalized dynamic equation. The payoff is the greatest lower bound of the distances between the evader and the pursuers when the game is terminated. The pursuers' goal, which contradicts that of the evader, is to minimize the payoff. We obtained sufficient conditions for completion of pursuit and evasion as well. To this end, players' attainability domain and optimal strategies are constructed.

Keywords Grönwall's inequality · Pursuit · Evasion · Optimal strategy · Attainability domain

Mathematics Subject Classification 91A23 · 91A10 · 49N70 · 49N90 · 34H05

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Published online: 05 February 2023

1 Introduction

Differential game problem of many players has received attention from numerous researchers due to its applications in various field of study (see, for example Ibragimov and Salimi 2009; Badakaya et al. 2021; Ibragimov et al. 2012; Ibragimov 2002; Ibragimov et al. 2018; Ibragimov et al. 2021b; Ibragimov 2005; Ibragimov and Hussin 2010; Levchenkov and Pashkov 1990; Alias et al. 2015; Ferrara et al. 2017; Ibragimov et al. 2021a; Kornev and Lukoyanov 2016; Dar'in and Kurzhanskii 2003; Rilwan and Badakaya 2018; Ivanov and Ledyaev 1981; Pashkov and Terekhov 1987; Sun and Tsiotras 2014; Jin and Qu 2010). In most of the research works, players' dynamics are governed by

$$
\begin{aligned}
&\dot{x}_i(t) = a(t)u_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \dots ;\\
&\dot{y}_j(t) = a(t)v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, 2, \dots
\end{aligned}
\tag{1}
$$

where $u_i(\cdot), v_j(\cdot)$ are the control functions of the i^{th} pursuer P_i and j^{th} evader E_j respectively, usually subjected to integral constraints (Ibragimov and Salimi 2009; Badakaya et al. 2021; Ibragimov et al. 2012; Ibragimov 2002; Ibragimov et al. 2018; Ibragimov et al. 2021b), geometric constraints (Ibragimov 2005; Ibragimov and Hussin 2010; Levchenkov and Pashkov 1990; Alias et al. 2015; Ferrara et al. 2017; Ibragimov et al. 2021a), or mixed constraints (Kornev and Lukoyanov 2016; Dar'in and Kurzhanskii 2003; Rilwan and Badakaya 2018). The function $a(\cdot)$ is a specified or arbitrary scalar function.

Simple motion differential games (i.e. the case $a(t) = 1$) were studied in Ivanov and Ledyaev (1981); Pashkov and Terekhov (1987); Sun and Tsiotras (2014); Jin and Qu (2010), and the conditions for completion of pursuit and also evasion was obtained by constructing optimal strategies of the players in each problem. The case $a(t) = \theta - t$ (θ is the duration of the game) is studied in the papers (Ibragimov and Hussin 2010) and (Ibragimov and Salimi 2009) with geometric and integral constraints imposed on the player control functions respectively. The authors Ibragimov and Salimi (2009); Ibragimov and Hussin (2010) obtained sufficient conditions for the game value (which involves finding conditions for completion of pursuit and also evasion) of the game using a certain half-space. Recently, Badakaya et al. (2021); Badakaya et al. (2022) extended the results in Ibragimov and Salimi (2009); Ibragimov and Hussin (2010) to the case of n^{th} order dynamic equations (but reduced to the dynamics (1) with $a(t) = (\theta - t)^{n-1}/(n-1)!,\ i = m, j = 1$). The authors (Badakaya et al. 2021) and (Badakaya et al. 2022) also obtained the game value of the game with integral constraints and geometric constraints (respectively) imposed on the players control functions. Following the works in Ivanov and Ledyaev (1981); Pashkov and Terekhov (1987); Sun and Tsiotras (2014); Jin and Qu (2010); Ibragimov and Salimi (2009); Ibragimov and Hussin (2010); Badakaya et al. (2021); Ibragimov and Satimov (2012), where $a(\cdot)$ is specified, pursuit and evasion problems described by a more general dynamic equations (1) have been studied by a handful of authors (Ahmed et al. 2019; Ibragimov and Satimov 2012; Ibragimov and Rikhsiev 2006; Rilwan et al. 2020; Rilwan et al. 2020) mostly with integral constraints. Moreover, pursuit and evasion problem with geometric constraints has received less attention compared to the integral constraints to the best of our knowledge. Hence the need for further research.

Recently, a generalization of the geometric constraint (the Grönwall-type constraint) is introduced in Samatov et al. (2020b), where the authors (Samatov et al. 2020b) considered a simple pursuit problem of one pursuer one evader in the space $\mathbb{R}^n$. The constraints are given as follows

$$\|u(t)\|^2 \le \rho^2 + 2k \int_0^t a(s)\|u(s)\|^2 ds; \tag{2}$$

$$\|v(t)\|^2 \le \sigma^2 + 2k \int_0^t a(s)\|v(s)\|^2 ds, \tag{3}$$

where $a(t) = 1, \rho$ and σ are given positive numbers, and k is a given non-negative number. They constructed optimal strategies for the players and obtained the optimal pursuit time of the game. The problems considered in Samatov et al. (2020a); Samatov et al. (2020b) brought forth interesting research questions such as: for an arbitrary scalar function $a(t)$, can we find conditions for completion of pursuit in the game described by (1) with the Grönwall-type constraints (2–3)? what conditions can guarantee evasion in the game described by (1) with the constraints (2–3)?

The answers to these questions will indeed generalize the results on pursuit and evasion problems considered in Samatov et al. (2020b); Ibragimov and Hussin (2010); Alias et al. (2015); Ferrara et al. (2017); Ibragimov et al. (2021a).

Summarizing, the main objective of this research is to address the research questions stated above. That is, finding conditions for completion of pursuit and also for evasion. To this end, we will construct the players' attainability domain and optimal Grönwall-type strategies.

2 Problem formulation

Let the dynamics of the pursuer P and evader E be governed by the equations (1) (with $i = j = 1$), where $x, y, x_0, y_0, u, v \in \mathbb{R}^n$, and also let the function $a(\cdot)$ be a positive scalar function on the interval $[0, \infty)$. The duration of the game, denoted θ, is fixed. The payoff function is the infimum of the distances between the evader and the pursuers at θ :

$$\gamma(\theta) := \inf_{u,v} \|x(\theta) - y(\theta)\| \tag{4}$$

The pursuer's goal is to minimize the payoff, and the evader's goal is to maximize it.

Definition 1 Samatov et al. (2020b) Functions $u(\cdot) = (u_1(\cdot), \ldots, u_n(\cdot))$ and $v(\cdot) = (v_1(\cdot), \ldots, v_n(\cdot))$ satisfying conditions (2) and (3) are called the admissible controls of the pursuer and evader respectively.

Given the players admissible controls $u(\cdot)$ and $v(\cdot)$, the corresponding paths $x(t)$, $y(t)$, at any time $t > 0$ of the players determined by $u(\cdot), v(\cdot)$, for any initial positions x_0, y_0, (respectively) are given by

$$x(t) = x_0 + \int_0^t a(s)u(s)ds; \tag{5}$$

$$y(t) = y_0 + \int_0^t a(s)v(s)ds. \tag{6}$$

Lemma 1 *Let $\eta(t),\ t \geq 0$ be a measurable function, β and k be non-negative real numbers. Then*

$$\|\eta(t)\| \leq \beta e^{k\int_0^t a(s)ds} \tag{7}$$

whenever

$$\|\eta(t)\|^2 \leq \beta^2 + 2k\int_0^t a(s)\|\eta(s)\|^2 ds. \tag{8}$$

Proof Let the assumptions of the lemma and (8) hold. Then

$$\frac{a(s)\|\eta(s)\|^2}{\beta^2 + 2k\int_0^s a(r)\|\eta(r)\|^2 dr} \leq a(s).$$

It follows that

$$\frac{1}{2k}\frac{d}{ds}\ln\left(\beta^2 + 2k\int_0^s a(r)\|\eta(r)\|^2 dr\right) \leq a(s)$$

or

$$d\ln\left(\beta^2 + 2k\int_0^s a(r)\|\eta(r)\|^2 dr\right) \leq 2ka(s)ds. \tag{9}$$

The conclusion (7) follows by integrating both sides of (9) from $s = 0$ to $s = t$. □

According to the lemma 1, if $u(\cdot)$ and $v(\cdot)$ are admissible controls then we must have

$$\|u(t)\| \leq \rho e^{k\int_0^t a(s)ds},\ \|v(t)\| \leq \sigma e^{k\int_0^t a(s)ds},\ t \geq 0.$$

Denote by $B(O, r)$ the ball of radius r centered at the origin O.

Definition 2 A continuous function $U(x_0, y_0, t, v)$,

$$U : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times B\left(O, \sigma e^{k\int_0^t a(s)ds}\right) \to B(O, \rho e^{k\int_0^t a(s)ds}),$$

such that the system

$$\begin{aligned} &\dot{x}(t) = a(t)U(x_0, y_0, t, v),\ \ x(0) = x_0, ; \\ &\dot{y}(t) = a(t)v(t),\ \ y(0) = y_0, \end{aligned} \tag{10}$$

has a unique solution for an admissible control $v(t)$ of the evader is called a strategy of the pursuer. The strategy U is said to be admissible if each control generated by this strategy is admissible.

Definition 3 A continuous function $V(x_0, y_0, t, x, y)$,

$$V : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to B(O, \sigma e^{k\int_0^t a(s)ds}),$$

is called a strategy of the evader if the following initial valued problem

$$\begin{aligned} &\dot{x}(t) = a(t)u(t), \quad x(0) = x_0, ; \\ &\dot{y}(t) = a(t)V(x_0, y_0, t, x, y), \quad y(0) = y_0, \end{aligned} \tag{11}$$

has a unique solution $(x(t), y), t \geq 0$. The strategy V is said to be admissible if each control generated by this strategy is admissible.

Definition 4 The strategy $U = U(x_0, y_0, t, v(t))$ guarantees the completion of pursuit at time θ if, for any admissible control of the evader $v(t),\ t \geq 0$, we have $x(\tau) = y(\tau)$ at some time $\tau \in [0, \theta]$, where $(x(\cdot), y(\cdot))$ is the solution of the initial value problem

$$\begin{aligned} &\dot{x}(t) = a(t)U(x_0, y_0, t, v(t)), \quad x(0) = x_0, ; \\ &\dot{y}(t) = a(t)v(t), \quad y(0) = y_0. \end{aligned} \tag{12}$$

Definition 5 The strategy $V(x_0, y_0, t)$ guarantees evasion in the game (1)-(3) with an initial positions x_0, y_0, if, for any admissible control of the pursuer $u(t),\ t \geq 0$, the relation $x(t) \neq y(t)$ holds for all $t \geq 0$.

3 Main results

3.1 The extended Π_{Gr}–strategy

Given $x_0 \neq y_0$, let $\xi_0 = \frac{x_0 - y_0}{\|x_0 - y_0\|}$ and $\delta = \rho^2 - \sigma^2$. The following is an extension of the Π_{Gr}–strategy constructed in Samatov et al. (2020a) for the simple motion differential game of one-pursuer-one-evader

$$U_{Gr}(t, v) := v(t) - \langle v(t), \xi_0 \rangle \xi_0 + \sqrt{\delta e^{2k \int_0^t a(s)ds} + \langle v(t), \xi_0 \rangle^2}\, \xi_0, \tag{13}$$

where $v(\cdot) \in \mathbb{R}^n$ is an admissible control of the evader. It can be verified that the strategy (13) satisfies

$$a(t)U_{Gr}(t, v) = a(t)v(t) - \lambda(t)\xi_0, \tag{14}$$

$$\|U_{Gr}(t, v)\|^2 = \|v(t)\|^2 + \delta e^{2k \int_0^t a(s)ds}, \text{for all } t \geq 0, \tag{15}$$

where $\lambda(t) = a(t)\left(\langle v(t), \xi_0 \rangle \pm \sqrt{\delta e^{2k \int_0^t a(s)ds} + \langle v(t), \xi_0 \rangle^2} \right)$. The following lemma is crucial in establishing the admissibility of the strategy (13).

Lemma 2 *The relation*

$$e^{2k \int_0^t a(s)ds} = 1 + 2k \int_0^t a(s) e^{2k \int_0^s a(r)dr} ds \tag{16}$$

holds for all positive real-valued function $a(s)$.

Proof Proof Setting

$$F(s) := e^{2k \int_0^s a(r)dr}$$

yields

$$dF(s) = 2ka(s)e^{2k\int_0^s a(r)dr}ds. \tag{17}$$

The relation in the lemma 2 follows by integrating both sides of (17) from $s = 0$ to $s = t$. From (14), (15) and the lemma 2, we obtain the admissibility of (13) as follows.

$$\begin{aligned}
\|U_{Gr}(t, v)\|^2 &= \|v(t)\|^2 + \delta e^{2k\int_0^t a(s)ds} \\
&\leq \sigma^2 + \delta e^{2k\int_0^t a(s)ds} + 2k\int_0^t a(s)\|v(s)\|^2 ds \\
&= \sigma^2 + \delta + 2k\int_0^t a(s)\delta e^{2k\int_0^s a(r)dr}ds + 2k\int_0^t a(s)\|v(s)\|^2 ds \\
&= \rho^2 + 2k\int_0^t a(s)\Big(\|v(s)\|^2 + \delta e^{2k\int_0^s a(r)dr}\Big)ds \\
&= \rho^2 + 2k\int_0^t a(s)\|U_{Gr}(s, v(s))\|^2 ds.
\end{aligned}$$

That is, $\|U_{Gr}(t, v)\|^2 \leq \rho^2 + 2k\int_0^t a(s)\|U_{Gr}(s, v(s))\|^2 ds$.

3.2 Attainability domains of players

The attainability domain of the pursuer P at any given time θ from the initial state x_0 is the closed balls $B(x_0, r_P(0, \theta))$, where

$$r_P(0, \theta) := \left(\rho^2 + 2k\int_0^\theta a(s)\|u(s)\|^2 ds\right)^{\frac{1}{2}} \int_0^\theta a(s)ds. \tag{18}$$

Based on the classical method of showing players attainability domain (see, for example Ibragimov and Salimi 2009), we must establish the following i. $\|x(\theta) - x_0\| \leq r_P(0, \theta)$;

ii. given any point $\bar{x}$ in $B(x_0, r_P(0, \theta)$ there exists an admissible control of the pursuer P that guarantees $x(\theta) = \bar{x}$. We show (i) using (5) as follows.

$$\begin{aligned}
\|x(\theta) - x_0\| &\leq \int_0^\theta a(r)\|u(r)\|dr \\
&\leq \int_0^\theta a(r)\left(\rho^2 + 2k\int_0^r a(s)\|u(s)\|^2 ds\right)^{\frac{1}{2}} dr \\
&\leq \left(\rho^2 + 2k\int_0^r a(s)\|u(s)\|^2 ds\right)^{\frac{1}{2}} \int_0^\theta a(r)dr \\
&\leq \left(\rho^2 + 2k\int_0^\theta a(s)\|u(s)\|^2 ds\right)^{\frac{1}{2}} \int_0^\theta a(r)dr =: r_P(0, \theta).
\end{aligned}$$

Thus, $\|x(\theta) - x_0\| \leq r_P(0, \theta)$. Note that to obtain the third inequality, we truncated some negative terms resulting from the integral in the second inequality. To show (ii), let $\bar{x} \in B(x_0, r_P(0, \theta))$. That is,

$$\|\bar{x} - x_0\| \leq r_P(0, \theta), \ . \tag{19}$$

We now establish $x(\theta) = \bar{x}$ using the pursuers' control

$$u(t) := \frac{\bar{x} - x_0}{\int_0^\theta a(s)ds}, \ 0 \leq t \leq \theta, \ . \tag{20}$$

as follows.

$$x(\theta) = x_0 + \int_0^\theta a(r)u(r)dr = x_0 + \frac{\bar{x} - x_0}{\int_0^\theta a(s)ds} \int_0^\theta a(r)dr = \bar{x}.$$

Hence $x(\theta) = \bar{x}$. Moreover, the admissibility of (20) follows easily from (19). That is

$$\|u(t)\|^2 = \frac{\|\bar{x} - x_0\|^2}{\left(\int_0^\theta a(s)ds\right)^2} \leq \rho^2 + 2k \int_0^t a(s)\|u(s)\|^2 ds.$$

Using similar argument, it can be shown that the attainability domain of the evader E at time θ from the initial state y_0 is the closed ball $B(y_0, r_E(0, \theta))$, where

$$r_E(0, \theta) := \left(\sigma^2 + 2k \int_0^\theta a(s)\|v(s)\|^2 ds\right)^{\frac{1}{2}} \int_0^\theta a(s)ds. \tag{21}$$

3.3 Conditions for completion of pursuit

To state the conditions, we introduce the following notations. Consider the game problem (1–3) and let:

$$\begin{aligned}
\Delta r &:= r_P^2(0, \theta) - r_E^2(0, \theta) \\
&= \left[(\rho^2 - \sigma^2) + 2k \int_0^\theta a(s)(\|u(s)\|^2 - \|v(s)\|^2)ds\right] \left(\int_0^\theta a(s)ds\right)^2 \\
&= \delta\left(e^{k\int_0^\theta a(s)ds} \int_0^\theta a(s)ds\right)^2,
\end{aligned} \tag{22}$$

and the half space X be defined as follows

$$X := \left\{z \in \mathbb{R}^n : 2\langle y_0 - x_0, z\rangle \leq \Delta r + \|y_0\|^2 - \|x_0\|^2\right\}. \tag{23}$$

Theorem 1 *If $\delta \geq 0$ and $y(\theta) \in X$, then the Π_{Gr}-strategy (13) guarantees the completion of pursuit in the game (1)-(3) for the pursuer.*

Proof Let the assumptions of the theorem hold and the strategy (13) be defined for all t in the interval $[0, \tau]$. For t in $(\tau, \theta]$, we set

$$U_{Gr}(t, v) = v(t), \tag{24}$$

where τ is the time instant at which $x(\tau) = y(\tau)$. Indeed, the strategy (24) is admissible in the time interval $(\tau, \theta]$ since

$$
\begin{aligned}
\|U_{Gr}(t,v)\|^2 &\le \sigma^2 + 2k\int_\tau^t a(s)\|v(s)\|^2 ds \\
&\le \rho^2 + 2k\int_0^t a(s)\Big(\|U_{Gr}(s,v)\|^2 - \delta e^{2k\int_0^s a(r)dr}\Big)ds \\
&\le \rho^2 + 2k\int_0^t a(s)\|U_{Gr}(s,v(s))\|^2 ds.
\end{aligned}
$$

Now let $x_0 \ne y_0$. By (5) and (6) we have $y(t) - x(t) = \xi_0 f(t)$ where

$$
f(t) := \|y_0 - x_0\| + \int_0^t a(s)\langle v(s), \xi_0\rangle ds - \int_0^t a(s)\Big(\langle v(s), \xi_0\rangle^2 + \delta e^{2k\int_0^s a(r)dr}\Big)^{\frac{1}{2}} ds.
$$

Since $f(0) = \|y_0 - x_0\| > 0$, then the conclusion of the Theorem 1 follows if we can establish $f(\theta) \le 0$. That is, $f(\theta) \le 0$ implies the existence of $\tau \in [0, \theta]$ such that $f(\tau) = 0$. To this end, we further introduce the function

$$
\mu(t) := \Big(\delta^{\frac{1}{2}} a(t) e^{k\int_0^t a(s)ds},\ a(t)\langle v(t), \xi_0\rangle\Big).
$$

Observe that

$$
\begin{aligned}
\int_0^\theta a(s)\Big(\langle v(s), \xi_0\rangle^2 + \delta e^{2k\int_0^s a(r)dr}\Big)^{\frac{1}{2}} ds &= \int_0^\theta \|\mu(s)\| ds \\
&\ge \left\|\int_0^\theta \mu(s)ds\right\| \\
&= \sqrt{K},
\end{aligned}
\tag{25}
$$

where $K := \delta\Big(\int_0^\theta a(s)e^{k\int_0^s a(r)dr}ds\Big)^2 + \Big(\int_0^\theta a(s)\langle v(s), \xi_0\rangle ds\Big)^2$.

It follows that

$$
f(\theta) \le \|y_0 - x_0\| + \int_0^\theta a(s)\langle v(s), \xi_0\rangle ds - \sqrt{K}. \tag{26}
$$

The assumption $y(\theta) \in X$ implies

$$
2\langle y_0 - x_0, y(\theta)\rangle \le \Delta r + \|y_0\|^2 - \|x_0\|^2. \tag{27}
$$

Since $y(\theta) = y_0 + \int_0^\theta a(s)v(s)ds$ and $y_0 - x_0 = \|y_0 - x_0\|\xi_0$, then we have

$$
\langle \xi_0, y(\theta)\rangle = \frac{1}{\|y_0 - x_0\|}\langle y_0 - x_0, y(\theta)\rangle \le \frac{\Delta r + \|y_0\|^2 - \|x_0\|^2}{2\|y_0 - x_0\|} =: d. \tag{28}
$$

Hence

$$
\langle \xi_0, y(\theta)\rangle = \langle \xi_0,\ y_0\rangle + \int_0^\theta a(s)\langle v(s), \xi_0\rangle ds
$$

implies

$$\int_0^\theta a(s)\langle v(s), \xi_0\rangle ds \leq d - \langle \xi_0,\ y_0\rangle. \tag{29}$$

In view of the fact that the function $\phi(t) = \|y_0 - x_0\| + t - (\Delta r + t^2)^{1/2}$ is an increasing function of t, then it follows from (26) and (29) that

$$f(\theta) \leq \|y_0 - x_0\| + d - \langle \xi_0,\ y_0\rangle - \left(\Delta r + (d - \langle \xi_0,\ y_0\rangle)^2\right)^{1/2}. \tag{30}$$

It can be verified that $(\|y_0 - x_0\| + d - \langle \xi_0,\ y_0\rangle)^2 = \Delta r + (d - \langle \xi_0,\ y_0\rangle)^2$ using the equations.

$$\begin{aligned} \|y_0 - x_0\|^2 &= \|y_0\|^2 + \|x_0\|^2 - 2\langle y_0, x_0\rangle \\ 2\|y_0 - x_0\|^2 &= \Delta r + \|y_0\|^2 - \|x_0\|^2 \\ 2\|y_0 - x_0\|\langle y_0, \xi_0\rangle &= 2\|y_0\|^2 - 2\langle y_0, x_0\rangle. \end{aligned} \tag{31}$$

Hence $f(\theta) \leq 0$. Consequently $f(\tau) = 0$ for some $\tau,\ 0 \leq t \leq \tau$. That is, $x(\tau) = y(\tau)$.

Since we already have $U_{Gr}(t, v) = v(t), \tau < t \leq \theta$ then it follows that

$$x(\theta) = x(\tau) + \int_\tau^\theta a(s)U_{Gr}(s, v(s))ds = y(\tau) + \int_\tau^\theta a(s)v(s)ds = y(\theta).$$

That is, $x(\theta) = y(\theta)$. □

3.4 Conditions for evasion

Theorem 2 *If* $\delta < 0$ *for all , then evasion is possible in the game* (1–3).

Proof Let the hypothesis hold. Consider the evader's strategy

$$V(t) := -\sigma e^{k\int_0^t a(s)ds}\xi_0. \tag{32}$$

Indeed, the strategy (32) is admissible since from lemma 2 we have

$$\begin{aligned} \|V(t)\|^2 &= \sigma^2 e^{2k\int_0^t a(s)ds} \\ &= \sigma^2\left(1 + 2k\int_0^t a(s)e^{k\int_0^s a(r)dr}ds\right) \\ &= \sigma^2 + 2k\int_0^t a(s)\sigma^2 e^{2k\int_0^s a(r)dr}ds \\ &= \sigma^2 + 2k\int_0^t a(s)\|V(s)\|^2 ds. \end{aligned}$$

Let $u(\cdot) \in \mathbb{R}^n$ be any admissible control of the pursuer, we show evasion using lemma 1 as follows.

$$\begin{aligned}
\|x(t)-y(t)\| &= \|x_0-y_0-\int_0^t a(s)V(s)ds+\int_0^t a(s)u(s)ds\| \\
&\geq \|x_0-y_0-\int_0^t a(s)V(s)ds\| - \|\int_0^t a(s)u(s)ds\| \\
&\geq \|x_0-y_0-\int_0^t a(s)V(s)ds\| - \int_0^t a(s)\|u(s)\|ds \\
&= \|x_0-y_0\| + \sigma\int_0^t a(s)e^{k\int_0^s a(r)dr}ds - \int_0^t a(s)\|u(s)\|ds \\
&\geq \|x_0-y_0\| + \sigma\int_0^t a(s)e^{k\int_0^s a(r)dr}ds - \rho\int_0^t a(s)e^{k\int_0^s a(r)dr}ds \\
&\geq \|x_0-y_0\| - \delta\int_0^t a(s)e^{k\int_0^s a(r)dr}ds > 0
\end{aligned}$$

Hence, $x(t) \neq y(t)$ for all $t \geq 0$. This completes the proof. □

4 Concluding remarks and suggestions for further research

We have studied a fixed duration pursuit-evasion differential game problem with the Grönwall-type constraints on players control functions. By virtue of the constraints on the players control functions, we constructed the players attainability domains. For the pursuit problem, we constructed the admissible Π_{Gr} strategy which is an extension of the well-known P–strategy, and proved under mild conditions on a certain half-space that the Π_{Gr} strategy can guarantee completion for pursuit. For the evasion problem, we proved that if the total energy resources of the pursuer is less than that of the evader, then evasion is guaranteed through out the game. The problem studied in this paper with multiple players, and also estimating the game value for the game (1) with the Grönwall-type constraints (2)-(3) are open problems for further research.

Author contributions All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by JR, MF and AJB. The first draft of the manuscript was written by BAP. All authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Funding Open access funding provided by Università degli Studi Mediterranea di Reggio Calabria within the CRUI-CARE Agreement. This work was partially supported by a grant from the Simons Foundation.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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