symmetry

MDPI

Article

On the Existence of Radial Solutions to a Nonconstant Gradient-Constrained Problem

Sofia Giuffrè *,† and Attilio Marcianò †

D.I.I.E.S, Mediterranea University of Reggio Calabria, Loc. Feo di Vito, 89122 Reggio Calabria, Italy; attilio.marciano@unirc.it

* Correspondence: sofia.giuffre@unirc.it; Tel.: +39-096-5169-3245

† These authors contributed equally to this work.

Abstract: In this paper, we study a variational problem with nonconstant gradient constraints. Several aspects related to problems with gradient constraints have been studied in the literature and have seen new developments in recent years. In the case of constant gradient constraint, the problem is the well-known elastic–plastic torsion problem. A relevant issue in this type of problem is the existence of Lagrange multipliers. Here, we consider the equivalent Lagrange multiplier formulation of a nonconstant gradient-constrained problem, and we investigate the class of solutions having a radial symmetry. We rewrite the problem in the radial symmetry case, and we analyse the different situations that may arise. In particular, in the planar case, we derive a condition characterizing the free boundary and obtain the explicit radial solution to the problem and the L^p Lagrange multiplier. Some examples support the results.

Keywords: nonconstant gradient constraints; Lagrange multipliers; radial solutions

check for updates

Citation: Giuffrè, S.; Marcianò, A. On the Existence of Radial Solutions to a Nonconstant Gradient-Constrained Problem. *Symmetry* **2022**, *14*, 1423. https://doi.org/10.3390/sym14071423

Academic Editor: Alexander Zaslavski

Received: 10 June 2022
Accepted: 6 July 2022
Published: 11 July 2022

1. Introduction

In this paper, we deal with a nonconstant gradient-constrained problem and the Lagrange multipliers associated with the problem.

In particular, we aim at studying the existence of radial solutions to its equivalent Lagrange multiplier formulation and investigating the free boundary.

The gradient-constrained problem is a classical problem that was subject to intense study a few decades ago. In its variational form, associated with the Laplacian, it reads

$$\text{Find } u \in K = \left\{ v \in H_0^{1,2}(\Omega) : |Dv|^2 = \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \leq G(x), \text{ a.e. in } \Omega \right\} \text{ such that:}$$

$$\int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \geq \int_\Omega F(v-u)dx \quad \forall v \in K, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded convex set with Lipschitz boundary $\partial\Omega$, $F \in L^p(\Omega)$, $p > 1$ (see [1,2] and references therein for several applications related to the problem).

An existence and uniqueness result in $W^{2,p}_{loc}(\Omega) \cap W^{1,\infty}_0(\Omega)$, with $1 < p < \infty$, for general linear elliptic equations with a nonconstant gradient constraint $G(x) \in C^2(\overline{\Omega})$ is proved in [3]. Regularity results for the solutions to the same problem are contained in [4–7].

A possible tool for studying this problem under a suitable condition for the constraint G is to rewrite the problem as a bi-obstacle problem, where the obstacles solve a Hamilton–Jacobi equation in the viscosity sense.

Following this method, in [8], the author studies a nonconstant gradient-constrained problem formulated by means of the variational inequality:

$$\text{Find } u \in K = \left\{ v \in H_0^{1,2}(\Omega) : |Dv|^2 = \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \leq G(x), \text{ a.e. in } \Omega \right\} \text{ such that:}$$

$$\int_\Omega \mathcal{L}u(v-u)\,dx \geq \int_\Omega F(v-u)\,dx, \quad \forall v \in K, \tag{2}$$

where

$$\mathcal{L}u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + cu$$

and $G(x) \in C^2(\overline{\Omega})$, $G(x) > 0$. The author proves that the nonconstant gradient-constrained problem admits a Lagrange multiplier, which is a Radon measure if the free term of the equation $f \in L^p$, $p > 1$, whereas, if f is a positive constant and under a suitable condition for G, the Lagrange multipliers belong to L^2.

We refer to [9–12] for other studies on the Lagrange theory and its application to variational models in which the existence of a Lagrange multiplier, that is always a relevant issue in this type of problem, is investigated.

In [2], the author considers a stationary variational inequality associated with the Laplacian, with nonconstant gradient constraint $|Du| \leq G(x)$, and proves the existence of a Lagrange multiplier assuming that the bounded open not-necessarily-convex set Ω has a boundary $\partial\Omega \in C^2$. In particular, the author proves that if $F \in L^\infty(\Omega)$, $G \in C^2(\Omega)$, $G > 0$, and $\Delta G^2 \leq 0$, the problem

$$\begin{cases} -\Delta u - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\overline{\mu} \frac{\partial u}{\partial x_i} \right) = F & a.e.\ in\ \Omega \\ u = 0 & on\ \partial\Omega \\ |Du| \leq G & a.e.\ in\ \Omega \\ \overline{\mu} \geq 0 & a.e.\ in\ \Omega; \\ \overline{\mu}(G - |Du|) = 0 & a.e.\ in\ \Omega \end{cases} \tag{3}$$

has a solution $(\mu, u) \in L^q(\Omega) \times W^{1,\infty}(\Omega)$, for any $q > 1$.

Let us stress that, in the case of $G = 1$, problem (1) is the well-known elastic–plastic torsion problem (see [13–16] and the references therein).

An interesting issue in the field of partial differential equations is the investigation of the class of solutions with symmetry. This study may provide some information on the problem under consideration, and it is also justified by the applications of symmetric solutions in problems that appear in mathematical physics.

The class of solutions with radial symmetry for the elastic–plastic torsion problem associated with the Laplacian, was studied in [17], where the author finds the formal explicit radial solution to the elastic–plastic problem in the case $F = const > 0$ and $n = 2$.

This result is generalized in [18] to the case $F \in L^p(\Omega)$, $p > 2$.

In this paper, we investigate for $n = 2$ the existence of radial solutions to (3), that is, the equivalent Lagrange multiplier formulation of the nonconstant gradient-constrained problem (2). In particular, we show, under a suitable condition, the existence of solutions $(\overline{\mu}, u) \in L^p(\Omega) \times W^{2,p}(\Omega)$ to (3) that are of the radial type, namely $\overline{\mu}(x) = \mu(|x|) = \mu(\rho)$, $u(x) = \varphi(|x|) = \varphi(\rho)$.

Moreover, by denoting

$$E = \{\rho \in [0,1] : \ |\varphi'(\rho)| < g(\rho)\}$$

and

$$P = \{\rho \in [0,1] : \ |\varphi'(\rho)| = g(\rho)\},$$

we determine a necessary and sufficient condition for the region P to exist, and we characterize the free boundary.

Let us recall that in the case $g \equiv 1$, namely, when conditions (3) represent the elastic–plastic torsion problem, the region E is the elastic region and P is the plastic region. Analogously, in the case of nonconstant gradient constraints, we again denote the plastic region by P.

The paper is organized in the following way. In Section 2, we state the assumptions and the main results of the paper. In Section 3, we determine, under condition (5), the existence of two regions $E = [0, \overline{\rho}]$ and $P = [\overline{\rho}, 1]$, in which $|Du| < G(x)$ and $|Du| = G(x)$, respectively. Moreover, we verify that the solution $u(x)$ of (3) belongs to $W^{2,p}(\Omega)$. Condition (5) is a necessary and sufficient condition for the region P to exist. Indeed, in Section 4, we study the cases in which condition (5) is not satisfied and show that, in this situation, only the region E may exist. Moreover, in Section 5, we provide some examples in different situations, and the explicit solutions are determined. Finally, in Section 6, we highlight the relationship between the problem under consideration and the obstacle problem, and in Sections 7 and 8 we summarize our results and consider future work.

2. Statement of the Problem

Now, we assume that Ω is the ball of $\mathbb{R}^n$, $n \geq 2$, of radius 1 centred at the origin, $F \in L^p(\Omega)$, $p > n$, and $G(x) \in C^2(\Omega)$, $G > 0$. Moreover, F and G are of radial type, namely $F(x) = f(|x|) = f(\rho)$ and $G(x) = g(|x|) = g(\rho)$, with $|x| = \rho$, $\rho \in [0, 1]$.

Our aim is to investigate the existence of solutions $(\overline{\mu}, u) \in L^p(\Omega) \times W^{2,p}(\Omega)$ to (3) that are of radial type, namely $\overline{\mu}(x) = \mu(|x|) = \mu(\rho)$, $u(x) = \varphi(|x|) = \varphi(\rho)$.

In these settings, since $u(x) = \varphi(\rho)$, $\frac{\partial u}{\partial x_i} = \varphi'(\rho)\frac{x_i}{\rho}$, $\Delta u = \varphi'' + \frac{n-1}{\rho}\varphi'(\rho)$, $|Du| = \varphi'(\rho)$, bearing in mind that $|Du| \leq G$, conditions (3) become

$$\begin{cases} |\varphi'(\rho)| \leq g(\rho) \quad a.e.\ in\ [0,1]; \\ \mu(\rho) \geq 0 \quad a.e.\ in\ [0,1]; \\ \mu(\rho)\big(g(\rho) - |\varphi'(\rho)|\big) = 0 \quad a.e.\ in\ [0,1]; \\ -\varphi''(\rho) - \dfrac{n-1}{\rho}\varphi'(\rho) - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\mu \frac{\partial u}{\partial x_i}\right) = f(\rho) \quad a.e.\ in\ [0,1]. \end{cases} \tag{4}$$

Now, we are able to obtain our main results that hold for $n = 2$.

We suppose that there exists $\overline{\rho} \in (0, 1)$ such that

$$\int_0^{\overline{\rho}} \rho f(\rho) d\rho = \overline{\rho} g(\overline{\rho}). \tag{5}$$

We show that condition (5) is a necessary and sufficient condition for the plastic region P to exist, and we characterize the free boundary.

For this purpose, we also assume that:

$$\begin{cases} \rho f(\rho) \geq 0 \ \text{ is a nondecreasing function in } [0,1]; \\ g(\rho) \text{ is a nonincreasing function in } [0,1]; \\ \rho g(\rho) g'(\rho) \leq 0 \ \text{ is a nonincreasing function in } [0,1]. \end{cases} \tag{6}$$

Let us emphasize that the condition "$\rho g(\rho) g'(\rho)$ is a nonincreasing function" ensures the extra assumption on G, $\Delta G^2 \leq 0$, required for the existence of a Lagrange multiplier in $L^q(\Omega)$ (see [2]).

The main results of the paper are formulated in the following theorems.

Theorem 1. *Let Ω be the ball of $\mathbb{R}^2$ of radius 1 centred at the origin. Under conditions (5) and (6), the region*

$$E = \{\rho \in [0,1] : \ |\varphi'(\rho)| < g(\rho)\} = [0, \overline{\rho}]$$

and the region

$$P = \{\rho \in [0,1] : \ |\varphi'(\rho)| = g(\rho)\} = [\overline{\rho}, 1].$$

Moreover, the solution φ to (4) is

$$\varphi(\rho) = \begin{cases} \displaystyle\int_\rho^{\overline{\rho}} \frac{1}{t} \int_0^t \sigma f(\sigma) d\sigma dt + \int_{\overline{\rho}}^1 g(t) dt & \rho \in [0, \overline{\rho}] \\ \int_\rho^1 g(t) dt & \rho \in (\overline{\rho}, 1], \end{cases}$$

and it results in $u(x) = \varphi(|x|) \in W^{2,p}(\Omega)$.

Finally,

$$\mu(\rho) = \begin{cases} 0 & \rho \in [0, \overline{\rho}] \\ \displaystyle\frac{1}{\rho g(\rho)} \int_0^\rho \sigma f(\sigma) d\sigma - 1 & \rho \in [\overline{\rho}, 1] \end{cases}$$

and $\mu(\rho) \in L^p(0,1)$.

Moreover, we are able to prove that if condition (5) is not satisfied, namely, $\forall \rho \in (0,1)$

$$\frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma < g(\rho) \quad or \quad \frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma > g(\rho),$$

the region P does not exist.

Due to the gradient constraints $|\varphi'(\rho)| < g(\rho)$, the case

$$\frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma > g(\rho) \quad \forall \rho \in (0,1)$$

is not admissible in our framework, since it implies

$$\varphi'(\rho) < -g(\rho) \quad \forall \rho \in (0,1).$$

Then, we may state the following theorem.

Theorem 2. *Under conditions (6), if*

$$\frac{1}{\rho} \int_0^\rho \sigma f(\sigma) d\sigma < g(\rho) \quad \forall \rho \in (0,1),$$

then

$$|\varphi'(\rho)| < g(\rho) \quad \forall \rho \in [0,1],$$

namely,

$$E = [0,1].$$

Moreover, the solution φ to (4) is

$$\varphi(\rho) = \int_\rho^1 \frac{1}{t} \int_0^t \sigma f(\sigma) d\sigma dt \quad \forall \rho \in [0,1].$$

This results in $u(x) = \varphi(|x|) \in W^{2,p}(\Omega)$.

3. Investigation of the Free Boundary

This section contains the proof of Theorem 1.

In particular, first, for $n = 2$ we investigate, under condition (5), the existence of two regions $E = [0, \overline{\rho}]$ and $P = [\overline{\rho}, 1]$, in which $|\varphi'(\rho)| < g(\rho)$, $\mu(\rho) = 0$ and $|\varphi'(\rho)| = g(\rho)$, $\mu(\rho) > 0$, respectively.

Then, in $[\overline{\rho}, 1]$, we should have $\varphi(\rho) = \int_0^\rho g(t) dt + c$ or $\varphi(\rho) = -\int_0^\rho g(t) dt + c$. Since we are searching for solutions $\varphi(\rho) \geq 0$ in $[0,1]$ (indeed, when $f = const. > 0$, $\varphi(\rho)$ is non-negative) and taking into account the fact that $\varphi(1) = 0$, it follows that

$$\varphi(\rho) = \int_{\rho}^{1} g(t)dt \geq 0 \ \ in \ [\overline{\rho}, 1]. \tag{7}$$

Furthermore, since we are searching for $\varphi(\rho)$ such that $\varphi'(\rho)$ is continuous, this results in

$$\varphi'(\overline{\rho}) = -g(\overline{\rho}). \tag{8}$$

In $[0, \overline{\rho}]$ $\mu(\rho) = 0$, then, from (4), we obtain

$$-\varphi''(\rho) - \frac{n-1}{\rho}\varphi'(\rho) = f(\rho),$$

that is, for $n = 2$,

$$-\varphi''(\rho) - \frac{\varphi'(\rho)}{\rho} = f(\rho),$$

$$D(\rho\varphi'(\rho)) = -\rho f(\rho),$$

and by integration between ρ and $\overline{\rho}$, requiring (8), we obtain

$$\rho\varphi'(\rho) = \int_{\rho}^{\overline{\rho}} \sigma f(\sigma)d\sigma - \overline{\rho}g(\overline{\rho}) \quad \forall \rho \in [0, \overline{\rho}]. \tag{9}$$

If $\rho = 0$ we regain condition (5), whereas for $\rho \in (0, \overline{\rho}]$

$$\varphi'(\rho) = \frac{1}{\rho}\int_{\rho}^{\overline{\rho}} \sigma f(\sigma)d\sigma - \frac{\overline{\rho}}{\rho}\, g(\overline{\rho}) \quad \forall \rho \in (0, \overline{\rho}]. \tag{10}$$

By virtue of (5), we have:

$$\begin{aligned} \varphi'(\rho) &= \frac{\int_{\rho}^{\overline{\rho}} \sigma f(\sigma)d\sigma - \overline{\rho}g(\overline{\rho})}{\rho} \\ &= \frac{\int_{0}^{\overline{\rho}} \sigma f(\sigma)d\sigma - \int_{0}^{\rho} \sigma f(\sigma)d\sigma - \overline{\rho}g(\overline{\rho})}{\rho} = -\frac{\int_{0}^{\rho} \sigma f(\sigma)d\sigma}{\rho}. \end{aligned} \tag{11}$$

Let us remark that, from the assumption $F(x) \in L^p(\Omega)$, $p > 2$, it follows that

$$\lim_{\rho \to 0^+} \rho f(\rho) = 0. \tag{12}$$

Then, $\varphi'(\rho)$ is continuous in $(0, \overline{\rho}]$, $\varphi'(\rho) \leq 0$ in $(0, \overline{\rho}]$. Moreover, if we check the behaviour for $\rho = 0$, we have

$$\begin{aligned} \lim_{\rho \to 0^+} \varphi'(\rho) &= \lim_{\rho \to 0^+} -\frac{\int_{0}^{\rho} \sigma f(\sigma)d\sigma}{\rho} = \\ &= \lim_{\rho \to 0^+} \rho f(\rho) = 0, \end{aligned} \tag{13}$$

and $\varphi'(\rho)$ is continuous in $[0, \overline{\rho}]$, assuming that $\varphi'(0) = 0$.

We must verify that

$$\varphi'(\rho) = -\frac{1}{\rho}\int_{0}^{\rho} \sigma f(\sigma)d\sigma \geq -g(\rho) \quad \forall \rho \in (0, \overline{\rho}], \tag{14}$$

namely,

$$\frac{1}{\rho}\int_{0}^{\rho} \sigma f(\sigma)d\sigma \leq g(\rho) \quad \forall \rho \in (0, \overline{\rho}]. \tag{15}$$

The assumption (6) implies that the function $\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma$ is also a nondecreasing function in $(0,\overline{\rho}]$. Indeed, it results in

$$D\left(\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma\right)=\frac{1}{\rho^2}\left(\rho^2 f(\rho)-\int_0^\rho \sigma f(\sigma)d\sigma\right),$$

and since $\rho f(\rho)$ is a nondecreasing function in $(0,1)$, we obtain

$$\int_0^\rho \sigma f(\sigma)d\sigma \leq \int_0^\rho \rho f(\rho)d\sigma = \rho^2 f(\rho),$$

namely,

$$D\left(\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma\right)\geq 0.$$

Then, by virtue of (5) and (6),

$$\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma \leq \frac{1}{\overline{\rho}}\int_0^{\overline{\rho}} \sigma f(\sigma)d\sigma = g(\overline{\rho}) \leq g(\rho) \quad \forall \rho\in(0,\overline{\rho}],$$

and (15) is achieved.

Now, we may derive the explicit solution $\varphi(\rho)$ in $[0,\overline{\rho}]$.

From (11), we obtain $\forall\rho\in(0,\overline{\rho}]$

$$\varphi(\rho)=-\int_0^\rho \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt + c. \tag{16}$$

By virtue of (7),

$$\varphi(\overline{\rho})=\int_{\overline{\rho}}^1 g(t)dt = -\int_0^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt + c, \tag{17}$$

we have

$$c=\int_{\overline{\rho}}^1 g(t)dt - \int_{\overline{\rho}}^0 \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt$$

and hence, in $(0,\overline{\rho}]$

$$\varphi(\rho)=\int_\rho^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt + \int_{\overline{\rho}}^1 g(t)dt. \tag{18}$$

Then, from (11) and (18) it follows that the solution $\varphi(\rho)\geq 0$ in $(0,\overline{\rho}]$ is nonincreasing in $(0,\overline{\rho}]$. It remains to study the behaviour of φ at $\rho=0$.

We have already verified that $\varphi'(\rho)=-\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma$ is continuous. Hence, we obtain

$$\lim_{\rho\to 0^+}\int_\rho^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt = \int_0^{\overline{\rho}} \frac{1}{t}\int_0^t \sigma f(\sigma)d\sigma dt < +\infty,$$

and $\varphi(\rho)$ is continuous in $[0,\overline{\rho}]$.

Finally,

$$\varphi''(\rho)=\frac{1}{\rho^2}\int_0^\rho \sigma f(\sigma)d\sigma - f(\rho) \quad a.e\ in\ [0,\overline{\rho}].$$

Let us observe that by virtue of (6),

$$\varphi''(\rho)=\frac{1}{\rho^2}\int_0^\rho \sigma f(\sigma)d\sigma - f(\rho) \leq \frac{1}{\rho^2}\int_0^\rho \rho f(\rho)d\sigma - f(\rho) = 0 \quad a.e\ in\ [0,\overline{\rho}].$$

In the same way as in [18], it is possible to verify that $u(x)=\varphi(|x|)\in W^{2,p}(\Omega)$. It can also be noted that this method is applicable to the study of evolutionary equations (see [19]).

Let us now investigate the region $P = [\overline{\rho}, 1]$. We have already found the explicit solution in $[\overline{\rho}, 1]$, namely $\varphi(\rho) = \int_\rho^1 g(t)dt$.

In the interval $[\overline{\rho}, 1]$, since $\varphi'(\rho) = -g(\rho)$, $\varphi''(\rho) = -g'(\rho)$, $\frac{\partial u}{\partial x_i} = \varphi'(\rho)\frac{x_i}{\rho}$, we obtain

$$g'(\rho) + \frac{n-1}{\rho}g(\rho) + \frac{1}{\rho^{n-1}}D(\rho^{n-1}\mu(\rho)g(\rho)) = f(\rho) \quad a.e.\ in\ [\overline{\rho}, 1]$$

If we assume $n = 2$, we obtain

$$g'(\rho) + \frac{1}{\rho}g(\rho) + \frac{1}{\rho}D(\rho\mu(\rho)g(\rho)) = f(\rho) \quad a.e.\ in\ [\overline{\rho}, 1]$$

and

$$D(\rho g(\rho) + \rho\mu(\rho)g(\rho)) = \rho f(\rho) \quad a.e.\ in\ [\overline{\rho}, 1].$$

By integration, since $\mu(\overline{\rho}) = 0$, we obtain

$$\mu(\rho) = \frac{1}{\rho g(\rho)}\int_0^\rho \sigma f(\sigma)d\sigma - 1 \quad a.e.\ in\ [\overline{\rho}, 1].$$

We must verify that $\mu(\rho) \geq 0$, namely,

$$\frac{1}{\rho g(\rho)}\int_0^\rho \sigma f(\sigma)d\sigma \geq 1. \tag{19}$$

Repeating the same arguments as above, we may prove that the function $\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma$ is nondecreasing in $[\overline{\rho}, 1]$. Then, by virtue of (5) and (6), it results in

$$\frac{1}{g(\rho)}\left(\frac{1}{\rho}\int_0^\rho \sigma f(\sigma)d\sigma\right) \geq \frac{1}{g(\rho)}\left(\frac{1}{\overline{\rho}}\int_0^{\overline{\rho}} \sigma f(\sigma)d\sigma\right) = \frac{g(\overline{\rho})}{g(\rho)} \geq 1 \quad \forall \rho \in [\overline{\rho}, 1],$$

that is, the desired condition (19).

It is easily seen that $\mu(\rho) \in L^p([0, 1])$.

In order to complete the investigation of the free boundary, let us check whether it could happen that the region $P = [0, \overline{\rho}]$ and the region $E = [\overline{\rho}, 1]$. In this case, from (4), it follows that

$$g'(\rho) + \frac{n-1}{\rho}g(\rho) + \frac{1}{\rho^{n-1}}D(\rho^{n-1}\mu(\rho)g(\rho)) = f(\rho) \quad a.e.\ in\ (0, \overline{\rho}]$$

If we assume $n = 2$, we obtain

$$g'(\rho) + \frac{1}{\rho}g(\rho) + \frac{1}{\rho}D(\rho\mu(\rho)g(\rho)) = f(\rho) \quad a.e.\ in\ (0, \overline{\rho}]$$

and

$$D(\rho g(\rho) + \rho\mu(\rho)g(\rho)) = \rho f(\rho) \quad a.e.\ in\ (0, \overline{\rho}]. \tag{20}$$

Integrating (20) in $[\rho, \overline{\rho}]$, $\rho \in [0, \overline{\rho}]$, and assuming $\mu(\overline{\rho}) = 0$, since $\mu(\rho) = 0$ in $[\overline{\rho}, 1]$, we have

$$\begin{aligned}
\mu(\rho) &= \frac{1}{\rho g(\rho)}\left[\overline{\rho}g(\overline{\rho}) - \int_\rho^{\overline{\rho}} \sigma f(\sigma)d\sigma\right] - 1 = \\
&= \frac{1}{\rho g(\rho)}\left[\int_0^{\overline{\rho}} \sigma f(\sigma)d\sigma - \int_\rho^{\overline{\rho}} \sigma f(\sigma)d\sigma\right] - 1 = \\
&= \frac{1}{\rho g(\rho)}\int_0^\rho \sigma f(\sigma)d\sigma - 1.
\end{aligned} \tag{21}$$

Since the function $\frac{1}{\rho} \int_0^\rho \sigma f(\sigma)d\sigma$ is nondecreasing in $(0,\overline{\rho}]$ and by virtue of (5) and (6), this results in

$$\frac{1}{g(\rho)}\left(\frac{1}{\rho} \int_0^\rho \sigma f(\sigma)d\sigma\right) \leq \frac{1}{g(\rho)}\left(\frac{1}{\overline{\rho}} \int_0^{\overline{\rho}} \sigma f(\sigma)d\sigma\right) = \frac{g(\overline{\rho})}{g(\rho)} \leq 1 \quad \forall \rho \in (0,\overline{\rho}],$$

namely,

$$\mu(\rho) \leq 0 \quad \forall \rho \in (0,\overline{\rho}].$$

In conclusion, if we assume (5), the case $P = [0,\overline{\rho}]$ and $E = [\overline{\rho},1]$ cannot happen.

4. Proof of Theorem 2

In this section we investigate, for $n = 2$, what happens if Equation (5) does not admit any solution $\overline{\rho} \in (0,1]$, namely $\forall \rho \in (0,1]$

$$\frac{1}{\rho} \int_0^\rho \sigma f(\sigma)d\sigma < g(\rho) \quad or \quad \frac{1}{\rho} \int_0^\rho \sigma f(\sigma)d\sigma > g(\rho). \tag{22}$$

Let us stress that due to the gradient constraints $|\varphi'(\rho)| < g(\rho)$, the case

$$\frac{1}{\rho} \int_0^\rho \sigma f(\sigma)d\sigma > g(\rho) \quad \forall \rho \in (0,1] \tag{23}$$

is not admissible in our framework, since it implies

$$\varphi'(\rho) < -g(\rho) \quad \forall \rho \in [0,1]. \tag{24}$$

From now on, we suppose that

$$\frac{1}{\rho} \int_0^\rho \sigma f(\sigma)d\sigma < g(\rho) \quad \forall \rho \in (0,1]. \tag{25}$$

As a consequence of the investigation of the free boundary in Section 3, in this case we obtain the result that the region P does not exist.

In the same way as in [18], we obtain that

$$\varphi'(\rho) = \frac{1}{\rho} \int_\rho^1 \sigma f(\sigma)d\sigma + \frac{\varphi'(1)}{\rho} \quad \forall \rho \in (0,1]. \tag{26}$$

Assuming

$$\lim_{\rho\to 0^+} \int_\rho^1 \sigma f(\sigma)d\sigma = \int_0^1 \sigma f(\sigma)d\sigma = -\varphi'(1), \tag{27}$$

$\varphi'(\rho) \in L^p([0,1])$.

Indeed, by virtue of (27), we have

$$\begin{aligned}\lim_{\rho\to 0^+} \varphi'(\rho) &= \lim_{\rho\to 0^+} \frac{\int_0^1 \sigma f(\sigma)d\sigma - \int_0^\rho \sigma f(\sigma)d\sigma + \varphi'(1)}{\rho} \\ &= \lim_{\rho\to 0^+} \frac{-\int_0^\rho \sigma f(\sigma)d\sigma}{\rho} = 0.\end{aligned}$$

Then,

$$\varphi'(\rho) = -\frac{1}{\rho} \int_0^\rho \sigma f(\sigma)d\sigma \quad \forall \rho \in (0,1] \tag{28}$$

is continuous in $[0,1]$, assuming $\varphi'(0) = 0$. Moreover,

$$|\varphi'(\rho)| < g(\rho) \quad \forall \rho \in [0,1],$$

that is, $E = [0,1]$.

It follows, since $\varphi(1) = 0$, that

$$\varphi(\rho) = \int_\rho^1 \frac{1}{t} \int_0^t \sigma f(\sigma) d\sigma dt. \tag{29}$$

5. Examples

Example 1. *Let us consider a first example, namely, $f = const = k > 0$ and $g(\rho) = e^{-\frac{\rho}{2}}$.*

Obviously, condition (6) is verified.

If we consider a first case, $f = k > 2e^{-\frac{1}{2}}$, the region P exists, since Equation (5)

$$\int_0^\rho k\sigma d\sigma = \rho g(\rho),$$

namely,

$$\frac{k}{2}\rho = e^{-\frac{\rho}{2}},$$

admits a solution $\overline{\rho} \in (0,1)$.

Then, by (18), we obtain the continuous function

$$\varphi(\rho) = \begin{cases} \int_\rho^{\overline{\rho}} \frac{1}{t} \int_0^t k\sigma d\sigma dt + \int_{\overline{\rho}}^1 e^{-\frac{t}{2}} dt = \frac{k}{4}(\overline{\rho}^2 - \rho^2) + 2(e^{-\frac{\overline{\rho}}{2}} - e^{-\frac{1}{2}}) & in\ E = [0, \overline{\rho}) \\ \int_\rho^1 e^{-\frac{t}{2}} dt = 2(e^{-\frac{\rho}{2}} - e^{-\frac{1}{2}}) & in\ P = [\overline{\rho}, 1]. \end{cases}$$

It is easily seen that $u, Du \in L^p(\Omega)$ and

$$\begin{aligned} \int_\Omega |D^2 u|^p \, dx &= \int_0^1 \left| \sum_{i,j=1}^2 \varphi''(\rho) \frac{x_i x_j}{\rho^2} + \sum_{i,j=1}^2 \varphi'(\rho) \, \frac{\delta_{ij}\rho^2 - x_i x_j}{\rho^3} \right|^p \rho \, d\rho \\ &\leq c(p) \left[\int_0^{\overline{\rho}} k^p \rho d\rho + \int_{\overline{\rho}}^1 \rho^{1-p} d\rho \right] < \infty. \end{aligned}$$

Finally, the Lagrange multiplier $\mu(\rho)$ exists and belongs to $L^p([0,1])$:

$$\mu(\rho) = \begin{cases} \frac{1}{\rho e^{-\frac{\rho}{2}}} \int_0^\rho k\sigma d\sigma \; - \; 1 = \frac{k}{2}\rho e^{\frac{\rho}{2}} \; - \; 1 \geq 0 & in\ P = [\overline{\rho}, 1] \\ 0 & in\ E = [0, \overline{\rho}). \end{cases}$$

If we consider the other case, $f = const = k$, $0 < k \leq 2e^{-\frac{1}{2}}$, the region P does not exist, since $\overline{\rho} \geq 1$ is the solution to Equation (5).

Then, $E = [0,1]$. In fact, from (28) we obtain

$$\varphi'(\rho) = -\frac{k}{2}\rho \quad \forall \rho \in [0,1],$$

then

$$-e^{-\frac{\rho}{2}} \leq \varphi'(\rho) \leq e^{-\frac{\rho}{2}} \quad \forall \rho \in [0,1]$$

and (27) is verified.

Then, from (29), we obtain the continuous function

$$\varphi(\rho) = \frac{k}{4}(1 - \rho^2). \tag{30}$$

Here, $\varphi(\rho)$ as in (30) and $\mu = 0$ verify conditions (4) in $[0,1]$.

Moreover, $u \in W^{2,p}(\Omega)$. In fact, it is easily seen that u, $Du \in L^p(\Omega)$ and

$$\int_\Omega |D^2u|^p\, dx = \int_0^1 \left| \sum_{i,j=1}^{2} \varphi''(\rho)\frac{x_ix_j}{\rho^2} + \sum_{i,j=1}^{2} \varphi'(\rho)\, \frac{\delta_{ij}\rho^2 - x_ix_j}{\rho^3} \right|^p \rho\, d\rho$$
$$\leq c\int_0^1 \rho d\rho < \infty.$$

Example 2. *Let us now consider problem (4) with* $f(\rho) = \dfrac{k}{\rho^\alpha}$, $0 < \alpha < 1$.

The condition $\alpha < 1$ *ensures that* $F(x) \in L^p(\Omega)$, $2 = n < p < \frac{2}{\alpha}$. *Moreover, condition (6) is verified.*

If we consider the case $k > \frac{2-\alpha}{e^{\frac{1}{2}}}$, *the region* P *exists, since Equation (5)*

$$\int_0^\rho \sigma \frac{k}{\sigma^\alpha}\, d\sigma = \int_0^\rho k\sigma^{1-\alpha} d\sigma = \rho g(\rho),$$

namely,

$$\frac{k}{2-\alpha}\rho^{1-\alpha} = e^{-\frac{\rho}{2}},$$

admits a solution $\overline{\rho} \in (0,1)$.

Then, from (18), we obtain the continuous function

$$\varphi(\rho) = \begin{cases} \dfrac{k}{(2-\alpha)^2}\left(\overline{\rho}^{2-\alpha} - \rho^{2-\alpha}\right) + 2(e^{-\frac{\overline{\rho}}{2}} - e^{-\frac{1}{2}}) & in\ E = [0,\overline{\rho}) \\ \\ 2(e^{-\frac{\rho}{2}} - e^{-\frac{1}{2}}) & in\ P = [\overline{\rho}, 1]. \end{cases}$$

It is easily seen that u, $Du \in L^p(\Omega)$ *and*

$$\int_\Omega |D^2u|^p\, dx = \int_0^1 \left| \sum_{i,j=1}^{2} \varphi''(\rho)\frac{x_ix_j}{\rho^2} + \sum_{i,j=1}^{2} \varphi'(\rho)\, \frac{\delta_{ij}\rho^2 - x_ix_j}{\rho^3} \right|^p \rho\, d\rho$$
$$\leq c\int_0^{\overline{\rho}} \rho^{1-\alpha p} d\rho < \infty.$$

Moreover, the Lagrange multiplier $\mu(\rho)$ *exists and belongs to* $L^p([0,1])$:

$$\mu(\rho) = \begin{cases} \dfrac{1}{\rho e^{-\frac{\rho}{2}}} \displaystyle\int_0^\rho k\sigma^{1-\alpha} d\sigma\ -\ 1 = \dfrac{k}{2-\alpha}\,\dfrac{\rho^{1-\alpha}}{e^{-\frac{\rho}{2}}}\ -\ 1 \geq 0 & in\ P = (\overline{\rho}, 1] \\ \\ 0 & in\ E = [0,\overline{\rho}]. \end{cases}$$

Finally, if we consider the other case $0 < k \leq \frac{2-\alpha}{e^{\frac{1}{2}}}$, *the region* P *does not exist, since the hypotheses of the zero theorem are not satisfied in* $[0,1]$ *and* $\overline{\rho} \geq 1$ *is the solution to (5).*

Then, $E = [0,1]$. *In fact, from (28), we obtain*

$$\varphi'(\rho) = -\frac{k}{2-\alpha}\rho^{1-\alpha} \quad \forall \rho \in [0,1],$$

then

$$-e^{-\frac{\rho}{2}} \leq \varphi'(\rho) \leq e^{-\frac{\rho}{2}} \quad \forall \rho \in [0,1]$$

and (27) is verified.

Then, from (29), we obtain the continuous function

$$\varphi(\rho) = \frac{k}{(2-\alpha)^2}(1 - \rho^{2-\alpha}) \quad \forall \rho \in [0,1]. \tag{31}$$

Here, $\varphi(\rho)$ as in (31) and $\mu = 0$ verify conditions (4) in $[0,1]$.
Moreover, $u \in W^{2,p}(\Omega)$. In fact, it is easily seen that $u, Du \in L^p(\Omega)$ and

$$\int_\Omega |D^2u|^p \, dx = \int_0^1 \left| \sum_{i,j=1}^{2} \varphi''(\rho) \frac{x_i x_j}{\rho^2} + \sum_{i,j=1}^{2} \varphi'(\rho) \frac{\delta_{ij}\rho^2 - x_i x_j}{\rho^3} \right|^p \rho \, d\rho$$
$$\leq c \int_0^1 \rho^{1-\alpha p} d\rho < \infty,$$

since $p < \frac{1}{\alpha}$.

6. Relationship with the Obstacle Problem

In this section, we highlight the relationship between the radial solutions to the non-constant gradient-constrained problem under consideration and to the obstacle problem.

In the case $F = const > 0$ in [8], the author proves, under the extra condition

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial G}{\partial x_j} \right) \geq 0 \quad \text{in } \Omega, \tag{32}$$

the equivalence between the problem

Find $u \in K = \left\{ v \in H_0^{1,2}(\Omega) : |Dv|^2 = \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i} \right)^2 \leq G(x), \text{ a.e. in } \Omega \right\}$ such that:

$$\int_\Omega \mathcal{L}u(v-u)\, dx \geq \int_\Omega F(v-u)\, dx, \quad \forall v \in K,$$

and the obstacle problem

Find $u \in K_W = \left\{ v \in H_0^{1,2}(\Omega) : |v(x)| \leq W(x) \text{ a.e. in } \Omega \right\}$ such that:

$$\int_\Omega \mathcal{L}u(v-u)\, dx \geq \int_\Omega f(v-u)\, dx, \quad \forall v \in K_W. \tag{33}$$

According to the definition by P.L. Lions in [20], the obstacle $W \in H^{1,\infty}(\Omega)$ is the viscosity solution to the Hamilton–Jacobi equation

$$\begin{cases} |DW| = \sqrt{G(x)} & a.e.\ in\ \Omega \\ W = 0 & on\ \partial\Omega \end{cases} \tag{34}$$

defined by

$$W(x) = inf_{x_0 \in \partial\Omega} L(x, x_0) \tag{35}$$

with

$$\begin{aligned} L(x,x_0) = inf \Big\{ \int_0^{T_0} \sqrt{G(\xi(s))} ds : \xi : [0,T_0] \to \overline{\Omega}, \\ \xi(0) = x,\ \xi(T_0) = x_0,\ |\xi'(s)| \leq 1\ a.e.\ in\ [0,T_0] \Big\}. \end{aligned} \tag{36}$$

In our settings, that is, the gradient constraint $|Du| \leq G(x)$, condition (6) implies that the extra condition on G is verified, namely $\Delta G^2 \leq 0$. Then, from Theorem 1 in [8], it follows that the solution u to the gradient-constrained problem coincides with the solution to the obstacle problem, with the obstacle function

$$W(x) = w(\rho) = \int_\rho^1 g(t) dt.$$

In the case $g = 1$, that is, the elastic–plastic torsion problem, the obstacle is $w(\rho) = 1 - \rho$, namely, the distance function.

Let us remark that in [8], the author provides an example that shows that the obstacle problem is not always equivalent to problem (2), even for $n = 1$.

7. Discussion

In this paper, we studied a nonconstant gradient-constrained problem. In this context, several results have been obtained in the literature, concerning different aspects such as the existence and regularity of solutions, the relationship with the obstacle problem, the existence of Lagrange multipliers, and so on.

Moreover, in the framework of partial differential equations, an interesting research direction is the study of solutions with symmetry.

In this paper, we focused on the existence of solutions with radial symmetry to the Lagrange multiplier formulation of a nonconstant gradient-constrained problem.

We rewrote the Lagrange multiplier problem in the radial setting and analysed all the possible cases.

We investigated the free boundary and determined a necessary and sufficient condition that ensures the existence of an elastic region and a plastic region. If this condition is not satisfied, we verified that the plastic region does not exist.

The results were supported by some numerical examples. Finally, we provided some comments on the relationship with the obstacle problem in the radial setting.

8. Conclusions

In the literature, the nonconstant gradient-constrained problem has been deeply investigated and has seen some recent developments. As it concerns the interesting issue of the existence of radial solutions, only the elastic–plastic torsion problem in the planar case was investigated, i.e., when the gradient constraint is constant.

We studied the case of nonconstant gradient constraint and found the explicit Lagrange multiplier and the explicit solution in the possible cases, that may arise.

In the future, we would like to generalize the result of the existence of radial solutions to the problem under investigation in a ball of R^n, $n > 2$. Moreover, we will investigate the class of solutions to the same problem with axial symmetry, as in [21].

Author Contributions: Conceptualization, S.G. and A.M.; Formal analysis, S.G. and A.M.; Investigation, S.G. and A.M.; Methodology, S.G. and A.M.; Supervision, S.G.; Validation, S.G. and A.M.; Writing—original draft, S.G. and A.M.; Writing—review & editing, S.G. and A.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: This research was partly supported by GNAMPA of the Italian INdAM (National Institute of Higher Mathematics).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Figalli, A.; Shahgholian, H. An overview of unconstrained free boundary problems. *Philos. Trans. Roy. Soc. A* **2015**, *373*, 20140281. [CrossRef] [PubMed]
2. Santos, L. Lagrange multipliers and transport densities. *J. Math. Pures Appl.* **2017**, *108*, 592–611.
3. Evans, L. A second order elliptic equation with gradient constraint. *Comm. Part. Diff. Equ.* **1979**, *4*, 555–572. [CrossRef]
4. Cimatti, G. The plane stress problem of Ghizetti in elastoplasticity. *Appl. Math. Optim* **1976**, *3*, 15–26. [CrossRef]
5. Ishii, H.; Koike, S. Boundary regularity and uniqueness for an elliptic equation with gradient constraint. *Comm. Partial Diff. Equ.* **1983**, *8*, 317–346. [CrossRef]
6. Jensen, R. Regularity for elastoplastic type variational inequalities. *Indiana Univ. Math. J.* **1983**, *32*, 407–423. [CrossRef]
7. Wiegner, M. The $C^{1,1}$-character of solutions of second order elliptic equations with gradient constraint. *Comm. Part. Diff. Equ.* **1981**, *6*, 361–371. [CrossRef]
8. Giuffrè, S. Lagrange multipliers and non-constant gradient constrained problem. *J. Differ. Equ.* **2020** *269*, 542–562. [CrossRef]
9. Daniele, P.; Giuffrè, S.; Lorino, M. Functional inequalities, regularity and computation of the deficit and surplus variables in the financial equilibrium problem. *J. Glob. Optim.* **2016**, *65*, 575–596. [CrossRef]

10. Giuffrè, S.; Idone, G.; Maugeri, A. Duality Theory and Optimality Conditions for Generalized Complementary Problems. *Nonlinear Anal.* **2005**, *63*, e1655–e1664. [CrossRef]
11. Giuffrè, S. ; Marcianò, A. Duality Minimax and Applications. *Minimax Theory Appl.* **2021**, *6*, 353–364.
12. Santos, L. Variational problems with non-constant gradient constraints. *Port. Math.* **2002**, *59*, 205–248.
13. Brezis, H. Moltiplicateur de Lagrange en Torsion Elasto-Plastique. *Arch. Rational Mech. Anal.* **1972**, *10*, 32–40. [CrossRef]
14. Brezis, H. Problèmes Unilatéraux. *J. Math. Pures Appl.* **1972**, *51*, 1–168.
15. Brezis, H.; Stampacchia, G. Sur la régularité de la solution d'inéquations elliptiques. *Bull. Soc. Math. France* **1968**, *96*, 153–180. [CrossRef]
16. Von Mises, R. Three remarks on the theory of the ideal plastic body. In *Reissner Anniversary Volume*; Edwards: Ann Arbor, MI, USA, 1949.
17. Rodrigues, J.F. *Obstacle Problems in Mathematical Physics*; Mathematics Studies n. 134; Elsevier Science Publishers B.V.: Berlin/Heidelberg, Germany, 1987.
18. Giuffrè, S.; Pratelli, A.; Puglisi, D., Radial solutions and free boundary of the elastic-plastic torsion problem. *J. Convex Anal.* **2018**, *25*, 529–543.
19. Shang, Y. The Limit Behavior of a Stochastic Logistic Model with Individual Time-Dependent Rates. *J. Math.* **2013**, *2013*, 1–7. [CrossRef]
20. Lions, P.L. *Generalized Solutions for Hamilton-Jacobi Equations*; Research Notes in Mathematics, Volume 69; Pitman Advanced Publishing Program: Boston, MA, USA, 1982.
21. Talenti, G. Soluzioni a simmetria assiale di equazioni ellittiche. *Ann. Mat. Pura Appl.* **1966**, *73*, 127–158. [CrossRef]