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DELAY DYNAMICAL SYSTEMS:

ENERGY GROWTH MODELS AND EPIDEMIC ISSUES

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DELAY DYNAMICAL SYSTEMS:
ENERGY GROWTH MODELS AND EPIDEMIC ISSUES

SISTEMI DINAMICI CON RITARDO:
MODELLI DI CRESCITA ENERGETICA E PROBLEMI EPIDEMICI

Έτσι, δεν γνωρίζω
Socrate (Platone)

ABSTRACT

Using the powerful mathematical tools consisting of vector calculus, differential equations and complex functions, the dissertation describes extensively, by simulation, the scientific problems relating to the introduction of determining factors, the variation of pre-established parameters and the modulation of specific aspects in the context of symmetrical dynamic systems represented by experimental models. The discussion is divided into two sections, dedicated to the chosen instruments and means of investigation (tools) respectively to the selected applications and to the study substrates (works), and is divided into three parts, distinguished according to the prevailing spheres of interest (since a segregation of the topics into completely separate compartments would be ontologically impossible, due to the very dense logical embrications and the very rich technical implications), including seven documents, for the most part already published in relevant sectorial scientific journals. In the first part on expounds the reference works: Stability and Hopf Bifurcation Analysis of a Distributed Time Delay Energy Model for Sustainable Economic Growth; Bifurcation Analysis of a Transportation Network for Energy with Distributed Delayed Carrying Capacity; and Dynamics of a Delayed Mathematical Model for One Predator Sharing Teams of Two Preys, highlighting the consequences of the insertion of delays and the effects of bifurcations in symmetrical dynamic systems represented by models of sustainable economic growth, of energy distribution networks with delayed transport capacity and of closed ecosystems with one predator and two preys struggling for survival. In the second part on examines the reference works: Complex Dynamics of a Model with R&D Competition; and Hopf Bifurcation Analysis in a Modified R&D Model with Delay; underlining the effects of the insertion of delays and the variability of dichotomous solutions in models with two competing departments subjects to opportunistic collaboration according to complex Dynamics of game theory. In the third part on explains the reference works: Multi-Attribute Decision Making Based on Interval-Valued Trapezoidal Neutrosophic Number and its Application in the Diagnosis of Viral Flu; and Analysis on Additional Environmental Stress - PPE Kit Disposal During Pandemic, A Dual Hesitant q-rung Orthopair MARCOS Methodology under Uncertainty, by interconnecting a multidisciplinary heterogeneous polymorphous miscellany, which unites disparate subjects (diagnosis of viral flu based on indistinct clinical symptoms, additional environmental pollution in the disposal of personal protective equipment or difficulties in taking collegiate top decisions in conditions of uncertainty) connected or reconnectable with the present ones pandemic issues.

COURTESY ABSTRACT

Utilizzando i potenti strumenti matematici costituiti dal calcolo vettoriale, dalle equazioni differenziali e dalle funzioni complesse, la dissertazione descrive diffusamente, per simulazione, le problematiche scientifiche relative all'introduzione di fattori determinanti, alla variazione di parametri prestabiliti e alla modulazione di aspetti specifici nell'ambito di sistemi dinamici simmetrici rappresentati attraverso modelli sperimentali. La trattazione si articola in due sezioni, dedicate alle strumentazioni e ai mezzi di indagine prescelti (tools) rispettivamente alle applicazioni e ai substrati di studio selezionati (works), e si suddivide in tre parti, distinte in base alle prevalenti sfere di interesse (poiché una segregazione degli argomenti in compartimenti completamente separati risulterebbe ontologicamente impossibile, per le fittissime embricazioni logiche e per le ricchissime implicazioni tecniche), comprendenti sette documenti, in massima parte già pubblicati su rilevanti riviste scientifiche settoriali. Nella prima parte si espongono i lavori di riferimento: Stability and Hopf Bifurcation Analysis of a Distributed Time Delay Energy Model for Sustainable Economic Growth; Bifurcation Analysis of a Transportation Network for Energy with Distributed Delayed Carrying Capacity; e Dynamics of a Delayed Mathematical Model for One Predator Sharing Teams of Two Preys, evidenziando le conseguenze degli inserimenti dei ritardi e gli effetti delle biforcazioni in sistemi dinamici simmetrici rappresentati attraverso modelli matematici di crescita economica sostenibile, di reti di distribuzione energetica con ritardi nella capacità di trasporto e di ecosistemi chiusi con un predatore e due prede in contrasto. Nella seconda parte si esaminano i lavori di riferimento: Complex Dynamics of a Model with R&D Competition; e Hopf Bifurcation Analysis in a Modified R&D Model with Delay, sottolineando gli effetti dei ritardi e delle biforcazioni nello sviluppo di dinamiche complesse in modelli con due reparti in contrapposizione, ma non restii a collaborazioni opportunistiche, secondo la teoria dei giochi. Nella terza parte si esplicano i lavori di riferimento: Multi-Attribute Decision Making Based on Interval-Valued Trapezoidal Neutrosophic Number and its Application in the Diagnosis of Viral Flu; e Analysis on Additional Environmental Stress - PPE Kit Disposal During Pandemic, A Dual Hesitant q-rung Orthopair MARCOS Methodology under Uncertainty, interconnettendo una miscellanea multidisciplinare eterogenea polimorfa di attualità, che accomuna temi disparati (diagnosi di influenza virale basate su sintomatologie cliniche evanescenti, inquinamento ambientale aggiuntivo nello smaltimento di equipaggiamenti protettivi personali o difficoltà nell'assunzione di decisioni apicali collegiali in condizioni di incertezza) collegati o ricollegabili con le presenti problematiche pandemiche.

Summary

INCIPIT

"Dicebat Bernardus Carnotensis nos esse quasi nanos gigantium homerus insidentes, ut possimus plura eis et remotiora videre, non utique proprii visus acumine, aut eminentia corporis, sed quia in altum subvehimur et extollimur magnitudine gigantea", "Bernard of Chartres said we are like dwarves on the shoulders of giants, so that we can see more things than them and farther away, certainly not because of the acuity of sight or the height of our body, but because we are lifted and carried aloft from the stature of giants" (John of Salisbury, *Metalogicon*, III, 4).

The famous medieval aphorism, attributed to Bernard of Chartres, a french scholar of the twelfth century, which metaphorically includes, condenses and raises the essence of the philosophical conception of the erudition of antiquity, assigning to the learning of old knowledge acquired by the ancestors the possibility of obtaining new notions learned from posterity, underlines the advantage of being able to exploit the cultural heritage of those who preceded us (in credit) to transmit knowledge (enriched by our contributions) to those who will follow us (in debt) and inspired the composition of this dissertation, based on the verification of innovative scientific awareness carried out with the application of mathematical instruments consolidated by the daily experimentation and conceived to try to reach the predetermined destinations in collaboration with older teachers, with peers of the same age and with younger learners.

For the unusual complexity of the subtle logical fil rouge that unites them, in a serial way, superficially, with different points of contact, and due to the uncommon abundance of notable technical implications that connect them, in a parallel manner, deeply, to different levels of investigation, the editing of the essays that make up this complex work involved a special commitment, in terms of intellectual concentration and professional application, given the youthful experience, mainly for the completion of the ordered collection of preliminary bibliographic documentation, for the widening of the mathematical modeling checks, for the improvement of the precise enunciation of the axiomatic scientific conclusions and for the supplement of the precise generalization of the information experimentally acquired, respecting the correct and perfect concinnitas between λόγοι e τόποι (criss cross contour shaping) in terms of paradigmatic conceptions, of pragmatic correspondences and of paragramatic constitutions.

Furthermore, before publication, requiring particular methodological consideration, especially to detect the major operational strategies to be used during the procedures for spatial arrangement of the phenomena examined, and demanding a singular systematic weighting, particularly to determine the best executive tactics to be used during the temporal organization procedures of the analyzed events, the relational expository references were subjected to radical processes of formal revision and substantial integration tending to make them more congruent and fitting both regarding the

preliminary designation of the proposed objectives, concerning chosen indicators determination and respecting collect data processing.

Today, in the context of current cultural reality, the planning, design and programming of organic and rational coherent scientific researches cannot disregard the use of the most advanced mathematical tools, as they are the most flexible, versatile and functional means for conducting thorough methodical investigations and for the formulation of comprehensive systematic assessments concerning the notable aspects of the various matters in terms of issues, problems and troubles.

Moreover, recently, the modern digital methods for symbolic representation of measurable quantities founded on discrete and finite numerical encodings, combined with the classic analog systems for conventional expression of the evaluable entities based on linear and infinite continuous modifications, was assuming a preponderant role in the framework of inductive and deductive practices essentials for technological development and fundamentals for scientific progress, because the cybernetic equipments introduction, informatic components diffusion and telematic instruments evolution supplies contemporary experts with a formidable power of calculation, which increases in logarithmic proportion to the performance improvements of the hardware devices and to the productivity flankins of the hardware applications.

Whether they are powerful videoterminals working in time sharing, electronic calculators, digital elaborators, dedicated workstations, informatic operators, personal computers or simple auxiliary processors, currently machines, with different computation skills and with very variable response speeds, they represent an essential aid for the organization of the information to be classified, and constitute an irreplaceable support for the reworking of the data to be examined during the management of composite, difficult and heterogeneous interventions, while the electronic functions of artificial brains, built by Man, perfectly integrate the neural activities of biological encephalons, created by Nature, admirably combining advanced digital concepts with rooted analog visions (until it’s possible and as long as mankind don’t lose her, his or its humanity, because we haven’t seen, heard or felt nothing yet) to achieve the complete synthesis between logical creativity and creative logic, providing the necessary automatic executive supports for the unattainable speculative lucubrations of the human mind and transforming the abstract subjective internal theoretical ideas into concrete external practical realizations.

However, a global approach to the problems of thought included in the panorama of philosophical sciences, rather than the philosophy of science, pertains to the philosophy of scientific discovery, investing the field of pure abstraction, with which every researcher must confront before being able to proceed with specific scientific experiments, projecting his transcendent inferences (with

functions concerning non-algebraic mathematical entities) into the sphere of immanent phenomenology (based on metaphysical concepts regarding sensitive realities) to draw irrefutable certifications destined to abandon the evanescent limbo of individual hypotheses, deriving from subjective conjectures inspired by instinctive intuitions, to transit in the tangible world of proven truths, descending from objective confirmations supported by an iron logic, assuming the value of universal laws, characterized by - documental, documentary and documented - evidence with cosmic range.

The importance of mathematical tools in scientific research assumes a leading relief especially during the practices of modular realization of general schemes, descriptive (referential) or predictive (inferential), based on the formalization of the themes to be represented, determined through logical structures exposed with expressions, functions and equations, on the construction of relational models, carried out using codified languages as precise, rational relationships and on the experimental reliability tests conducted through verification of existence (presence of an effective solution to the problem faced), optimality (persistence of an efficient resolution of the question posed) and stability (subsistence of an ability to effectively respond to changing system variables) fulfilled with the aid of suitable selection algorithms elaborated with computerized means to achieve the validation of the model by simulation.

In principle, the work proposes the objectives of addressing, clarifying and solving, with the mathematical tools constituted by vector calculus, differential equations and complex functions, the articulated scientific problems attributable to the introduction of determining factors, to the variation of pre-established parameters and the modulation of specific aspects on symmetrical dynamic systems represented through extraordinarily adaptable, flexible and versatile mathematical models, while, specifically, the research neatly exposes, organically examining and minutely explaining (highlighting natural links as interlock crossfader) the related logical applications characterized by multiple interactions between cofactors with complex implications, in the name of a perfectly balanced morphological equilibration in terms of conceptual concinnity carefully weighted between tools (detecting subjects) and works (detected objects).

Examining it superficially, without knowing its motivational implications, the particular internal organization of the dissertation (founded on many distinct subjects articulated in seven papers) may seem somewhat incongruous structurally and rather functionally incongruent, due to the variety and vastness of the topics considered, which make it appear culturally fractioned and frayed; however, the discussion faithfully reflects the breadth and depth of scientific research agreed with the Academic Coordinators over the three-year period and underlies the hidden logical trait d’union that connects, at various levels (ibidem et passim), but not like sundry articles, the studies concerning

the first group of works - Stability and Hopf Bifurcation Analysis of a Distributed Time Delay Energy Model for Economic Growth (with analysis of various types of kernels in possible combinations), Bifurcation Analysis of a Transportation Network for Energy with Distributed Delayed Carrying and Capacity (with stability analysis and Hopf bifurcation in different cases of kernel occurrences) - (I year), the in-depth studies pertaining to the second block of arguments - Dynamics of a Delayed Mathematical Model for One Predator Sharing Teams of Two Preys (with Hopf bifurcation and local stability analysis on respective specific properties and complex dynamics of a model with R&D competition explanation) and Hopf Bifurcation Analysis in a Modified R&D Model with Delay (with local stability and bifurcation analysis) - (II year), and the experiments concerning the third section of matters (after an introductive synthetical description of CoViD-19 or SARS-2 pandemic) - Multi-Attribute Decision Making Based on Interval-Valued Trapezoidal Neutrosophic Number and its Application in the Diagnosis of Viral Flu (with Multi-Criteria Decision Making strategy based upon the Interval-Valued Trapezoidal Neutrosophic Numbers and an illustrative numerical example for the application of the proposed GRPT methodology based on IVTrNN representation frameworks) and Analysis on Additional Environmental Stress-PPE Kit Disposal during Pandemic, a Dual Hesitant q-Rung Orthopair MARCOS Methodology (with the Conventional MARCOS Method and an analysis of sensibility and comparison) - (III year).

In the first part the dissertation expounds, examines and explains the reference works: Stability and Hopf Bifurcation Analysis of a Distributed Time Delay Energy Model for Sustainable Economic Growth; Bifurcation Analysis of a Transportation Network for Energy with Distributed Delayed Carrying Capacity; and Dynamics of a Delayed Mathematical Model for One Predator Sharing Teams of Two Preys, highlighting the consequences of the insertion of delays and the effects of bifurcations in symmetrical dynamic systems represented by models of sustainable economic growth, of energy distribution networks with delayed transport capacity and of closed ecosystems with one predator and two preys struggling for survival.

In the second part the dissertation expounds, examines and explains the reference works: Complex Dynamics of a Model with R&D Competition; and Hopf Bifurcation Analysis in a Modified R&D Model with Delay; underlining the effects of the insertion of delays and the variability of dichotomous solutions in models with two competing departments subjects to opportunistic collaboration according to complex dynamics of game theory.

In the third part the dissertation expounds, examines and explains the reference works: Multi-Attribute Decision Making Based on Interval-Valued Trapezoidal Neutrosophic Number and its Application in the Diagnosis of Viral Flu; and Analysis on Additional Environmental Stress - PPE

Kit Disposal During Pandemic, A Dual Hesitant q-rung Orthopair MARCOS Methodology under Uncertainty, by interconnecting a multidisciplinary heterogeneous polymorphous miscellany, which unites disparate subjects (diagnosis of viral flu based on indistinct clinical symptoms, additional environmental pollution in the disposal of personal protective equipment or difficulties in taking top collegiate decisions in conditions of uncertainty) connected or reconnectable with the present ones pandemic issues.

Given the particular organization of the contents, the dissertation is subject to various reading methods based on the prevailing contingent needs (exploration ratio, interpretation filters, acquisition integrations), therefore it can be considered with a sequential (horizontal) vision, following the logical arrangement of the topics addressed, with a rational (vertical) scan, according to the scientific connections of the articles reported, or by full consultation (diagonal), based on the different points of contact and in relation to the several connection plans that characterize the discussion.

Initially, the issues related to the introduction of delays into dynamic systems included in the mathematical representation of models function of the resolution of accessory problems are identified, studying them with the tools of vector calculus applied to isomorphism relations with reflexive, symmetric and transitive properties, then we proceed with topological spaces evaluation (without calculating their connectivity), by distinguishing the convergent sequences from the fundamentals, we provide the complete normed spaces description related to the metric and of the vector spaces generalizing the notion of euclidean space considered through the functional analysis and we continue with the determination of the parametric values by differential equations.

Then we study the effects of a delay distributed over time in an energy model concerning sustainable economic growth and the results deriving from its bivalent impact on the dynamic behavior of the system, which can be stabilized or destabilized based on the variations in the equilibrium point of dichotomous bifurcations, supercritical or subcritical, it continues with the analysis of the consequences of a delayed distribution of the transport capacity on the functioning of an energy supply network in local asymptotic stability, and ends with the prediction of the repercussions of a delay on the possible interactions between a predator and two prey inserted in a closed ecosystem.

Then the complex dynamics linked to the reaction delays of the competing research and development sections, but opportunistically conniving, are identified in the simulation of models related to systems with symmetrical structure, characterized by extreme variability induced by continuous succession of interpretable crucial junctions as intertwined divergent acquittals or cross convergent dissolves (crossfading).

Finally on consider the mind clearness coefficients relating to the discriminating capacities of multifactor evaluation of inaccurate, uncertain, inconsistent, incomplete, unimportant, indeterminate or unidentified clinical informations in viral flu diagnostics, the additional environmental pollution caused by massive use (with consequent overdone disposal) of plastic individual protective equipments and the importance of disturbing elements in collective apical specialistic decisions.

Moreover, as supplementary corollaries, it is possible to determine extensively fundamental nosological, statistical and demographic implications, based on the interactions between habitat, agent and patient, useful for the construction of mathematical epidemiological models (necessarily imperfect), deterministic or stochastic, with a closed or open option, homogeneous or heterogeneous, with descriptive or predictive value, assigning subjects to subgroups corresponding to different stages of contagion with transition points expressed as derivatives, using differential equations, defining the possible variables as probabilities, quantified in percentage terms, or segregating the subjects in isolated uniform compartments not susceptible to transfers, to obtain projections concerning space-time profiles of morbid events activation (origin), evolution (growth), attenuation (weakness) and extinction (ending).

From collation of the three parts into which the dissertation is divided, it is easy to go back to the technical motivations, to the didactic impulses and to the implementation themes that inspired the research work, articulated in a balanced tripartite form combining philosophical conceptions, gnoseological notions and epistemological cognitions on an equal footing in a synchronic cultural synchrasis aimed at harmonizing natural sciences (biology and medicine) with social sciences (sociology and economics) through suitable mathematical applications (statistics and demography) in a polymorphic whole capable of fully expressing the interests vastness animating a three-year period of studies, forcibly protracted due to the CoViD-19 (SARS-2) pandemic, which caused multiple deferments, postponements and extensions during the studies conduction and embarrassing complications, difficulties and impediments in the course of the investigations completion, but ended too soon, unfortunately, considering the wealth of human experiences involved.

SECTION A
(Tools)

CHAPTER 1

Preliminaries and Tools

1.1 Introduction

In this chapter we gonna introduce some theoretical elements, tools, definitions, theorems, very useful for the sequel in giving to the reader a trace to better understand the new results contained in the second part of this thesis. It will be organized into three sub paragraphs in which will be arranged basic elements from algebra, vector calculus and some functional analysis ingredients.

1.2 Some Algebraic Tools and Structures

The aim of this thesis is mainly to introduce the potential reader to its central scope: the delay differential equations. In doing this we need some useful definitions.

Let two sets A and B the *Cartesian Product* $A \times B$ is the set whose elements are ordered pairs (a, b) of elements such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A \quad b \in B\} \qquad (1)$$

The set $A \times A$ is indicated with A^2.

More generally we can define A^n the set whose elements are the ordered n-tuples $(a_1, \dots, a_n)$ with $a_1, \dots, a_n \in A$.

$$A^n = \{(a_1, \dots, a_n) \mid a_i \in A \quad \forall_i = 1, \dots, n\} \qquad (2)$$

Fixed a subset F of the Cartesian Product $A \times B$, we can define a *relationship* the correspondence $f: A \to B$ such that:

$$\forall a \in A \; \exists b \in B \mid (a, b) \in F \qquad (3)$$

We can say that a and b are in relationship or in other words that b is the image of a through f. The subset of elements of A for which an element b exists such that the pair $(a, b) \in F$ is called *domain*.

If the relationship $f: A \to B$ is such that

$$\forall a \in A \; \exists! \; b \in B \mid (a, b) \in F \qquad (4)$$

then this relationship is called *application* or *function*; the domain is A.

If $A = B$ then we must speak about a relationship in the set A.

An *algebraic structure* is any non-empty set S upon which "arithmetic like" operations have been defined. So, in order to give an algebraic structure to the sets we must define in them some operation.

A *binary internal* operation on a set S is an application:

$$o: S \times S \to S \mid (a, b) \to a \; o \; b \qquad (5)$$

In other words, it assigns exactly an element to each pair of elements all belonging to S.

A scaling operation, also known as *external binary* operation on a set S is an application:

$$o': \mathbb{K} \times S \to S \mid (k, b) \to k \; o' b \qquad (6)$$

where $\mathbb{K}$ denotes a field[1].

It assigns to each pair of elements $k \in \mathbb{K}$ and $b \in S$ one element $k \; o' b \in S$.

Let us consider, for example, the set V of vectors on real space and two operations in it: scalar product and vector product. The first is an application:

$$V \times V \to \mathbb{K}$$

where K is a *division ring* ($\mathbb{R}$ or $\mathbb{C}$)[2]. So, it isn't a binary internal operation.

On the contrary vector product is an application.

$$V \times V \to V$$

So, it is a binary internal operation[3].

An example of external binary operation can be scalar multiplication of n-tuples in $\mathbb{R}^n$: given any $k \in \mathbb{R}$ and any n-tuple $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the scalar multiplication on $\mathbb{R}^n$ is defined as it follows:

$$\big(k, (x_1, x_2, \dots, x_n)\big) \longrightarrow k(x_1, x_2, \dots, x_n) = (kx_1, kx_2, \dots, kx_n) \qquad (7)$$

It results in a new n-tuple in which each component has just been multiplied by k.

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[1] To see page 19.

[2] To see page 19.

[3] Scalar product is defined as it follows:

$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$

where (u_x, u_y, u_z) and $(v_x, v_{y,} v_z)$ are Cartesian components of the two vectors, so it is a real number. The vector product is defined as it follows:

$$\vec{u} \wedge \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \vec{\imath} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \vec{\jmath} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \vec{k}$$

where $\vec{\imath} \, \vec{\jmath} \, \vec{k}$ are the unit vectors for the axes x, y, z. So, it is a *vector*.

A *group* G is a non-empty set of elements on which we can define a binary internal operation "o" which assigns exactly an element belonging to G and denoted with $a\ o\ b$ and to each pair of elements $a, b \in G$; this operation has the following properties:

1) *Associativity*

   Given three elements $a, b, c \in G$

$$a\ o\ (b\ o\ c) = (a\ o\ b)\ o\ c \qquad (8)$$

2) *Existence of an identity element*

   There is an element $u \in G$ such that

$$\forall a \in G \Rightarrow a\ o\ u = u\ o\ a = a \qquad (9)$$

3) *Existence of inverse element*

   For each $a \in G$ there is an element $a' \in G$ such that

$$a\ o\ a' = a'\ o\ a = u \qquad (10)$$

These elements u and a' are unique.

If the binary operation is the multiplication $(o = \times)$, u is called *unit element* or *group identity*; it is denoted by "1", while a' is called *inverse* or *reciprocal* and it is denoted by "a^{-1}".

If the binary operation is the addition $(o = +)$, then u is called *neutral element* or *group identity* and it is denoted by "0", while a' is called opposite and denoted by "$-a$".

If the binary operation defined in the group is the multiplication, it is called *multiplicative group*; when the operation is the addition we speak about *abelian* or *commutative group*. In the last case the commutative property also applies as regarding the sum, that is

$$\forall a, b \in G \ \Rightarrow a + b = b + a \qquad (11)$$

It is possible that the previous property is valid only for some elements of G; in this case we can't speak about an abelian group, but we must state that those elements commute with each other.

A group G is called *finite* if it contains a finite number of elements, on the contrary it is called *infinite*. In a finite group we can define order of G its *cardinality*, that is the number of elements it consists of; as regarding an infinite group we say that it has infinite order.

Let G a group and H a non-empty subset of G. We can say that H is a *subgroup* of G if and only if it is a group as regarding the same binary operation defined on G. To verify if H is a subgroup of G we can apply the following lemma:

<u>*Lemma 1*</u>: A non-empty subset H of G is a subgroup of G if and only if

$$\forall g, h \in H \ \Rightarrow g \cdot h^{-1} \in H \qquad (12)$$

In an abelian group the previous lemma becomes:

<u>*Lemma* 2</u>: A non-empty subset H of G is a subgroup of G if and only if

$$\forall g, h \in H \ \Rightarrow g - h \in H \qquad (13)$$

An algebraic structure is a set in which we define some operations. We can define *homomorphism* between two algebraic structures, an application which preserves the operations. In other words, if A and B are two algebraic structures, the application

$$f: A \to B$$

is an homomorphism if:

$$\forall a, b \in A \ \Rightarrow f(a\ o\ b) = f(a)\ o'\ f(\ b) \qquad (14)$$

we observe that generally the operations can be different.

So, if G, G' are two groups, an homomorphism of G in G' is an application

$$f: G \to G'$$

which preserve the operation. If the operation in both groups is the multiplication we have

$$f(x \cdot y) = f(x) \cdot f(y) \quad \forall x, y \in G \qquad (15)$$

So, if G is an abelian group and G' is multiplicative group, f is an homomorphism if and only if it results

$$f(x + y) = f(x) \cdot f(y) \quad \forall x, y \in G \qquad (16)$$

If f is injective the homomorphism is called *injective*; if f is surjective it is called *surjective homomorphism*; finally, if f is bijective it is called *isomorphism*[4].

Let A be a set in which we can define two binary operations called sum and product. We say that A is a *ring* if:

1) A is an abelian group respect to the sum;
2) The multiplication is associative, that is $a(b \cdot c) = (a \cdot b)c \quad \forall a, b, c \in A$;
3) The two distributive laws are valid of the product respect to the sum

$$\forall a, b, c \in A \ \Rightarrow a(b + c) = a \cdot b + a \cdot c$$
$$\forall a, b, c \in A \ \Rightarrow (a + b)c = a \cdot c + b \cdot c$$

The definition of *ring* implies in A the existence of zero and the opposite $\forall a \in A$.

A subset $B \subset A$ is called a *subring* of A if it has the structure of a ring with respect to the same composition laws defined in A. It is demonstrable that a subset B of a ring A is a subring if and only if $a - b \in B$ and $a \cdot b \in B \ \ \forall a, b \in B$.

There are various types of rings:

1) A ring is called with *identity* if it has a neutral element respect to the product, that is an element u such that

$$u \cdot a = a \cdot u = a \ \ \forall a \in A$$

[4] A function $f: A \to B$ is injective if $\forall x_1, x_2 \in A$ if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. It is surjective$f(A) \equiv B$. It is bijective if it is both injective and surjective.

Generally, the identity is indicated with "1";

2) A ring is called *commutative* when the product is also commutative

$$\forall a, b \in A \ \Rightarrow \ a \cdot b = b \cdot a$$

The cancellation law of the product is valid in the set of real numbers only if one of the factors is null; in the rings it can also be satisfied for elements other than zero.

In a ring A, an element $a \in A$, with $a \neq 0$, is called divisor of zero if $\exists b \neq 0$ such that $a \cdot b = 0$ or $b \cdot a = 0$ with $b \in A$.

A ring with no divisors of zero is called *integrity domain*.

So, in a integrity domain A, if $a, b \in A$ and $a \cdot b = 0$ it must be $a = 0$ or $b = 0$.

The ring definition doesn't require the existence of the inverse of an element. In other words, if we consider a ring with the identity, not all elements have the inverse, or it is possible that the inverse doesn't exist for any element of A.

The elements of A for which the inverse exists, that is $\exists b \in A$: $a \cdot b = b \cdot u$, are called *units* or *invertible*. So, an element $a \in A$ is called invertible if exists an element $a' \in A$ such that

$$a \cdot a' = a' \cdot a = 1$$

This element is unique, and it is called the inverse of *a* and it is indicated with a^{-1}.

Examples:

1) $A = 0$ is a banal ring;
2) $(\mathbb{Z}+,\times)$ is a ring respect to the ordinary operations of sum and product. It is a commutative ring with identity, and it is also a domain of integrity;
3) $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, with the sum and product operations are all commutative rings with identity in which each non-null element admits the inverse and are also integrity domains.

Let A be a set in which we can define two binary operations called sum and product**.** A is a division ring if:

1) A is an abelian group with respect to the sum;
2) $A - \{0\}$ with the product is a multiplicative group;
3) The two distributive laws are valid of the product respect to the sum:

$$\forall a, b, c \in A \ \Rightarrow a(b + c) = a \cdot b + a \cdot c$$

$$\forall a, b, c \in A \ \Rightarrow (a + b)c = a \cdot c + b \cdot c$$

If the product is commutative, the division ring is called *field*.

A division ring is a ring, or a domain of integrity with identity in which however each element is invertible.

The difference between a ring and a division ring is that in the first not all elements have the inverse, while in the second the elements all have inverse.

Examples:

1) $\mathbb{Z}$ is neither division ring nor field. $\mathbb{Z}$ is an abelian group respect to the sum, but it isn't a multiplicative group;
2) $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields.

1.3 Vector Calculus

Let V be a set on a field K in which we can define two operations: the first is internal, it is called "*sum*" and it is indicated with $(+)$; the second is external, it is called "*product by a scalar*" and it is denoted with (o).

We can say that $(V, +, o)$ is a *vector space* on K if it enjoys the following properties:

1) V is an abelian group in relation to the "*sum*";
2) The *product by a scalar* (o) instead enjoys the following properties:

$$\forall a, b \in k \quad \text{and} \quad \forall u, v \in V \quad \Rightarrow$$

- $(a, b)v = a(b, v)$
- $(a + b)v = a \cdot v + b \cdot v$
- $a(u + v) = a \cdot u + b \cdot u$
- $1 \cdot v = v$

An example of vector space is the set F:

$$F = \{f: [a, b] \to \mathbb{R} \; or \; \mathbb{C}\}$$

of all the functions mapping an interval $[a, b]$ into $\mathbb{R}$ or $\mathbb{C}$, with respect to the operations of sum of functions and the product of a scalar for a function defined as follows:

– $[f + g](x) = f(x) + g(x) \quad \forall f, g \in F \quad \forall x \in [a, b]$
– $(\lambda f)(x) = \lambda f(x) \quad \forall f \in F \quad \forall x \in [a, b] \quad \forall \lambda \in \mathbb{R} \; or \; \mathbb{C}$

in fact, this set with these operations enjoys all the properties of vector space.

1.3.1 Elementary properties of vector spaces

I) Uniqueness of the null vector

Let V be a vector space on a field K; we can demonstrate that in V there is only one null element.

Proof: We indicate with O_v the null vector in order to distinguish it from the zero of the field, indicated with $O_{\mathbb{K}}$. We suppose that two null different vectors exists, O_v and O_v'. We can write

$$O_v + O_v' = O_v$$

because O'_v is a neutral element; on the other hand, we can write

$$O'_v + O_v = O'_v$$

because O_v is a neutral element, too. It follows that

$$O_v = O'_v$$

II) Uniqueness of the opposite vector

Let V be a vector space on a field K and $v \in V$. The opposite of v exists and it is unique.

Proof: The existence of the opposite of v is ensured by the definition of abelian group. To demonstrate its uniqueness we suppose that v' and v'' are two different opposite vectors of v. We can write:

$$v' = v' + O_v$$

where O_v is the neutral element. Since v'' is the opposite v we have:

$$O_v = v + v''$$

So

$$v' = v' + O_v = v' + (v + v'') = (v' + v) + v'' = O_v + v'' = v''$$

and finally

$$v' = v''$$

The only one opposite vector of v is denoted by $-v$.

III) Cancellation of the product

Let V be a vector space on a field K, we can demonstrate that $\forall a \in K$ and $\forall v \in V$, we have that:

$$a \cdot v = O_v \iff a = O_k \vee v = O_v$$

IV) Opposite of the vector $a \cdot v$

Let V be a vector space on a field K, we can demonstrate that the opposite of the vector $a \cdot v$ is obtained by multiplying *v* by the opposite of *a*. So

$$\forall a \in K \text{ and } \forall v \in V \Rightarrow -(a \cdot v) = (-a)v$$

1.3.2 Vector Subspaces

Let V be a vector space on a field K and S one of his subsets; we can say that S is a *vector subspace* of V when it is a vector space with respect to the same operations defined in V and restricted to S.

To check if a subset S of a vector space V is a subspace of V we should check all the properties of vector spaces with respect to the operations defined in V. in other words, we should verify if S is an abelian group with respect to the sum and if the external product for a scalar enjoys the four properties of vector spaces. It's clear that in S all the properties are valid because the operations defined in S are the same operations defined in V. Thus, the only question is to verify the closure of these operations, in particular if given two elements belonging to S their sum belongs to S, too, if the product of an element of S by a scalar is an element of S and finally if every element of S has its opposite in S.

We can use a property which characterizes the subsets S of V; it gives a necessary and sufficient condition for them to be subspaces of V.

Before stating this property, we have to give an important definition:

<u>*Definition 1*</u>: Let $v_1, v_2, \dots, v_n$ be n vectors belonging to a vector space V. We can define *linear combination* of these vectors with coefficients in the field K the vector:

$$v = \lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \cdots + \lambda_n \cdot v_n \qquad (17)$$

So, the linear combination of n vectors is a vector obtained by the sum of the products of each vector $v_1, v_2, \dots, v_n$ by the scalars $\lambda_1, \lambda_2, \dots, \lambda_n$.

Taking into account this definition the above property is the following:

<u>*Property I*</u>: *Let* V be a vector space on a field K and S one of his subsets. We can say that S is a vector subspace of V if and only if $\forall u, v \in S$ and $\forall \alpha, \beta \in k$ it results that $\alpha u + \beta v \in S$.

In other words, a subset S of a vector space V is a subspace of V if and only if every linear combination of vectors belonging to S is an element of S yet.

It is clear that a subset S of a vector space V, to be a subspace of V, must contain the null vector. So, if $O_v \notin S$ we can say that $S \subseteq V$ isn't a vector subspace; on the other hand, if $O_v \in S$ we can state that S is a vector subspace only if we're able to verify the closure of the operations.

Let's give some example of importance vector subspaces:

- Let V be a vector space on a field K; fixed an element $v \in V$, let's consider the set $< v >$ whose elements are multiple of v by the scalar λ:

  $$< v >: \{\lambda \cdot v \quad | \quad \lambda \in K\}$$

  We can demonstrate that $< v >$ is a vector subspace of V, called *subspace generated* by $v \in V$.

This type of subspace can be generalized:

- Let V be a vector space on a field K; fixed r vectors $v_1, v_2, \dots, v_r$, let's consider the set $< v_1, v_2, \dots, v_r >$ whose elements are all possible linear combinations of the r vectors with coefficients in K:

$$< v_1, v_2, \dots, v_r >: \ \{\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \cdots + \lambda_r \cdot v_r \ \mid \lambda_i \in K\}$$

We can demonstrate that $< v_1, v_2, \dots, v_r >$ is a vector subspace of V, called *subspace generated* by the r vectors $(v_1, v_2, \dots, v_r)$.

1.3.3 Intersection, sum and direct sum of subspaces

Let V be a vector space and let $V_1, V_2, \dots, V_p$ p vector subspaces of V. We can define *intersection* of V_i the set

$$G = \bigcap_{i=1}^{n} V_i \qquad (18)$$

whose elements are the vectors v_i such that v_i belongs to the subspace V_i with $i = 1,2, \dots, p$.

We can demonstrate that G is a vector subspace. To this end it is sufficient to verify that every linear combination of elements of G belongs to G.

Let's consider two vectors $v_i, u_i \in V_i \ \ \forall\, i = 1, 2, \dots, p$. Considering that V_i is a vector subspace, we have:

$\lambda \cdot v_i \in V_i$ and $\mu \cdot u_i \in V_i \ \ \forall\, i = 1, 2, \dots, p$ and $\forall\, \lambda, \mu \in K$.

So, we have:

$$\lambda \cdot v_i + \mu \cdot u_i \in V_i \ \ \forall\, i = 1, 2, \dots, p$$

that is:

$$\lambda \cdot v_i + \mu \cdot u_i \in G$$

Let V be a vector space and let $V_1, V_2, \dots, V_p$ p vector subspaces of V. We can define *sum* of V_i the set

$$H = V_1 + V_2 + \cdots + V_p$$

whose elements $v \in V$ can be written as the sum of an element of each V_i.

$$H = V_1 + V_2 + \cdots + V_p = \{v \in V \mid \exists v_i \in V_i \ \ v = v_1 + v_2 + \cdots + v_p\}$$

we can demonstrate that H is a vector subspace.

Let V be a vector space and let V_1, V_2two of its subspaces. We can say that $V_1, V_2$are disjoint if their intersection is the null vector.

$$V_1 \cap V_2 = \{O_v\}$$

Let's give the following definition:

<u>*Definition 2*</u>: Let V be a vector space and let $V_1, V_2, \dots, V_p$ p vector subspaces of V. The *sum*

$$H = V_1 + V_2 + \cdots + V_p$$

is a *direct* one, and we can write:

$$H = V_1 \oplus V_2 \oplus \ldots \oplus V_p$$

if $v_1 + v_2 + \cdots + v_p = 0$ only when $v_1 = v_2 = \cdots = v_p = 0$

Being:

$$v_1 \in V_1, v_2 \in V_2, \ldots, v_p \in V_p$$

Let us now enunciate some theorems that give important properties of the direct sum of subspaces.

Theorem 1: Let $H = \bigoplus_{i=1}^{p} V_i$, then $\forall v \in H$ is expressed in a unique way in the following form:

$$v = v_1 + v_2 + \cdots + v_p$$

In other words, there is only one vector for each subspace whose sum equals v.

Theorem 2: Let V be a vector space and V_1, V_2 two of its subspaces. Then:

$$V = V_1 \oplus V_2 \quad \Leftrightarrow \quad V_1 \cap V_2 = \{O_o\}$$

1.3.4 Linear dependence and independence

Let $\boldsymbol{v_1, v_2, \ldots, v_p}$ be $\boldsymbol{p}$ vectors of a vector space $\boldsymbol{V}$; these vectors $\boldsymbol{v_1, v_2, \ldots, v_p}$ are called *linearly dependent* if there are not all null scalars $\boldsymbol{\lambda_1, \lambda_2, \ldots, \lambda_p} \in \boldsymbol{K}$ such that:

$$\lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \cdots + \lambda_p \cdot v_p = 0 \qquad (19)$$

Otherwise, the vectors are called *linearly independent*. In other words, the vectors $v_1, v_2, \ldots, v_p$ are linearly independent if the relation (19) is valid only when all λ_i are equal to zero.

Property II: Let V be a vector space on a field $\mathbb{K}$; let's consider $v_1, v_2 \in V$; we can say that v_1, v_2 are linearly dependent if and only if they are proportional.

Property III: Let's consider r vectors $v_1, v_2, \ldots, v_r$ of a vector space V; if they are linearly dependent, then one of the r vectors can be expressed by means of the remaining $r - 1$ vectors.

Property IV: Given r vectors $v_1, v_2, \ldots, v_r$ of a vector space V, if one of them is expressed as a linear combination of the remaining ones $r - 1$, then all r vectors are linearly dependent.

Property V: Given s vectors $v_1, v_2, \ldots, v_s$ of a vector space V, if r of the s vectors $(r \leq s)$ are linearly dependent, then all s vectors are linearly dependent.

Property VI: If s vectors $v_1, v_2, \ldots, v_s$ of a vector space V are linearly independent, then every subset of the s given vectors is a set of linearly independent vectors.

Property VII: If in a set of vectors there is the null vector, then all vectors are linearly dependent.

Property VIII: We consider the Cartesian vector space K^n; then three vectors of K^n are linearly dependent if and only if they are coplanar.

1.3.5 Set of generators of a vector space and basis of a vector space

Let V be a vector space on the field $\mathbb{K}$ and $\{v_1, v_2, \dots, v_n\}$ a set of vectors of V; we can state that $\{v_1, v_2, \dots, v_n\}$ is a set of generators for V if each vector of V can be expressed as a linear combination of $v_1, v_2, \dots, v_n$, i.e. if:

$$\exists\, \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{K} \;\mid\; v = \lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \cdots + \lambda_n \cdot v_n \quad \forall v \in V \qquad (20)$$

Let V be a vector space on the field $\mathbb{K}$ and $(v_1, v_2, \dots, v_n) = \Sigma$ n vectors of V. Σ is a basis for V when:

a) Σ generates V; that is $< v_1, v_2, \dots, v_n >$ coincides with V; this means that each $v \in V$ is expressed as a linear combination of $v_1, v_2, \dots, v_n$.
b) $v_1, v_2, \dots, v_n$ are linearly independence.

It is also said that Σ is a free vector system.

Finally, a basis is a free system of generators for V.

1.3.6 *Fundamental or unitary versors*

Let's consider the vector space K^n and its vectors:

$$e_1 = (1,0, \dots,0); \quad e_2 = (0,1, \dots,0); \quad \dots \quad e_n = (0,0, \dots,1)$$

that is e_i has all null components except the *i*-th one, which is equal to 1.

We know that the *modulus* of a vector v in terms of its components $(v_1, v_2, \dots, v_n)$ is defined by

$$|v| = \sqrt{(v_1^2 + v_2^2 + \cdots + v_n^2)} \qquad (21)$$

The vector whose modulus is equal to 1 is called *versor*.

The subspace generated by the vectors $e_1, e_2, \dots, e_n$, denoted by $< e_1, e_2, \dots, e_n >$, coincides with all K^n: in fact, every vector $x \in K^n$ can be expressed has a linear combination of $e_1, e_2, \dots, e_n$. Moreover, these vectors are linearly independent. So, we can say that $(e_1, e_2, \dots, e_n)$ is a basis of K^n called *canonical basis*. We note that:

$$|e_i| = 1$$

The vectors $e_1, e_2, \dots, e_n$ are called *fundamental versors* of K^n.

1.3.7 *Dimension of a vector space*

Let V be a vector space on the field $\mathbb{K}$ and $\Sigma = (e_1, e_2, \dots, e_n)$ a basis of V. We can demonstrate that each vector $v \in V$ can be uniquely written as linear combination of $e_1, e_2, \dots, e_n$, that is

$$v = x_1 \cdot e_1 + x_2 \cdot e_2 + \cdots + x_n \cdot e_n \qquad (22)$$

So, each vector $v \in V$ identifies an ordered n-tuple $(x_1, x_2, \dots, x_n) \in K^n$; vice versa each n-tuple $(x_1, x_2, \dots, x_n) \in \mathbb{K}^n$ identifies a single vector $v \in V$ such that $v = x_1 \cdot e_1 + x_2 \cdot e_2 + \cdots + x_n \cdot e_n$. Therefore, we can give the following definition:

Definition 3: If $(e_1, e_2, \dots, e_n)$ is a basis of V and $v \in V$, the n scalars $x_1, x_2, \dots, x_n \in \mathbb{K}$ such that

$$v = x_1 \cdot e_1 + x_2 \cdot e_2 + \cdots + x_n \cdot e_n \qquad (23)$$

taken in the given order are called the *components of v respect to the basis* $(e_1, e_2, \dots, e_n)$.

This definition isn't ambiguous since a basis is an ordered set; if we change the order of the elements of the basis, we obtain another basis and consequently the components change.

Let's enunciate a theorem of simple proof:

Theorem 3: Let V be a vector space on the field $\mathbb{K}$ and $\Sigma = (e_1, e_2, \dots, e_n)$ a basis of V.

If $v_1, v_2, \dots, v_p,\ p \leq n$ are p linear independent vectors of V, then we can find in Σ $n - p$ vectors that added to the given p, form a new basis for V.

As a consequence of this theorem, we can state that if V is a vector space and $\Sigma = (e_1, e_2, \dots, e_n)$ is a basis of V, then n linear independent vectors of V are also a basis for it.

In view of this, we can therefore say that:

I) Each free system has at most n elements, that is n is the maximum number of linearly independent vectors in V.

II) All the basis of V have the same number n of elements.

For this reason, we can give the following definition:

Definition 4: The dimension of a vector space on a field $\mathbb{K}$ is called *the number of vectors which constitute any basis of V*.

It's useful to extend the notion of dimension to the null vector space as well: it has zero dimension. In fact, we can consider as basis of this vector space the null set, that is a set formed by zero elements.

If the vector space has a basis formed by finite number of elements it's said that the vector space is finite, or it has finite dimension; on the other hand, if the vector space hasn't a finite basis it's called infinite.

1.4 Linear Maps

We know that a function f of a set A in a set B is a law that associates to each element $x \in A$ one and only one element $f(x) \in B$. It is denoted by $f: A \to B$; the element $f(x) \in B$ is called the *image* of x. Given two vector spaces V and W on the same field $\mathbb{K}$, among the functions $f: V \to W$ there are those that retain the operations. Let's therefore give the following definition:

Definition 5: Let V, W be two vector spaces on the same field $\mathbb{K}$; we say that the function $f: V \to W$ is a *linear map* or *homomorphism* if it enjoys the following properties:

1. The image of the sum of two elements is always equal to the sum of their images
$$\forall u, v \in V \quad \Rightarrow \quad f(u+v) = f(u) + f(v) \qquad (24)$$
2. The image of the product of a scalar a for an element v is always equal to the product of a for the image of v.
$$\forall a \in \mathbb{K}, \forall v \in V \quad \Rightarrow \quad f(a \cdot v) = a \cdot f(v) \qquad (25)$$

So, $f: V \to W$ is linear if it preserves the two basic operations in a vector space, that is vector addition and product of a scalar by a vector[5].

The terminology concerning linear maps is analogous to that seen in homomorphisms between two algebraic structures; in particular:

- If f is injective it is called *monomorphism*;
- If f is surjective it is called *epimorphism*;
- If f is biunivocal it is called *isomorphism*.

If $V \equiv W$, then the linear map $f: V \to V$ is called *endomorphism*. A biunivocal endomorphism is finally called *automorphism*.

The following properties can be demonstrated:

Property IX: Each linear map brings the null vector into the null vector.

In other words, given the linear map $f: V \to W$, we have:
$$f(O_V) = O_W \qquad (26)$$

Property X: Each linear map brings linearly dependent vectors into linearly dependent vectors.

So, if $v_1, v_2, \dots, v_p \in V$ are linear dependent vectors, then $f_1, f_2, \dots, f_p \in W$ are linear dependent vectors.

Property XI: Each linear map brings subspaces in subspaces.

So, given the linear map $f: V \to W$, if E is a subspace of V, then $f(E)$ is a subspace of W.

Property XII: Each linear map brings a system of generators into a system of generators.

This means that if $v_1, v_2, \dots, v_p$ is a system of generators for V, then $f_1, f_2, \dots, f_p$ is a system of generators for W.

[5] Let V and W be two vector spaces of dimensions n and m respectively on the same field $\mathbb{K}$. Let us indicate with $\Sigma = (e_1, e_2, \dots, e_n)$ a basis of V and $\Sigma' = (e'_1, e'_2, \dots, e'_m)$ a basis of W. Given the linear map $f: V \to W$, we can demonstrate that there is a matrix $A \in \mathcal{M}(m,n)$, where $\mathcal{M}(m,n)$ is vector space of matrix $m \times n$ with constant coefficients, associated to f and called transformation matrix. The columns of this matrix are the components of the vector images $f(e_1), f(e_2), \dots, f(e_n)$ of base Σ with respect to base Σ'.

1.4.1 Isomorphisms between vector spaces

Definition 6: Taken two vector spaces V and W in the set of all possible vector spaces on the same field $\mathbb{K}$, we say that V and W are isomorphic vector spaces, and we write $V \simeq W$ if there is an isomorphism of one in the other:

$$V \simeq W \quad \Leftrightarrow \quad \exists \text{ an isomorphism } f: V \to W$$

The isomorphism relation introduced in the set of vector spaces on the same field $\mathbb{K}$ is an equivalence relation, that is it enjoys the properties:

1) *Reflexive property* $V \simeq V$;
2) *Symmetric property* if $V \simeq W \quad \Rightarrow \quad W \simeq V$;
3) *Transitive property* if $V \simeq W$ and $\mathrm{W} \simeq \mathrm{U} \quad \Rightarrow \quad V \simeq U$.

We state the following theorems, the proof of which we omit.

Theorem 4: Let V be a vector space of dimension n on a field $\mathbb{K}$; then it is isomorphic to $\mathbb{K}^n$.

Theorem 5: Let V, W be two vector spaces on the same field $\mathbb{K}$; then V is isomorphic to W if and only if they have the same dimension:

$$V \simeq W \quad \Leftrightarrow \quad \dim V = \dim W$$

Having established that the isomorphism is an equivalence relation, it induces a *partition* on the family of vector spaces on the same field $\mathbb{K}$.[6] In other words all vector spaces are divided into equivalence classes; each of them contains all vector spaces that are isomorphic to each other, so having the same dimension and therefore also the corresponding Euclidean space $\mathbb{K}^n$. Each equivalence class can be identified by a representing vector space V_i and it is indicated by the notation $[V_i]$. According to what we said above we can state it's possible to assume as representative of each equivalence class the Euclidean space $\mathbb{K}^n$. This means that any vector space can be isomorphic to a Cartesian one by means of a linear map that associates to each vector of V the n-tuples of its components with respect to a base. Thus, we can understand the importance of Euclidean spaces compared to all vector spaces; although a generic vector space has a different structure from a Euclidean one, they have similar properties. Therefore, it is possible to study Cartesian spaces, extending the results obtained to vector spaces in general.

[6] Let A be a set and $\mathcal{X} = \{x_i\}_{j \subset \mathbb{N}}$ a family of its subsets; we say that $\mathcal{X}$ is a partition of A if it enjoys the following properties:

1) $\bigcup_{i \in j} \{x_i\} = A$ that is for each element $a \in A$ there is at least one subset $\{x_i\} \in \mathcal{X}$ that contains it;
2) $\{x_i\} \cap \{x_k\} = \emptyset \quad \forall\, i \neq k$ that is the subsets are two by two disjointed.

1.5 Metric Spaces

A particular abstract algebraic structure that finds its usefulness in several branches of Mathematics is the *metric space*. A metric space is a set X endowed with a metric which associates to each pair of elements (points) of X a distance. The metric is axiomatically defined; axioms are suggested by some simple properties of distance, as it is familiarly defined between points on the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. However, it is a very general concept.

Definition 7: A metric space is a pair $(X; d)$, where X is a set and d is a metric on X, that is a function

$$d: X \times X \to [0; +\infty[$$

such that, for each choice of $x, y, z \in X$ it satisfies the following properties:

i) $d(x, y) \geq 0 \quad \forall\, x, y \in X$ (positivity)

ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$ (cancellation property)

iii) $d(x, y) = d(y, x)$ (symmetry)

iv) $d(x, z) \leq d(x, y) + d(x, z)$ (triangle inequality)

X is generally called the *set underlying* $(X; d)$; its elements are called *points*. For fixed points $x, y \in X$ the non-negative number $d(x, y)$ is called the *distance* between x and y. The properties *i), ii), iii), iv),* are the axioms of the metric; in particular, the term "triangle inequality" comes from elementary geometry.

A subset $(Y, \tilde{d})$ of $(X; d)$ is obtained by taking a subset $Y \subset X$ and restricting d to $Y \times Y$; then the metric on Y is given by the restriction

$$\tilde{d} = d|_{Y \times Y}$$

$\tilde{d}$ is called *metric induced* on Y by d.

Examples:

1. Let be $X = \mathbb{R}$ and $d(x, y) = |x - y|$; then $(\mathbb{R}; |\cdot|)$ is a metric space.
2. Let $X = \mathbb{R}^n$ be the set all ordered n-tuples and

$$d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} \qquad (28)$$

We can state that $\mathbb{R}^n$ is a metric space called *n-dimensional Euclidean space* (or simply *Euclidean n-space*). In fact, the properties *i), ii), iii)* are satisfied for the same definition of distance; moreover, it is easy to demonstrate that (28) satisfies the "triangle inequality"[7].

[7] The demonstration uses the inequality

3. Let $X = \mathcal{C}_{[a,b]}$ be the set of all continuous functions defined on a closed interval $[a, b]$ and

$$d(f, g) = \max_{a \le t \le b} |f(t) - g(t)| \qquad (29)$$

Then $(\mathcal{C}, d)$ is a metric space; it is very important in analysis. We can denote this metric space and the underlying set of "points" with the symbol $\mathcal{C}_{[a,b]}$ or simply $\mathcal{C}$; it is often called a function space to emphasize that its elements are functions.

1.5.1 Norm and Normed Space

Let V be a vector space on $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is a *function* (or application) $\|\cdot\|$:

$$V \to \mathbb{R} \text{ (or } \mathbb{C})$$

which satisfies the following properties:

a) $\|v\| \geq 0 \quad \forall v \in V$;

b) $\|v\| = 0 \quad \Leftrightarrow \quad v = 0$;

c) $\|\lambda v\| = |\lambda| \|v\| \quad \forall \lambda \in \mathbb{R}$ and $v \in V$ (linearity property);

d) $\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in W$ (triangle inequality).

For each v belonging to V the real (or complex) number $\|v\|$ is called *norm* of v. A vector space V is called *normed space* if a norm is defined on it. In a normed space we can introduce a distance by putting:

$$\rho(v, w) = \|v - w\| \qquad (30)$$

this means that ρ satisfies the following properties:

1) $\rho(v, w) \geq 0 \qquad \forall v, w \in V$;

2) $\rho(v, w) = 0 \qquad \Leftrightarrow \quad v = w$;

3) $\rho(v, w) = \rho(w, v) \quad \forall u, w \in V$;

4) $\rho(u, v) \leq \rho(u, w) + \rho(w, v) \quad \forall u, v, w \in V$ (triangle inequality).

Every normed space is a *metric space*; the converse is false. It's important to say that a metric space is a vector space in which a distance is defined.

A normed space $(V; \|\cdot\|)$ is *complete* if it is complete as a metric space with the induced metric.

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$$

called *Cauchy-Schwarz inequality*.

1.6 Elements of Topology

1.6.1 Topological spaces and their properties

In order to be able to analyse the topology of metric spaces and its properties it is useful to give the following definitions.

Definition 8: Let X be a non-empty set. A topological structure, or simply topology on a set X is a family of sets $\mathcal{T} \subseteq \wp(X)$ where $\wp(X)$ is a power set, such that the following properties apply:

a. the empty set $\emptyset$ and X belong to $\mathcal{T}$

$$\emptyset, X \in \mathcal{T}$$

b. $\mathcal{T}$ is closed for any union; this means that the union of any family of elements of $\mathcal{T}$ still belongs to $\mathcal{T}$

$$\forall \{A_i\}_{i \in N} \subseteq \mathcal{T} \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{T}$$

c. $\mathcal{T}$ is closed for finite intersections; this means that the intersection of a finite number of elements of $\mathcal{T}$ still belongs to $\mathcal{T}$

$$\forall A_1, A_2, \dots, A_n \in \mathcal{T} \quad \Rightarrow A_1 \cap \dots \cap A_n \in \mathcal{T}$$

The elements of $\mathcal{T}$ are called *openings* of the topology $\mathcal{T}$.

Example: *Let* be $X = \{a, b, c, d\}$;

$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}\}$

$\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}\}$

$\mathcal{T}_3 = \{X, \emptyset, \{a, b, c\}, \{b, c, d\}\}$

$\mathcal{T}_4 = \{X, \emptyset, \{b, c\}, \{a, b, c\}, \{c, d\}\}$

$\mathcal{T}_5 = \{\emptyset, \{b\}, \{b, c\}, \{b, c, d\}\}$

The families $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies, while the families $\mathcal{T}_3$, $\mathcal{T}_4$ and $\mathcal{T}_5$ are not topologies. In fact:

$$\{a, b, c\} \cap \{b, c, d\} \notin \mathcal{T}_3$$

$$\{b, c\} \cup \{c, d\} \notin \mathcal{T}_4$$

$$X \notin \mathcal{T}_5$$

Definition 9: A topological space is a pair $(X, \mathcal{T})$ where X is a non-empty set and $\mathcal{T}$ is a topology on X. The elements of X are called *points*.

If there is no possibility of confusion, we can denote a topological space simply by X, without mentioning $\mathcal{T}$.

Theorem 6: Let X be a set. A family $\mathcal{T} \subseteq \wp(X)$ is a topology on X if and only if $\emptyset, X \in \mathcal{T}$, it is closed for any unions and for simple intersections.

The *trivial topology* on a set X is $\mathcal{T} = \{\emptyset, X\}$: the only openings are therefore the empty set and X.

The *discrete topology* on a set X is $\mathcal{T} \subseteq \wp(X)$. Every set is open.

<u>*Definition 10*</u>: Let $(X, \mathcal{T})$ be a topological space, a subset $A \subseteq X$ is called *closed* (or closed of $\mathcal{T}$) if its complementary set is an opening, i.e. it is an element of $\mathcal{T}$:

$$A \text{ is closed} \quad \Leftrightarrow \quad A^c \in \mathcal{T}$$

The empty set and space X are both openings and closed in every topology.

In the trivial topology the only closed are the empty set and X (which are opening, too).

In the discrete topology every set is closed; it is opening, too.

<u>*Theorem 7*</u>:. Let $(X, \mathcal{T})$ be a topological space. Let $\sigma \subseteq \wp(X)$ be the family of closed of $\mathcal{T}$, the following properties apply:

1. $\emptyset, X \in \sigma$;
2. σ is closed for any intersection: this means that for every family $\{A_i\}_{i \in N} \subseteq \sigma$ the intersection $\cap_i A_i$ is an element of σ;
3. σ is closed for finite unions: this means that for each $A_1, \dots, A_n \in \sigma$ we have

$$A_1 \cup \dots \cup A_n \in \sigma$$

Conversely, if σ is a family of subsets of X which satisfies the conditions $1.; 2.; 3.$, then the family $\mathcal{T} = \{A^c : A \in \sigma\}$ whose elements are the complementary sets of the elements of σ is a topology on X.

Let X be a set. The *cofinite topology* is the one for which the closed are X and all subsets of X with a finite number of elements. That is:

$$\mathcal{K} = \{\emptyset\} \cup \{A \subseteq X \mid A^c \text{ is finite}\}$$

This means that the openings are the complementary of the finite sets. If X is a finite set, the cofinite topology coincides with the discrete topology.

<u>*Theorem 8*</u>: Let $\{\mathcal{T}_i\}_{i \in N}$ be a family of topologies on X. Then $\cap_i \mathcal{T}_i$ is a topology of X.

However, it is important to emphasise that the union of two topologies is not necessarily a topology.

Let $(X, \mathcal{T})$ be a topological space and let be $A \subseteq X$.

- $x \in X$ is called *internal* to A if there exists $B \in \mathcal{T}$ such that $x \in B \subseteq A$;
- $x \in X$ is an *adherence point* of A if for each $B \in \mathcal{T}$, it follows that if $x \in B$ then $B \cap A \neq \emptyset$. This means that x is an adherence point if it is not an internal point of A^c;
- $x \in X$ is an *accumulation point* of A if for each $B \in \mathcal{T}$ such that $x \in B$ there exists $y \neq x$ such that $y \in A \cap B$. In other words, x is an accumulation point if it is an adherence point of $A \backslash \{x\}$. (We note that accumulation points are adherence points, too);
- An adherence point of A that is not an accumulation point is called an *isolated point*.

The definition immediately shows that the internal points of A are points of A. The adherence points of A, on the other hand, may not be points of A, but all points of A are adherence points. If A is open, all its points are internal. If A is closed, it contains all its adherence points (because all the points of A^c are internal to A^c). We note that if $A = X$, a point $x \in A$ is isolated if and only if $\{x\}$ is open (and it is different from X).

Definition 11: Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X.

- The *internal part* of A is the union of all openings contained in A. It is denoted by $Int(A)$ or Å;
- The *closure* of A is the intersection of all closed containing A. It is denoted by $\bar{A}$;
- The *frontier* of A denoted by ∂A, is the set $\bar{A}\backslash$Å.

We can demonstrate the following:

Theorem 9: Let $(X, \mathcal{T})$ be a topological space and let be $A \subseteq X$. Then:

- $Int(A)$ is the greatest open contained in A;
- $\bar{A}$ is the smallest closed containing A;

We note that A is opening in $(X, \mathcal{T})$ if and only if $Int(A) = A$ and it is closed if and only if $\bar{A} = A$.

The following characterisations of internal part, closure and frontier may be useful:

Theorem 10: Let $(X, \mathcal{T})$ be a topological space and let be $A \subseteq X$. Then:

- $Int(A) = (\overline{A^c})^c$ and it coincides with the set of internal points of A;
- $\bar{A} = \left(Int(A^c)\right)^c$ and it coincides with the set of adherence points of A;
- $\partial A = \partial A^c = \left(Int(A^c) \cup Int(A)\right)^c$.

The set of the adherence points is indicated by $Ad(A)$. By definition it is the complementary of the set of internal points of A^c, so:

$$Ad(A) = \left(Int(A^c)\right)^c = \bar{A}$$

As far as the frontier we have:

$$\partial A = \bar{A}\backslash\text{Å} = \bar{A} \cap \left(\text{Å}\right)^c = \left(Int(A^c)\right)^c \cap Int(A)^c = \left(Int(A^c) \cup Int(A^c) \cup Int(A)\right)^c$$

This expression is symmetrical in A and A^c; this means that if we apply the same reasoning to A^c we obtain:

$$\partial A^c = \left(Int(A) \cap Int(A^c)\right)^c = \left(Int(A^c) \cup Int(A)\right)^c = \partial A$$

In mathematics, we often need to study a topological space locally; in particular, we want to study what happens near a given point. For this reason, it is appropriate to formalise the concept of the neighbourhood of a point.

Definition 12: Let $(X, \mathcal{T})$ be a topological space and let be $x \in X$. A neighbourhood of x is any subset N of X such that x is an internal point of N.

This definition does not require that the neighbourhood is open.

The classic drawing used to represent neighbourhoods is as follows:

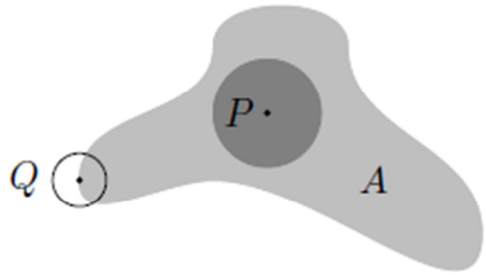

Figure 1: ***A*** is neighbourhood ***P*** but not of ***Q***.

Theorem 11: Let X be a topological space, let be $x \in X$ and let A, B be two neighbourhoods of x. Then $A \cap B$ is a neighbourhood of x. If $C \supseteq A$ then C is a neighbourhood of x.

The family of all neighbourhoods of x is denoted by $\mathcal{N}(x)$.

1.6.2 Topology of metric spaces and of $\mathbb{R}^n$

The concepts of neighbourhoods, internal and external points, open or closed sets, accumulation points and others can be extended to any metric space.

Definition 13: Let $(X; d)$ be a metric space and let x_0 be one of its points. A neighbourhood of x_0 and radius r is defined as the set of all points $x \in X$ that are from x_0 less than r.

Definition 14: Let X_0 be a point of $\mathbb{R}^n$. A *spherical neighbourhood* of centre X_0 and radius r is the set of points Y of $\mathbb{R}^n$ whose distance from X_0 is less than r.

$$B(X_0, r) = \{Y \in \mathbb{R}^n \mid \|X_0 - Y\| < r\} \qquad (31)$$

The set of all spherical neighbourhood of $\mathbb{R}^n$ is called the *Standard Topology* of $\mathbb{R}^n$.

If $n = 1$, $\mathbb{R}^n \equiv \mathbb{R}$, $X_0 \equiv x_0$ and the spherical neighbourhood is an open interval of centre x_0 and radius r formed by all points of $\mathbb{R}$ having a distance from x_0 less than r.

$$B(x_0, r) = \{y \in \mathbb{R} \mid |x_0 - y| < r\}$$

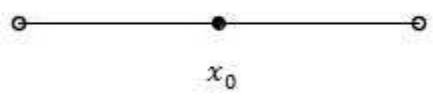

Figure 2: open interval of centre x_0 and radius r

If $n = 2$, $\mathbb{R}^n \equiv \mathbb{R}^2$, and the spherical neighbourhood is the circle of centre X_0 and radius r excluding the corresponding circumference.

$$B(X_0, r) = \{Y \in \mathbb{R}^2 \quad | \quad (x_1^0 - y_1)^2 + (x_2^0 - y_2)^2 < r^2\}$$

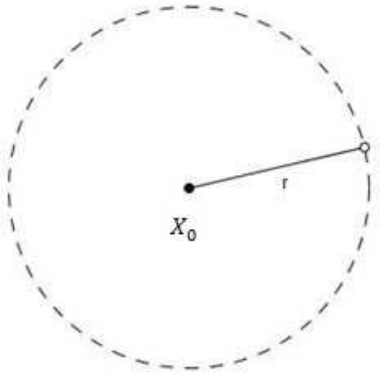

Figure 3: circle of centre $\boldsymbol{X_0}$ and radius $\boldsymbol{r}$

If $n = 3$, $\mathbb{R}^n \equiv \mathbb{R}^3$, and the spherical neighbourhood is the solid sphere of centre X_0 and radius r deprived of its shell and denoted by "ball".

$$B(X_0, r) = \{Y \in \mathbb{R}^3 \quad / \quad (x_1^0 - y_1)^2 + (x_2^0 - y_2)^2 + (x_3^0 - y_3)^2 < r^2\}$$

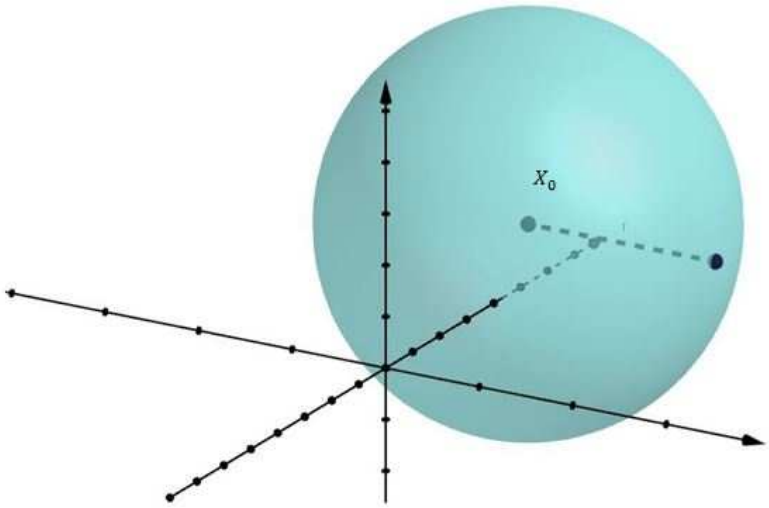

Figure 4: solid sphere of centre $\boldsymbol{X_0}$ and radius $\boldsymbol{r}$

<u>*Definition 15*</u>: Let be $A \subseteq X$, a point x_0 is *internal* to A if exists a neighbourhood of x_0 that is totally contained in A.

<u>*Definition 16*</u>: A point x_0 is called a *frontier point* for a set $A \subseteq X$ if every neighbourhood of x_0 contains both points that belong to A and points that don't belong to it.

The frontier of A, (it is denoted by ∂A) is the set of frontier points of A. Often the frontier of a set is also called a *boundary*.

Definition 17: A point x_0 is called an *external point* for a set $A \subseteq X$ if there is a neighbourhood of x_0 which has no points in common with A but which is contained in its complementary.

Definition 18: A set is called *open* if all its points are internal ones; this means that an open set contain no frontier points. A set is called *closed* if it contains all its frontier points, $\partial A \subseteq A$.

The definitions of accumulation point, and isolated point of a metric space are analogous to the same definitions of topological spaces; in particular:

Definition 19: A point x_0 is called an *accumulation point* for $A \subseteq X$ if all its neighbourhood contain points of A other than x_0 itself; the points of A that are not accumulation points for A are called *isolated points*.

An accumulation point may belong to the set A or not belong to it.

All internal points of A are accumulation point for A.

If A has accumulation points, it necessarily contains infinite elements. Conversely, if a set contains a finite number of elements (that is a finite set), then all its points are isolated.

The same definitions apply to $\mathbb{R}^n$.

1.6.3 Convergent Successions, Cauchy Successions and Banach Space

Let (V, ρ) be a metric space. A succession[8] $\{x_n\}$ of elements belonging to V is convergent if V contains an element x such that:

$$\lim_{n \to +\infty} \|x_n - x\| = 0 \qquad (32)$$

This means that:

$$\forall \varepsilon > 0 \; \exists \, n(\varepsilon) \in \mathbb{N} \mid \forall n > n(\varepsilon) \quad \Rightarrow \|x_n - x\| < \varepsilon$$

We can write $x_n \to x$; x is called limit of $\{x_n\}$.

A succession $\{x_n\}$ of elements belonging to V is called Cauchy Succession if $\forall \varepsilon > 0$ we can find an integer number $n(\varepsilon)$ such that if $n, m \geq n(\varepsilon) \;\Rightarrow\; \|x_n - x_m\| < \varepsilon$. This means that:

$$\lim_{n,m \to +\infty} \|x_n - x_m\| = 0 \qquad (33)$$

We can prove that every convergent succession is a Cauchy one.

Having introduced the notion of Cauchy Succession, we can remember some definitions and properties related to metric spaces:

Property XIII: A metric space (V, ρ) is called *complete* if every Cauchy Succession converges in V. This means that the limit of every Cauchy Succession belongs to V.

---

[8] A numerical succession is a function f that associates with each natural number n a real number x_n:

$$f: \mathbb{N} \to \mathbb{R} \;\mid\; \forall n \in \mathbb{N} \;\mapsto\; x_n \in \mathbb{R}$$

n is the independent variable called *succession index*, x_n is the dependent variable called *generic term* of the succession.

Example: The spaces $\mathbb{C}$ and $\mathbb{R}$ are complete; the space $\mathbb{Q}$ of the rational numbers is not complete.

Definition 20: A subset B of a metric space (V, ρ) is called *dense* (in V) if every x of V is the limit of a Succession of elements of B; that is, if

$$\forall x \in \quad \exists \{x_n\}_n \subset B \mid x_n \to x$$

Example: $\mathbb{Q}$ is a metric space dense in $\mathbb{R}$.

Definition 21: A normed complete space is called *Banach Space*.

Example: Let $C[a, b]$ be the set of the continuous functions in the closed and limited interval $[a, b]$ with values in $\mathbb{R}$ (or $\mathbb{C}$). Let us assume

$$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

We can demonstrate that the previous relationship defines a norm in $C[a, b]$. This set with this norm is a *Banach Space*.

In order to determine whether a normed space is complete, and therefore is a Banach Space, the following criterion is useful:

Theorem 12: *(Completeness criterion for a normed vector space)*: A normed vector space is complete if and only if every absolutely convergent series in the space is convergent.[9]

1.6.4 Banach-Caccioppoli (fixed point) theorem

There are a large number of fixed-point theorems and many mathematicians have dealt with this subject. The first mathematician to state such a theorem regarding algebraic curves was Henri Poincarè in 1886. In 1912 the mathematician Luitzen Brouwer formulated one of them for the spaces of finite dimension, while in 1941 Shizuo Kakutani has dealt with it in the within of

[9] Given a sequence of numbers $a_1, a_2, \dots, a_n, \dots$, the succession

$s_1 = a_1$

$s_2 = a_1 + a_2$

...

$s_n = a_1 + a_2 + \cdots + a_n$

…

is called *number series*. It is indicated briefly by the notation $\sum_{n=1}^{+\infty} a_n$. The sums $s_1, s_2, \dots, s_n, \dots$, are called *partial sums* or *reductions of the series*. The symbol $\sum_{n=1}^{+\infty}$ is called *summation*; we read it summation over n ranging from 1 to plus infinity.

A series $\sum_{n=1}^{+\infty} a_n$ is convergent if the succession of its reductions has a finite limit, that is

$$\lim_{n \to +\infty} s_n = s, \quad s \in \mathbb{R}$$

A series $\sum_{n=1}^{+\infty} a_n$ is called *absolutely convergent* if the series of absolute values of the succession

$$|a_1| + |a_2| + \cdots + |a_n| + \cdots$$

is *convergent*.

palindromic functions. The theorem of this family of greater interest in economy is that of Banach contractions; in fact, it has important applications in game theory, in particular for the definition of general economic equilibrium and was first applied by Kenneth Arrow in 1972, then by Gérard Debreu in 1983 and finally by John Forbes Nash in 1994.

Before stating the Banach-Caccioppoli theorem, it is essential to give the concepts of fixed point and contraction.

Definition 22: Let X be a set, with $X \neq \emptyset$; let $f: X \to X$ be a function of X in itself. A point $x \in X$ is called a *fixed point* for the function f if it coincides with its imagine or in other words if $f(x) = x$.

Examples:

- Let's consider the function $f: \mathbb{R} \to \mathbb{R}$ such that each x corresponds to its square, that is:

$$f: x \to x^2$$

  It is easy to verify that this function has two fixed points, namely $x = 0$ and $x = 1$. In fact:

$$f(0) = 0^2 = 0 \qquad f(1) = 1^2 = 1$$

- Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f: x \to x^3$$

  It has three fixed point, that is $= 0$, $x = 1$ and $x = -1$. In fact:

$$f(0) = 0^3 = 0 \qquad f(1) = 1^3 = 1 \qquad f(-1) = (-1)^3 = -1$$

Definition 23: Let (X, d) be a metric space and let $T: X \to X$ be an application of X in itself. We say that T is a *contraction* if there exists a positive number λ less than 1, i.e. such that $0 < \lambda < 1$, which satisfy the following property:

$$d\big(T(x), T(y)\big) \leq \lambda d(x, y) \qquad \forall x, y \in X$$

This means that the distance between the images of two elements of X is less than the distance between the elements themselves.

The contractions are *Lipschitzian functions* and therefore they are uniformly continuous[10].

At this point we can state the following theorem:

Theorem 13: Let (X, d) be a complete metric space and let $T: X \to X$ a contraction. Then there exists and there is only one fixed point for T, that is:

$$\exists!\ \bar{x} \in X \ \mid\ \bar{x} = T(\bar{x})$$

Moreover, whatever $x_0 \in X$, the succession defined by:

$$x(n+1) = T\big(x(n)\big), \quad x(0) = x_0$$

converges to the fixed point $\bar{x}$ for $n \to +\infty$.

The Banach theorem just stated is a simple and powerful tool because it is widely applicable, for example for the resolution, through the iterative method, of linear and non-linear equations. Other applications can also be found in Mathematical Analysis, particularly in the field of ordinary differential equations, partial differential equations and integral equations.

1.6.5 Pre-Hilbertian Spaces and Hilbert Spaces

Let us preface this by giving the following definition.

Definition 24: Let $\mathbb{K}$ be a field and let V, W be two vector spaces on it with dimension n and m respectively. Let the map be

$$F: V \times W \to \mathbb{K}$$

which associates the ordered pair of vectors $(v, w) \in V \times W$ with the scalar $F(v, w) \in \mathbb{K}$

$$(v, w) \mapsto F(v, w)$$

It is called *bilinear form* if it is linear both in V and W; this means that

$$\forall\ \alpha, \beta \in \mathbb{K}, v, v_1, v_2 \in V, w, w_1, w_2 \in W$$

[10] Lipschitzianity is a concept that is placed between the concepts of derivability and continuity; in other words, if f is a function defined in an interval I with values in $\mathbb{R}$ we have:
f derivable in $I \ \Rightarrow f$ lipschitzian in $I \ \Rightarrow f$ uniformly continuous in $I \ \Rightarrow f$ continuous in I
As we known, a function f is uniformly continuous on an interval I if

$$\forall \varepsilon > 0 \ \ \exists \delta > 0 \ \mid \ \ |f(y) - f(x)| < \varepsilon \quad \forall x, y \ \mid \ \ |y - x| < \delta$$

where δ depends only on ε and not on x, y, unlike continuous functions for which δ also depends on x, y.
A function is lipschitzian when it has a growth value limited by a constant, i.e. the ratio between the variation of the function on the ordinates and on the abscissae never exceeds a certain fixed value. This value is called *Lipschitz constant*. In more rigorous mathematical terms, a function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is called *Lipschitzian* if there exists a constant $K > 0$ such that $\|f(x) - f(y)\| \leq K\|x - y\|$.

the following properties apply:

(a) Additivity and homogeneity with respect to the first component

$$\mathrm{F}(\alpha v_1 + \beta v_2, w) = \alpha F(v_1, w) + \beta F(v_2, w)$$

(b) Additivity and homogeneity with respect to the second component

$$\mathrm{F}(v, \alpha w_1 + \beta w_2) = \alpha F(v, w_1,) + \beta F(v, w_2)$$

In particular if $V = W$ the map $F: V \times V \to \mathbb{K}$ is called *bilinear form* on V.

So, a bilinear form is a function defined on the Cartesian product between two vector spaces (or a Cartesian product defined on the same vector space) with values in a field $\mathbb{K}$ such that it is linear in both components.

We know that in a vector space two operations are defined: the sum between vectors and the product of a scalar for a vector. If in a vector space we also define a scalar product, it becomes an abstract set where we can define a concept of orthogonality analogous to the Euclidean one in $\mathbb{R}^n$. This allows us to introduce a privileged reference system in which the calculations are particularly simple, and a concept of orthogonal projection that makes possible to approximate a generic element of a vector space by means of elements of a particular subspace. In Banach spaces the completeness property allows to demonstrate theorems similar to those that apply in $\mathbb{R}^n$. This synthesis of ideas give rise to the concept of Hilbert space, which is a Banach space where a scalar product is defined and therefore a concept of orthogonality.

Definition 25: Let V be a vector space on a field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$). We say that V is a vector space with an internal product or scalar product, if in addition to the two operations of a vector space (sum of vectors and product of a scalar for a vector) a third operation is defined. It is called scalar product or internal product, it is indicated by:

$$(“,”) : V \times V \to \mathbb{K}$$

and it enjoys the following properties:

(i) $(x, x) \geq 0 \quad \forall x \in V$ (positivity)

(ii) $(x, x) = 0 \iff x = 0$ (definiteness)

(iii) $(\lambda x + \mu y, z) = \lambda(x, z) + \mu(y, z) \ \forall x, y, z \in V, \lambda, \mu \in \mathbb{K}$ (linearity on the first component)

(iv) If $\mathbb{K} = \mathbb{R} \to (x, y) = (y, x) \quad \forall x, y \in V$ (commutativity)

(v) If $\mathbb{K} = \mathbb{C} \to (x, y) = \overline{(y, x)} \quad \forall x, y \in V$ (symmetry)

In the last equality the supersign $\bar{}$ indicates the *conjugate complex*.

If $\mathbb{K} = \mathbb{R}$, linearity on the second component also follows from the commutativity property and the scalar product is called bilinear; on the other hand, if $\mathbb{K} = \mathbb{C}$ from (iii) and (v) it follows:

$$(z, \lambda x + \mu y) = \bar{\lambda}(z, x) + \bar{\mu}(z, y) \ \ \forall x, y, z \in V, \lambda, \mu \in \mathbb{C}$$

and the complex scalar product is called *sesquilinear*.

The vector space V with this operation is called *Pre-Hilbertian space.*

Examples:

1. The space $\mathbb{R}^n$ with the scalar product

$$(x, y) = \sum_{j=1}^{n} x_j y_j$$

2. The space $\mathbb{C}^n$ (on $\mathbb{C}$) with the scalar product

$$(x, y) = \sum_{j=1}^{n} x_j \overline{y_j}$$

3. The space $C[a, b]$ of real valued functions with the scalar product

$$(f, g) = \int_a^b f(t)\, g(t)\, dt$$

4. The space $C[a, b]$ of functions with complex values with the scalar product

$$(f, g) = \int_a^b f(t)\, \overline{g(t)}\, dt$$

In a Pre-Hilbertian space X we can define in a natural way for each $x \in X$ the norm

$$\|x\| = \sqrt{(x, x)} \qquad (34)$$

It is called the *internal product norm* and denoted by $\|\cdot\|$.

The following theorem is easy to prove.

Theorem 14: Let V be a Pre-Hilbertian space; then the following inequalities apply:

I. $|(x, y)| \leq \sqrt{(x, x)} \cdot \sqrt{(y, y)} \quad \forall x, y \in V \qquad (35)$ (Cauchy-Schwarz inequality)

This inequality can be written in an equivalent way using the norm

$$|(x, y)| \leq \|x\| \cdot \|y\| \quad \forall x, y \in V \qquad (36)$$

II. $\|x + y\| \leq \|x\| + \|y\| \qquad \forall x, y \in V \qquad (37)$ (Triangular inequality)

III. $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad \forall x, y \in V \qquad (38)$ (Parallelogram identity)

According to what we have shown so far, a Pre-Hilbertian space is a normed vector space whose norm comes from a scalar product and satisfies the parallelogram identity. Conversely, given any normed vector space, we may wonder if there is a scalar product which induces that norm. From this point of view, we can say that a necessary condition is that parallelogram identity is valid. More precisely, it can be shown that if V is a normed vector space whose norm satisfies the parallelogram identity, then the product

$$(x, y) = \frac{1}{2}[\|x + y\|^2 - \|x\|^2 - \|y\|^2] \qquad (39)$$

is actually a scalar product that induces the norm $\|\cdot\|$. Thus, the parallelogram identity characterises the norms of Pre-Hilbertian spaces.

In a vector space with internal product, we can define in a natural way the concept of orthogonality.

Definition 26: Let V be a vector space with internal product and let x, y be two vectors of V; we can say that $x, y \in V$ are orthogonal to each other, and we write $x \perp y$, if and only if $(x, y) = 0$.

The concept of orthogonality allows to rewrite even for Pre-Hilbertian spaces a generalized Pythagoras theorem. More precisely we have:

Theorem 15: Let $x_1, x_2, \dots, x_n$ be elements of a Pre-Hilbertian space V; we suppose that they are orthogonal two by two. Then the square of the norm of the sum is equal to the sum of the squares of the norms:

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 = \sum_{j=1}^{n} \|x_j\|^2 \qquad (40)$$

Definition 27: A Pre-Hilbertian vector space complete with respect to the norm induced by the scalar product is called *Hilbert space*.

A generic Hilbert space is denoted by $\mathscr{H}$.

So, a Hilbert space is a Banach one whose norm comes from an internal product.

Example: The vector space $\mathbb{R}^n$ with the scalar product:

$$(x, y) = \sum_{j=1}^{n} x_j y_j$$

and the induced norm

$$\|x\| = \sqrt{(x, x)}$$

is a Hilbert space.

In Hilbert spaces, the Pythagoras theorem can be extended to infinite sums in the following way:

Theorem 16: Let $\{x_j\}_{j=1}^{\infty}$ be a succession of elements of a Hilbert space $\mathscr{H}$ which are orthogonal two by two and such that the numerical series $\sum_{j=1}^{\infty} \|x_j\|^2$ converges. Then the series $\sum_{j=1}^{\infty} x_j$ converges in $\mathscr{H}$ and the relation applies:

$$\left\| \sum_{j=1}^{\infty} x_j \right\|^2 = \sum_{j=1}^{\infty} \|x_j\|^2 \qquad (41)$$

1.7 Elements of Functional Analysis: definitions and tools

The theory of normed space and Banach spaces represents a large part covered a branch of mathematical analysis called *functional analysis*.

Let us consider the set C of continuous functions mapping an interval $[a, b]$ into $\mathbb{R}$ (or $\mathbb{C}$); it is a subset of the vector space of all the functions mapping the interval $[a, b]$ into $\mathbb{R}$ or $\mathbb{C}$

$$F = \{f: [a, b] \to \mathbb{R} \; or \; \mathbb{C}\}$$

We know that if f and g are two continuous functions and α and β are two scalars, the function:

$$\alpha f + \beta g$$

is continuous yet. So, the set C is a vector subspace of the vector space F, that is a vector space itself. In C we can define the length, or norm of a continuous function, as well as Euclidean distance between two continuous functions.

Let us consider the so called "sup norm" or uniform norm, defined as:

$$\|f\|_\infty \sup_{a \le x \le b} |f(x)| \qquad (42)$$

It is simple to prove that this norm has the following properties:

i) $\|f\|_\infty > 0$ if $f \neq 0$;

ii) $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty$;

iii) $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ (triangle inequality).

The last one is equivalent to:

$$\sup_{a \le x \le b} |f(x) + g(x)| \le \sup_{a \le x \le b} |f(x)| + \sup_{a \le x \le b} |g(x)|$$

We know that a vector space in which a norm is defined, that is a function with positive values, that enjoys the previous properties, is a normed vector space. So, the vector space C with the norm defined above is a normed space.

In C we can also define the Euclidean distance between functions by putting:

$$d(f, g) = \|f - g\|_\infty \qquad (43)$$

it has the following properties:

i) $d(f, g) \ge 0$ (positiveness);

ii) $d(f, g) = 0 \Leftrightarrow f = g$;

iii) $d(f, g) = d(g, f)$ (symmetry);

iv) $d(f, h) \le d(f, g) + d(g, h)$ (triangle inequality).

1.7.1 Pointwise and uniform convergence of sequences of functions

Let I an interval into $\mathbb{R}$ (or $\mathbb{C}$), for each n belonging to $\mathbb{N}$ we can consider a function $f_n: I \to \mathbb{R}$ (or $\mathbb{C}$) . In this way we realize a sequence of functions $\{f_n(x)\}$, that is an application that for each integer n associates a function defined in I. For each fixed x belonging to I $\{f_n(x)\}$ is a numerical sequence, to which we can apply the usual concept of convergence. So, the numerical sequence is convergent when it admits finite limit, which depends on the choice of x.

As x varies we find a subset A of I that includes all x values for which the sequence converges.

As x varies in this set A, we obtain a new function $f(x)$ defined into A, that gives point by point the limit of the sequence.

We can give the following:

Definition 28: A sequence of functions $\{f_n(x)\}$ *converges punctually* to a function $f(x)$ in a subset A of I, if

$$\forall x \in A \;\Rightarrow\; \lim_{n\to+\infty} f_n(x) = f(x) \qquad (44)$$

this means that:

$$\forall x \in A \;\text{ and }\; \forall \varepsilon > 0 \;\; \exists \bar{n}(\varepsilon, x) \mid \forall n > \bar{n}(\varepsilon, x) \;\Rightarrow\; |f_n(x) - f(x)| < \varepsilon \qquad (45)$$

Observe that $\bar{n}(\varepsilon, x)$ is depending both on ε and x. $f(x)$ is pointwise limit of the sequence, while A is called *point convergent set*.

As regarding a sequence of continuous functions, pointwise convergence doesn't guarantee the continuity property of its limit. For this reason, we must introduce a different notion of convergence, stronger than the punctual one. This new type of convergence is uniform one. We can say that the sequence of functions $\{f_n(x)\}$ converge uniformly in A to a function $f(x)$ if for each $\varepsilon > 0$ we can find an integer $\bar{n}(\varepsilon)$ (only depending on ε) such that:

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall x \in A, \;\; n \geq \bar{n}(\varepsilon) \qquad (46)$$

In a similar way we can say that the sequence $\{f_n(x)\}$ converge uniformly in A if

$$\lim_{n\to+\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0 \qquad (47)$$

The function $f(x)$ is called *uniform limit*, while A is called *uniform convergence set*.

The main different between pointwise converge and uniform one is that in the last we can find an integer $\bar{n}(\varepsilon)$ only depending on ε, which fits for every point x belonging to A, while in the pointwise convergence the integer $\bar{n}(\varepsilon, x)$ depends both on ε and x.

In other words, the pointwise convergence requires that $f_n(x) \to f(x)$ point by point, while the uniform convergence requires that f_n approaches the function f "everywhere simultaneously".

We can demonstrate that if a sequence of function $f_n: I \to \mathbb{R}$ (or $\mathbb{C}$) converge uniformly on A then it converges punctually, too.

The converse isn't true.

The uniform convergence allows us to deduce the continuity of the limit function f from the continuity of the single elements of the sequence $\{f_n\}$. We can demonstrate a theorem which states that if $f_n: I \to \mathbb{R}$ (or $\mathbb{C}$) is a sequence of continuous functions in I uniformly convergent to a limit function f, then this limit f is continuous in I.

An immediate consequence of the previous theorem is that if the limit function of a sequence of continuous functions isn't continuous, then the sequence of functions doesn't converge uniformly.

Another important property of the uniform convergence is that if the sequence of function $\{f_n\}$ convergences uniformly to f and every element of the sequence is limited in I, then the limit function f is also limited in I.

CHAPTER 2

Basic Theory of Delay Differential Equations

2.1 Differential Equations: an introduction

2.1.1 Ordinary Differential Equation

An *Ordinary Differential Equation* (ODE) of the order n is an equation in which a numerical y function of the variable x (real or complex) appears as unknown. It establishes a link between the independent variable x, the function y and its first n derivatives $y', y'', \dots, y^{(n)}$. Such an equation is therefore of the type:

$$f\left[x, y(x), y'(x), y''(x), \dots, y^{(n)}(x)\right] = 0 \qquad (1)$$

in which f is an assigned numerical function of the $n+2$ variables $x, y, y', y'', \dots, y^{(n)}$, considered as independent, and defined in a certain set of the space $\mathbb{R}^{n+2}$ (or $\mathbb{C}^{n+2}$).

The order of ODE is that of the highest derivative of the unknown function that actually appears in the equation. It's clear that one of the $n+2$ variables may not appear in the given equation; $y^{(n)}$ must always be explicitly stated if the equation is to be of the order n and not of lower order.

The adjective «ordinary» refers to the circumstance that the unknown function y depends on a single variable x. Naturally, we can also consider differential equations in which the unknown function depends on two or more variables; these equations establish a link among these variables, the unknown function and its partial derivatives, up to a certain order n, with respect to the same variables. Such equations are called *Partial Differential Equations* (PDE).

There are other types of equations that have a function as unknown: *integral equations* and *integro-differential equations*. An integral equation has an unknown function which appears under the sign of integral; if the unknown function is only under the sign of the integral, the equation is called *first species*, otherwise it is called *second species*. Among these equations we can find the *Fredholm integral equation*, in which the extremes of the integral are constant, and the *Volterra integral equation* in which one extreme is constant while the other is variable.

An integro-differential equations has an unknown function which appears both under the sign of integral and the sign of derivation. This type of equation is found, for example, in the study of hereditary phenomena, that are dependent not only on the current state of the institution and any initial conditions, but on all previous states; in physics we find these equations in the analysis of electrical circuits.

Going back to the Ordinary Differential Equations of order n, let's consider a function $y = \varphi(x)$ continuous with its first n derivatives in a certain range I of $\mathbb{R}$ (or $\mathbb{C}$); if the following identity applies:

$$f\left[x, \varphi(x), \varphi'(x), \dots, \varphi^{(n)}(x)\right] \equiv 0 \qquad \forall x \in I$$

then it is called a *solution* or *integral* of the *differential equation* (1).

The graph of the function:

$$y = \varphi(x)$$

is called the *integral curve* of the equation itself.

A differential equation of order n is said in normal form when it is written as:

$$y^{(n)} = g\left[x, y(x), \dots, y^{(n-1)}(x)\right] \qquad (2)$$

where g is an assigned function of $n+1$ variables $x, y, y', \dots, y^{(n-1)}$, that is, when it is given resolved with respect to the derivative of the maximum order $y^{(n)}$.

Solving a differential equation of order n means determining all its solutions. In practice, when we solve a differential equation, we must always integrate; the obtained solution is a function of the variable x and of certain arbitrary constants: this solution is called the *general integral* of the given differential equation.

Let's clarify the concept with some example.

The problem of finding the primitive functions $y(x)$ of a given function $f(x)$, continuous in an interval I of $\mathbb{R}$ (or $\mathbb{C}$) leads to solving a first-order differential equation in normal form:

$$y' = f(x) \qquad (3)$$

All the solutions of the equation (3) are obtained by an indefinite integral given by the following formula:

$$y = \int f(x)dx = \varphi(x) + c \qquad (4)$$

where φ represents any primitive of the function f and c is an arbitrary constant. The general integral of the equation (3) is therefore represented by expression (4) as the parameter c varies. Thus, the equation (3) admits infinite integrals, given by (4), which depend on an arbitrary constant; we can say that the differential equation (3) admits a family of ∞^1 integral curves dependent on parameter c. This circumstance is general, that is the general integral of a first-order differential equation depends on a single constant.

We observe that, arbitrarily fixed a point $P_0 \equiv (x_o, y_0)$, with $x_o \in$ I, there is always one and only one value c_o of the constant c such that the integral curve whose equation is:

$$y = \varphi(x) + c_o \qquad (5)$$

passes through P_0. The integral corresponding to the c_o value of the constant, that is the integral (5) is called the *particular integral* of the differential equation (3).

Generally, a particular integral of a first-order differential equation is obtained by forcing the general integral to satisfy a certain condition: this is the so-called *Cauchy problem*.

Similarly, we can say that the general integral of a second-order differential equation depends on two constants c_1 and c_2, that is a second-order differential equation will admit ∞^2 solutions, which depend on the infinite values of the two constants.

Here too, it's possible to detail the two constants c_1 and c_2, that is, to solve the Cauchy problem, by requiring the general integral to satisfy two conditions.

Given an ordinary differential equation of order n in the normal form (2) it's generally integral is a function of the type:

$$y = y(x, c_1, c_2, \dots, c_n) \qquad (6)$$

it depends on n arbitrary constants $c_1, c_2, \dots, c_n$, so a differential equation of order n admits ∞^n solutions, that is, a family of ∞^n integral curves. Here too, the Cauchy problem is solved by imposing that the general integral satisfies n conditions: this makes it possible to obtain n particular values for the constants $\overline{c_1}, \overline{c_2}, \dots, \overline{c_n}$ and consequently a particular integral:

$$y = y(x, \overline{c_1}, \overline{c_2}, \dots, \overline{c_n}) \qquad (7)$$

So, we can define general integral of an ordinary differential equation of order n the solution which has the maximum number of arbitrary constants, equal to the order of the equation itself. Any integral obtained from the general integral by taking particular values $\overline{c_1}, \overline{c_2}, \dots, \overline{c_n}$ for the constants is called the *particular integral* of the differential equation.

In the classification of the solutions of an ordinary differential equation of order n, besides the general integral and the particular integrals, there is another type of integral, called *singular integral*. More precisely, the singular integral of a differential equation is a solution of this equation which doesn't correspond to any value of the arbitrary constants of the general integral[11].

[11] The equation $y' = xy^2$ has the solution $y = 0$; its general integral is $y = -\frac{2}{x^2+c}$. The integral $y = 0$ is obtained by the previous integral as limit for $c \to \infty$, so it is a particular integral.

The equation $y' = x^{\frac{2}{3}}$ has the solution $y = 0$ and its general integral is $y = \left(\frac{x+c}{3}\right)^3$. The integral $y = 0$ doesn't correspond to any value of the constant c, so it is a singular integral.

2.1.2 Differential Difference Equations

The Differential Difference Equations, or more simply *Difference Equations* are the discrete equivalent of differential equations in the continuum. A difference equation leads to a sequence of numbers generated recursively using a rule that binds each number in the sequence to the previous ones. An example in economics concerns the calculation of the amount of a bank deposit with a certain annual interest i. If C is the initial deposited capital and if neither withdrawals nor deposits are made, we want to calculate the amount of the deposit after a certain number of years n. To solve the problem mathematically, it is necessary to identify the function of the problem itself, the variable on which it depends and the relation to which it must satisfy (e.g., the equation of which it is the solution). In this case, the function is the amount of the deposit which depends on the year in question, while the variable is the year n. If we denote by $x(n)$ the amount of the deposit at year n, considered from when we opened it, in order to calculate the amount at year $n+1$ we must add to $x(n)$ its percentage at the rate i:

$$x(n+1) = x(n) + i\,x(n)$$

that is

$$x(n+1) = (1+i)\,x(n) \qquad (8)$$

The (8) is a recurrence relation, i.e., a relation which makes it possible to generate the value of the function $x(n)$ recursively from an initial given value $x(0)$. In this case, the initial deposited capital.
We can read the former (8) in a different way: it can be considered a difference equation, that is an equation whose unknown is a numerical sequence; it establishes a relationship between the value of the $(n+1)$-th term of the sequence and the n-th term (or a number of preceding terms). We have obtained a mathematical model for a wide category of phenomena called evolutionary, that is those which change over time[12]; it's called *discrete-time dynamic model*; this denomination is due to the circumstance that we record its variations only at a certain countable deadline. Similar models are used in the so-called *Population Theory* in problems such as population growth.
Let's indicate with $p(t)$ the number of individuals of a fixed population at time t. We suppose that, in a small interval of time h, the increase of the number of individuals in a population is proportional to the time elapsed and to the number of individuals at the initial instant. For a certain constant of proportionality α we have:

$$p(t+h) - p(t) = \alpha\,h\,p(t) \qquad (9)$$

with $h \neq 0$. If we divide by h we obtain:

[12] The word "time" can indeed indicate the physical time of watches, but also any variable such as the position in space, or, in a pressed arrangement, the page of a book, etc.

$$\frac{p(t+h)-p(t)}{h} = \alpha\, p(t)$$

Turning to the limit for $h \to 0$ in the previous relation we get:

$$\lim_{h\to 0} \frac{p(t+h)-p(t)}{h} = \lim_{h\to 0} \alpha\, p(t)$$

Strictly speaking, the function p only takes on discrete values; if we assume that the function $p(t)$ is continuous (this approximation is permissible when the population is very large so fluctuations of one unit can be thought of as very small), the first member of the previous relation represents the limit of an incremental ratio, i.e. the first derivative of the function $p(t)$; at the second member the product $\alpha\, p(t)$ doesn't depend on h, so the limit operation returns the product itself. The final result is:

$$p'(t) = \alpha\, p(t) \qquad (10)$$

This is an example of differential equation; the unknown is the function $p(t)$ and the equation expresses a condition on the first derivative of the function $p(t)$. The constant α can be interpreted as the (instantaneous) growth rate of the population.

In cases of practical interest (for example in annual surveys of the number of individuals, or at fixed time intervals), as the function is discrete, it isn't derivable; therefore, it has no meaning to pass the limit for $h \to 0$. The equation (9) can be rewritten thinking that the survey is done year by year ($h = 1$ year):

$$p(n+1) - p(n) = \alpha\, p(n) \qquad (11)$$

In this way we obtained a *difference equation*.

Let $\{x_n\}$ be a real-valued sequence[13], let $k \geq 1$ be an integer and let $f(y_k, y_{k-1}, \dots, y_0, m)$ a function with real values of $k+2$ variables.

<u>*Definition 1*</u>: A *difference equation* is a relation such as:

$$f(x_{n+k}, x_{n+k-1}, \dots, x_n, n) = 0 \qquad (12)$$

which must be satisfied by the sequence $\{x_n\}$ for every integer n. k is called the order of the difference equation.

<u>*Definition 2*</u>: A *difference equation* is called *linear* if the function f is a polynomial of first degree in the variables $y_k, y_{k-1}, \dots, y_0$, that is when it is in the form:

$$a_{k,n}x_{n+k} + a_{k-1,n}x_{n+k-1} + \cdots + a_{0,n}x_n = g_n \qquad (13)$$

[13] A real sequence is called an application whose domain is the set $\mathbb{N}$ of natural numbers as its domain and that takes on values in $\mathbb{R}$.

$$f: \mathbb{N} \to \mathbb{R} \quad \forall x \in \mathbb{N} \;\Rightarrow f \in \mathbb{R}$$

The independent variable is denoted by the letter n and the corresponding value of the function is indicated by a notation such as x_n (i.e., the variable n is placed as a subscript).

where g_n is called *known term*. The equation is said to have *constant coefficients* if $a_{k,n}, a_{k-1,n}, \dots, a_{0,n}, g_n$ don't depend on n. The equation is called *homogeneous* if $g_n = 0$ for each n. Finally, given a linear non-homogeneous equation in the form (13), the equation obtained by setting equal to zero the known term g_n, that is:

$$a_{k,n}x_{n+k} + a_{k-1,n}x_{n+k-1} + \cdots + a_{0,n}x_n = 0 \qquad (14)$$

is called the *associated homogeneous equation* of (13).

According to the previous definitions, the equation (8) derived above is an example of a linear first-order difference equation.

We finally pose the following:

Definition 3: A *difference equation* is said to be in *normal form* if the function f has the shape:

$$y_k - g(y_{k-1}, \dots, y_0, m)$$

that is if the *algebraic equation* $f(y_k, y_{k-1}, \dots, y_0, m) = 0$ can be solved with respect to the variable y_k for each m.

Definition 4: A *difference equation* is said to be *autonomous* if the function f doesn't depend on m, that is it has the form:

$$f(y_k, y_{k-1}, \dots, y_0)$$

In order to solve difference equations, we can use the following theorem:

Theorem 1: Let f be a function defined on $\mathbb{R}^k \times \mathbb{N}$; the difference equation in normal form:

$$x_{n+k} = f(x_{n+k-1}, x_n, n) \qquad (15)$$

has one and only one solution that satisfies the initial conditions:

$$x_{k-1} = d_{k-1}, \dots, x_0 = d_0 \qquad (16)$$

where $d_{k-1}, \dots, d_0$ are k arbitrary fixed numbers.

2.1.3 Delay Differential Equations

The analysis of real-world problems involves the study of quantities which evolve over time and that generally also depend on other variables of various kinds. These problems are modelled by one or more functions of these variables; the equations describing the corresponding mathematical model bind the unknown function $y(t)$, its first derivative $\dot{y}(t)$ and/or its subsequent derivatives up to a certain order. These are therefore ordinary differential equations of the type:

$$\dot{y}(t) = f\big(t, y(t)\big) \qquad (17)$$

where

$$y(t)\colon [t_0, T] \to \mathbb{R}^n$$

is an unknown function which is supposed to be known at a point t_0 called the initial instant:

$$y(t_0) = y_0$$

while

$$f(t,y)\colon [t_0,T]\times\mathbb{R}^n\to\mathbb{R}^n$$

is an appropriate function of t and y that characterises the problem. This type of equation admits only one solution if $f(t,y)$ is a *Lipschitzian function* with respect to the second argument in the interval $[t_0,T]$, that is there exists a $K>0$ such that

$$\|f(t,v)-f(t,w)\|\le K\|v-w\| \qquad \forall\, v,w\in\mathbb{R}^n \ \text{ and } \ \forall\, t\in[t_0,T]$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

An example is given by the equation:

$$\dot{y}(t)=\lambda y(t) \qquad t\in[t_0,T] \qquad (18)$$

with the initial condition $y(t_0)=y_0$. It is obtained from the general case (17) with $n=1$ and $f(t,y)=\lambda y$; we can demonstrate that f is a lipschitzian function. The solution of the previous equation is $y(t)=y_0e^{\lambda t}$. These types of problems are called *Initial Value Problems* (IVP).

The differential equation (18) is the mathematical model of the evolution of a system whose variation in time, expressed by the first derivative, is proportional to the function itself. It happens, for instance, in a financial capital that pays off over time at a constant rate and whose fruits are immediately re-invested. The same equation is used to describe a simplified model of population development, seen as a continuous function of time, in which there are no deaths and the growth due to new births is proportional to the number of individuals and each new birth is immediately able to reproduce. This mathematical model, proposed by *Thomas Robert Malthus* in 1798, represent the evolution of a population in the presence of unlimited resources and in the absence of predators or antagonists for resource use. More precisely let be:

- $N(t)$ number of individuals at time t;
- λ birth rate (number of births per individual per unit of time);
- μ death rate (number of deaths per individual per unit of time).

The differential equation describing the evolution of the population is:

$$\dot{N}(t)=(\lambda-\mu)N(t) \qquad (19)$$

Placed $r=\lambda-\mu$ the *growth rate* (or *biological potential*) of the population and $N(t_0)$ the number of individuals in the population at time t_0, the solution of the previous equation is:

$$N(t)=N(t_0)e^{r(t-t_0)} \qquad (20)$$

If births exceed deaths, i.e., $r>0$, then the population is growing exponentially; if births are fewer than deaths, i.e., $r<0$, then the population tends to become rapidly extinct.

In this model we assume that $N(t)$ varies continuously as t varies; this assumption is reasonable if the number of the individuals is very high, and changes (birth, deaths, or migrations) occur randomly.

Malthus' law describes fairly well the real evolution of the population as long as the number of individuals isn't very large. When the population grows beyond a certain limit, it no longer adheres to reality; just think that the earth's itself would not be enough to contain the entire world population (from the equation (20) it can be deduced that if we were to live on water, in 2625 there would barely be enough room for all of us to stand, and in 2660 we would have to stand one on the other's shoulders).

As the population grows the individuals of the same species compete with each other for the exploitation of available resources, therefore it becomes necessary to introduce into the equation a term that takes into account the competition.

This is what the Dutch mathematical-biologist *Pierre François Verhulst* did in 1838 who proposed the following differential equation, called *logistic equation*:

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{K}\right) \qquad t_0 < t \leq t_f \qquad (21)$$

with the initial condition $N(t_0) = N_0$. In it N is the number of individuals of the species, r is the growth rate, K is the carrying capacity of the population, i.e., the maximum number of individuals that can be in a given environment.

Also in this model, the number $N(t)$ of individuals in a population at time t is treated as if it were a continuous and derivable function of time, so the variation of a population (increase or decrease in the number of individuals) is expressed by the derivative of $N(t)$ with respect to time.

The equation expresses the fact that the variation of the population is proportional to the total number of individuals and to the distance between the current value of the population and the maximum attainable value. The proportionality with respect to the number of individuals is natural since each individual can reproduce or become extinct. The proportionality with the difference $K - N(t)$ expresses the capacity of the environment to receive new individuals. If the available resources, for example food and water, are not sufficient to feed the species it is reasonable to assume that the species has less capacity for development, and this happens to a greater extent the closer the population is to the limit value.

It is possible to give an explicit form to the solution of this equation in the following way:

$$N(t) = \frac{KN_0e^{rt}}{K + N_0(e^{rt} - 1)} \qquad (22)$$

The *logistic function* (22) applies to solve the differential equation that defines a demographic trend dependent on the availability of vital resources.

Graphically, the logistic function is represented by a symmetric sigmoid curve that expresses an almost exponential initial growth rate, slowing progressively, for the reduction of available resources, until it reaches an asymptotic position where development is null for the achievement of a settled demographic equilibrium.

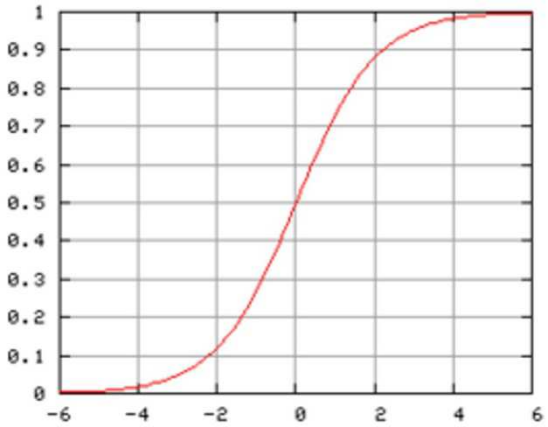

Figure 1: logistic curve

The logistic equation in demography is accompanied by a dynamic one-dimensional model of population formalized by the differential equation:

$$\dot{N}(t) = -rN \ln\left(\frac{N}{K}\right) \qquad (23)$$

where N represents the anthropic community, while the constants r and K correspond to the growth rate and the carrying capacity or the asymptotic term of development (defined by the resources available in the environment).

The general solution of this differential equation expresses the so-called *law of Gompertz*, which mathematically establishes the trend of population development in relation to the size of the anthropic community (for small populations compared to available environmental resources there is an exponential growth that stabilizes slowly until it becomes almost linear with the approach to the value of asymptotic balance while in the opposite case there is a rapid decrease in numbers until the point of equilibrium is reached stabilized).

In nature, many systems show a delay in reacting to external conditions and more generally to the state of system. Mathematical models in which the reaction of the system subjected to a certain stimulus isn't always immediate but may occur with a certain delay are described by the differential

equations with delay. Sometimes this delay is negligible, while in other cases neglecting it can lead to significant errors.

An example is given by population models, in which the lag corresponds to the gestation period; in such models the birth rate at time t is affected both by the population's level at the same time t and by the gestation period T. Therefore, it should take into account not only the population level $p(t)$ at the same time, but also the population level at time $t - T$ at which the individuals who come into existence at time t were conceived.

In general, a differential equation with delay, known as Delay Differential Equation (DDE) is a problem of the form:

$$\dot{y}(t) = f\big(t, y(t), y(t-\tau)\big) \quad t_0 < t \leq T \quad y(t) = \phi(t) \qquad t \leq t_0 \quad (24)$$

where $\tau = \tau\big(t, y(t)\big)$, which in general is itself a function, is called *delay*, often also called *lag*[14], while ϕ is called *memories*.

The IVP can be solved for delayed differential equations too, that is to determine among the infinite solutions that integral which satisfies a particular initial condition. It is important to observe that in this case it is not enough to know an initial value of the solution as in ordinary initial value problems, but we must know the entire history of the solution before to the initial instant. In fact, there are a lot of differential equations with delay that have different solutions but the same initial value. Anyway, the solution of these problems exists and is unique if the following conditions are satisfied:

- the function f is continuous and a lipschitzian one with respect to the last two arguments;
- the function $\tau\big(t, y(t)\big)$ is non-negative, continuous and lipschitzian with respect to the last argument;
- the function ϕ is continuous and lipschitzian.

We can find an example of DDE in Biology. We have already described the logistic equation developed by *Verhulst* in 1838; in this equation is shown a model to study population dynamics in which the environment, endowed with a limited number of resources, conditions the growth of the population. The function (22) is increasing and limited: individuals tend to reproduce until they reach the saturation value K; observations of some phenomena in biology show that the number of the individuals of some species $N(t)$ in some periods decreases and then increases again, exceeding K, following an oscillatory trend. On the other hand, this model takes for granted that if the

[14] It's useful to clarify the existing differences between the two approaches: the economists usually talk about "lags" to define the lapse-time in which the effects produced by an event act. A mathematician, on the other side, talks about a "delay" to analyze, from a different point of view, a similar aspect that in this case can be exactly computed.

environment changes, for instance when the population grows and consumes more resources the individuals of the species notice it instantaneously. Another limitation of the logistic equation is that it implies a birth/death rate of individuals of the species that responds instantaneously to changes in population size. However, there are cases in which individuals of a certain species feel the urge to reproduce; they don't do so immediately, but with a certain delay: this happens, for example, if they are aware of the availability of food or can be conditioned by the environment. In reality this is not exactly the case.

For all these reasons it is necessary to change the model or to modify it, in order to impose that the individuals of the species realize with delay that the habitat that hosts them is changed. *Hutchinson* was the first mathematician in 1948 to introduce the delay factor into the logistic equation to explain the oscillatory phenomena. His model is described by the *logistic equation with delay*.

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right) \quad t_0 < t \leq t_f \quad N(t) = \phi(t) \qquad (25)$$

In this formulation, the change due to the births of individuals remains instantaneous: it is represented by the proportionality with respect to $N(t)$; the influence of the environment now occurs with some delay: this lag is obtained by imposing the proportionality with respect to the factor $1 - \frac{N(t-\tau)}{K}$.

The trend of the solution function of the equation (25) changes considerably as the delay τ varies, although the other parameters remain constant. By way of example, assuming $\phi(t) = 3$ and $K = 100$, the graphs corresponding to $\tau = 10$ and $\tau = 20$ are the following:

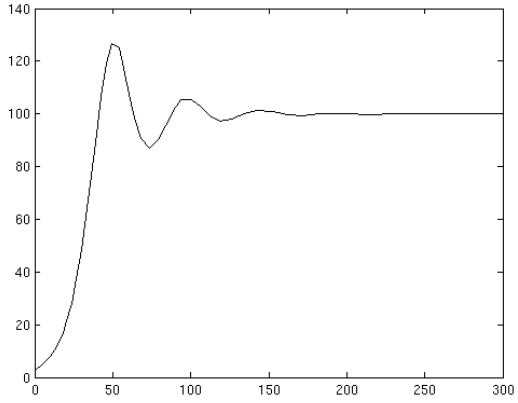

Figure 2: $\boldsymbol{\tau = 10}$ for $\boldsymbol{N > K}$ there are initial oscillations, after a certain value $\boldsymbol{t}$ the solution stabilizes at the value $\boldsymbol{K}$.

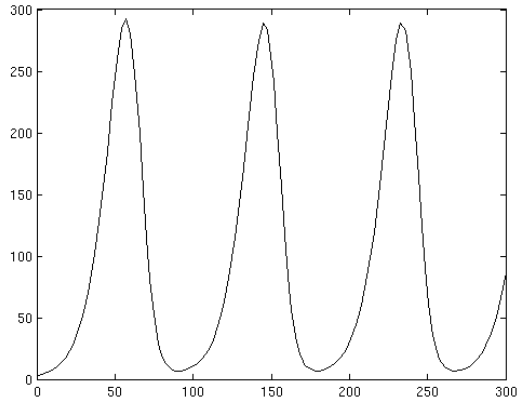

Figure 3: $\boldsymbol{\tau = 20}$ $\boldsymbol{N > K}$ and the solution is periodic.

2.1.4 Retarded Delay Differential Equation – A general initial value problem

Let a, b be two real numbers, with $b > a$; let's consider the Banach space of continuous functions defined in the interval $[a, b]$ into $\mathbb{R}^n$, denoted by $C([a, b], \mathbb{R}^n)$, that enjoy the property of uniform convergence. If $[a, b] = [-r, 0]$, in which r is a positive constant, this space is denoted by $C = C([-r, 0], \mathbb{R}^n)$; in it the norm of an element φ is written as:

$$\|\varphi\| = \sup_{-r \le \vartheta \le 0} |\varphi(\vartheta)| \qquad (26)$$

We consider the upper bound instead of the maximum of the interval in order to include also the case of unlimited interval below ($r = +\infty$).

If $\sigma \in \mathbb{R}$, $A \geq 0$, let x be an element of the Banach space C mapping the interval $[\sigma - r,\ \sigma + A]$ into $\mathbb{R}^n$:

$$x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$$

For any $t \in [\sigma, \sigma + A]$, let $x_t \in C$ be defined by

$$x_t(\vartheta) = x(t + \vartheta) \quad \vartheta \in [-r, 0]$$

Let $f: \mathbb{R} \times C \to \mathbb{R}^n$ be a given function. A Retarded Delay Differential Equation (RDDE) or Retarded Functional Differential Equation (RFDE) is given by the following relation:

$$\dot{x}(t) = f(t, x_t) \qquad \text{for } t \geq \sigma \qquad (27)$$

where " $\dot{}$ " represents the right-hand derivative.

Definition 5: A function x is said to be a solution of (27) on $[\sigma - r, \sigma + A]$ if

$x \in C([\sigma - r, \sigma + A], \mathbb{R}^n), (t, x_t) \in \mathbb{R} \times C$

and x_t satisfies (27) for $t \in [\sigma,\ \sigma + A]$.

Given $\sigma \in \mathbb{R}$, $\varphi \in C$, $x(\sigma, \varphi)$ is said to be a solution of (27) with initial value φ at σ if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of (27) on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \varphi) = \varphi$.

Under the assumption that the solution is unique, it is possible to define the function for each $t \geq 0$:

$$T(t): \varphi \to x_t(\sigma, \varphi) \qquad (28)$$

whose domain and codomain is C. This function is the solution map of the following IVP:

$$\begin{cases} \dot{x}(t) = f(t, x_t) & \text{for } t \geq \sigma \\ x_\sigma = \varphi \end{cases} \qquad (29)$$

The previous equation is a general type which includes, as special cases ordinary differential equations, differential difference equations of the type:

$$\dot{x}(t) = f\left(t, x(t), x\big(t - \tau_1(t)\big), \dots, x\left(t - \tau_p(t)\right)\right) \qquad 0 \leq \tau_i \leq r \quad i = 1, 2, \dots, p \qquad (30)$$

but also the integro-differential equations of the type:

$$\dot{x}(t) = \int_{-r}^{0} g\big(t, \vartheta, x(t+\vartheta)\big)\, d\vartheta \qquad (31)$$

If the function $f(t, x_t)$ in (27) is given by a relation of the type:

$$f(t, x_t) = L(t, x_t) + h(t) \qquad (32)$$

where $L(t, x_t)$ is linear in x_t, we say that the equation (27) is a *linear delay differential equation*. In particular, it is called *linear homogeneous* if $h(t) \equiv 0$ and *linear nonhomogeneous* if $h(t) \not\equiv 0$. We define equation (27) *autonomous* if

$$f(t, x_t) = g(x_t)$$

in which g doesn't depend on t. If not, it is called *non-autonomous*.

2.2 Dynamical Systems: definitions and modelling

2.2.1 A brief introduction to Systems of Differential Equations

Let $y_1, y_2, \dots, y_n$ be n unknown numerical functions of the real (or complex) variable x; a differential system is an expression of the type

$$\begin{cases} F_1(x, y_1, y_1', y_2, y_2' \dots, y_n, y_n') = 0 \\ F_2(x, y_1, y_1', y_2, y_2' \dots, y_n, y_n') = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_n(x, y_1, y_1', y_2, y_2' \dots, y_n, y_n') = 0 \end{cases} \qquad (33)$$

where F_i $(i = 1, 2, \dots, n)$ are numerical functions of $2n + 1$ variables defined in the same subset of the space $\mathbb{R}^{2n+1}$ (or $\mathbb{C}^{2n+1}$).

The equations of the system are all of the first order, but as its order of the system we assume the number n of the component equations, that is, the sum of the orders of the single equations.

A solution or integral of the system is any n-tuple of numerical functions $\varphi_1, \varphi_2, \dots, \varphi_n$ which can be derived in a certain interval I of $\mathbb{R}$ (or $\mathbb{C}$) such that it results identically in I

$$F_i\big(x, \varphi_1(x), \varphi_1'(x), \varphi_2(x), \varphi_2'(x), \dots, \varphi_n(x), \varphi_n'(x)\big) = 0 \quad (i = 1, 2, \dots, n) \qquad (34)$$

The integral $\{\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)\}$ is represented geometrically by a regular line Γ[15] of space $S_{n+1}(x, \varphi_1, \varphi_2, \dots, \varphi_n)$; the Cartesian equations of Γ are

$$\varphi_i = \varphi_i(x) \quad (i = 1, 2, \dots, n) \qquad (35)$$

Solving or integrating the system (33) means determining all its solutions.

Naturally we can consider differential systems in which derivatives of a higher order than the first appear for some or all the unknowns. For example, a system of form

[15] A line Γ: $\{\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)\}$ is a regular one if $\varphi_i(x)$ are continuous functions with their first derivative and $|\varphi_i'(x)| \neq 0 \quad \forall\, i = 1, 2, \dots, n$.

$$\begin{cases} f(x,y,z,y',z'') = 0 \\ g(x,y,z,y',z'') = 0 \end{cases} \qquad (36)$$

is of the first order in y and the second order in z: in total it is called order three, the sum of the orders in relation to the two unknowns y and z. It should be noted that the system (36) is equivalent to a system of three equations of the first order in three unknowns. In fact we can assume as the third unknown the derivative $z'(x)$ placing:

$$z'(x) = u(x)$$

thus the system (36) becomes.

$$\begin{cases} f(x,y,z,y',u,u') = 0 \\ g(x,y,z,y',u,u') = 0 \\ z' - u = 0 \end{cases} \qquad (37)$$

which is a system of three equations of the first-order in the unknowns $y(x), z(x), u(x)$.

This procedure is quite general, i.e. it is valid for any differential equation of order n. In fact, given the differential equation

$$f(x,y,y',y'',\dots,y^n) = 0 \qquad (38)$$

we can consider the derivatives of y as successive unknowns, that is we can place:

$$y'(x) = z_1(x), \quad y''(x) = z_2(x), \dots, y^{n-1}(x) = z_{n-1}(x)$$

In this way we obtain the system of n equations of the first-order:

$$\begin{cases} f(x,y,z_1,z_2,\dots,z_{n-1},z'_{n-1}) = 0 \\ y' - z_1 = 0 \\ z'_1 - z_2 = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ z'_{n-2} - z_{n-1} = 0 \end{cases} \qquad (39)$$

in the unknowns $y(x), z_1(x), \dots, z_{n-1}(x)$. In fact, if φ is a solution of equation (38) it is immediately seen that the n-tuple of functions

$$y_1 = \varphi, \qquad z_1 = \varphi', \qquad z_2 = \varphi'', \dots, z_{n-1} = \varphi^{n-1}$$

is a solution of system (39). Conversely, if the n-tuple of functions

$$y = \psi_1(x), \qquad z_1 = \psi_2(x), \dots, z_{n-1} = \psi_{n-1}(x)$$

is the solution of the system (39), it is easy to verify that the function $\psi_1(x)$ is a solution of the equation (38). Therefore the theory of differential equations and differential systems of any order is entirely contained in the theory of differential systems of the first order with respect to all unknowns.

A differential system of the first order with respect to each unknown function is said to be of normal form when it is resolved with respect to the first derivatives of the unknown functions, that is when it is written in the following form

$$\begin{cases} y_1' = f_1(x, y_1, y_2, \dots, y_n) \\ y_2' = f_2(x, y_1, y_2, \dots, y_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_n' = f_n(x, y_1, y_2, \dots, y_n) \end{cases} \qquad (40)$$

where the functions $f_1, f_2, \dots, f_n$ are all defined in the same region Ω of the space $\mathbb{R}^{n+1}$ (or $\mathbb{C}^{n+1}$), while $y_1, y_2, \dots, y_n$ are the unknown functions. Introducing vector notations, that is, placing $Y(x) = \big(y_1(x) \dots\dots y_n(x)\big)$ and $F = (f_1 \dots\dots f_n)$ the system (40) can be rewritten in the vector form

$$Y'(x) = F\big(x, Y(x)\big) \qquad F: \Omega \to \mathbb{R}^n \ (\mathbb{C}^n)$$

If the functions $f_1 \dots\dots f_n$ are polynomials of the first degree, the system is called linear and has the form[16]

$$\begin{cases} y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + \dots\dots + a_{1n}(x)y_n + b_1(x) \\ y_2' = a_{21}(x)y_1 + a_{22}(x)y_2 + \dots\dots + a_{2n}(x)y_n + b_2(x) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n' = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \dots\dots + a_{nn}(x)y_n + b_n(x) \end{cases} \qquad (41)$$

where $a_{ij}(x)$ and $b_i(x)$ are continuous functions defined in a certain interval I of $\mathbb{R}$ (or $\mathbb{C}$). Introducing the matrices

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix} \quad Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} \quad B(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{pmatrix}$$

the system (41) can be written in the compact form

$$Y'(x) = A(x)Y(x) + B(x) \qquad (42)$$

The matrix $A(x)$ is called coefficient matrix, while $B(x)$ is the column of known terms. If $B = 0$ the system is said homogeneous, otherwise it is called affine. Besides, if A and B are constants the system is linear with constant coefficients, also called autonomous.

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[16] There are also non-linear differential systems, such as the following:

$$\begin{cases} x' = -x(a + by) \\ y' = -y(cx + d) \end{cases}$$

We can see it in population dynamics problems (Lokta-Volterra equation).

When the linear differential system is associated with an appropriate set of initial conditions we talk about the *Cauchy problem*; thus, set a point $x_0 \in I$ and a constant vector $Y_0 \in \mathbb{R}^n$ (or $\mathbb{C}^n$) Cauchy problem is a type writing

$$\begin{cases} Y'(x) = A(x)Y(x) + B(x) & \forall x \in I \\ Y(x_0) = Y_0 \end{cases} \qquad (43)$$

A solution of this problem is a vector Y which in addition satisfies the initial condition Y_0. The following important theorem of existence and uniqueness applies:

<u>*Theorem 2*</u>: Let $A: I \to \mathcal{M}^n, \quad B: I \to \mathbb{R}^n$ (or $\mathbb{C}^n$) be matrices of continuous functions and let be $x_0 \in I, Y_0 \in \mathbb{R}^n$ (or $\mathbb{C}^n$); then there is one and only one solution of the *Cauchy problem*.

2.2.2 Dynamical Systems – Historical Notes

The term *Dynamical System* refers to a mathematical model that describes a situation evolving over time with a deterministic law, that is. an evolutionary one for which the state of the system at any instant of time depends uniquely on the state in which it was in the immediately preceding instant. Historically the concept of *Dynamical System* has its origin in Newtonian mechanics. In a letter sent to Leibniz in 1677, Newton wrote that he had found an important method that he could not reveal in an overt way. He did so by proposing an anagram, in a form almost impossible to decipher, known as the *fundamental anagram of the calculus*. The decoded sentence is as follows: «Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa» which means: «Given an equation that contains any number of flowing quantities (today we call them derivatives) find the flussions (today primitives) and vice versa». The method indicated by Newton became a privileged tool for the study of natural systems that evolve over time. In 1776 Pierre Simon de Laplace hypothesized the existence of a dynamic system able to regulate, in a rigidly deterministic and predictable way, the entire Universe. According to Laplace the state of a natural system at a certain moment depends on that existing at a previous instant. Therefore, knowing at a fixed time the relations between what he called the entity of the universe (today we call them state variables) it is possible to know the configuration of such entities in any other future instant. Laplace was aware that it is not possible to know precisely the different entities at a certain moment, since they are the result of measurement processes. However, he assumed that a small uncertainty in the values of the initial conditions had equally small consequences on the entire trajectory and that the calculation of the future state was slightly altered. Ultimately, it is possible to represent the evolution of a real system through a dynamic system, so that it is predictable and without any uncertainty. At that time it was known that when the equations of motion were non-

linear the temporal evolution of the system could not be accurately determined; a classic example is the dynamics of fluids, in the assumptions of turbulent motion: it has been one of the problems which most stimulated the studies on non-linear dynamic systems. At the beginning of the 20th century, interest in the study of dynamic systems declined considerably. It was resumed thanks to the studies of the French mathematician, physicist and philosopher Henri Poincaré who can be considered the founder of the qualitative theory of dynamic systems, that is a way of studying the laws of motion that renounces any claim to analytical or numerical knowledge of solutions and which is based on geometric-visual methods. Poincaré clearly described the phenomenon of so-called deterministic chaos, stating that, although natural laws are known, the evolution of a system can be known only by approximation. Moreover, small variations in the initial conditions may produce large and not negligible changes in the final situation. Poincaré's achievements were probably too far ahead of his time and were therefore underestimated. After the important contributions to the qualitative theory of dynamical systems provided by the Russian school in the 1930s and by Birkhoff's studies in the United States, a decisive contribution to the diffusion of this field of Mathematics came with a series of articles: the one in 1963 by the American meteorologist Edward Lorenz and the other in 1976 by the English physicist Robert May, who, studying models for Ecology, illustrated the onset of chaotic dynamics through the study of a discrete time dynamical system obtained by iterating a second degree polynomial.

The previous year, in the December issue of the American Mathematical Monthly, a short article by James A. Yorke and his doctoral student Tien-Yien Li appeared whose title was: "Period three implies chaos". The topic of the article and the results obtained are actually stated on the first page:

> «The way phenomena or processes evolve or change in time is often described by differential equations or difference equations. One of the simplest mathematical situations occurs when the phenomenon can be described by a single number ... [and this] number x_{n+1} at the beginning of the $n+1st$ year (or time period) can be written $x_{n+1} = F(x_n)$, where F maps an interval J into itself. These models are highly simplified, yet even this apparently simple equation ... may have surprisingly complicated dynamic behavior. We approach these equations with the viewpoint that irregularities and chaotic oscillations of complicated phenomena may sometimes be understood in terms of the simple model, even if that model is not sufficiently sophisticated to allow accurate numerical predictions. Lorenz ... took this point of view in studying turbulent behavior in a fascinating series of papers ... In this paper we analyze a situation in which the sequence $\{F(x_n)\}$ is non-periodic and might be called "chaotic". [Our main] theorem shows that chaotic behavior will result in any situation in which a "population" of size x can grow for two or more successive generations and then having reached an unsustainable height, a population bust follows to the level x or below ... May has recently discovered other strong properties of these maps in his independent study of how the behavior changes as a parameter is varied»[17].

In this article for the first time the term chaos has been explicitly used to characterize the behavior of those deterministic dynamical systems that together show aperiodicity of trajectories in the phase

[17] Landmark paper: T.Y. Li, J.A. Yorke, Period Three Implies Chaos, American Mathematical Monthly N$_o$ 82 (1975).

space that describes them and a sensitive dependence of these trajectories from the initial conditions of the system. This means that a small variation in the initial state of the system results in much larger variations in the states it will reach during its temporal evolution. In real physical systems it is very rare to know exactly the initial conditions and consequently, when they manifest chaotic behaviors, this places severe limitations to the predictability of their long-term evolution, even if the physical laws that govern them are fully understood. It may also happen that the behavior defined as chaotic has the appearance of a phenomenon of a random nature and that therefore such a physical system is considered non-deterministic at a fundamental level. This article, therefore, takes up Poincaré's theories.

2.2.3 Dynamical Systems – Fundamental Concepts

A dynamical system is any physical system (also in a broad sense) whose state is subject to change over time according to an evolutionary law of mathematical character, that is, the state of the system in any instant of time depends uniquely on the state in which it was in the immediately preceding instant. The concept of the dynamical system is now used in every field of study to analyze the evolutionary behavior of any entity that can be described by a mathematical model. Dynamical systems are also studied in the abstract, as mathematical objects, in mathematical physics and systems theory. The state of a system is in general the set of values of the physical quantities that are necessary to describe it completely with respect to a certain class of phenomena that occur within it or in which it participates. In general, a dynamical system is defined by the values assumed at a certain moment of time by n real scalar variables $x_1, x_2, \dots, x_n$ which are called state variables or dynamical variables. The number n of state variables defines the order of the system. The state of a system can be synthetically described by means of a vector, defined in an appropriate abstract mathematical space, whose components are neatly given by the values of all the quantities that contribute to define that state:

$$X = (x_1, x_2, \dots, x_n)$$

The vector X is therefore an element of a subset W of the vector space $\mathbb{R}^n$called *phase space*; it can be discrete, that is made up of isolated points in finite or infinite numbers, or it can be a continuous space. According to the mathematical structure of the phase space, we talk about:

- *Topological Space*, in which a metric is not necessarily defined.
- *Measurable Space*, in which the state is generally associated with a notion of measurement.

- *Differentiable Variety*[18]: this is a particular type of topological space, which allows the use of metric and differential instruments.
- *Complex Variety*: this is a further extension to the differentiable varieties in that the dynamic variables considered are at complex values.

In a dynamical system the state variables (at least one of them) and therefore the state vector are functions of a real non-negative parameter t representing time. Time can flow by taking values in a discrete set or in a continuous interval, depending on the intrinsic properties of the considered system or on analytical choices or even on experimental reasons. Depending on the way time flows within it, a dynamical system is said to be either *discrete-time* or *continuous-time*.

The evolution of a dynamical system can take place in a deterministic way, that is, when each initial state uniquely determines the immediately following state, or stochastic. In the latter case, the succession of the system states over time is defined by means of a time-dependent random variable that takes values in spaces more general than that of real numbers. The evolution of a dynamical system is described mathematically by an operator – defined according to the laws that regulate the behavior of the system as a function of time – which acts on the state vector at a certain moment, transforming it into the state vector that describes the system at the desired moment. The sequence of the states constituting the evolution over time of a dynamical system starting from any of its initial states, constitute a temporally ordered set of points in its phase space, called the *orbit* of the system; it obviously has nothing to do with the trajectory of its possible motion in real physical space. Orbits can be divided into three classes:

1. *Constant Orbits*, consisting of a single point, such that $X_k = X_0, k = 0, 1, 2, ...$ or $\Phi(X_0 = X_0)$; they are also called fixed or equilibrium points of the system.
2. *Recurring Orbits*: they are those that continually return to "visit" already visited areas of the phase space. Particularly important are closed orbits (cycles), including periodic orbits, and quasi-periodic orbits.
3. *Non-Recurring Orbits*: they are those that don't return to regions of the phase space that have already been visited.

A dynamical system is specified by an algebraic relationship involving time t, the value of the individual components of the vector $X(t)$ and the time derivative of the vector function $X(t)$. The nature and properties of this relationship depend on the system considered (deterministic or stochastic) and the way time flows.

[18] The term n-dimensional variety refers to the formalisation of the n-dimensional surface concept. For example, for $n = 2$ are varieties the cylinder, the sphere, any plane $\mathbb{R}^2$ or any of its open subset.

In a discrete time stochastic dynamical system and with a continuous phase space the evolution operator is given by a set of recursive functions dependent on a parameter whose value is selected using a probability distribution.

In a deterministic discrete-time system with a continuous phase space W, the evolution operator is a map or endomorphism $f: W \to W$, while the law governing the dynamical system is a finite difference equation or recursive function having the form:

$$X_{k+1} = f(X_k) = f\left(x_{1_k}, x_{2_k}, \dots, x_{n_k}\right) \qquad (44)$$

for each integer $k \geq 0$, such that every value taken from the map is a function of the immediately preceding one. If the map is invertible with continuity, it is called homeomorphism; if, on the other hand, it is invertible with regularity (that is it is continuous and differentiable) it is called diffeomorphism. To uniquely determine an orbit, it is necessary to specify the initial value

$$X_0 = X(t_0) = [x_1(t_0), x_2(t_0), \dots, x_n(t_0)]$$

of the succession. Once X_0 is known, the existence and uniqueness of an orbit of the discrete dynamical system are guaranteed by the *Induction Principle*[19].

In a deterministic system with continuous time and whose phase space is a differentiable variety the dynamics of the system is described by an ordinary vector differential equation in $\mathbb{R}^n$, or a system of n ordinary scalar differential equations in normal form and autonomous (that is, with a second member independent of time) of the type

$$\frac{dX}{dt} = F(X) \qquad (45)$$

where $F(X)$ is a vector differentiable field (of class C^1); an orbit of a continuous dynamical system is a function $X(t) \in W$, solution of the differential equations (45).

The fact that differential equations (45) are of the first order doesn't constitute an effective restriction, since any system of differential equations can be expressed in this form by introducing auxiliary variables. Moreover, it is not a restriction either that the system of equations is

[19] The Induction Principle is a proof rule used when the property or theorem has a statement formulated as a function of natural numbers. It is useful to establish the validity of the thesis for each integer n, or possibly starting from a certain $n_0 \in \mathbb{N}$, from the verification of two conditions: the validity of the *zero step* and that of the *inductive step*. It can be stated as follows:

Let $n_0 \geq 0$ be an integer and let $P(n)$ be a predicate defined for every integer $n \geq n_0$. Suppose that the following two conditions are verified:

(i) $P(n_0)$ is true;

(ii) for every $n \geq n_0$, if $P(n)$ is true then $P(n+1)$ is true.

Then $P(n)$ is true for every $n \geq n_0$.

Condition (i) is called *induction base* or *inductive base*, condition (ii) is called *inductive step*, and the assumption that $P(n)$ is true is called *inductive hypothesis*. In practice, the Induction Principle is used as follows: first we check that $P(n_0)$ is true; then we assume that $P(n)$ is true for a generic n and, using this information, we show that $P(n+1)$ is also true.

autonomous, since any system of differential equations can be put in this form by still introducing appropriate auxiliary variables. If the flow is defined on the entire real axis it constitutes a *diffeomorphism* and has the properties of a group of regular transformations dependent on a parameter.

The problem of the continuous dynamical system (45) and an initial condition $X(0) = X_0$ is called the *initial value problem* or *Cauchy problem*. Although a continuous dynamical system is part of the theory of ordinary differential equations, the existence and uniqueness of a solution is not a trivial result[20].

The integral flow (or general integral) of a continuous dynamical system can be defined as the family of applications $\Phi_t: W \to W, (t \in [0, +\infty[)$ defined as follows: given $X_0 \in W$ the image $\Phi_t(X_0)$ is given by the vector value $X(t)$ assumed by the solution of the system (45), having initial data X_0, calculated at time t.

A point $X_0 \in W$ is an equilibrium point (or fixed point) of the dynamical system (45) if the only orbit corresponding to the initial value X_0 is the constant $X(t) \equiv X_0$.

In order to calculate the solutions of a dynamical system, that is to determine analytically its orbits, it is necessary to solve exactly the equations (44) or (45) that describe it. Even for the simplest systems, this is only possible in a very limited number of situations and often requires very sophisticated mathematical techniques. Indeed, the numerical determination of the orbit of a system is sufficient to effectively define its temporal evolution only when the system is extremely simple. On the other hand, when it is a complex system, that is characterized by many state variables, or it is non-linear with respect to some of its variables or not all the laws that contribute to describe it are known or even when some of its variables cannot be measured with the necessary precision, approximate methods inevitably mean that the validity and relevance of the solutions found are called into question. In this case the main problem is the stability of the calculated orbits, that is the property of an orbit not to change qualitatively significant for small variations in the initial state of the system. On the other hand, it is often more important to know the common properties of different orbits than the individual details. In many application situations, it is necessary to classify orbits or it is required that the system evolves along only one type of orbit. Qualitative analysis of dynamical systems deals with the classification of orbits; however, it is generally possible only for linear systems and those whose phase space has only two dimensions. In other cases it is important to know the behavior of the orbits of a system as a function of some parameter and in particular to

[20] A continuous dynamical system admits only one solution under the hypothesis that the vector field F is of class C^1; in this case, in fact, the following theorem applies:

Theorem: G*iven* a point $x \in W$ there exists an interval $[0, t^*[$ and a curve $X: [0, t^*[\to W$ such that $X(t)$ is a local solution of the dynamical system (45) under the initial condition $X(0) = X_0$.

find out if the latter has bifurcation values, that is those values through which the qualitative behavior of the system changes. Finally, there are situations in which all the orbits of a system appear irregular, as if it were governed by a random dynamics. In these cases it may be necessary to calculate average values of the state variables, using either a very long orbit or many different orbits, and thus the so-called ergodic systems become important, for which the average values are practically independent of the initial state of the system and this "forgets" its past in its evolution.

2.2.4 Dynamical Systems – States of Equilibrium and Stability

The first integral of a dynamical system is defined as a regular function, that is a single-valued and continuously differentiable one, that remains constant at all points in the phase space belonging to the same orbit. The knowledge of the first integrals of a dynamical system is fundamental for the qualitative analysis of the system itself and also to look for exact solutions of the equations that describe it. A dynamical system is said to be integrable if it admits a number of functionally independent prime integrals equal to the dimensions of its phase space. This is equivalent to saying that it admits an explicit formulation of the integral flow.

There are relatively few integrable dynamical systems; in general, the solutions of the dynamical system can be calculated explicitly when the vector field F is linear. Let us analyze the simple cases of a one-dimensional dynamical system ($n = 1$) and a two-dimensional decoupled linear dynamical system.

- *One-dimensional dynamical system* Let be $W = \mathbb{R}$ and $a \in \mathbb{R}$; let us consider the linear one-dimensional dynamical system described by the differential equation

$$\frac{dx}{dt} = ax$$

  Its solution is given by the function

$$x(t) = x_0 e^{at}$$

  with the initial condition $x(0) = x_0$.

- *Two-dimensional decoupled linear dynamical system* Let be $W = \mathbb{R}^2$ and $a, b \in \mathbb{R}$; let us consider the dynamical system described by the system of differential equation

$$\begin{cases} \frac{dx}{dt} = ax \\ \frac{dy}{dt} = by \end{cases}$$

  The solution $X(t) = \big(x(t), y(t)\big)$, with the initial condition $X_0 = (x_0, y_0)$ is found solving the two equations separately. We obtain:

$$x(t) = x_0 e^{at} \;\; y(t) = y_0 e^{bt}$$

A problem that is often posed is to determine the limit state of the dynamical system for $t \to +\infty$; this involves calculating the following limit

$$\lim_{t \to +\infty} X(t)$$

In the previous case, and in general in all cases where the solution can be calculated explicitly, this problem results in the simple calculation of a limit of a function of a variable. The behavior at infinity obviously depends on the constants a and b and the initial conditions.

Now we can give the following definition:

Definition 6: A dynamical system is called *linear* when the vector field F in (45) is linear, that is the system is of the form

$$\frac{dX}{dt} = AX \qquad (46)$$

where $A \in \mathcal{M}(n, n)$. Since the vector field F, described by the formula $F(X) = AX$, is of class C^1, the conditions of existence and uniqueness of the solutions of the problem to the initial values apply. Let's take again the one-dimensional case written in the form:

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) = x_0 \end{cases} \quad a \in \mathbb{R} \qquad (47)$$

Integrating both members in the interval $[0, t]$ we obtain:

$$x(t) = x_0 + \int_0^t \frac{dx(s)}{ds} ds = x_0 + a \int_0^t x(s) ds \qquad (48)$$

The differential equation that regulates the system has therefore been reformulated in integral form. Although we already know the solution, we can solve this equation with the method of successive approximations, which consists in obtaining the solution $x(t)$ as an approximation of a succession of functions $x_k(t),\ k \in \mathbb{N}$ recursively defined as follows:

$$\begin{cases} x_0(t) = x_0 \\ x_{k+1}(t) = x_0 + a \int_0^t x_k(s) ds, \quad k \geq 0 \end{cases} \qquad (49)$$

Explicitly calculating the first two values of the succession we have:

$$x_1(t) = x_0 + a \int_0^t x_0 ds = (1 + at) x_0$$

$$x_2(t) = x_0 + a \int_0^t (1 + as) x_0 ds = \left(1 + at + \frac{a^2 t^2}{2}\right) x_0$$

Generalizing for each integer k, we have the following formula

$$x_k(t) = \left(\sum_{i=0}^{k} \frac{a^i t^i}{i!}\right) x_0 \qquad (50)$$

which can be proved strictly using the Induction Principle (see note[18]).

If we consider the limit for $k \to +\infty$ of the approximating equation (49), we can state that this limit:

$$\lim_{k \to +\infty} x_k(t) = x(t)$$

satisfies the integral equation (48); in fact we can demonstrate that $x_k(t)$ converges uniformly to $x(t)$ over the entire interval $[0, t]$. This statement derives directly from the theorem of the passage to the limit under the sign of integral[21]. We must observe that the summation (50) is the succession of partial sums relative to the Taylor series of the exponential function e^{at} that is uniformly convergent on each interval $[0, t] \quad t \geq 0$. Therefore the solution to problem (47) is represented by

$$x(t) = \left(\sum_{i=0}^{\infty} \frac{a^i t^i}{i!}\right) x_0 = e^{at} x_0 \qquad (51)$$

Now let us consider the general case, with an initial condition:

$$\begin{cases} \dfrac{dX}{dt} = AX \\ X(0) = X_0 \end{cases} \qquad A \in \mathcal{M}(n, n) \qquad (52)$$

Here too we can reformulate the problem in an integral[22] by posing

$$X(t) = X_0 + \int_0^t AX(s)ds \qquad (53)$$

We then define the successive approximations

$$\begin{cases} X_0(t) \equiv X_0 \\ X_{k+1}(t) = X_0 + \displaystyle\int_0^t AX_k(s)ds, \quad k \geq 0 \end{cases} \qquad (54)$$

Similarly to the one-dimensional case, the following formula is obtained for the approximations

$$X_k(t) = \left(\sum_{i=0}^{k} \frac{A^i t^i}{i!}\right) X_0 \qquad (55)$$

In contrast to the one-dimensional case, the solution in the general case isn't known a priori. There are in fact two problems: the first is that of the convergence of the succession $X_k(t)$ (we must

[21] Let $\{f_n\}_{n \in \mathbb{N}}$ be a succession of functions defined in the interval $[a, b] \quad a, b \in \mathbb{R}$ such that f_n converges uniformly to f. If each f_n is integrable according to Riemann in $[a, b]$, f is also Riemann integrable in $[a, b]$ and we have:

$$\lim_{n \to +\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx = \int_a^b \lim_{n \to +\infty} f_n(x)dx$$

This means that we can arbitrarily invert the sign of the limit with that of the integral.

[22] The integral of a vector is the vector consisting of the integrals of the individual components.

calculate the limit for $k \to +\infty$ of a series of functions with values in the space of square matrices); the second is that of the consistency of the limit: that is it is necessary to verify that the eventual limit of the series is the solution of Cauchy's problem.

To solve both problems we can put:

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$$

where the second member represents an infinite series whose generic term is a matrix. Thus, if the matrix A of the system (52) is in the form

$$A = \lambda \mathbb{I}$$

where λ is a real number and $\mathbb{I}$ is the identity matrix, it can easily be verified that the solution of the dynamical system is $X(t) = e^{\lambda t} X_0$. More generally, if A is a diagonal matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

with $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$, the solution of the corresponding dynamical system is $X(t) = \big(x_1(t), \dots, x_n(t)\big) = \big(e^{\lambda_1 t} x_{0,1}, \dots, e^{\lambda_n t} x_{0,n}\big)$, in which $X_0 = \big(x_{0,1}, \dots, x_{0,n}\big)$ is the initial condition[23].

Now let us state the following theorem:

Theorem 3: Let A be a square matrix $n \times n$. Then the exponential series

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[23] Given a matrix B $n \times n$ the exponential of B is "formally" defined by the series:

$$e^B = \sum_{i=0}^{\infty} \frac{B^i}{i!}$$

By way of examples, let be $A = \lambda \mathbb{I}$, where $\lambda \in \mathbb{R}$ and $\mathbb{I}$ is the identity matrix. Considering that $A^k = \lambda^k \mathbb{I}$, we have:

$$e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!} = \sum_{i=0}^{\infty} \frac{\lambda^i t^i \mathbb{I}}{i!} = e^{\lambda t} \mathbb{I}$$

Similarly, let be

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

In this case we have:

$$A^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

Then we obtain:

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

$$\sum_{i=0}^{\infty} \frac{A^i t^i}{i!} = e^{At}$$

converges punctually for every $t \in \mathbb{R}$ and uniformly over every compact interval of $\mathbb{R}$. Moreover, the limit e^{At} is a continuous function.

This theorem allows to solve the problem of the convergence of approximations X_k defined by (55); in fact, according to it, we can state that:

$$\lim_{k \to +\infty} X_k(t) = e^{At} X_0 =: X(t)$$

Really, the uniform convergence of the exponential series implies the uniform convergence of the term $AX_k(s)$ in (54); therefore it is possible to pass this limit under the sign of integral (see note[20]): this also allows to solve the problem of consistency. Then we obtain, in the limit for $k \to +\infty$:

$$X(t) = X_0 + \int_0^t AX(s)ds$$

More generally, if A is a diagonalizable matrix[24], it can be verified that all the orbits of the dynamic system (46) can be written as linear combinations of exponential functions. In fact, given the real eigenvalues of A $\lambda_1, \lambda_2, \dots, \lambda_n$ and $V_1, V_2, \dots, V_n$ the corresponding eigenvectors[25], let V be the matrix whose columns are the vectors V_i. We can write:

$$X(t) = Ve^{Dt}V^{-1}X_0$$

where X_0 is the initial data and

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

We obtain:

$$X(t) = V \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} V^{-1}X_0$$

Thus each $X(t)$ coordinate is a linear combination of scalar exponential functions.

[24] A square matrix A of order n with coefficients in a field $\mathbb{K}$ is said to be diagonalizable if it is similar to a diagonal matrix D. This is equivalent to stating that $A \in \mathcal{M}(n,n)$ is diagonalizable if and only if there exists an invertible matrix P such that $P^{-1}AP = D$. P is called the diagonalizing matrix of A.

[25] Let A be a square matrix of order n with coefficients in a field $\mathbb{K}$ ($A \in \mathcal{M}(n,n)$). A scalar λ_0 is an eigenvalue of matrix A if there is a non-null vector v such that $Av = \lambda_0 v$. The vector v is called a eigenvalue relative to the eigenvalue λ_0. The eigenvalues of A are the solutions of equation $\det(A - \lambda \mathbb{I}_n) = 0$, where $\det(A - \lambda \mathbb{I}_n)$ is called the characteristic polynomial associated with the matrix A.

Let us analyze the case when $A \in \mathcal{M}(2,2)$; the solution $X(t) = \big(x(t), y(t)\big)$ can be written as:

$$\begin{cases} x(t) = e^{at} x_0 \\ y(t) = e^{bt} y_0 \end{cases}$$

where a, b are the eingvalues of A and $X_0 = (x_0, y_0)$ is the initial data. The qualitative behaviour of the system depends on the sign of a and b. The significant cases are as follows:

- $a < 0, b < 0$. All orbits have as their limit the origin for $t \to +\infty$. If $x_0 \neq 0$ the orbits are written in Cartesian form (trajectory) as:

  $$y = y_0 \left(\frac{x}{x_0}\right)^{b/a}$$

  The point of equilibrium in the origin is called *attractive node*.
- $a > 0, b > 0$. All orbits tend to infinite for $t \to +\infty$. The trajectories coincide with those of the previous point, but travelled in the opposite direction. The equilibrium point in the origin is called *repulsive node*.
- $a < 0 < b$. All orbits with $x_0 \neq 0$ tend to infinite for $t \to +\infty$. The trajectories are:

  $$y = y_0 \left(\frac{x}{x_0}\right)^{-b/a}$$

  The equilibrium point in the origin is called *saddle point*.

When a dynamical system

$$\dot{X} = F(X) \qquad F: W \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

with W open set, is non-linear, it isn't always possible to explicitly determine its integral flow. However, certain qualitative properties of orbits can be determined from some structural properties of the system. For example, under fairly general conditions it is possible to establish the limit behaviour for large times, that is the limit of orbits for $t \to +\infty$. The following theorem gives a necessary condition for a point to be a limit state.

Theorem 4: If the solution $X(t)$ has a limit for $t \to +\infty$, that is. If

$$\lim_{t \to +\infty} X(t) = X_l \in W$$

where W is the open on which F is defined, then X_l is an equilibrium point of the system, namely $F(X_l) = 0$.

Now let us give the following definitions:

Definition 7: Given the dynamical system

$$\dot{X} = F(X) \qquad F: W \to \mathbb{R}^n$$

- A point $S \in W$ is said to be *attractive* if there exists an open sphere U, $S \in U \subset W$ such that for every initial condition $X_0 \in U$, the solution $X(t)$ corresponding to it has S as its limit. The previous theorem allows us to state that every attractive point must be of equilibrium.
- If S is an equilibrium point, we can define *basin of attraction* of S the set U of initial conditions such that the corresponding solutions have S as their limit.
- A point $S \in W$ is said to be *stable* if for every open sphere U containing S there exists an open sphere V containing S ($U, V \subset W$) such that every solution with initial data in V is all contained in U for every $t \geq 0$.
- A stable and attractive point is said to be *asymptotically stable*.
- A point S is *unstable* if and only if it is not stable. In other words, S is said to be unstable if it is contained in an open sphere U such that for every open sphere V containing S there is an initial condition $X_0 \in V$ whose solution doesn't lie in U for some $t \geq 0$.

To determine the stability or instability of an equilibrium point of a non-linear dynamical system, we can basically use two methods: the *linearization method* and the *Lyapounov functional method*.

2.2.5 Linearization Method

The linearization method consists in studying the dynamical system obtained by linearizing the vector field F, that is considering only the linear part. More precisely, in the hypothesis of vector field F of class C^1, if A is its Jacobian matrix[26] in his point X_0, we have:

$$F(X) = A(X - X_0) + R$$

Where the term R is an infinitesimal of order higher than the first in X_0, that is:

$$\lim_{X \to X_0} \frac{R}{\|X - X_0\|} = 0$$

$A(X - X_0)$ is said "the linear part of F near X_0".

[26] The Jacobian of a vector function of several real variables is a matrix whose elements are the first partial derivatives of the function. In particular, given the vector function $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ defined in a subset Ω of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ such that
x $= (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ associates $f(x) = \big(f_1(x), f_2(x), \dots, f_m(x)\big) \in \mathbb{R}^m$
the Jacobian matrix associated with the function f is given by:

$$J_f(\mathrm{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathrm{x}) & \frac{\partial f_1}{\partial x_2}(\mathrm{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathrm{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathrm{x}) & \frac{\partial f_2}{\partial x_2}(\mathrm{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathrm{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathrm{x}) & \frac{\partial f_m}{\partial x_2}(\mathrm{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathrm{x}) \end{bmatrix}$$

In general this is a rectangular matrix with m rows (where m is the dimension of the function's arrival set) and n columns (with n dimension of the f definition set).

To better understand what we will say later, let's press some definitions and theorems.

Definition 8: A matrix $A \in \mathcal{M}(n,n)$ is called *semi-simple* if it is diagonalizable in the complex sense, that is. if there exists a base of $\mathbb{C}^n$ formed by eigenvectors $\{V_1, V_2, \dots, V_n\}$ such that

$$AV_k = \lambda_k V_k \qquad \lambda_k \in \mathbb{C}, k = 1,2, \dots, n$$

Complex numbers λ_k are called eigenvalues of A.

Theorem 5: If the matrix A is semi-simple, then all orbits of the dynamical system $\dot{X} = AX$ can be expressed by linear combinations of exponential functions $e^{a_k t}$ (where a_k are the real parts of the eigenvalues of A) multiplied by trigonometric functions $\cos(b_k t)$and $\sin(b_k t)$ (where b_k are the imaginary parts of the eigenvalues of A).

Definition 9: A matrix $N \in \mathcal{M}(n,n)$ is said *nilpotent* if there is a positive integer m such that $N^m = 0$.

The smallest m for which the previous equality is verified is called the nilpotency order of A. All eigenvalues of a nilpotent matrix are null.

Theorem 4: Every matrix $A \in \mathcal{M}(n,n)$ can be written in one and only one way as the sum of a semi-simple matrix S and a nilpotent matrix N that commute with each other, i.e.

$$A = S + N \qquad NS = SN$$

The previous theorem allows us to express the exponential of At in the form:

$$e^{At} = e^{St} e^{Nt}$$

Now we can briefly study the stability of a dynamical system by the linearization method.

Definition 10: Given an equilibrium point X_l and called A the Jacobian matrix of F in X_l, the linear system

$$\dot{Y} = AY$$

is the linearized system of $\dot{X} = AX$ in X_l[27]. The real parts of the eigenvalues of A are called *Lyapounov exponents* of the system.

If all the exponents of Lyapounov are negative the equilibrium point X_l is said a *well*. If all the exponents of Lyapounov are positive, X_l is said to be a *source*.

As we've seen before, the matrix A can be written as $A = S + N$, (where S is semi-simple matrix, while N is a nilpotent one) so the exponential of At is equal to the product $e^{St}e^{Nt}$. The first factor is conjugated of a diagonal matrix block, each of which is the exponential $e^{\lambda t}$, where λ is a eigenvalue of A (real or complex), expressed in matrix form; he second term is conjugate of the exponential of a nilpotent matrix in canonical form, and therefore is a matrix containing monomials in t. The following linear well theorem applies:

[27] The linearized system always admits 0 as the equilibrium point.

Theorem 5: Let the linear dynamical system $\dot{X} = AX$ be given. If the eigenvalues of A all have negative real part then the equilibrium point in the origin is asymptotically stable.

The question remains whether the stability of the linearized system can also be extended to the starting system. This is not always true, as is evidenced by the following theorem, which provides many useful indications

Theorem 6: If all the eigenvalues of the matrix A have strictly negative real parts, or if there is at least one with a strictly positive real part, then the equilibrium of the linearized system has the same stability or instability properties as the equilibrium of the starting system.

In particular, it is possible to state an analogous theorem for a non-linear system:

Theorem 7 (Non-linear Well Theorem): Let X_l be a well for the dynamical system $\dot{X} = F(X)$, where the vector field F is defined on $W \subset \mathbb{R}^n$and is of class C^1. Then let A be the matrix of the linearized system in X_l. If c is a positive real number such that every eigenvalue λ of A has real part

$$Re(\lambda) < -c$$

then there exists an open sphere U containing X_l such that:

(a). the integral flow $\Phi^t(X)$ is defined for every X in U and for every $t \geq 0$;

(b). there is a constant $B > 0$ such that, for every $X \in U$ and for every $t \geq 0$ we have

$$\|\Phi^t(X) - X_l\| \leq Be^{-ct}\|X - X_l\|$$

In particular. X_l is asymptotically stable.

Similarly, if X_l is a source, then the equilibrium point is unstable.

2.2.6 Lyapounov Functional Method

Consider a dynamical system $\dot{X} = F(X)$, where F is a vector field defined on an open $W \subset \mathbb{R}^n$ and of class C^1. Let a functional $V: W \to \mathbb{R}$ of class C^1 be given. The *total derivative* of V (along the vector field F) is the function

$$t \mapsto \dot{V}\big(X(t)\big) = \frac{d}{dt}V\big(X(t)\big) = \nabla V(X) \cdot F(X)$$

Therefore, the total derivative of V along F is the derivative of the compound function $t \mapsto V \circ \Phi^t$.

Definition 11: A function V defined on an open $U \subset W$ containing the equilibrium point X_l and of class C^1 is said *Lyapounov functional* for the dynamical system with respect to the equilibrium X_l if the two conditions apply:

(a). $\dot{V}\big(X(t)\big) \leq 0$ for every orbit $X(t)$ having initial datum in U;

(b). $V(X_l) = 0$ *e* $V(X) > 0\ \forall X \in U, X \neq X_l$.

If they are worth the property (b) and the following:

(c). $\dot{V}\big(X(t)\big) < 0$ for every orbit $X(t)$ having initial datum in U other than the equilibrium X_l then V is said *Lyapounov functional narrow*.

A Lyapounov functional has a strong minimum at the equilibrium point, and its value never grows along the solutions. A Lyapounov functional narrow is strictly decreasing along the solutions.

Theorem 8 (Lyapounov's first stability theorem): If the equilibrium point X_l possesses a Lyapounov functional $V(X)$ in an open U containing it, then X_l is stable.

If property (a) of Lyapounov functionals is satisfied on the whole open W, it is possible to obtain stronger properties of stability.

Definition 12: A set $P \subset W$ is said *positively invariant* if, for every $X_0 \in P$ the solution with initial datum X_0 exists for every $t > 0$ and is contained in P.

Theorem 9 (Lyapounov's second stability theorem): Let X_l be an equilibrium point and let $V(X)$ be a Lyapounov functional with respect to X_l defined over all of W and such that property (a) of Definition 11 is satisfied on all of W. If P is a compact (of nonzero measure), containing X_l, positively invariant and such that the function V is strictly decreasing along the orbits contained in P (except in X_l), then X_l is asymptotically stable and P is contained in the basin of attraction of X_l.

2.2.7 Bifurcation Theory

When the vector field F describing the dynamical system $\dot{X} = F(X)$ depends on parameters, the stability of the fixed points changes as these parameters change. It becomes therefore interesting to determine, if they exist, those values of the parameters by passing through which the number and/or the properties of the fixed points of the system change and consequently its evolution (that is, the shape of the orbits) undergoes a qualitative transformation. These values are called bifurcation points. For the sake of simplicity, suppose that F depends on only one parameter μ, in addition to X, and that the existence, uniqueness and continuous dependence theorems of the initial data are valid

$$\dot{X} = F(X, \mu)$$

The possible equilibrium configurations will depend on μ: in fact the equation

$$F(X, \mu) = 0 \qquad (56)$$

The previous equation implicitly defines a curve in the space (X, μ) that represents the locus Γ of the equilibrium points (i.e. the solutions of (56)). In general, the analytical expression of this locus in the explicit form $X_e = X_e(\mu)$ will only be available if it is possible, from (56), to derive X as a function of μ or vice versa. In general, the character (stable or unstable) of equilibrium configurations changes with the variation of μ for all points of Γ. If, in addition to μ, other parameters are present, the simplest way to proceed is to analyze the behaviour of the equilibrium

solutions as a function of each of these parameters, keeping the others fixed. The analysis is always done in the space (X, μ); if $X \in \mathbb{R}^2$ it is the three-dimensional space, if $X \in \mathbb{R}^1$ (that is X is a scalar function) it is the ordinary Cartesian plane. More generally, if $X \in \mathbb{R}^n, n \geq 3$, it is more convenient to analyze, instead of X_e, its particular scalar functions, such as

$$|X_e(\mu)| = \sqrt{\sum_{k=1}^{n} X_{e_k}^2} \qquad \text{or} \qquad |X_e(\mu)| = \max_{1 \leq k \leq n} |X_{e_k}|$$

so as to continue to have a simple view of the link between X_e and μ. The diagrams thus obtained are called *bifurcation diagrams*. The locus Γ is usually composed of several branches or offshoots that are stretches (open or closed) of curves that may intersect each other and terminate at points at the finite μ. The number of pairs $(\mu, X_e(\mu)) \in \Gamma$ usually changes with the variation of μ: in fact, a line parallel to the X axis can cut the equilibrium curve at one, none or in many points depending on the configuration of the curve itself. The points at which the number of equilibrium solutions changes are called *branch points*.

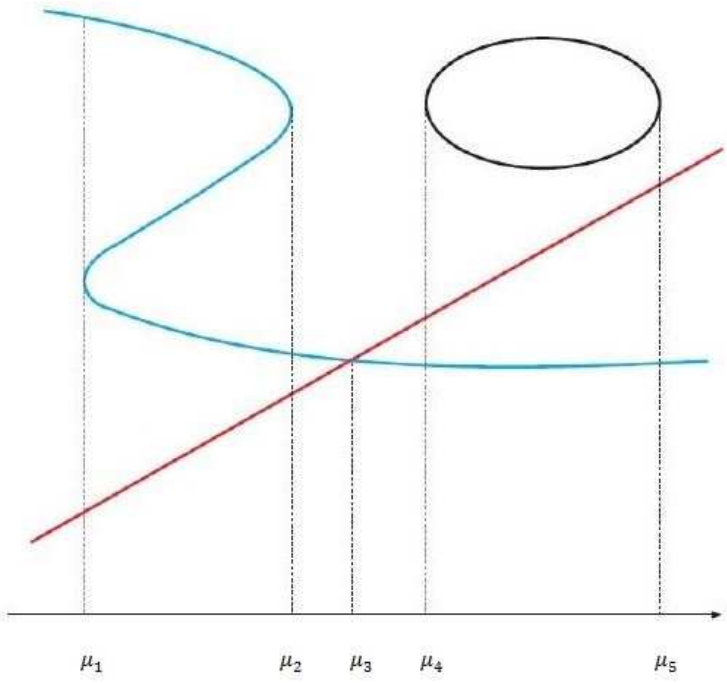


Figure 5 The equilibrium curve consists of several branches. The number of equilibrium solutions changes as $\boldsymbol{\mu}$ varies. There are three solutions if $\boldsymbol{\mu = \mu_1}$, $\boldsymbol{\mu = \mu_2}$, $\boldsymbol{\mu = \mu_4}$ and $\boldsymbol{\mu = \mu_5}$, two if $\boldsymbol{\mu < \mu_1}$, $\boldsymbol{\mu_2 < \mu < \mu_3}$, $\boldsymbol{\mu_3 < \mu < \mu_4}$ **or** $\boldsymbol{\mu > \mu_5}$ and finally four if $\boldsymbol{\mu_1 < \mu < \mu_2}$ **or** $\boldsymbol{\mu_4 < \mu < \mu_5}$. So $\boldsymbol{\mu_1, \mu_2, \mu_4, \mu_5}$ are branch points while $\boldsymbol{\mu_3}$ is a bifurcation point.

With reference to Fig. 5 the point corresponding to $\mu = \mu_3$ has a particular nature: for $\mu = \mu_3$ two distinct branches of the locus Γ meet and at that point the tangent to Γ is not univocally defined.

These points are generically called *bifurcation points*. More precisely we can give the following definition:

Definition 13: We say that one equilibrium solution "bifurcates" from another in $\mu = \mu_b$ if there exist two distinct equilibrium solutions $\hat{X}_e(\mu)$ and $\check{X}_e(\mu)$, continuous with respect to μ and such that $\hat{X}_e(\mu) = \check{X}_e(\mu)$. The common point $(\mu_b, \hat{X}_e(\mu))$ is called *bifurcation point*.

It is convenient to classify certain types of bifurcations and adopt the convention of indicating the lines that form the locus of equilibria with solid strokes in the stable case and with dashed lines in the unstable case. Bifurcation points are normally characterized by stable-instable variation moving from one branch to the other.

- *Saddle-Node Bifurcation*. This bifurcation, also called *tangent bifurcation*, occurs when, for μ, tending to μ_b two different equilibrium solutions $\hat{X}_e(\mu)$ and $\check{X}_e(\mu)$, one of stable equilibrium and the other of unstable equilibrium, tend to each other, and for $\mu = \mu_b$ they disappear by collapsing. It occurs when a real eigenvalue of the Jacobian matrix A of Ftends to zero (without crossing the imaginary axis).
- *Transcritical Bifurcation*. This bifurcation, more commonly called *bifurcation with exchange of stability*, occurs when, for μ, tending to μ_b two different equilibrium solutions $\hat{X}_e(\mu)$ and $\check{X}_e(\mu)$, one of stable equilibrium and the other of unstable equilibrium, tend to each other, and for $\mu = \mu_b$ they merge, then separating immediately after exchanging stability. It occurs when an eigenvalue real eigenvalue of A crosses the imaginary axis and becomes positive.
- *Pitchfork Bifurcation*. This bifurcation, also called *Period-doubling*, can be of two types: *direct* (or *supercritical*) or *reverse* (or *subcritical*). Direct bifurcation occurs when, for $\mu = \mu_b$ a stable equilibrium solution $\bar{X}_e$ becomes unstable originating two stable and attractive solutions $\hat{X}_1$ and $\hat{X}_2$ for $\mu > \mu_b$. Conversely, reverse bifurcation occurs when, in addition to $\bar{X}_e$, for $\mu < \mu_b$there are also two points $\check{X}_1$ and $\check{X}_2$ unstable, which collapse to $\bar{X}_e$ for $\mu = \mu_b$, making it unstable. In both cases this bifurcation occurs when a real eigenvalue of A crosses the imaginary axis and becomes positive.

2.2.8 Hopf Bifurcation

In dynamical systems of at least order 2 there is also another type of bifurcation: the *Hopf Bifurcation*. Such bifurcation occurs when a stable equilibrium state bifurcates into an unstable

equilibrium state and a stable limit cycle[28]. Hopf bifurcation therefore explains how a limit cycle can arise by continuous transformation of an equilibrium with the variation of a parameter. This phenomenon is very frequent, not only in electromechanical systems, but also in economic and biological ones. Hopf Bifurcation can be *direct* (or *supercritical*) or *reverse* (or *subcritical*).

In the direct case the equilibrium state X_e generates for $\mu = \mu_b$ a stable and attractive limit cycle for $\mu > \mu_b$ and when the parameter varies the equilibrium state becomes unstable.

In the reverse case, instead, an unstable limit cycle "surrounds" the equilibrium state X_e for $\mu < \mu_b$ and narrows on it as μ tends to μ_b. For $\mu = \mu_b$ the cycle collapses on X_e making it unstable.

A sufficient condition for this type of bifurcation is the existence, for the Jacobian matrix A of F calculated in an equilibrium state, of purely imaginary eigenvalues that "cross the imaginary axis with non-null velocity", as is indicated by the following theorem.

Theorem 10 (Hopf's theorem): Let $X_e = X_e(\mu)$ be an equilibrium solution asymptotically stable for $\mu < \mu_b$ and unstable for $\mu > \mu_b$; if at the point $\mu = \mu_b$ the Jacobian matrix A of F has one and only one pair of purely imaginary simple eigenvalues, that is:

$$Re\left(\lambda_{|\mu=\mu_b}\right) = 0 \qquad \text{and} \qquad Im\left(\lambda_{|\mu=\mu_b}\right) \neq 0$$

and furthermore

$$v = \left[\frac{d}{d\mu} Re(\lambda)\right]_{|\mu=\mu_b} \neq 0$$

then, $(X_e(\mu_b), \mu_b)$ is a bifurcation point and for μ sufficiently close to μ_b, there is a limit cycle. It is stable if $X_e(\mu_b)$ is locally asymptotically stable.

In any case, the initial period of the limit cycle is

$$T = \frac{2\pi}{Im\left(\lambda_{|\mu=\mu_b}\right)}$$

The radius r of the cycle generated by a Hopf bifurcation for $\mu \to \mu_b$ behaves as follows:

$$r \simeq a\sqrt{|\mu - \mu_b|}$$

where a is a constant.

[28] A limit cycle or cyclic attractor is a closed orbit in a plane (two-dimensional) phase space with the following property:

- no constant orbit (or fixed point) is contained in the limit cycle;
- any orbit beginning at a point in the limit cycle must always remain within the limit cycle;
- any trajectory starting in a limited neighbourhood of the limit cycle must approach asymptotically to the limit cycle;
- there is at least one orbit within it that approaches the limit cycle asymptotically.

If all the orbits contained in a limited neighbourhood of the limit cycle converge to it in the future, the limit cycle is called *stable* (or *attractive*); if they converge to it in the past, it is said to be *unstable*.

CHAPTER 3

Modelling

3.1 A Brief Introduction to Mathematical Models

Mathematical modelling is a relatively young discipline: it dates back to the early decades of the Twentieth century. While it has traditionally been fundamental in Physics, in recent decades it has also become important in other fields of Science, such as Biology and Medicine, becoming even more relevant in Economics and Finance. The reasons for this success are due to flexibility and universality of mathematical tools, which are sometimes the only ones that allow satisfactory answers to increasingly complex problems. This is also thanks to the computing power of increasingly sophisticated machines now available.

A mathematical model is the construction of an articular *reproduction* of *reality* in order to study it *mathematically*; it can be considered as the interface between a part of the real world and the world of mathematical theories. In order to explain this concept let's take an example: suppose we want to study the evolution of the price of a derivative instrument[29], in order to make also some forecasts. If we want to use a mathematical model, we must first establish the essential and characteristic factors in the evolution; for these variables we will look for the fundamental relationships that quantitatively describe the dynamics, obtaining one or more equations of various types that constitute the mathematical model. It must then be analyzed in order to obtain the information required.

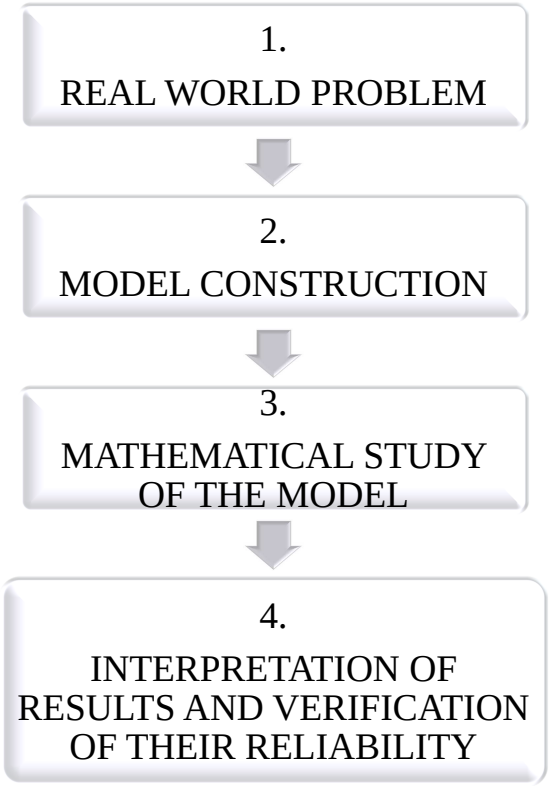
So, the definition of a mathematical model contains itself some essential elements:

- The initial and fundamental reference to *reality*, which can consist of a physical, biological, economic phenomenon, etc.; it may simply be a situation of daily life;
- The word *reproduction*: however, it must not be considered as a faithful duplication of reality, but as a simplified description of it. In fact, the reality is very complex and therefore rich in details, in order to be able to be fully studied; this is why we prefer to make a drawing of it, as if it were a "caricature", that simplifies it, highlighting only the central aspects;

[29] Derivatives are securities or financial contracts whose value depends on the market value of one or more assets called underlying. They are particularly popular in the so-called over-the-counter markets and their rapid development nowadays has been influenced by the possibility of on-line transactions. Derivatives include futures, options and swaps, while underlying assets commonly include commodities, shares or bonds.

- The adverb *mathematically*: it reminds us that the simplification of reality is fundamental in order to be able to translate it “in formula”, thus making possible its subsequent study with the language and tools of Mathematics.

The main phases in which the realization of a mathematical model can be synthetized in the following flow diagram:



1. *Analysis of the phenomenon*: at the origin of a model there is always a real phenomenon that the researcher wants to describe and of which he wants to foresee the evolution.
2. *Model construction*: taking into account the available mathematical tools, we identify the quantities and relationships that are considered essential, neglecting instead all those aspects that, considered secondary, are a source of trouble. Of course, the choices made for the same problem can be different depending on the scholar who elaborates the model.
3. *Mathematical study of the model*: in this phase the structure of the model is analyzed, clarifying its terms and specifying quantitatively and logically the relationships considered. This allows us to understand how the different aspects vary when we modify one of them, but also to preview new fields of applicability.
4. *Experimental verification*: this is the phase of empirical control of the mathematical model. To this end, simulations are carried out by introducing some sets of data and the results are

analyzed. This makes it possible to establish the quality of the model: if it proves too simplified or even inadequate, it is modified appropriately or even it is completely changed.

It follows from the above that the credibility of a mathematical model depends on its goodness and its ability to represent an acceptable compromise between completeness and analysability. In fact, completeness would require taking into account a very large number of components but also relationships between them that could be realistic; on the other hand, the greater the number of variables and the more realistic the relationships between them, the more complicated the model is: this puts at risk the capacity for analysis. A model can be considered good if it achieves an effective compromise between these two requirements, however, this evaluation requires back-testing, using already known cases. It is important to emphasise that a mathematical model cannot and should not claim to be universal, it should always be used "*cum grano salis*".

Some models, such as those in Physics, Chemistry, Medicine and Finance, allow quantitative analyses and predictions; in others, typically those in Economics, the analyses are necessarily qualitative and are only used to test the influence of certain factors on the evolution of a given system.

3.2 Prey-Predator Model

Among the most widely studied mathematical models are those of predator-prey type, in which one species is the main source of food for the other. Denoting prey with x and predators with y, we can construct a model of the type

$$\begin{cases} \dot{x} = xf(x,y) \\ \dot{y} = yg(x,y) \end{cases} \qquad (1)$$

where the growth rate of prey $f(x,y)$ must be a decreasing function in y while the growth rate of predators $g(x,y)$ must be an increasing function in x.

Among the prey predator models, the most famous is undoubtedly the Lotka and Volterra's one. Italian mathematician Vito Volterra was the one who made a major contribution to the development of mathematical models applied to the biological sciences in the 1920s, in particular by introducing the prey-predator cycle and the principle of competitive exclusion. He delved into the study of a strange phenomenon that had affected two species of fish in the upper Adriatic Sea during the First World War at the request of zoologist Ugo D'Ancona; the latter, in fact, had observed a consistent increase in selaci (predatory fish) and hypothesized that this was related to the decrease in fishing activity caused by naval war in the Adriatic. According to D'Ancona's hypothesis, fishing in peacetime disrupted the natural balance between species; the war, with the consequent suspension

of fishing, had restored a new natural balance in favour of predatory fish. Volterra approached the problem by initially assuming that predators and prey coexisted in the same habitat, without any external action (fishing) that would produce indiscriminate collection among the species. Volterra's model predicted a population growth according to the logistic equation influenced by the so-called "encounter theory", that is from the hypothesis that the effect of predation was proportional to the number of encounters between specimens of the two species.

Similar conclusions to Volterra's had been reached in the same years also by the American mathematician and physic-chemist Alfred J. Lotka, first employed at the U.S. Patent Office and the National Bureau of Standards, then university teacher and finally a statistical supervisor for an insurance company but interested all his life in biology. He wrote the first textbook on mathematical biology, in which he proposed that natural selection was basically a competition among organisms for access to the energy available in the environment: organisms that survive and thrive are those that are able to capture in greater quantities and use energy more efficiently. Extending these conceptions to human evolution, Lotka suggested that the historical transition from relying only on solar energy to societies based on non-renewable forms of energy posed fundamental challenges to social formations. In 1910 Lotka had proposed a predator-prey model in the study of autocatalytic chemical reactions, essentially obtaining a logistic evolutionary equation. In 1920 he extended this model to organic systems using as examples various species of plants and herbivorous animals, and in 1925 he used his equations in his mathematical biology textbook.

The population dynamics model developed independently by Volterra and Lotka allows to describe the interaction between two species in ideal conditions. The general idea is to consider a scenario in which two species coexist: predators and their prey. Assuming that the predators present at a certain instant (y) can feed only on the prey population present at the same instant (x), the total quantity of food consumed by predators (that is the quantity of preys eaten) per unit time is proportional to the number of encounters between prey and predators, which will be proportional to both populations and therefore to their product. There is also a minimum amount of food per unit time (m) needed by a predator to survive long enough to reproduce; it can therefore be assumed that the rate of growth of the predator population is proportional, in addition to the population already present, also to the gap between the food available and the food needed for subsistence. The growth rate of the prey population in the ideal situation where there are no predators and the resources provided by the environment are inexhaustible is proportional to the population already present according to the logistic model. This growth rate is slowed by the action of predators, which is proportional to the number of encounters between the two populations and thus the product of their instantaneous sizes.

This results the two autonomous and non-linear ordinary differential equations in $\mathbb{R}^2$ that describe the dynamics of the system:

$$\begin{cases} \dot{x} = \alpha x - \beta xy \\ \dot{y} = -\gamma y + \delta xy \end{cases} \qquad (2)$$

where α, β, γ and δ are constant parameters (defined as positive) and are a measure:

- α of the biological potential of prey;
- β of the predation rate;
- γ, δ of the biological potential of predators.

The interpretation is simple: in a closed environment live two species x and y, prey and predator respectively. In the absence of predators, prey would have Malthusian growth $x(t) = x_0 e^{\alpha t}$; similarly, predators would become extinct by law $y(t) = ye^{-\gamma t}$. The encounter of a prey with a predator, which has a frequency proportional to the product xy, produces a decrease in the number of prey and an increase in the number of predators, with rate (effectiveness of the encounter in making a new individual appear or disappear) governed by the constants β and δ.

It is immediately seen that $\dot{x}$ becomes null on the lines $x = 0$ and $y = \frac{\alpha}{\beta}$ and similarly $\dot{y}$ becomes null on the lines $y = 0$ and $x = \frac{\gamma}{\delta}$. There are therefore two equilibrium points, the origin and the nontrivial point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$. The first of them clearly corresponds to the impossibility of spontaneous generation and is unstable (saddle point). This instability is important: otherwise, a non-zero population might be attracted to this fixed point, and the dynamics of the system would therefore always lead to the extinction of both species for many initial conditions. Instability in the origin thus ensures that an evolution toward extinction of both species is disadvantaged in this type of model. In fact, it could occur only if an external event led to the complete elimination of the preys. If, on the other hand, predators were eliminated, the prey population would return to an exponentially. The second fixed point corresponds instead to a situation in which predators devour only the minimum amount of prey necessary for their survival while the preys reproduce in such a way as to simply compensate the losses. This fixed point is not of the hyperbolic[30] and therefore no general conclusion can be drawn about its stability. Outside the equilibrium points there are particular solutions on the axes $y = 0$ (Malthusian growth of x) and $x = 0$ (extinction of y).

[30] A hyperbolic equilibrium point is an equilibrium point of a continuous dynamical system such that the matrix of the linearized system has no eigenvalues with zero real part (that is no zero Lyapounov exponents). In $\mathbb{R}^2$ there are only three types of hyperbolic equilibrium points: wells, saddles and sources, depending on whether the number of negative Lyapounov exponents is 2, 1 or 0. It is also shown that a hyperbolic equilibrium point is either a well or unstable.

For a generic initial datum with x and y both positive, a somewhat exceptional fact occurs: in the system there exists a constant of motion, that is, a first integral of the system of equations that describes it; precisely the quantity

$$F(x, y) = \delta x - \gamma \log x + \beta y - \alpha \log y \qquad (3)$$

We can easily verify that $\dot{F} = 0$. The orbits thus coincide with the level curves of F in the xy plane. From Lyapounov's first theorem applied to the constant of motion F around the equilibrium point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$, we can see that the equilibrium is stable, and therefore small perturbations give small motions. In addition, the orbits are periodic. Prey and predator populations each have ups and downs that alternate periodically with the passage of time. Everything happens endogenously, that is, without any intervention from outside. In other words, the system sways, producing what are called, in a very explanatory term, *self-sustained oscillations*: if there are many preys, predators increase, which then causes them to decrease; but then the predators themselves without food decrease, at which point prey grows again.

In this model, then, in the situations of non-equilibrium we have that predators thrive when there is an abundance of preys but, in the long run, they find themselves without enough food for everyone and begin to become extinct. As the predator population decreases the prey population increases again. This dynamic continues in an unlimited cycle of increases and decreases. If, on the other hand, one or more parameters are assumed to vary over time, the dynamics of the system changes in a radical way and exhibits characteristics of strong dependence on initial conditions.

3.3 Mathematical Models of Economic Growth

Lotka and Volterra's prey-predator model set up the basic “paradigm” with which to translate the phenomenology of populations into a mathematical object, thus applying modelling to a discipline, such as Biology, that traditionally saw Mathematics as only the statistical methods for handling experimental data. There is no doubt that the mathematical study of population problems has important precedents:, the first deterministic models are in fact those of Malthus and Verhulst, who both dealt with population dynamics. More specifically, the English demographer Thomas Malthus, in his work «Essay on the Principles of Population» of 1798, studied for the first time the evolution of the English population using mathematical methods, so constructing a mathematical model of exponential growth of the population. This model, however, was based on assumptions far from reality, so much so that it was corrected by Verhulst with the logistic growth model, obtained assuming that the environment has a carrying capacity beyond which it can no longer provide

nourishment and resources, thus leading to asymptotic population growth, as a result of which the number of the same tends toward equilibrium.

Among the disciplines that use modelling of phenomena for a better understanding of them, as well as to be able to make predictions about their future development, a prominent place is occupied by Economics, especially in the field of economic growth. In a nutshell, economic growth is concerned with the expansion of a country's income and its underlying factors, including the availability and dynamics of work, capital and natural resources, the choices of operators (companies and families) between consumption, savings and investment, technical and technological progress, as well as a very wide range of other elements from the functioning of markets to the institutional context. These problems were first posed and analyzed extensively by classical economists and in particular by Adam Smith (1776) and David Ricardo (1817-1823). These authors were followed by many other economists interested in the phenomenon of growth, which was certainly central until 1870. However, none of these economists used mathematical methods of analysis, although in contemporary times mathematical formulations of their theories have been proposed.
The reasons for economic growth began to fascinate scholars after the great crisis that involved the world economy in the late 1930s, cracking the thesis that the market was capable by itself of feeding a continuous development. The focus on growth arises, therefore, from Keynesian theory to which we owe the first demonstration of the possibility that an economic system can evolve at a rate lower than that necessary to ensure full employment of resources. According to many scholars, Keynes' main contribution concerns effective demand theory. Neoclassical theory holds that the flexibility of the interest rate in the financial market ensures that on average – i.e., taking into account cyclical phenomena of a transitory nature, such as crises of confidence, errors of monetary authorities – the level of investment adjusts to the supply of full-employment savings. In effective demand theory, on the other hand, it is the level of savings that adjusts, via income multiplier, to the level of investment decided by entrepreneurs. There is no reason why the level of investment decided by entrepreneurs should be that corresponding to full employment, so that the multiplier (or actual demand) analysis indicates the possibility of underemployment balances. Keynesian analysis is generally considered a short-term analysis because the equilibria determined in it are on the basis of the given productive capacity assumption. This raises the question of whether its conclusions can be extended to the long run.

The use of mathematical methods in the analysis of economic growth also takes place in the 1930s by Von Neumann (1937) and Harrod (1939) whose contributions are, respectively, the basis of multisector and macroeconomic models of growth. The initial contributions were followed by

increasingly more articulated and complex models, in which the mathematical method assumed a significant role. It should be remembered that economic theory is based on three components: the analytical-formal component, which makes wide use of mathematical methods; the historical-quantitative component, which uses institutional and historical elements; and the economic-quantitative component, which uses statistical data by means of econometric methods; for this reason there are many lines of analysis of growth phenomena that don't use mathematical methods, but which have made a very important contribution to the analysis of growth and dynamics.

3.3.1 Multisector Dynamical Models - J. Von Neumann's Model

The first author to deal with a dynamic multisector model was mathematician John von Neumann who, in 1937, published in German his famous work on regular economic growth; however, it had to take another ten years for economists to recognize the importance of his model. Neumann considers an economy in which n different goods are produced by means of m linear and additive production processes, which use the same produced goods as factors of production (so-called circular production); the linearity of production processes implies that returns to scale are everywhere constant. We observe that in the representation given by von Neumann there can be production processes with multiple productions and both fixed and circulating capital. Let's denote with a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ the nonnegative quantity of the i-th good employed by the j-th technology activated at the unit scale, and with bb_{ij} the symbol concerning the corresponding quantity produced (also nonnegative). These quantities are considered stationary and exogenously assigned, that is, they constitute the model data. Synthetically, these technological data can be represented by means of a pair of matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}; \qquad B = \begin{pmatrix} b_{11} & b & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix}$$

Subdivided time into periods of equal duration, and denoted by t the generic time period, let $x(t) = (x_1(t), \ldots, x_m(t))$ be the vector, non-negative, whose component $x_j(t)$ indicates the scale of activation of the j-th production process during the t-th period. The production of goods requires a certain amount of time to elapse between the use of the productive factors and the obtaining of the products, and precisely it must pass a whole period in order to obtain the products from the factors. In absence of a final demand, the goods produced at the end of the generic period t become the factors of production available to start a new production cycle in the period $t + 1$. Therefore the quantities of goods must satisfy the inequalities:

$$\sum_{j=1}^{m} a_{ij}x_j(t+1) \leq \sum_{j=1}^{m} b_{ij}x_j(t) \qquad (i = 1, 2, \ldots, n)$$

namely

$$Ax(t+1) \leq Bx(t) \quad (t = \cdots, -1, 0, 1, 2, \ldots.) \qquad (4)$$

Introduced the (non-negative) prices of goods in the period t, represented by the vector $p(t) = (p_1(t),\ p_2(t), \ldots, p_n(t)\ )$ von Neumann, following the theorists of the general equilibrium, writes that if in the period t the good i-th is used in smaller quantities than available then it must have in equilibrium price zero, i.e. $p_i(t) = 0$. Therefore synthetically we can write

$$p(t+1)Ax(t+1) = p(t+1)Bx(t) \qquad (5)$$

Considering now the productive processes, these must not allow extra profits; this means that in every productive process the unitary cost cannot be less than the unitary revenue; therefore the following inequalities must apply

$$\sum_{i=1}^{n} p_i(t)a_{ij} \geq \sum_{i=1}^{n} p_i(t+1)b_{ij} \qquad (j = 1, 2, \ldots, m)$$

that is

$$p(t)A \geq p(t+1)B \qquad (6)$$

If, however, for the productive process j-th the unitary cost exceeds the unitary revenue then that process must not be activated, that is it must be put $x_j(t) = 0$. Therefore we have

$$p(t)Ax(t) = p(t+1)Bx(t) \qquad (7)$$

The relations $(4) - (7)$ represent von Neumann's model of multisector growth; the author limited his analysis to the consideration of particular solutions of the following type

$$x(t+1) = \alpha x(t); \qquad p(t+1) = \frac{1}{\beta}p(t) \qquad (8)$$

where αa and β are two positive numbers, the first representing the growth factor (common to all goods) of the quantities of goods, while the second can be interpreted as a factor of interest. Given x the vector of activation scales in the first period, and p the vector of prices in the first period, the model can be synthetically written substituting as follows:

$$\alpha Ax \leq Bx$$
$$\alpha pAx = pBx$$
$$\beta pA \geq pB$$
$$\beta pAx = pBx$$

The first relation states that if a good is not fully utilised, an equilibrium price equal to zero follows in the second relation. The third relation states that the maximum profit factor is β so that, from the fourth relation, if a process achieves a profit factor of less than β, then that process (whose costs,

multiplied by β, exceed revenues) is not activated. The data of the model are the technology matrices A and B, while the unknowns are x, p, α, β. The existence of non-negative solutions obviously depends on the assumptions about A and B.

Von Neumann assumed that $a_{ij} + b_{ij} > 0$ for each pair of indices i, j (economically opaque assumption); this assumption was later modified with the following economically more significant and still universally used assumption: at least one factor is required in every productive process; each good is produced by at least one technology that is, mathematically

$$sA > 0; \qquad Bs > 0$$

where s represents the sum vector $s = (1, 1, \ldots, 1)$. Von Neumann proved the existence of a solution by employing his own generalisation of Brouwer's fixed point theorem[31]. Thanks to improvements to the model, it is possible to prove that there exists a positive number, $\hat{\alpha}$, to which is associated (at least) an $\hat{x} > O$, solutions to the problem.

$$\hat{\alpha} = \sup\{\alpha \mid \alpha Ax \leq Bx, x \in \Re_+^m\} \qquad (9)$$

It is also shown that in equilibrium we obtain $\hat{\alpha} = \hat{\beta}$, that is. growth factor and interest factor are equal to each other.

The orbits defined by the relationships (8). are only one among the possible types of orbits, but they have a special economic significance in that, from their characterisation in (9), they are the path along which the economy considered grows as rapidly as possible.

3.3.2 Macroeconomic models – Harrod-Domar and Solow-Swan models

The Harrod-Domar model is considered the prototype of aggregate growth models. In the most traditional formulation, it is a continuous-time model for an isolated economy, represented by the relationships between the following aggregate variables, to be interpreted as instantaneous intensities:

- ✓ Y denotes the aggregate output;
- ✓ K denotes the aggregate capital (expressed in the same units as output);
- ✓ L represents the amount of available labour;
- ✓ S denotes global savings;
- ✓ I represents the investment (homogeneous with output).

[31] In algebraic topology, Brouwer's fixed point theorem is part of the large family of fixed point theorems. It states that all compact and connected subsets X^n of a Euclidean space enjoy the so-called fixed point property. More precisely we have:

Theorem: Let $f: X^n \to X^n$ be a continuous function; then there exists a point x_0 of X^n such that $f(x_0) = x_0$

A first relationship expresses the instantaneous rate of production as a function of the quantity of capital and labour:

$$Y(t) = \min\{vK(t), uL(t)\} \qquad (10)$$

This means that output is obtained from capital and labour, according to fixed coefficients v, u.

Labour grows exponentially at the exogenous positive rate g:

$$L(t) = L_0 e^{gt} \qquad L_0 > 0 \qquad (11)$$

where L_0 represents the initially available labour.

Savings is a given fraction s, with $0 < s < 1$, of income, that is:

$$S(t) = sY(t) \qquad (12)$$

In equilibrium, saving equals investment

$$S(t) = I(t) \qquad (13)$$

The rate of capital growth coincides with the investment

$$\dot{K}(t) = \mathrm{I}(t) \qquad (14)$$

In the case of $\min\{vK(t), uL(t)\} = vK(t)$, from the previous equations we derive

$$\dot{K}(t) = \mathrm{I}(t) = S(t) = sY(t) = svK(t)$$

the first and last members form a simple differential equation whose general solution is:

$$K(t) = K_0 e^{svt} \qquad (15)$$

where K_0 represents the given initial capital. To obtain continuous full employment of both factors of production (capital and labour) it must be $vK(t) = uL(t)$ for each t, i.e.

$$vK_0 e^{svt} = uL_0 e^{gt} \qquad (16)$$

for each $t \geq 0$. From (16) we obtain first that $vK_0 = uL_0$, that is. capital and labour must initially be available in the ratio $\frac{u}{v}$ determined by the coefficients of production; moreover it must be verified, instant after instant, the equality

$$sv = g \qquad (17)$$

The previous relationship establishes the famous condition, also known as the "*razor's edge*" condition, according to which the exogenous quantities s, v, g must satisfy the (17) in order to have continuous full employment of both factors. Otherwise, one or the other of the two factors remains partially unemployed. At full employment, we also derive from relations (16) and (10) that

$$Y(t) = vK_0 e^{svt} \qquad (18)$$

Since the parameters $s, v\ g$ are all positive, clearly all quantities in the model grow over time at the instantaneous rate $sv > 0$. It is therefore observed that the higher the rate of growth of the economy the higher the saving rate s and the capital coefficient v. No influence is instead directly exerted by the labour coefficient u. It therefore appears that the economy's growth rate is favoured by the savings rate and technical progress, which is likely to increase the coefficient v to the extent that it

leads, ceteris paribus, to an increase in capital relative to labour. Of course, when (17) doesn't apply, then labour is partially unemployed, if it is $sv < g$, while in the opposite situation it is capital that is partially unemployed.

Harrod-Domar's growth model is followed by other strands of analysis, among which the neoclassical Solow-Swan model is certainly worth analysing; it focuses on the role of productive and technological factors of growth. The starting point of this model is the specification of a production function in which aggregate output (Y) depends on the amount of capital (K), employment (N) and time. Formally we have

$$Y = F(K, N, t) \qquad (19)$$

Output is divided between consumption and changes in the amount of capital (i.e. investment), so that we have

$$Y = Nc + \dot{K} \qquad (20)$$

where c is per capita consumption (and therefore Nc represents aggregate consumption) and $\dot{K} = \frac{dK}{dt}$. It is important to note that in this literature it is almost always assumed that employment is equal (or proportional) to the size of the labour force, so that all "coordination problems" disappear at the outset.

In the original model of Solow with technical progress labour augmenling, a closed economic system is described, endowed with competitive markets and formed by rational (and mutually identical) individuals in which technology is formalized by a production function characterised by decreasing substitution returns of labour and capital separately, and constant scale returns of the two factors. As far as total consumption, there are basically two versions. The first is the so-called "*behaviourist*" version of aggregate growth theory, which consists in thinking about a possible specification of the per capita consumption (or savings) function that is empirically plausible. The existence of a function of the type

$$c = c(K, N, t) \qquad (21)$$

so that the analysis of the model is reduced to the study of the differential equation

$$\dot{K} = F(K, N, t) - Nc(K, N, t) \qquad (22)$$

The second is the so-called "*optimizing*" version, in which it is assumed that the economy is composed of a single immortal individual who chooses the optimal consumption path in order to maximise the discounted and weighted sum (or rather the integral, since the formalisation is in continuous time) of instantaneous utilities (where the weights represent the size of the population).

The consumer therefore chooses its consumption over time in order to maximize the following integral

$$\int_0^{+\infty} e^{-\varrho t} u\big(c(t)\big)\, N(t) dt \qquad (23)$$

where $\varrho > 0$ measure the rate at which instantaneous utilities are discounted. The constraint that technology imposes on inter-temporal utility maximisation is given by

$$Nc + \dot{K} = F(K, Nt) \qquad (24)$$

the standard assumption made regarding the production function is that it can be written as

$$Y(t) = F\big(K, A(t)N(t)\big) \qquad (25)$$

with the further assumptions, after having c properly hosen the units

$$A(t) = e^{\mu t} \;\; \mu > 0 \qquad (26)$$

and

$$N(t) = e^{\lambda t} \;\; \lambda > 0 \qquad (27)$$

As regards the utility function is concerned, it has the form

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \quad \sigma \neq 1 \qquad (28)$$

where we have $\sigma > 0$, and $\frac{1}{\sigma}$ is the intertemporal substitution elasticity. Taking into account the hypotheses introduced, the integral to be maximised then becomes

$$\int_0^{+\infty} e^{-\varrho t} \left(\frac{c^{1-\sigma} - 1}{1 - \sigma}\right) N(t) dt \qquad (29)$$

subject to the constraint

$$N(t)c(t) + \dot{K}(t) = F\big(K(t), A(t)N(t)\big) \qquad (30)$$

In order to determine the solution to the problem, the assumption usually introduced is that the production function is of the Cobb-Douglas type

$$F(K, AN) = K^{\alpha}(AN)^{1-\alpha} \;\; 0 < \alpha < 1 \qquad (31)$$

In the standard model of aggregate theory, the growth rates of employment, population and technical progress are exogenous to the model, and no particular role is assigned to human capital or government policy. In these hypotheses the system converges to a uniform growth equilibrium (*steady state*) in which capital per capita is accumulated at the same rate as income (and consumption) per capita, and in which the savings rate and the real interest rate are constant along the equilibrium path. Moreover, the growth rate of income (and consumption) per capita are all equal to the (exogenous) rate of technical progress. Therefore technology, although exogenously

given, is the only force that produces growth in per capita income (by increasing labour efficiency), and thus is the engine of growth.

One of the empirical predictions of the model is that there is substantial equality in the development rates of different countries. These rates can certainly differ from each other, but such differences can only be of a transitory nature: in fact, since for each country the per capita growth rate is inversely dependent on initial income, different rates should cause convergence of levels and thus poor countries should tend to grow faster than rich ones. The aggregate growth model (with exogenous technical progress) therefore predicts the existence of a force that promotes convergence in the per capita incomes of different countries. However, the assumption that poor countries tend to grow faster than rich ones seems to be inconsistent with the "*cross-country*" evidence. In recent years, a large number of studies have been carried on the international comparison of growth rates. Contrary to the predictions of aggregate growth models, empirical evidence indicates that there is a large difference in both the levels and rates of economic growth of different countries.

The empirical evidence therefore allows us to reasonably conclude that development rates certainly differ substantially from country to country and they are not systematically correlated with initial income levels (regressions of the development rate of a given period on initial income alone usually yield non-significant results).

Endogenous growth models emerged precisely as a reaction to the unsatisfactory contribution of aggregate growth models to the interpretation of the differences between the development rates of different countries. Endogenous growth theory differs from aggregate growth theory in its emphasis on economic growth as an endogenous product of the economic system, and not as the result of forces acting from outside. These models then try to identify in capital (or other factors that can be reproduced) an engine of growth that cannot be exhausted. The merit of endogenous growth models is therefore that they succeed in showing how technical progress (resulting from the decisions of agents) allows to frame the evolution of the economic system together with the growth of the population.

SECTION B
(Works)

PART I

Stability and Hopf Bifurcation Analysis of a Distributed Time Delay Energy Model for Sustainable Economic Growth

This paper examines the consequences of including distributed delays in an energy model. The stability behaviour of the resulting equilibrium for our dynamic system is analyzed, including models with Dirac, weak and strong kernels. Applying the Hopf Bifurcation theorem we determine conditions under which limit cycle motion is born in such models. The results indicate that distributed delays have an ambivalent impact on the dynamical behaviour of systems, either stabilizing or destabilizing them.

1. Introduction

Domar (1946) and Solow (1956) introduced the systematic study of the growth process, where the accumulation of physical capital is seen as a key growth driver. This research unveiled key structural characteristics that affect the long-term determination of labor productivity: savings, population growth, technological change. In this regard, different augmented versions of the Solow model were developed over the decades that include among others the contribution of Mankiw *et al.* (1992) employing a physical and an intangible capital (human capital), and the Solow-Jorgenson-Griliches residual model (Jorgenson and Griliches 1967), framework used to address issues concerning intangibles such as the contribution of intangible capital to output growth and how does the inclusion of intangibles affect the allocation of output growth between capital formation and multifactor productivity growth (Corrado *et al.* 2009).

On the other hand, the basic neoclassical growth concept abstracts from a potentially important feature of the growth process: purchasing a piece of machinery at a specific time and place makes little sense unless the equipment can be supplied with energy and put to use. Indeed, it is of primary importance for the economy to be able to distribute electricity across the economy.

Natural science experiments by Banavar *et al.* (1999, 2002) and West *et al.* (1997, 1999) developed the energy distribution network as a living organism to validate the empirical results of biology recognized as. Kleiber's Law. Dalgaard and Strulik (2011) introduced this theory and were the first to integrate something as complicated as the electricity distribution network into a macroeconomic

framework. They claimed that the application of biological organism-related concepts on man-made energy networks is a true strategy for the following three reasons: the cardiovascular system functions in the same manner as the electricity network, facilitating the transfer of nutrients and power around the network; man-made and biological networks have different added properties due to the development process over time, as processes refine their natural selection in the case of a biological network as well as constant rework and updating of human-made networks; physicists use empirical methods from biological organisms to look for universal laws of scaling that affect human society. The relentless pressure to move towards an optimum distribution system allows for certain characteristics to be shared between biological and man-made networks. If the consumption per capita of electricity is the equivalent of metabolism and capital per capita is equal to body mass, it is assumed that the relationship between capital and electricity is concave and log linear. Dalgaard and Strulik (2011) demonstrated that energy is accessible at geographically dispersed locations through an economy's self-organization. Further, they investigated the relationship between capital per capita and energy per capita from the energy demand viewpoint and claimed that energy demand is driven by the need for capital management, maintenance and generation.

Bianca *et al.* (2013) modified the model by Dalgaard and Strulik (2011) with the assumption that the energy conservation formula would be influenced by a time delay, thereby characterizing the dynamics of the system by the following delay differential equation,

$$\dot{k}(t) = \frac{\varepsilon}{v}[k(t-\tau)]^{\alpha} - \frac{\mu}{v}k(t-\tau) \qquad (1)$$

where k denotes per capita capital, μ and v are the energy required to operate and maintain the generic capital good, and the energy costs to create a new capital good, respectively, $\alpha \in (0,1)$ and $\varepsilon > 0$ are real constants, and $\tau \geq 0$ represents a time delay. For $\tau = 0$, Eq. (1) reduces to a law of motion for capital which is structurally identical to that implied by the Solow model (Solow 1956). Since it shares its technical properties, there exists a unique stable steady-state k_* where $k_*^{\alpha-1} = \mu/\varepsilon$, to which the economy adjusts. For $\tau > 0$, by choosing time delay as a bifurcation parameter, Bianca *et al.* (2013) proved that the model loses stability and a Hopf bifurcation occurs when time delay passes through critical values. It is known that time delays in economic situations can be modeled in two different ways: discrete time delays, ideal when a fixed time delay for the agents involved is institutionally or socially defined, and continuously distributed time delays, appropriate when the delay is uncertain or varying periods of delays are spread across the agents.

In this paper, following Guerrini *et al.* (2019), we generalize the delay differential equation model with a discrete delay, namely (1), adopting continuously distributed time delays. Accordingly, the model may be written as follows

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t} g\,(t-r,S,m)k(r)dr\right]^{\alpha} - \frac{\mu}{v}\int_{-\infty}^{t} g(t-r,T,n)\,k(r)dr. \qquad (2)$$

The function g in (2) is a non-negative bounded function defined on $[0,+\infty)$ which reflects the influence of the past states on the current dynamics and it is called the delay kernel. Here, S, T are positive parameters associated with the average length of the continuous delay and $m,n \in \{0,1\}$ determine the shape of the weighting function. In line with Cushing (1977), we consider the following types of gamma distribution for g,

$$g(t-r,\zeta,0) = \left(\frac{1}{\zeta}\right)e^{-\frac{1}{\zeta}(t-r)} \qquad \text{and} \qquad g(t-r,\zeta,1) = \left(\frac{2}{\zeta}\right)^{2}(t-r)e^{-\frac{2}{\zeta}(t-r)},$$

where $\zeta = S,T$, which are named *weak delay kernel* and *strong delay kernel*, respectively. In the former case, weights are exponentially declining with the most weight being given to the most recent output; in the latter one, zero weight is assigned to the most recent output, rising to maximum weight at a point ζ time units in the past and declining exponentially to zero thereafter. Notice that as $\zeta \to 0$, the function g tends to a Dirac delta function, i.e. $\delta(t-r)$, so that one recovers the discrete delay case (1) (with $\tau = T$). Therefore, Eq. (2), that we are interested in, is more general than Eq. (1). Since time delay does not change the equilibria of the equation, Eq. (2) has exactly the same equilibrium point of the standard Solow model ($\tau = 0$) since time delay does not change the equilibria of the equation. An analysis of the model (2), using a combination of the previous expressions for g as well Dirac kernel, will be done via the so-called linear chain trick technique (MacDonald 1978), which transforms the integrodifferential system (2) into an equivalent system of ordinary or delay differential equations. In this context, the principal role of delays is in destabilizing an otherwise stable economy and depending upon a combination of two delays, the delay can also stabilize the economy. Our results will stress the importance of the theoretical modelling framework used as a device that may dramatically change the findings of the model in (1). In this context, the principal role of delay is in destabilizing an otherwise stable economy and the delay can, depending on the combination of two delays, also stabilize the economy. For future research, we propose to extend our model in a computational and experimental way and generalize it to include the intangible asset of human capital.

2. Weak Kernels

Let, $m = 0$ and $n = 0$. Eq. (2) rewrites as

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)\,dr\right]^{\alpha} - \frac{\mu}{v}\int_{-\infty}^{t}e^{-\frac{1}{T}(t-r)}k(r)\,dr. \qquad (3)$$

For convenience, we define the new variables $x(t)$ and $y(t)$ by

$$x(t) = \int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)\,dr, \qquad y(t) = \int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)\,dr.$$

Applying the linear chain trick technique, Eq. (3) can be transformed into the following third-dimensional system of ordinary differential equations

$$\begin{cases}\dot{k}(t) = \dfrac{\varepsilon}{v}[x(t)]^{\alpha},\\ \dot{x}(t) = \dfrac{1}{S}[k(t) - x(t)]\,,\\ \dot{y}(t) = \dfrac{1}{T}[k(t) - y(t)].\end{cases} \qquad (4)$$

The local stability of the unique positive equilibrium point (k_*, x_*, y_*) of system (4), where $x_* = y_* = k_*$ and k_* is the steady-state of (1), is governed by the roots of the corresponding characteristic equation for system (4). Linearizing this system at its equilibrium point we obtain the characteristic equation

$$\lambda^3 + b_1(S,T)\lambda^2 + b_2(S,T)\lambda + b_3(S,T) = 0, \qquad (5)$$

where

$$b_1(S,T) = \frac{S+T}{ST} > 0, \qquad b_2(S,T) = \frac{v + \mu(S - \alpha T)}{ST}, \qquad b_3(S,T) = \frac{(1-\alpha)\mu}{vST} > 0.$$

Lemma 1. *The equilibrium point k_* of (3) is locally asymptotically stable for $0 \le T < T_*$ and unstable for $T > T_*$, where*

$$T_* = \frac{v + \sqrt{v^2 + 4\alpha\mu S(v + \mu S)}}{2\alpha\mu}.$$

Proof. By the Routh-Hurwitz criteria, the equilibrium point is locally asymptotically stable if and only if $b_1(S,T) > 0, b_3(S,T) > 0$ and $b_1(S,T)b_2(S,T) > b_3(S,T)$. Thus, the stability condition is

confirmed if $b_1(S,T)b_2(S,T) > b_3(S,T)$. A direct calculation yields $\alpha\mu\, T^2 - vT - S(v+\mu S) < 0$. The statement follows solving this inequality. ■

We now return to the characteristic equation (5) and show the possibility of the birth of a limit cycle at $T = T_*$ by applying the Hopf bifurcation theorem. According to this theorem, we can establish the existence of a cyclic solution if the cubic characteristic equation has a pair of pure imaginary roots and the real parts of these roots change signs with a bifurcation parameter. At the critical value T_* one has $b_1(S,T_*)b_2(S,T_*) = b_3(S,T_*)$, and the characteristic equation can be rewritten as

$$[\lambda + b_1(S,T_*)][\lambda^2 + b_2(S,T_*)] = 0,$$

which has roots

$$\lambda_1 = -b_1(S,T_*) < 0, \qquad \lambda_{2,3} = \pm i\omega_{*,} \quad \text{with} \quad \omega_* = \sqrt{b_2(S,T_*)}.$$

Recalling that the equilibrium is locally asymptotically stable in absence of delays, if the transversality condition

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} > 0$$

holds, then a Hopf bifurcation occurs at the equilibrium point when T passes through the critical value T_*. Differentiating Eq. (5) with respect to T, and using (5), we have

$$\frac{d\lambda}{dT} = \frac{-b_1'(S,T)\lambda^2 - b_2'(S,T)\lambda}{3\lambda^2 + 2b_1(S,T)\lambda + b_2(S,T)},$$

where

$$b_1'(S,T) = -\frac{v+\mu S}{ST^2}, \qquad b_2'(S,T) = -\frac{(1-\alpha)\mu}{vST^2}.$$

Since $\omega_*^2 = b_2(S,T_*)$ and $b_1(S,T_*)b_2(S,T_*) = b_3(S,T_*)$, after some calculations, we get

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} = -\frac{b_1'(S,T_*)b_2(S,T_*) + b_1(S,T_*)b_2'(S,T_*)}{2[b_2(S,T_*) + b_1^2(S,T_*)]}.$$

Since the numerator in the above expression is equal to $S[\alpha\mu T_*^2 + S(v + \mu S)]/v > 0$, we conclude that the crossing direction of characteristic root through the imaginary axis is from right to left as T increases. Summarizing the above analysis, we have the following result.

Theorem 1. *Eq.* (3) *undergoes a Hopf bifurcation at its equilibrium point* k_* *when* $T = T_*$.

3. Weak and Strong Kernels

Let, $m = 0$ and $n = 1$. Eq. (2) becomes

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)\,dr\right]^{\alpha} - \frac{\mu}{v}\int_{-\infty}^{t}\left(\frac{2}{T}\right)^{2}(t-r)e^{-\frac{2}{T}(t-r)}k(r)\,dr. \qquad (6)$$

Defining the new variables $x(t)$, $y(t)$ and $z(t)$ by

$$x(t) = \int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)\,dr, \qquad y(t) = \int_{-\infty}^{t}\left(\frac{2}{T}\right)^{2}(t-r)e^{-\frac{2}{T}(t-r)}k(r)\,dr,$$

and

$$z(t) = \int_{-\infty}^{t}\left(\frac{2}{T}\right)e^{-\frac{2}{T}(t-r)}k(r)\,dr,$$

Eq. (6) is turned into the following fourth-dimensional system of ordinary differential equations

$$\begin{cases}\dot{k}(t) = \dfrac{\varepsilon}{v}[x(t)]^{\alpha} - \dfrac{\mu}{v}y(t),\\ \dot{x}(t) = \dfrac{1}{S}[k(t) - x(t)],\\ \dot{y}(t) = \dfrac{2}{T}[z(t) - y(t)],\\ \dot{z}(t) = \dfrac{2}{T}[k(t) - z(t)].\end{cases} \qquad (7)$$

The characteristic equation for system (7) at the equilibrium point (k_*, x_*, y_*, z_*), where $x_* = y_* = z_* = k_*$, takes the form

$$\lambda^4 + c_1(S,T)\lambda^3 + c_2(S,T)\lambda^2 + c_3(S,T)\lambda + c_4(S,T) = 0, \qquad (8)$$

where

$$c_1(S,T) = \frac{4S + T}{ST} > 0, \qquad c_2(S,T) = \frac{4v(S+T) - \alpha\mu T^2}{vST^2},$$

$$c_3(S,T) = \frac{4v + 4\mu\,(S - \alpha T)}{vST^2}, \qquad c_4(S,T) = \frac{4(1-\alpha)}{vST^2} > 0.$$

Now, it is necessary to investigate the distribution of roots of Eq. (8) in order to determine the stability of the equilibrium.

Lemma 2. *The equilibrium point k_* of* (6) *is locally asymptotically stable for $\alpha < (v + S\mu)/(\mu T)$ and*

$$\varphi(T) = c_1(S,T)c_2(S,T)c_3(S,T) - c_3^2(S,T) - c_1^2(S,T)c_4(S,T) > 0. \qquad (9)$$

Proof. Using the Routh-Hurwitz criteria, all roots of the polynomial in (8) are negative or have negative real parts if and only if the following conditions hold:

$c_1(S,T) > 0, c_3(S,T) > 0, c_4(S,T) > 0$ and $c_1(S,T)c_2(S,T)c_3(S,T) > c_3^2(S,T) + c_1^2(S,T)c_4(S,T)$, yielding the statement. ■

Remark 1. *Condition* (9) *is equivalent to*

$$(\alpha^2\mu^2)T^4 - (v\mu + 4\alpha v\mu + S\alpha\mu^2)T^3 + 4(S^2\alpha\mu^2 - Sv\mu - 2S\alpha v\mu + v^2)T^2 + 4S(-S^2\mu^2 - Sv\mu + 4v^2)T + 16vS^2(\mu S + v) > 0.$$

Next, we select T as the bifurcation parameter and show possibility of the birth of limit cycles when $T = T_*$. Let $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be the roots of the characteristic equation (8). Then, we have

$$\lambda_1+\lambda_2 + \lambda_3 + \lambda_4 = c_1(S,T), \qquad \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = c_2(S,T) \quad (10)$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = -c_3(S,T), \qquad \lambda_1\lambda_2\lambda_3\lambda_4 = c_4(S,T). \qquad (11)$$

If there is $T = T_*$ such that $\varphi(T_*) = 0$, then by the Routh-Hurwitz criterion at least one root, say λ_1, has real part equal to zero. From the second equation of (11) it follows that $Im\lambda_1 = \omega_1 \neq 0$, so that there is another root, say λ_2, such that $\lambda_2 = \bar{\lambda}_1$. Since $\varphi(T)$ is a continuous function of its roots, λ_1 and λ_2 are complex conjugates in an open interval including T_*. A a result, the equations in (10) and (11) have the following form at $T = T_*$,

$$\lambda_3 + \lambda_4 = -c_1(S,T_*), \qquad \omega_1^2 + \lambda_3\lambda_4 = c_2(S,T_*), \qquad (12)$$

$$\omega_1^2(\lambda_3 + \lambda_4) = -c_3(S,T_*), \qquad \omega_1^2\lambda_3\lambda_4 = c_4(S,T_*) \quad (13)$$

If λ_3 and λ_4 are complex conjugates, from the first equation of (12) we derive that

$2Re\lambda_3 = -c_1(S,T_*) < 0$. If λ_3 and λ_4 are real, from the first and the second equation of (12) and (13), respectively, we obtain that $\lambda_3 < 0$ and $\lambda_4 < 0$. According to the Hopf Bifurcation Theorem, it remains to verify the transversality condition. Finding the derivative on both sides of (8) with respect to T, we have

$$\frac{d\lambda}{dT} = -\frac{c_1'(S,T)\lambda^3 + c_2'(S,T)\lambda^2 + c_3'(S,T)\lambda + c_4'(S,T)}{4\lambda^3 + 3c_1(S,T)\lambda^2 + 2c_2(S,T)\lambda + c_3(S,T)}, \qquad (14)$$

where

$$c_1'(S,T) = -\frac{4}{T^2}, \qquad c_2'(S,T) = -\frac{4(2S+T)}{ST^3},$$

$$c_3'(S,T) = \frac{4\alpha\mu T - 8(v+\mu T)}{vST^3}, \qquad c_4'(S,T) = -\frac{8(1-\alpha)\mu}{vST^3}.$$

Since $\omega_* = c_3(S,T_*)/c_1(S,T_*)$, it follows from (14) that

$$\left(\frac{d\lambda}{dT}\right)_{T=T_*} = \frac{[c_1'(S,T_*)c_3(S,T_*) - c_1(S,T_*)c_3(S,T_*)]i\omega_* + c_2'(S,T_*)c_3(S,T_*) - c_1(S,T_*)c_4'(S,T_*)}{2\{[c_1(S,T_*)c_2(S,T_*) - 2c_3(S,T_*)]i\omega_* - c_1(S,T_*)c_3(S,T_*)\}} \qquad (15)$$

Multiplying both numerator and denominator of (15) by the conjugate of the denominator, and recalling that $\varphi(T_*) = 0$, i. e. $= c_3^{*2} = c_1^* c_2^* c_3^* - c_1^{*2} c_4^*$, after a long calculation we get

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*} = -\frac{c_1(S,T_*)\varphi'(T_*)}{2\{c_1^3(S,T_*)c_3(S,T_*) + [c_1(S,T_*)c_2(S,T_*) - 2c_3(S,T_*)]^2\}}.$$

As T increases, a positive (resp. negative) sign of (15) implies crossing of the imaginary axis from left to right (resp. from right to left). Thus, we have the following result.

Theorem 2. *Assume that $\alpha < (v + S\mu)/(\mu T)$ and $\varphi(T) > 0$, where $\varphi(T)$ is defined as in* (9). *If there exists $T = T_*$ such that $\varphi(T_*) = 0$ and $\varphi'(T_*) < 0$, then a Hopf bifurcation occurs at the equilibrium point k_* of* (6) *as T passes through T_*.*

4. Weak and Dirac Kernels

Let, $m = 0$ and $T \to 0$. Eq. (2) takes the form

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{1}{S}\right)e^{-\frac{1}{S}(t-r)}k(r)\,dr\right]^{\alpha} - \frac{\mu}{v}\,k(t-T). \qquad (16)$$

Setting

$$x(t) = \int_{-\infty}^{t} \left(\frac{1}{S}\right) e^{-\frac{1}{S}(t-r)} k(r)\, dr,$$

Eq. (16) takes the form of a second-dimensional system of delay differential equations

$$\begin{cases} \dot{k}(t) = \dfrac{\varepsilon}{v}[x(t)]^{\alpha} - \dfrac{\mu}{v} k(t-T), \\ \dot{x}(t) = \dfrac{1}{S}[k(t) - x(t)]. \end{cases} \qquad (17)$$

The associated characteristic equation of the linearization of (17) at the equilibrium point (k_*, x_*), where $x_* = k_*$, is

$$\lambda^2 + \frac{1}{S}\lambda - \frac{\alpha\mu}{vS} + \left(\frac{\mu}{vS} + \frac{\mu}{v}\lambda\right) e^{-\lambda T} = 0. \qquad (18)$$

Lemma 3. *Let* $T = 0$. *The equilibrium point* k_* *of* (16) *is locally asymptotically stable.*

Proof. In absence of delay, (18) reduces to

$$\lambda^2 + \left(\frac{1}{S} + \frac{\mu}{v}\right)\lambda + \frac{(1-\alpha)\mu}{vS} = 0.$$

The conclusion is a straightforward matter being the coefficients both positive. ■

For the case $T > 0$, we determine parameter values for which (18) may have pure complex roots. We seek $\omega > 0$ such that $\lambda = i\omega$ satisfies (17). Substituting into (17), recalling that $e^{-i\omega T} = \cos(\omega T) - \sin(\omega T)$, and equating real and imaginary parts, we find that ω must simultaneously satisfy

$$\begin{cases} \omega^2 + \dfrac{\alpha\mu}{vS} = \dfrac{\mu}{vS}\cos(\omega T) + \dfrac{\mu}{v}\omega \sin(\omega T), \\ \dfrac{1}{S}\omega = \dfrac{\mu}{vS}\sin(\omega T) - \dfrac{\mu}{v}\omega\cos(\omega T). \end{cases} \qquad (19)$$

Recalling that $\sin^2(\omega T) + \cos^2(\omega T) = 1$, squaring both sides of the equations in (19), adding and rearranging gives

$$\omega^4 - \left(\frac{\mu^2 S^2 - 2\alpha\mu v S - v^2}{v^2 S^2}\right)\omega^2 - \frac{(1-\alpha^2)\mu^2}{v^2 S^2} = 0.$$

This equation in ω^2 has a unique positive root, say ω_0, where

$$\omega_0^2 = \frac{(\mu^2S^2 - 2\alpha\mu vS - v^2)^2 + v^2S^2\sqrt{(\mu^2S^2 - 2\alpha\mu vS - v^2)^2 + 4(1-\alpha^2)\mu^2v^2S^2}}{2v^4S^4}. \qquad (20)$$

Solving for $\cos(\omega T)$ and $\sin(\omega T)$ in (19) yields

$$\cos(\omega T) = \frac{\alpha}{1+S^2\omega^2} > 0, \qquad \sin(\omega T) = \frac{vS^2\omega^3 + (v+\alpha\mu S)\omega}{\mu(1+S^2\omega^2)} > 0. \qquad (21)$$

Therefore, from (21) we see that

$$T_j = \frac{1}{\omega_0}\left[\cos^{-1}\left(\frac{\alpha}{1+S^2\omega_0^2}\right) + 2j\pi\right], \qquad j = 0,1,2,\ldots \qquad (22)$$

are the critical values of T for which the characteristic equation (18) has purely imaginary roots $\lambda = \pm i\omega_0$.

Let $\lambda(T) = u(T) + i\omega(T)$ be the root of (18) such that $u(Tj) = 0$ and $\omega(Tj) = \omega_0$. By differentiating (18) implicitly with respect to T, we get

$$\left[2vS\lambda + v + \mu Se^{-\lambda T} - \mu(1+S\lambda)Te^{-\lambda T}\right]\frac{d\lambda}{dT} = \mu(1+S\lambda)\lambda e^{-\lambda T}. \qquad (23)$$

Hence, we have

$$\left(\frac{d\lambda}{dT}\right)^{-1} = \frac{(2S\lambda+1)ve^{\lambda T} + \mu S}{\mu(1+S\lambda)\lambda} - \frac{T}{\lambda}.$$

Then,

$$sign\left[\left.\frac{d(Re\lambda)}{dT}\right|_{T=T_j}\right] = sign\left[\left.Re\left(\frac{d\lambda}{dT}\right)^{-1}\right|_{T=T_j}\right], sign\left[\sqrt{(\mu^2S^2 - 2\alpha\mu vS - v^2)^2 + 4(1-\alpha^2)\mu^2v^2S^2}\right] > 0.$$

We summarize the foregoing discussion in the form of the following theorem.

Theorem 3. *Let T_0 be defined as in* (20). *The equilibrium point k_* of* (16) *is locally asymptotically stable when $0 < T \le T_0$ and unstable when $T > T_0$. A Hopf bifurcation occurs at the equilibrium when $T = T_0$.*

5. Strong and Weak Kernels

Let $m = 1$ and $n = 0$. Eq. (2) is now governed by the following equation

$$\dot{k}(t) = \frac{\varepsilon}{v}\left[\int_{-\infty}^{t}\left(\frac{2}{S}\right)^2 (t-r)e^{-\frac{2}{S}(t-r)}k(r)\,dr\right]^{\alpha} - \frac{\mu}{v}\int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)\,dr. \qquad (24)$$

Setting $x(t)$, $y(t)$ and $z(t)$ by

$$x(t) = \int_{-\infty}^{t}\left(\frac{2}{S}\right)^2 (t-r)e^{-\frac{2}{S}(t-r)}k(r)\,dr, \qquad y(t) = \int_{-\infty}^{t}\left(\frac{2}{S}\right)e^{-\frac{2}{S}(t-r)}k(r)\,dr,$$

and

$$z(t) = \int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)\,dr.$$

Eq. (24) rewrites as a fourth-dimensional system of ordinary differential equations

$$\begin{cases} \dot{k}(t) = \dfrac{\varepsilon}{v}[x(t)]^{\alpha} - \dfrac{\mu}{v}z(t), \\ \dot{x}(t) = \dfrac{2}{S}[y(t) - x(t)], \\ \dot{y}(t) = \dfrac{2}{T}[k(t) - y(t)], \\ \dot{z}(t) = \dfrac{1}{T}[k(t) - z(t)]. \end{cases} \qquad (25)$$

In order to examine local dynamics of the above system in the neighborhood of the steady state (k_*, x_*, y_*, z_*), where $x_* = y_* = z_* = k_*$, we consider its linearized version and get the following characteristic equation

$$\lambda^4 + p_1(S,T)\lambda^3 + p_2(S,T)\lambda^2 + p_3(S,T)\lambda + p_4(S,T) = 0, \qquad (26)$$

where

$$p_1(S,T) = \frac{S+4T}{ST} > 0, \qquad p_2(S,T) = \frac{S^2\mu + 4vS + 4v}{S^2Tv} > 0,$$

$$p_3(S,T) = \frac{4v + 4\mu\,(S - \alpha T)}{vS^2T}, \qquad p_4(S,T) = \frac{4(1-\alpha)\mu}{vS^2T} > 0.$$

Stability of (25) can be examined by finding the locations of the eigenvalues of Eq. (26).

Comparing Eq. (26) with Eq. (8) reveals the similarities among them, and, consequently, analytical methodologies are similar. To avoid unnecessary repetition, the analysis of (26) is simplified.

Using first the Routh-Hurwitz criterion and then the Hopf Bifurcation Theorem we arrive at the following results.

Theorem 4. *Let* $\psi(T) = p_1(S,T)p_2(S,T)p_3(S,T) - p_3^2(S,T) - p_1^2(S,T)p_4(S,T)$.

1) *The equilibrium point* k_* *of* (24) *is locally asymptotically stable if* $0 < \alpha < (v + \mu S)/(\mu T)$ *and* $\psi(T) > 0$.
2) *If* $\alpha < (v + S\mu)/(\mu T), \psi(T) > 0$ *and there exists* $T = T_*$ *such that* $\psi(T_*) = 0$ *and* $\psi'(T_*) < 0$, *then the equilibrium point bifurcates to a limit cycle through a Hopf Bifurcation at* T_*.

6. Dirac and Weak Kernels

Let, $S \to 0$ and $n = 0$ Eq. (2) turns to be

$$\dot{k}(t) = \frac{\varepsilon}{v}[k(t-S)]^{\alpha} - \frac{\mu}{v}\int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{T}(t-r)}k(r)\,dr. \qquad (27)$$

Introducing the variable

$$x(t) = \int_{-\infty}^{t}\left(\frac{1}{T}\right)e^{-\frac{1}{S}(t-r)}k(r)\,dr$$

allows (27) to be changed into the following two-dimensional system of delay differential equations

$$\begin{cases}\dot{k}(t) = \dfrac{\varepsilon}{v}[k(t-S)]^{\alpha} - \dfrac{\mu}{v}x(t),\\ \dot{x}(t) = \dfrac{1}{T}[k(t) - x(t)].\end{cases}$$

A straightforward calculation yields the following characteristic equation

$$\lambda^2 + \frac{1}{T}\lambda + \frac{\mu}{vT} - \frac{\alpha\mu}{v}\left(\frac{1}{T} + \lambda\right)e^{-\lambda S} = 0 \qquad (28)$$

Lemma 4. *Let* $S = 0$. *The equilibrium point* k_* *of* (27) *is locally asymptotically stable if* $0 \le T < v/(\alpha\mu)$ *and unstable if* $T \ge v/(\alpha\mu)$.

Proof. When $S = 0$, (28) reduces to

$$\lambda^2 + \left(\frac{1}{T} - \frac{\alpha\mu}{v}\right)\lambda + \frac{(1-\alpha)\mu}{vT} = 0.$$

Hence, we have the statement noticing that all the coefficients of this equation are positive when $T < v/(\alpha\mu)$. ■

Now, let us take $S > 0$. We shall investigate the roots of the transcendental equation (28) that lie in the left half of the complex plane. Suppose that $\lambda = i\omega, \omega > 0$, is a root of (28) for some S. Substituting this root into (28) and separating the real and imaginary parts implies

$$\omega^2 - \frac{\mu}{vT} = -\frac{\alpha\mu}{v}\cos(\omega S) - \frac{\alpha\mu}{v}\omega\sin(\omega S), \qquad \frac{\omega}{T} = -\frac{\alpha\mu}{v}\sin(\omega S) + \frac{\alpha\mu}{v}\omega\cos(\omega S). \qquad (29)$$

Adding squares of these equations we obtain the following equation in ω^2,

$$\omega^4 + \left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{\alpha^2\mu^2}{v^2}\right)\omega^2 + \frac{(1-\alpha^2)\mu^2}{v^2T^2} = 0. \qquad (30)$$

Proposition 1. *Let*

$$T^2 = \frac{v\left(-1 + \sqrt{1-\alpha^2} + \sqrt{2}\sqrt{1-\sqrt{1-\alpha^2}}\right)}{\alpha^2\mu}. \qquad (31)$$

Then Eq. (28) *has a pair of pure imaginary roots* $\lambda = i\omega_{\pm}$, *with* $0 < \omega_- < \omega_+$, for $T_2 < T < v/(\alpha\mu)$, *where*

$$\omega_{\pm}^2 = \frac{T^2\alpha^2\mu^2 + 2Tv\mu - v^2 \pm \sqrt{\alpha^4\mu^4T^4 + 4\alpha^2v\mu^3T^3 + 2\alpha^2v^2\mu^2T^2 - 4v^3\mu T + v^4}}{2T^2v^2}. \qquad (32)$$

Proof. We start noticing that the constant term of (30) is positive. Therefore, solving for potential positive roots of (30) using the quadratic formula leads to the existence of two positive roots

$$\omega_{\pm}^2 = -\frac{1}{2}\left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{\alpha^2\mu^2}{v^2}\right) \pm \frac{1}{2}\sqrt{\left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{\alpha^2\mu^2}{v^2}\right)^2 - \frac{4(1-\alpha^2)\mu^2}{v^2T^2}}$$

under the conditions

$$\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{\alpha^2\mu^2}{v^2} < 0,$$

and

$$\Delta = \left(\frac{1}{T^2} - \frac{2\mu}{vT} - \frac{\alpha^2\mu^2}{v^2}\right) - \frac{4(1-\alpha^2)\mu^2}{v^2T^2} > 0.$$

The former condition gives

$$\alpha^2\mu^2T^2 + 2\mu vT - v^2 > 0,$$

whose solution is given by

$$T > \frac{v\left(-1+\sqrt{1-\alpha^2}\right)}{\alpha^2\mu} = T_1.$$

The latter condition is instead verified when

$$\Delta > 0 \quad \Leftrightarrow \quad \frac{1}{T^2} - \frac{2\mu}{vT} - \frac{\alpha^2\mu^2}{v^2} < -\frac{2\sqrt{1-\alpha^2}\mu}{vT} \quad \Leftrightarrow \quad \alpha^2\mu^2T^2 + 2\mu v\left(1-\sqrt{1-\alpha^2}\right)T - v^2 > 0,$$

leading to $T > T_2$. This together with the fact that $T_1 < T_2 < v/(\alpha\mu)$ completes the proof. ■

To find the corresponding critical values $S_j^{\pm}$ of S where the pure imaginary roots $i\omega_{\pm}$ exist, we solve (29) for $\sin(\omega S)$ and $\cos(\omega S)$, and get

$$\sin(\omega S) = \frac{\omega(-v+\mu T - vT^2\omega^2)}{\alpha\mu(1+T^2\omega^2)}, \qquad \cos(\omega S) = \frac{1}{\alpha(1+T^2\omega^2)} > 0.$$

By (32) one has

$$\sin(\omega S) = -\frac{\omega(v^2+\alpha^2\mu^2T^2) \pm \sqrt{\alpha^4\mu^4T^4 + 4\alpha^2v\mu^3T^3 + 2\alpha^2v^2\mu^2T^2 - 4v^3\mu T + v^4}}{4\alpha v\mu(1+T^2\omega^2)}$$

It is now immediate that $\sin(\omega_+ S) < 0$. On the other hand,

$$sign[\sin(\omega_- S)] = sign\left[-v^2 - \alpha^2\mu^2T^2 + \sqrt{\alpha^4\mu^4T^4 + 4\alpha^2v\mu^3T^3 + 2\alpha^2v^2\mu^2T^2 - 4v^3\mu T + v^4}\right]$$

A direct calculation shows this sign to be also negative as $T < v/(\alpha\mu)$. Hence, $S_j^{\pm}$ $(j = 0,1,2,\dots)$ are defined by

$$S_j^{\pm} = \frac{1}{\omega_{\pm}}\left\{2\pi - \cos^{-1}\left[\frac{1}{\alpha(1+T^2\omega_{\pm})}\right] + 2j\pi\right\}.$$

Next, we check the validity of the transversality result. Differentiating (28) with respect to S, we have

$$\left\{2v\lambda + \frac{1}{T} - \alpha\mu\left[1 - S\left(\frac{1}{T}+\lambda\right)\right]e^{-\lambda S}\right\}\frac{d\lambda}{dS} = -\alpha\mu\lambda\left(\lambda+\frac{1}{T}\right)e^{-\lambda S}. \qquad (33)$$

Then, using (28), it follows that

$$\left(\frac{d\lambda}{dS}\right)^{-1} = -\frac{\alpha\mu - v\left(2\lambda+\frac{1}{T}\right)e^{\lambda S}}{\alpha\mu\lambda\left(\lambda+\frac{1}{T}\right)} - \frac{S}{\lambda}.$$

Therefore,

$$sign\left[\frac{d(Re\lambda)}{dS}\Big|_{S=S_j^{\pm}}\right] = sign\left[Re\left(\frac{d\lambda}{dS}\right)^{-1}\Big|_{S=S_j^{\pm}}\right], sign\left[\sqrt{\alpha^4\mu^4T^4 + 4\alpha^2 v\mu^3T^3 + 2\alpha^2v^2\mu^2T^2 - 4v^3\mu T + v^4}\right].$$

Hence the sign is positive for ω_+ and negative for ω_-. This implies that all the roots that cross the imaginary axis at $i\omega_+$(resp. $i\omega_-$) cross from left to right (resp. from right to left) as S increases. It remains to prove that $\lambda = i\omega_\pm$ are simple roots for (28). Suppose $\lambda = i\omega_+$ (similarly for $i\omega_-$) is repeated, then from (33) one must have

$$-\alpha\mu i\omega_+\left(i\omega_+ + \frac{1}{T}\right)\left[\cos\left(\omega_+S_j^{\pm}\right) - i\sin\left(\omega_+S_j^{\pm}\right)\right] = 0.$$

which clearly implies a contradiction. Thus, the conditions for a Hopf Bifurcation are met and the Hopf bifurcation theorem holds. The above analysis is now summarized as follows.

Theorem 5. *Let T_2 be define as in* (31).

1) *Let $0 \le T \le T_2$. The equilibrium point k_* of* (27) *is locally asymptotically stable for all S.*

2) *Let $T_2 < T < v/(\alpha\mu)$. Stability switches occur as the time delay S increases from zero to the positive infinity, with the occurrence of a Hopf bifurcation at each switch. If*

3) *Let $T \ge v/(\alpha\mu)$. The equilibrium point k_* of* (27) *is unstable for all S.*

7. Concluding Remarks

We have considered different modelling approaches to study an energy model for sustainable economic growth when time delay is replaced by way of distributed delays. The system is modeled by gamma distributed delays, which includes the differential equations model with a discrete delay and the ordinary differential equations model as special cases. Employing the Routh-Hurwitz criterion and the results on distribution of the zeros of transcendental functions, we get a set of conditions to determine the stability of the equilibrium point and the existence of Hopf bifurcations. The choice of continuously distributed lag over a fixed time interval also yield the complex behaviour of emerging stability loss and gain which may repeat alternatively.

Bifurcation Analysis of a Transportation Network for Energy with Distributed Delayed Carrying Capacity

Recently, [10] have developed a mathematical model of an economy viewed as a transport network for energy. Based on [24], [5] have adapted their ideas and proposed a generalization by introducing a logistic-type equation for population with delayed carrying capacity. This study examines the consequences of replacing time delays with distributed time delays in their model. The local asymptotic stability of the equilibrium point is studied by analyzing the corresponding characteristic equation. It is found that the destructive impact of the agents on the carrying capacity leads the system dynamic behavior to exhibit stability switches and Hopf Bifurcations to occur.

1. Introduction

The Solow model ([19]) has been a point of reference of economic growth since the 1950s. It attempted to explain how increased capital stock generates greater per capita production and revealed the role of savings, population growth and technological change in the long-term determination of labor productivity. In this regard, various augmented versions of this model were built over the decades, including among others the contribution of [17], employing a physical and an intangible capital (human capital), and the Solow residual model, addressing issues concerning intangibles, such as the contribution of intangible capital to output growth and how does the inclusion of intangibles affects the allocation of output growth between capital formation and multifactor productivity growth (see e.g. [7]).

[10] shed some light on the relationship between energy distribution and economic growth, viewing the economy as a transportation network for electricity and getting a model whose dynamics are formally the same as in the Solow model. Specifically, energy originates from a power plant and is diffused across the economy to the sites at which it is used via a power grid. Energy is needed to run, maintain and create capital. Without an electricity supply, any investment in machinery at a particular place or time will not lead to economic growth. They set a supply relationship between electricity consumption per capita and the size of the economy measured by capital per capita. Techniques taken from biological sciences were used in their modelling: a power law association between consumption per capita of electricity and capital per capita with an exponent assigned to capital bounded between 1/2 and 3/4 with the final exponent dependent on the efficiency of the network (see [1]). Notice that the ability to harness energy for the betterment of society and how efficiently it can be deployed was first mentioned in 1862 by [20]. To prevent limits to growth as a

result of energy shortages will require technological change even if the supply of energy increases. The distribution of energy through a network is therefore crucial.

It is worth stressing that in Dalgaard and Strulik's model population increases at a positive constant rate ([16] growth), which is a totally unrealistic assumption because it doesn't consider the environmental limits that have consequences on the population. Exponential growth may happen for a while in environments where there are few individuals and plentiful resources. But when the number of individuals gets large enough, resources will be depleted, slowing the growth rate. Eventually, the growth rate will level off, making an S-shaped curve. This population size at which it levels off represents the maximum population size a particular environment can support and is called the carrying capacity. Therefore, to model more realistic population growth, scientists developed the logistic growth model (see [21]). [13] pointed out that the process of reproduction is not instantaneous, there is a lag in some of the processes involved, and so the logistic model is inappropriate for the description of population growth. For this reason, he proposed to introduce a time delay in the logistic model. The resulting model is known as the delayed logistic equation or Hutchinson's equation or, under a suitable change of variables, Wright's equation ([23]).

Many authors (e.g. [11]) have stressed that, due to the influence on the carrying capacity resulting from the existing populations, its original definition implying a constant value has lost its meaning. The society activity does influence its own carrying capacity that can be either enhanced by producing new goods, materials, knowledge, and so on, or can be destroyed by unreasonable exploitation of resources, e.g., by deforestation, polluting water, and spoiling climate. The idea that the carrying capacity may be not a constant but a function of population fractions has repeatedly appeared in the literature in the form of general discussions. The article by [24] suggests an interesting approach to the transformation of logistic equations with certain time lags and variable carrying capacity of the biosphere to the shape that allows to model punctuated staircase-like growth (or decline), which corresponds better to real processes in natural and social macrosystems. Taking into account the fact that the carrying capacity is not a simple constant describing the available resources, but that these resources are subjected to the change due to the activity of the system individuals, who can either increase the carrying capacity by creative work or decrease it by destructive actions, [5] have improved the Dalgaard-Strulik model ([10]). However, they used a discrete delay, which, sometimes, might be seen as a rough approximation in modeling the delay distribution over a large size population since it is implicitly assumed that each individual within the population is subject to the same maturation delay. According to this fact, [22] introduced distributed delay into models, taking the length of the delay from a probability distribution (see e.g. [8], [9], [15]). Since the delay might not be exactly the same for every member, this will provide a

more appropriate description. There is also some experimental evidence which indicates that such continuously distributed delays are more accurate than those with instantaneous time lags (see [6]).
In this paper, we propose and study an extension of the model introduced by [5], whose principal feature is the functional dependence of the population carrying capacities on the population species and delay maturation delay distributed by the continuous Gamma distributions [18]. The stability and Hopf bifurcation analysis of the equilibrium under different conditions are carried out and the influence on the system behaviour of the destroyed or created carrying capacity is pointed out. The organization of this paper is as follows. In Section 2, the model is presented. In Sections 3 and 4, local stability of the positive equilibrium and existence of the local Hopf bifurcation are discussed in case of weak or strong kernel, respectively. Some main conclusions are drawn in Section 5.

2. The Model

We begin by considering the model developed by [5], who introduced a logistic-type equation with delayed carrying capacity in the transport network model for energy of [10] and obtained the following system of delay differential equations

$$\begin{cases} \dot{k}(t) = \dfrac{\varepsilon}{v}[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A + BL(t-\tau)}\right]k(t) \\ \dot{L}(t) = \gamma L(t) - \dfrac{C[L(t)]^2}{A + BL(t-\tau)} \end{cases} \qquad (2.1)$$

where k is capital per capita, L denotes labor/population, μ and v represent the energy required to operate and maintain the generic capital good and the energy costs to create a new capital good, respectively, $a \in (0,1)$ is a real constant proportional to the dimension and efficiency of the network, $\mu > 0$ is a real constant. C defines the balance between competition and cooperation.
$A > 0$ is a natural carrying capacity, provided by Nature, i.e. it is the pre-existing carrying capacity. The parameter B is the created or destroyed capacity, depending on whether the society activity is constructive or destructive. It is a production factor if it is positive ($B > 0$) and a destruction factor in case it is negative ($B < 0$). Since it is well known that dynamical systems with distributed delays are more general than those with discrete delay, we now assume that the time delay is not the same for all members of the population, but rather is distributed according to the gamma distribution function [18]

$$g(u,T,n) = \left(\frac{n}{T}\right)^n \frac{u^{n-1}e^{-\frac{n}{T}u}}{(n-1)!} \qquad (2.2)$$

with n a positive integer that determines the shape of the weighting function and $T \geq 0$ a parameter associated with the mean time delay of the distribution. Accordingly, we derive that the dynamics of our model are then governed by the following integro-differential equations system.

$$\begin{cases} \dot{k}(t) = \dfrac{\varepsilon}{v}[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A + BF(t,T,n)}\right]k(t) \\ \dot{L}(t) = \gamma L(t) - \dfrac{C[L(t)]^2}{A + BF(t,T,n)} \end{cases} \tag{2.3}$$

where

$$F(t,T,n) = \int_{-\infty}^{t} g(t - r - T, n)L(r)\, dr.$$

Notice that as $T \to 0$ the distribution function approaches the Dirac distribution, and thus system (2.3) is reduced to system (2.1), i.e. one recovers the discrete delay case. In order to analyse the local behaviour of system (2.3) one should consider the characteristic equation of the linearized system at the equilibrium point for which it is known to be hard to derive general stability conditions. For this reason, we follow the standard procedure that consists in drawing attention to some special cases and examine stability of the equilibrium analytically by applying the Routh-Hurwitz theorem, which provides conditions that are both necessary and sufficient for this polynomial to have roots with negative real parts. Henceforth, we will concentrate on the two particular cases: $n = 1$ and $n = 2$. The first case corresponds to a weak delay kernel in the sense that the maximum weighted response of the growth rate is to current population density while past densities have exponentially decreasing influence; the second case instead represents a strong delay kernel in the sense that the maximum influence on growth rate response at any time t is due to population density at the previous time $t - T$.

3. Stability Analysis and Hopf Bifurcation (Case Weak Kernel)

Let $g(\cdot)$ in (2.2) be a weak kernel, i.e.

$$g(t - r, T, 1) = \frac{1}{T} e^{-\frac{1}{T}(t-r)}$$

For convenience, a new variable is introduced and defined as follows

$$x(t) = \int_{-\infty}^{t} \left(\frac{1}{T}\right) e^{-\frac{1}{T}(t-r)} L(r) dr.$$

Applying the linear chain trick technique (see [15]), system (2.3) is then transformed into the following system of three dimensional ODEs

$$\begin{cases} \dot{k}(t) = \dfrac{\varepsilon}{v}[k(t)]^a - \left[\dfrac{\mu}{v} + \gamma - \dfrac{CL(t)}{A + Bx(t)}\right]k(t) \\ \dot{L}(t) = \gamma L(t) - \dfrac{C[L(t)]^2}{A + Bx(t)} \\ \dot{x}(t) = \dfrac{1}{T}[L(t) - x(t)] \end{cases} \tag{3.1}$$

The equilibrium points of (2.3) are the same as those for (2.1). Hence, it follows from [5] the existence of a unique non-trivial equilibrium (k_*, L_*, x_*), where

$$\varepsilon k_*^{a-1} = \mu \qquad x_* = L_* = \frac{\gamma A}{C - \gamma B} \tag{3.2}$$

when

$$C - \gamma B \neq 0 \quad \text{and} \quad sign(C - \gamma B) \tag{3.3}$$

It is well known that the local stability of (k_*, L_*, x_*) is governed by the roots of the associated characteristic equation for (3.1). By linearizing (3.1) at the equilibrium point, we derive that the characteristic equation of system (3.1) is given by

$$det\begin{bmatrix} -(1-\alpha)\dfrac{\mu}{v} - \lambda & \dfrac{\gamma k_*}{L_*} & -\dfrac{BCk_*L_*}{(A + BL_*)^2} \\ 0 & -\gamma - \lambda & \dfrac{\lambda^2 B}{C} \\ 0 & \dfrac{1}{T} & -\dfrac{1}{T} - \lambda \end{bmatrix} = 0$$

By simple calculation, one obtains

$$\left[-(1-\alpha)\frac{\mu}{v} - \lambda\right][\lambda^2 + a_1(T)\lambda + a_2(T)] = 0 \tag{3.4}$$

where the coefficients are

$$a_1(T) = \gamma + \frac{1}{T}, \quad a_2(T) = \frac{\gamma}{T}\left(1 - \frac{\gamma B}{C}\right)$$

The Routh-Hurwitz criterion implies that the equilibrium is locally asymptotically stable, i.e. all roots of the polynomial (3.4) are negative or have negative real parts, if and only if the following conditions hold

$$a_1(T) > 0, \qquad a_2(T) > 0 \qquad (3.5)$$

By (3.3), conditions (3.5) leads to

$$\gamma + \frac{1}{T} > 0, \qquad C > 0 \qquad (3.6)$$

We have two cases: $\gamma > 0$, in which case (3.6) reduces to $C > 0$ with $C > \gamma B$, and so we get $C > \max\{0, \gamma B\}$; $\gamma < 0$, then (3.6) gives $T < -\frac{1}{\gamma}, 0 < C < \gamma B$ with $B < 0$. In this latter case, the curve $T = -\frac{1}{\gamma}$ divides the parameter space into stable and unstable parts. In order to check the possibility of the emergency of a limit cycle at $T = -\frac{1}{\gamma}$, we use the Hopf Bifurcation theorem and check if the characteristic equation (3.4) possesses a pair of purely imaginary roots and the real parts of these roots change signs with a bifurcation parameter. Supposing that $\lambda = i\omega(\omega > 0)$ is a root of (3.4), then, at the critical value $T = -\frac{1}{\gamma}$, one in particular find

$$-\omega^2 + a_1\left(-\frac{1}{\gamma}\right)i\omega + a_2\left(-\frac{1}{\gamma}\right) = 0 \qquad (3.7)$$

Separating the real and imaginary parts of (3.7) yields the contradiction $0 = \gamma + 1/(-\frac{1}{\gamma}) > 0$. Thus, there is no imaginary root for (3.4). Summarizing the analysis above we arrive at the following stability results.

Proposition 3.1.

1) *The equilibrium point* (k_*, L_*, x_*) *of* (6) *is locally asymptotically stable for all* $T > 0$ *if* $\gamma > 0$ *and* $C > \max\{0, \gamma B\}$.
2) *The equilibrium point* (k_*, L_*, x_*) *of* (6) *is locally asymptotically stable for* $T < -\frac{1}{\gamma}$ *and unstable for* $T > -\frac{1}{\gamma}$ *if* $\gamma < 0$ *and* $0 < C < \gamma B,\ B < 0$. *There can be no Hopf bifurcation at the equilibrium.*

4. Stability Analysis and Hopf Bifurcation (Case Strong Kernel)

Let $g(\cdot)$ in (2.2) be a strong kernel, i.e.

$$g(t-r,T,2)=\left(\frac{2}{T}\right)^2(t-r)e^{-\frac{2}{T}(t-r)}$$

Introducing the new variables

$$z(t)=\int_{-\infty}^{t}\left(\frac{2}{T}\right)^2 e^{-\frac{2}{T}(t-r)}u(r)dr, \qquad z(t)=\int_{-\infty}^{t}\left(\frac{2}{T}\right)^2 e^{-\frac{2}{T}(t-r)}L(r)dr,$$

again an application of the linear chain trick technique leads system (2.3) to be rewritten as a four dimensional system of ODEs

$$\begin{cases}\dot{k}(t)=\dfrac{\varepsilon}{v}[k(t)]^a-\left[\dfrac{\mu}{v}+\gamma-\dfrac{CL(t)}{A+Bz(t)}\right]k(t)\\ \dot{L}(t)=\gamma L(t)-\dfrac{C[L(t)]^2}{A+Bz(t)}\\ \dot{z}(t)=\dfrac{2}{T}[u(t)-z(t)]\\ \dot{u}(t)=\dfrac{2}{T}[L(t)-u(t)]\end{cases} \tag{4.1}$$

The characteristic equation of the linearized system of (4.1) at the equilibrium (k_*,L_*,z_*,u_*), where $z_*=u_*=L_*$, and k_*,L_* are defined as in (3.2), is provided by

$$det\begin{bmatrix}-(1-\alpha)\dfrac{\mu}{v}-\lambda & \dfrac{\gamma k_*}{L_*} & -\dfrac{BCk_*L_*}{(A+BL_*)^2} & 0\\ 0 & -\gamma-\lambda & \dfrac{\gamma^2B}{C} & 0\\ 0 & \dfrac{1}{T} & -\dfrac{1}{T}-\lambda & \dfrac{2}{T}\\ 0 & \dfrac{2}{T} & 0 & -\dfrac{2}{T}-\lambda\end{bmatrix}=0 \tag{4.2}$$

By expanding (4.2), the characteristic equation becomes

$$\left[-(1-\alpha)\frac{\mu}{v}-\lambda\right][\lambda^3+b_1(T)\lambda^2+b_2(T)\lambda+b_3(T)]=0 \tag{4.3}$$

where

$$b_1(T) = \gamma + \frac{4}{T}, \qquad b_2(T) = \frac{4}{T}\left(\gamma + \frac{1}{T}\right), \qquad b_3(T) = \frac{4\gamma}{T^2}\left(1 - \frac{\gamma B}{C}\right)$$

Due to the Routh-Hurwitz criterion, we have local asymptotic stability for the equilibrium of system (4.1) if and only if the following conditions

$$b_1(T) > 0, \quad b_3(T) > 0, \quad b_1(T)b_2(T) - b_3(T) > 0 \qquad (4.4)$$

hold true. The first two conditions are equivalent to

$$\gamma + \frac{4}{T} > 0, \quad C > 0 \qquad (4.5)$$

A direct calculation changes the latter condition in (4.4) to

$$(\gamma T + 2)^2 + \frac{\gamma^2 BT}{C} > 0 \qquad (4.6)$$

Proposition 4.1

1) *The equilibrium point* (k_*, L_*, z_*, u_*) *of* (12) *is locally asymptotically stable for all* $T > 0$ *if* $B > 0, \gamma > 0, C > \gamma B$, *or if* $B < 0, \gamma > 0, C + \gamma B \geq 0$.

2) *The equilibrium point* (k_*, L_*, z_*, u_*) *of* (12) *is locally asymptotically stable for* $T < T_*$ *and unstable for* $T > T_*$*if* $B < 0, \gamma > 0, C + \gamma B < 0$ *or if* $B < 0, \gamma < 0, C + \gamma B > 0$, *where*

$$T_* = -\frac{2C}{\gamma(C + \gamma B)}$$

Proof. Let $B > 0$. Then (4.6) is always verified so that the conditions for stability reduce to (4.5). In addition, we must have $\gamma > 0$ since $C > \gamma B$. Next, let $B < 0$. From (4.6) we have

$$|\gamma T + 2| > -\frac{\gamma^2 BT}{C}$$

and so

$$\frac{\gamma^2 BT}{C} < \gamma T + 2, \qquad \gamma T + 2 > -\frac{\gamma^2 BT}{C}$$

The first inequality gives $\gamma(C-\gamma B)T+2C>0$, which is obviously true. The second one instead yields $\gamma(C+\gamma B)T+2C>0$. The conclusions now hold considering the cases $\gamma>0$ and $\gamma<0$. Notice that when $\gamma<0$ it is $C+\gamma B>0$ and $T_*<-4/\gamma$. □

We now return to the characteristic equation (4.3) and show the possibility of the birth of a limit cycle at $T=T_*$ by applying the Hopf bifurcation theorem. We start noticing that at the critical value T_*, one has $b_1(T_*)b_2(T_*)=b_3(T_*)$, so that the characteristic equation (4.3) factors as

$$\left[-(1-\alpha)\frac{\mu}{v}-\lambda\right][\lambda+b_1(T_*)][\lambda^2+b_2(T_*)]=0$$

This shows that two roots are real and negative

$$\lambda_1=-(1-\alpha)\frac{\mu}{v}<0, \qquad \lambda_2=-b_1(T_*)<0$$

and two are purely imaginary

$$\lambda_{3,4}=\pm i\omega_*, \qquad \text{with} \qquad \omega_*=\sqrt{b_2(T_*)}$$

Next, we need to check whether the real part of the conjugate complex root changes its sign as the bifurcation parameter T passes through its critical value T_*. Assuming $\lambda=\lambda(T)$, differentiating the characteristic equation (4.3) with respect to T, and arranging terms, we get

$$\frac{d\lambda}{dT}=\frac{-b_1'(T)\lambda^2-b_2'(T)\lambda}{3\lambda^2+2b_1(T)\lambda+b_2(T)}$$

where

$$b_1'(T)=-\frac{4}{T^2}, \qquad b_2'(T)=-\frac{4}{T^2}\left(\gamma+\frac{2}{T}\right), \qquad b_3'(T)=-\frac{8\gamma}{T^3}\left(1-\frac{\gamma B}{C}\right)$$

From being $\omega_*^2=b_2(T_*)$ and $b_1(T_*)b_2(T_*)=b_3(T_*)$, after some calculations, we have

$$Re\left(\frac{d\lambda}{dT}\right)_{T=T_*}=-\frac{b_1'(T_*)b_2(T_*)+b_1(T_*)b_2'(T_*)}{2[b_2(T_*)+b_1^2(T_*)]}$$

Conditions (4.4) in particular yield $b_2(T_*)>0$, i.e. $\gamma+\frac{1}{T}>0$. Thus,

$$-b_1'(T_*)b_2(T_*) + b_1(T_*)b_2'(T_*) = \frac{4}{T_*^2}\left[\frac{4}{T_*}\left(\gamma + \frac{1}{T_*}\right) + \left(\gamma + \frac{4}{T_*}\right)\left(\gamma + \frac{2}{T_*}\right)\right] > 0$$

Since the sign of (4.7) is positive, only crossing the imaginary axis from left to right is possible as T increases.

In conclusion we have the following theorem.

Theorem 1

The equilibrium point (k_*, L_*, z_*, u_*) of (4.1) loses stability at $T = T_*$ and bifurcates to chaos as T increases.

5. Conclusions

An extension of the model proposed by [5] is examined in which distributed time delays are assumed into a transportation network for energy model. Since the delay might not be exactly the same for every member of the population, but rather might vary according to some distribution, it is more appropriate to use a distributed delay or integrodifferential equation in their model. Conditions are given to the local asymptotical stability of the equilibrium and the appearance of Hopf bifurcations is investigated. It is shown that the creative processes impacting the carrying capacity do not affect stability, while the destructive processes may have a stabilizing effect as well as a destabilizing effect. In the latter case, chaotic behaviour emerges at the stability switch. In future research we will apply this methodology taking into account the possibility of including the conservation of global resources through the use of the thermos tatted kinetic theory for active particles proposed in [2] and generalized in [3], [4].

Dynamics of a Delayed Mathematical Model for One Predator Sharing Teams of Two Preys

A delayed mathematical model consisting of teams of two prey and one predator is studied. Existence of Hopf Bifurcation and local stability are obtained by linearizing the model at the positive equilibrium. By using the normal form method and center manifold theorem, Hopf Bifurcation properties are studied. Some numerical simulations and a qualitative analysis are presented.

1. Introduction

In a biological point of view, prey-predator dynamics are one of the most basic phenomena in the World for getting the equilibria of the species and a lot of applied mathematicians, economists and ecologists have in time paid much attention to this issue. In [21], Murray presented the first fundamental prey-predator system as Lotka-Volterra system which was proposed to study the dynamics oscillatory of certain fish species in the Adriatic sea. Many kinds of predator-prey systems have been investigated by scholars since the classical works by Lotka [15] and Volterra [28], especially the predator-prey systems describing the interaction among multiple species in nature due to the more complex relationships among species. In [18]-[16], Meng *et al.* formulated different two predator-one prey competition systems. In [4]-[8], Djomegin *et al.* studied two prey-one predator competition systems. In [17]-[13], Mbava *et al.* investigated food chain systems with different functional responses. In [6, 24]. authors introduced two advantages for some animals to form technically a team. The first is that it is more efficient to look for food in a team than doing that alone. The second is that it can reduce predation risk to live in a team. The pursing behavior and the prey reaction could be interesting in modeling observed dynamics to investigate prey-predator system with the assumption that during predation the members of preys help each other. In [6], Elettreby proposed a two-prey one-predator system with linear functional response in which the prey teams help each other. Elettreby studied the local and global stability of the system [6]. In [24], Tripathi *et al.* formulated a two-prey one-predator system with linear functional response in which teams of preys group help each other in the presence of predator while compete each other in the absence of predator and they studied local and global stability of the system. In [25], Tripathi *et al.* studied two preys one-predator competitive system with Beddington-DeAngelis functional response in which the two teams of prey help each other against the predator and the two teams of prey interact each other competitively in the absence of predator, and they investigated the existing of Hopf bifurcation of the system. Kundu and Maitra formulated a three species predator-prey system

with cooperation among the prey in which delays are taken just in the growth components for each of the species and the preys are picked such that there is no competition among them at any condition [14]. They studied the effects of the delays on the proposed system and derived the sufficient conditions for the existence of Hopf bifurcation by choosing the delays as the bifurcation parameter.

In [19], Mishra and Raw obtained a prey-predator system by considering teams of one predator and two preys with Monod-Haldane and Holling type II functional response based on the system proposed by Tripathi *et al.* [24] and the assumption that the first prey is dangerous and the second prey is harmless for predator:

$$\begin{cases} \dfrac{du(t)}{dt} = a_1 u(t) - b_1 u^2(t) - \dfrac{mu(t)p(t)}{iu^2(t) + c_1} + \sigma_1 u(t)v(t)p(t) \\ \dfrac{dv(t)}{dt} = a_2 v(t) - b_2 v^2(t) - \dfrac{nv(t)p(t)}{v(t) + c_2} + \sigma_2 u(t)v(t)p(t) \\ \dfrac{dp(t)}{dt} = \dfrac{m_1 u(t)p(t)}{iu^2(t) + c_1} + \dfrac{n_1 v(t)p(t)}{v(t) + c_2} - \delta_1 p(t) - \delta_2 p^2(t) \end{cases} \quad (1.1)$$

where $u(t)$ and $v(t)$ are the densities of the two preys at time t, respectively. $p(t)$ is the density of the predator at time t. a_1 and a_2 are the growth rates of the dangerous prey and the innocent prey, respectively; b_1 and b_2 are the intra-specific components of the dangerous prey and the innocent prey, respectively; c_1 and c_2 are the intra-specific components of the dangerous prey and the innocent prey, respectively; σ_1 and σ_2 are the coefficients of the help between teams of two prey; $\frac{m_1}{m}$ and $\frac{n_1}{n}$ are the conversion efficiencies from the dangerous prey and the innocent prey to the predator, respectively; i represents the inverse measure of inhibitory effect; in [19] authors studied the stability and Hopf bifurcation of system (1.1).

Time delays have been incorporated into predator-prey systems in many related works [30]-[5]. Specially, time delay due to the gestation is a common example in predator-prey systems, because that the increment in the predator population due to predator does not appear immediately after consuming the prey and the predator needs a certain period to reproduce their progeny. Therefore, the present birth rate of the predator depends upon the number of individuals present at time $t - \tau$, where τ should be regarded as the gestation period [30]. Based on the above discussions, we incorporate time delay due to gestation of the predator into system (1.1) and investigate the following delayed predator-prey system:

$$\begin{cases} \dfrac{du(t)}{dt} = a_1u(t) - b_1u^2(t) - \dfrac{mu(t)p(t)}{iu^2(t) + c_1} + \sigma_1u(t)v(t)p(t) \\ \dfrac{dv(t)}{dt} = a_2v(t) - b_2v^2(t) - \dfrac{nv(t)p(t)}{v(t) + c_2} + \sigma_2u(t)v(t)p(t) \\ \dfrac{dp(t)}{dt} = \dfrac{m_1u(t-\tau)p(t-\tau)}{iu^2(t-\tau) + c_1} + \dfrac{n_1v(t-\tau)p(t-\tau)}{v(t-\tau) + c_2} - \delta_1p(t) - \delta_2p^2(t) \end{cases} \quad (1.2)$$

where τ is the time delay due to gestation of the predator. The initial conditions for system (1.2) take the form

$$u(\theta) = \varphi_1(\theta) \geq 0, \qquad v(\theta) = \varphi_2(\theta) \geq 0, \qquad p(\theta) = \varphi_3(\theta) \geq 0 \qquad (1.3)$$

where $\theta \in [-\tau, 0[, \big(\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta)\big) \in C\{[-\tau, 0], R_+^3\}, R_+^3 = \{(u, v, p): u \geq 0, v \geq 0, p \geq 0\}$.

The rest of this paper is structured as follows. In the next section, local stability and existence of Hopf bifurcation are analyzed. In Section 3, direction and stability of the Hopf bifurcation are determined. In Section 4, simulations are presented in order to verify the correctness of the obtained results. Finally, our paper is end with conclusion in Section 5.

2. Hopf Bifurcation and Local Stability Analysis

The characteristic polynomial concerning the positive equilibrium E_* is the following:

$$\lambda^3 + K_2\lambda^2 + K_1\lambda + K_0 + (L_2\lambda^2 + L_1\lambda + L_0)e^{-\lambda\tau} = 0 \qquad (2.1)$$

where

$$K_0 = k_{33}(k_{12}k_{21} - k_{11}k_{22})$$

$$K_1 = k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}k_{21}$$

$$K_2 = -(k_{11} + k_{22} + k_{33})$$

$$L_0 = (k_{12}l_{33} - k_{13}l_{32})k_{21} + (k_{13}k_{22} - k_{12}k_{23})l_{31} + (k_{23}l_{32} - k_{22}l_{33})k_{11}$$

$$L_1 = l_{33}(k_{11} + k_{22}) - k_{13}l_{31} - k_{23}l_{32}L_2 = -l_{33}$$

and

$$k_{11} = \frac{2imu_*^2 p_*}{(iu_*^2 + c_1)^2} - b_1 u_*, k_{12} = \sigma_1 u_* p_*$$

$$k_{13} = \sigma_1 u_* p_* - \frac{mu_*}{iu_*^2 + c_1}$$

$$k_{21} = \sigma_2 v_* p_*, a_{22} = -b_2 v_* + \frac{nv_* p_*}{(v_* + c_2)^2}$$

$$k_{23} = \sigma_2 u_* v_* - \frac{mu_*}{iu_*^2 + c_1}, + k_{33} = -\delta_1 - 2\delta_2 p_*$$

$$l_{31} = \frac{m_1(c_1 - iu_*^2)p_*}{(iu_*^2 + c_1)^2}, l_{33} = \frac{n_1 c_2 p_*}{(v_* + c_2)^2}$$

$$l_{33} = \frac{m_1 u_*}{iu_*^2 + c_1} + \frac{n_1 v_*}{v_* + c_2}$$

Lemma 2.1 *[19] If the condition* (H_1) *holds, that is,*

$2imu_* p_* < b_1(iu_*^2 + c_1)^2,\ np_* < b_2(v_* + c_2)^2,\ v_*(iu_*^2 + c_1) < \frac{m}{\sigma_1}, u_*(v_* + c_2) < \frac{n}{\sigma_2}$ *and* $u_*^2 < \frac{c_1}{i}$,

then $E_*(u_*, v_*, p_*)$ *is locally asymptotically stable (l.a.s) if* $\tau = 0$.

When $\tau > 0$, substituting $\lambda = i\omega(\omega > 0)$ into Eq. (2.1), we obtain

$$-\omega^3 i - K_2\omega^2 + K_1\omega i + (-L_2\omega^2 + L_1\omega i + L_0)e^{-i\omega\tau} = 0 \qquad (2.2)$$

Then from (2.2) separating real and imaginary part we get

$$\begin{cases} L_1\omega \sin\omega\tau + (L_0 - L_2\omega^2)\cos\omega\tau = K_2\omega^2 - K_0 \\ L_1\omega \cos\omega\tau - (L_0 - L_2\omega^2)\sin\omega\tau = \omega^3 - K_1\omega \end{cases} \qquad (2.3)$$

Then, we get

$$\omega^6 + (K_2^2 - 2K_1 - L_2^2)\omega^4 + (K_1^2 - 2K_1K_2 - L_1^2)\omega^2 + K_0^2 - L_0^2 = 0 \qquad (2.4)$$

We assume that (H_2) from (2.4) has at lease one positive root ω_0. Next, on substituting $K_2^2 - 2K_1 - L_2^2 = K_{02}, K_1^2 - 2K_1K_2 - L_1^2 = K_{01}K_{00} = K_0^2 - L_0^2$ and $\omega^2 = z$, then Eq. (2.4) becomes

$$z^3 + K_{02}z^2 + K_{01}z + K_{00} = 0 \qquad (2.5)$$

Eliminating $\sin(\omega\tau)$ from (2.3) and substituting ω_0, where ω_0 is a positive root of (2.4), we get

$$\tau_0 = \frac{1}{\omega_0} \times \arccos\left\{\frac{(K_1 - K_2L_2)\omega_0^4 + (K_0L_2 - K_1L_1 + K_2L_1)\omega_0^2 - K_0L_2}{L_2^2\omega_0^4 + (L_1^2 - 2L_0L_2)\omega_0^2 + L_0^2}\right\}$$

Now, differentiating Eq. (2.1) with respect to τ and then substituting $\lambda = i\omega_0$, we have

$$Re\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{f'(z_0)}{L_2^2\omega_0^4 + (L_1^2 - 2L_0L_2)\omega_0^2 + L_0^2}$$

where $f(z) = z^3 + K_{02}z^2 + K_{01}z + K_{00}$ and $z_0 = \omega_0^2$. Thus, if $f'(z_0) \neq 0$, then

$$Re\left[\frac{d\lambda}{d\tau}\right]_{\tau=\tau_0}^{-1} \neq 0$$

Summarizing the above analysis, we can obtain the following results.

Theorem 2.2. For system (1.2), if the conditions (H_1)-(H_2) hold, then the positive equilibrium $E_*(u_*, v_*, p_*)$ is locally asymptotically stable when $\tau \in [0, \tau_0)$; system (1.2) undergoes a Hopf Bifurcation at $E_*(u_*, v_*, p_*)$ when $\tau = \tau_0$ and a family of periodic solutions bifurcate from $E_*(u_*, v_*, p_*)$.

3. Properties of the Hopf Bifurcation

According to [35] we can define the following statements. Let $\mu = \tau - \tau_0$ with $\mu \in R$, then $\mu = 0$ is the Hopf bifurcation value of system (1.2).

Let $u_1(t) = u(t) - u_*, u_2(t) = v(t) - v_*, u_3(t) = p(t) - p_*$, and normalize τ by $t \to (t/\tau)$. Then, system (1.2) can be transformed into the equation in the phase space $C = C([-1,0], R^3)$ as follows

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t)$$

where $u(t) = (u_1, u_2, u_3)^T \in R^3, L_\mu: C \to R^3, F: R \times C \to R^3$with

$$L_\mu\phi = (\tau_0 + \mu)\big(K\phi(0) + L\phi(-1)\big)$$

where

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ l_{31} & k_{32} & k_{33} \end{pmatrix}$$

and

$$F(\mu,\phi) = (\tau_0 + \mu)(F_1, F_2, F_3)^T$$

with

$$F_1 = k_{14}\phi_1(0)\phi_2(0) + k_{15}\phi_1(0)\phi_3(0) + k_{16}\phi_2(0)\phi_3(0) + k_{17}\phi_1^2(0) + k_{18}\phi_1^2(0)\phi_3(0) + k_{19}\phi_1^3(0) + k_{110}\phi_1(0)\phi_2(0)\phi_3(0) + \cdots,$$

$$F_2 = k_{24}\phi_1(0)\phi_2(0) + k_{25}\phi_1(0)\phi_3(0) + k_{26}\phi_2(0)\phi_3(0) + k_{27}\phi_2^2(0) + k_{28}\phi_2^2(0)\phi_3(0) + k_{29}\phi_2^3(0) + k_{210}\phi_1(0)\phi_2(0)\phi_3(0) + \cdots,$$

$$F_2 = k_{34}\phi_3^2(0) + l_{34}\phi_1(-1)\phi_3(-1) + l_{35}\phi_2(-1)\phi_3(-1) + l_{36}\phi_1^2(-1) + l_{37}\phi_2^2(-1) + l_{38}\phi_1^2(-1)\phi_3(-1) + l_{39}\phi_2^2(-1)\phi_3(-1) + l_{310}\phi_1^3(-1) + l_{311}\phi_2^3(-1) + \cdots$$

and

$$l_{14} = \sigma_1 p_*, l_{15} = \frac{m(iu_*^2 - c_1)}{(iu_*^2 + c_1)^2} + \sigma_1 v_*, l_{16} = \sigma_1 u_*$$

$$l_{17} = \frac{imu_* p_*(3c_1 - iu_*^2)}{(iu_*^2 + c_1)^3} - b_1, l_{18} = \frac{imu_*(3c_1 - iu_*^2)}{(iu_*^2 + c_1)^3}$$

$$l_{19} = \frac{imp_*}{(iu_*^2 + c_1)^2} - \frac{8i^2 mc_1 u_*^2 p_*}{(iu_*^2 + c_1)^4}, l_{110} = \sigma_1$$

$$k_{24} = \sigma_2 p_*, k_{25} = \sigma_2 v_*, k_{26} = \sigma_2 u_* - \frac{nc_2}{(v_* + c_2)^2}$$

$$k_{27} = \frac{nc_2 p_*}{(v_* + c_2)^3} - b_2, k_{28} = \frac{nc_2}{(v_* + c_2)^3}$$

$$k_{29} = \frac{nc_2 p_*}{(v_* + c_2)^4}, k_{210} = \sigma_2$$

$$k_{34} = -\delta_2, l_{34} = \frac{m_1(c_1 - iu_*^2)}{(iu_*^2 + c_1)^2}, l_{35} = \frac{n_1 c_2}{(v_* + c_2)^2}$$

$$l_{36} = \frac{im_1 u_* p_*(iu_*^2 - 3c_1)}{(iu_*^2 + c_1)^3}, l_{37} = -\frac{n_1 c_2 p_*}{(v_* + c_2)^3}$$

$$l_{38} = \frac{im_1u_*(iu_*^2 - 3c_1)}{(iu_*^2 + c_1)^3}, l_{39} = -\frac{n_1c_2}{(v_* + c_2)^3}$$

$$l_{310} = \frac{8i^2m_1c_1u_*^2p_*}{(iu_*^2 + c_1)^4} - \frac{im_1p_*}{(iu_*^2 + c_1)^2}, l_{311} = -\frac{n_1c_2p_*}{(v_* + c_2)^4}.$$

By using Riesz representation theorem there exists a function $\eta(\theta, \mu)$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \phi \in C \qquad (3.2)$$

Choosing

$$\eta(\theta, \mu) = (\tau_0 + \mu)\big(K\delta(\theta) + L\delta(\theta + 1)\big)$$

and $\delta(\theta)$ is the Dirac delta function. For $\phi \in C([-1,0], R^3)$, define the operator $A(\mu)$.

$$A(\mu_0)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & 1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu_0)\phi(\theta), & \theta = 0 \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0 \\ F(\mu_0, \phi), & \theta = 0 \end{cases}$$

Then, system (1.2) becomes

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t \qquad (3.3)$$

For $\varphi \in C^1([o, 1], (R^3)^*)$, define A^*

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s < 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s, 0)\varphi(-s), & s = 0 \end{cases}$$

and a bilinear product

$$< \varphi(s), \phi(\theta) > = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{\varphi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi \qquad (3.4)$$

with $\eta(\theta) = \eta(\theta, 0)$.

Then, we can conclude that $i\omega_0$ are eigenvalues of $A(0)$ and $A^*(0)$. Suppose that

$\rho(\theta) = (1, \rho_2, \rho_3)^T e^{i\omega_0\theta}$ and $\rho^*(s) = V(1, \rho_2^*, \rho_3^*)^T e^{i\omega_0\theta}$ are the corresponding eigenfunctions. By direct calculation we obtain

$$q_2 = \frac{i\omega_0 - a_{33} - b_{33}e^{-i\omega_0}}{a_{32}} q_3$$

$$q_3 = \frac{a_{32}(i\omega_0 - a_{11})}{a_{12}(i\omega_0 - a_{33} - b_{33}e^{-i\omega_0})}$$

$$q_2^* = -\frac{i\omega_0 + a_{11}}{a_{21}}$$

$$q_3^* = \frac{a_{32}(i\omega_0 + a_{11}) - a_{21}b_{13}e^{i\omega_0}}{a_{21}(i\omega_0 + a_{33} + b_{33}e^{i\omega_0})}$$

From $< \rho^*, \rho >= 1$ we have

$$\bar{V} = \left[1 + \rho_2\bar{\rho}_2^* + \rho_3\bar{\rho}_3^* + \tau_0 e^{-i\tau_0\omega_0}(b_{13} + b_{33}\bar{\rho}_3^*)\right]^{-1}$$

Using the algorithm given in [32] and the similar computation process to that in [28], we can obtain the expressions of g_{20}, g_{11}, g_{02} and g_{21} as follows:

$$g_{20} = 2\bar{V}\tau_0[a_{13}\rho_2 + a_{14}\rho_2^2 + \bar{\rho}_2^*(a_{24}\rho_2 + a_{25}\rho_2^2)]$$

$$g_{11} = \bar{V}\tau_0[a_{13}(\rho_2 + \bar{\rho}_2) + 2a_{14}\rho_2\bar{\rho}_2 + \bar{\rho}_2^*(a_{24}(\rho_2 + \bar{\rho}_2) + 2a_{25}\rho_2\bar{\rho}_2)]$$

$$g_{02} = 2\bar{V}\tau_0[a_{13}\bar{\rho}_2 + a_{14}\bar{\rho}_2^2 + \bar{\rho}_2^*(a_{24}\bar{\rho}_2 + a_{25}\bar{\rho}_2^2)]$$

$$\begin{aligned} g_{21} = 2\bar{V}\tau_0 \Bigg[& a_{13}W_{11}^{(1)}(0)\rho_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0) \\ & + a_{14}\left(2W_{11}^{(2)}(0)\rho_2 + W_{20}^{(2)}(0)\bar{\rho}_2\right) \\ & + \bar{\rho}_2^*\left(a_{24}\left(W_{11}^{(1)}(0)\rho_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\right)\right. \\ & \left. + a_{25}\left(2W_{11}^{(2)}(0)\rho_2 + W_{20}^{(2)}(0)\bar{\rho}_2\right)\right)\Bigg] \end{aligned}$$

with

$$W_{20}(\theta) = \frac{ig_{20}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}\bar{\rho}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_1e^{2i\tau_0\omega_0\theta}$$

$$W_{11}(\theta) = \frac{ig_{11}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{11}\bar{\rho}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_2$$

E_1 and E_2 can be solved by

$$E_1 = 2\begin{pmatrix} 2i\omega_0 - a_{11} & -a_{12} & -b_{13}e^{-2i\tau_0\omega_0} \\ -a_{21} & 2i\omega_0 - a_{22} & -a_{23} \\ 0 & -a_{32} & 2i\omega_0 - a_{33} - b_{33}e^{-2i\tau_0\omega_0} \end{pmatrix}^{-1} \times \begin{pmatrix} a_{13}\rho_2 + a_{14}\rho_2^2 \\ a_{24}\rho_2 + a_{25}\rho_2^2 \\ 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} a_{11} & a_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} + b_{33} \end{pmatrix}^{-1} \times \begin{pmatrix} a_{13}(\rho_2 + \bar{\rho}_2) + 2a_{14}\rho_2\bar{\rho}_2 \\ a_{24}(\rho_2 + \bar{\rho}_2) + 2a_{25}\rho_2\bar{\rho}_2 \\ 0 \end{pmatrix}$$

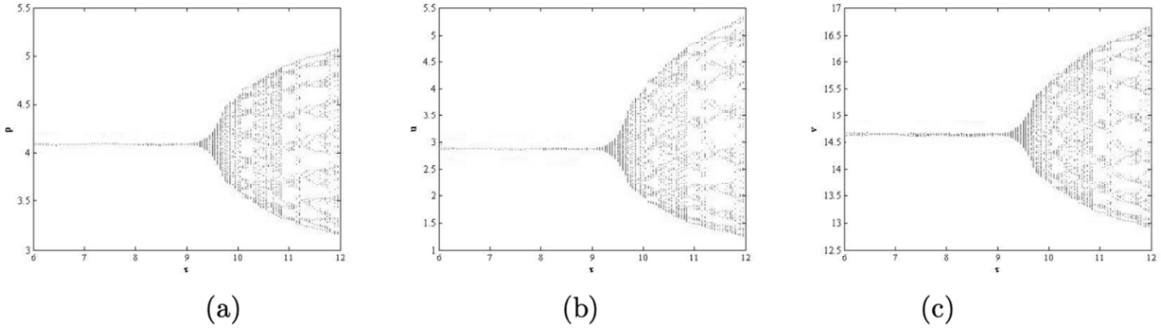
Thus, we have

$$C_1(0) = \frac{i}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \qquad \mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}}, \qquad \beta_2 = 2Re\{C_1(0)\}, \qquad T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\tau_0\omega_0} \tag{3.5}$$

where μ_2 determines the direction of the Hopf bifurcation; β_2 determines the stability of the bifurcating periodic solutions and T_2 determines the period of the bifurcating periodic solutions. In conclusion, we can obtain the following results based on the fundamental results about Hopf bifurcation in the literature [32].

According to [35], we have the following result:

Theorem 3.1. For system (1.2), the Hopf bifurcation is supercritical (or subcritical) if $\mu_2 > 0$ (or $\mu_2 < 0$); the bifurcating periodic solutions are stable (or unstable) if $\beta_2 < 0$ (or $\beta_2 > 0$); the period of the bifurcating periodic solutions increase (or decrease) if $T_2 > 0$ (or $T_2 < 0$).



(a) (b) (c)

Figure 1: Bifurcation diagram with respect to time delay

4. Numerical Simulation

Choosing

$$a_1 = 0.25, b_1 = 0.02, m = 0.45, i = 0.035, c_1 = 7, \sigma_1 = 0.001, a_2 = 0.3, b_2 = 0.02,$$
$$n = 0.3, c_2 = 8, \sigma_2 = 0.004, m_1 = 0.1, n_1 = 0.1, \delta_1 = 0.1, \delta_2 = 0.001.$$

Then, system (1.2) becomes

$$\begin{cases} \dfrac{du(t)}{dt} = 0.25u(t) - 0.02u^2(t) - \dfrac{0.45(t)p(t)}{0.035u^2(t) + c_1} + 0.001u(t)v(t)p(t) \\ \dfrac{dv(t)}{dt} = 0.3v(t) - 0.02v^2(t) - \dfrac{0.3v(t)p(t)}{v(t) + 8} + 0.004u(t)v(t)p(t) \\ \dfrac{dp(t)}{dt} = \dfrac{0.1u(t-\tau)p(t-\tau)}{0.035u^2(t-\tau) + c_1} + \dfrac{0.1v(t-\tau)p(t-\tau)}{v(t-\tau) + 8} - 0.101p(t) - \delta_2 p^2(t) \end{cases}$$

5. Conclusions

In this paper, we proposed a delayed mathematical model for one predator sharing teams of two preys by considering time delay due to the dynamics of the predator into the model formulated in [19]. The aim of this research is to investigate the effect of the time delay on stability of the proposed model. It is found that the time delay due to behavior of the predator is responsible for the stability of the model. When the time delay is suitable small, then the model is locally asymptotically stable. Once the delay passes through a certain critical value, the model will lose its stability and a Hopf Bifurcation occurs, which indicates that the species in the model will coexist periodic oscillation mode under some certain conditions. The obtained results in the present paper are supplements to [19].

Part II

Complex Dynamics of a Model with R&D Competition

The paper analyzes a two-stage oligopoly game of semi-collusion in production described by a system with a symmetric structure. We examine the local stability of a Nash equilibrium and the presence of bifurcations. We discover that the model is capable of exhibiting extremely complicated dynamic behaviors.

1. Introduction

Research and development (R&D) are critical to a company's growth. They enable the development of lower manufacturing costs, increase production productivity, and improve the quality of goods. The existing theories of oligopolies, particularly within R&D, have raised concerns about the position of R&D spillover in bringing about process innovation. Kamien *et al.* [10] and D'Aspremont and Jacquemin [7] are the most significant contributors to this sector. [7] first developed a two-stage game in which companies simultaneously aimed to reduce their R&D expenditures and obtain research development spillovers from their rivals.

Firms compete in a duopoly game in which both stages, each one with a spillover impact from the other in the outcome of R&D operation. Then, Kamien *et al.* [10] presented four possible versions. In these models, both companies reduce the expense of R&D acquisitions by the use of Research Joint Ventures (RJVs) or R&D cartelization. Following that, they commit themselves to compete in the product market through a Bertrand or Cournot game. The four models acknowledge that spillovers have impacted study works to varying degrees during the research and development period.

Joint venture establishment, which is linked to commodity export rivalry, is the most ideal and attractive strategy since it results in higher revenues and lower product costs. Qiu [13] and Amir *et al.* [1], had a similar approach to their studies. Shibata [14] recently investigated R&D investment spillovers in a variety of business systems. Specifically, he expanded Matsumura *et al.* [11] attempts to integrate R&D investment spillovers.

The variety of oligopoly games, in recent years, has grown substantially. Cavalli and Naimzada [5] analyzed oligopoly models with various decision-making structures and rationality degrees. Dubiel-Teleszynski [9] investigated heterogeneous Cournot games with adaptive and bounded participants.

Peng *et al.* [12] developed a duopoly Stackelberg rivalry model and explored the presence, steadiness and bifurcation of equilibrium points. Ding *et al.* [8] presented a dynamical Cournot game associated with rationality and time delay for marginal benefit maximization. Many complex dynamical behaviors were found.

Overall, single-stage games are analytically tractable but not useful for simulations. However, concerning technical commodities, several dynamic factors are critical. Currently, a two-stage oligopoly game has increased the attention of academics. Bischi and Lamantia [2,3] suggested a two-stage oligopoly game to model firms' R&D networks. Matsumura *et al.* [11] assumed a two-stage Cournot model in which companies decide on R&D spending first and then on production amounts. Shibata [14] analyzed the spillover effects of R&D spending through a variety of industry systems. He expands the function of Matsumura *et al.* [11] in particular by including R&D investment spillovers.

Throughout history, companies operating in oligopolistic economies have been strongly advised by traditional theory that collaboration is preferable to competition on quantity or price. However, Fershtman and Gandal's [15] research indicated that collusion in the commodity sector might lower the total income. This is due mainly to the rivalry in various aspects of contact.

Zhang *et al.* [16] built a dynamical two-stage duopoly game based on Fershtman and Gandal's research results, assuming that the business has a linear inverse demand function and the firms are bounded rational. They believed that companies compete in the R&D stage and allowed for spillover at this stage. Both companies engage in collusive behavior in the product sector. That is, each company determines its amounts by the aggregate income. Cao *et al.* [4] revisited Zhang *et al.* [16] model by extending the related discrete time to the case of continuous time with delay.

In this paper, by taking the model of Zhang *et al.* [16] as a starting point, we propose an alternative modeling approach to transform the discrete-time model into a continuous delayed time model. Through the study of stability of the stationary equilibrium point, we observe switches from stability to instability to stability and characterize the birth of Hopf Bifurcations.

The plan of this paper is as follows. Section 2 introduces the model with R&D competition augmented time delay. Section 3 provides an analysis of the stability properties of the dynamic system. Section 4 outlines our conclusions.

2. The Model

In this section, we revisit the model by Zhang *et al.* [16], where, instead of discrete time scales, continuous time scales are assumed. The model consists of a repeated two-stage game of semi-collusion in production with two companies in an oligopoly market, where the cost and inverse

demand functions are linear, starting with a repetitive two-stage game. In the first point, we presume that both companies compete in R&D, but that spillover is permitted. In the second level, we presume that companies organize their output in accordance with the shared benefit. The model that emerges is made up of a two-dimensional map that defines the time evolution of the dynamic game:

$$\begin{aligned} x_1(t+1) &= x_1(t) + v_1 x_1(t)[A_1 + Bx_1(t) + Cx_2(t)] \\ x_2(t+1) &= x_2(t) + v_2 x_2(t)[A_2 + Bx_2(t) + Cx_1(t)] \end{aligned} \qquad (1)$$

with x_1 and x_2 the R&D efforts of firm 1 and 2, respectively, $A_1 > 0,\ A_2 > 0$. A symmetry of parameters v_1 and v_2 exists in this system. System (1) has three boundary equilibria

$$E_0 = (0,0), \qquad E_1 = \left(-\frac{A_1}{B}, 0\right), \qquad E_1 = \left(0, -\frac{A_2}{B}\right)$$

and a Nash equilibrium

$$E_* = \left(\frac{A_2C - A_1B}{B^2 - C^2}, \frac{A_1C - A_2B}{B^2 - C^2}\right)$$

which is stable, provided that $(A_1, A_2, B, C) \in S_1$ where

$$S_1 = \{(A_1, A_2, B, C) \ : \ A_1 > 0, A_2 > 0, B < 0, |B| > |C|, \Delta_1 > 0, \Delta_2 > 0\},$$

which $\Delta_1 = A_2C - A_1B$ and $\Delta_2 = A_1C - A_2B$. In order to have a continuous version of system (1), we notice (1) is expressed as $x_l(t+1) = \phi(x_l(t)), l = 1, 2$. Rewriting this as

$$[x_l(t+1) - x_l(t)] = -x_l(t) + \phi(x_l(t))$$

assuming that the terms in brackets approximates $\partial x_l(t)/\partial t$ and the existence of a time delay between the awareness of the right-hand side expression and the knowledge of the left-hand side expression, we obtain the following time-delayed model:

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_1(t-\tau) + v_1 x_1(t-\tau)[A_1 + Bx_1(t-\tau) + Cx_2(t-\tau)] \\ \dot{x}_2(t) &= -x_2(t) + x_2(t-\tau) + v_2 x_2(t-\tau)[A_2 + Bx_2(t-\tau) + Cx_1(t-\tau)] \end{aligned} \qquad (2)$$

It is clear that the steady states of the delay model (2) are the same of (1). Since we are interested in considering the effect of delay τ on the system dynamics, for this purpose, we will analyze the linearized version of (2) about the equilibrium keeping in mind that the Nash equilibrium will be

locally asymptotically stable if and only if each of the characteristic roots has negative real parts (see, e.g., [15]).

3. Stability and Existence of Bifurcations

The linearization of (2) around E_* has a characteristic equation expressed by

$$\begin{vmatrix} -1-\lambda+(Mv_1+Bv_1x_1^*)e^{-\lambda\tau} & Cv_1x_1^*e^{-\lambda\tau} \\ Cv_2x_2^*e^{-\lambda\tau} & -1-\lambda+(Nv_2+Bv_2x_2^*)e^{-\lambda\tau}-\lambda \end{vmatrix} = 0$$

where $M = A_1 + Bx_1^* + Cx_2^* + Bv_1x_1^*$ and $N = A_2 + Bx_2^* + Cx_1^*$. Calculating the determinant gives

$$\lambda^2 + 2\lambda + 1 + (m + m\lambda)e^{-\lambda\tau} + ne^{-2\lambda\tau} = 0 \qquad (3)$$

where

$$m = -2 - B(v_1x_1^* + v_2x_2^*), \qquad n = 1 + B(v_1x_1^* + v_2x_2^*) + (B^2 - C^2)v_1v_2x_1^*x_2^*$$

We now employ the method proposed in Chen *et al*. [6] to analyze the distribution of characteristic roots. It is clear that the marginal stability is determined by the equations $\lambda = 0$ and $\lambda = i\omega(\omega > 0)$. The case $\lambda = 0$ is simple. Substituting $\lambda = 0$ into Equation (3), one obtains $1 + m + n = (B^2 - C^2)v_1v_2x_1^*x_2^* \neq 0$. Therefore, the characteristic Equation (3) has a zero root. The case $\lambda = i\omega(\omega > 0)$ is instead more complicated. Substituting $\lambda = i\omega$ into Equation (3) yields

$$-\omega^2 + 2i\omega + 1 + (mi\omega + m)e^{-i\omega\tau} + ne^{-2i\omega\tau} = 0 \qquad (4)$$

If $(\omega t)/2 \neq (\pi/2) + j\pi, j \in \mathbb{Z}$, then let $\theta = \tan[(\omega t)/2]$, and we have

$$e^{-i\omega\tau} = \frac{1 - i\theta}{1 + i\theta} \qquad (5)$$

Separating the real and imaginary parts, we obtain that θ satisfies

$$\begin{matrix} (\omega^2 - 1 + m - n)\theta^2 - 4\omega\theta = \omega^2 - 1 - m - n \\ (m-2)\omega\theta^2 - 2(\omega^2 - 1 + n)\theta = -(2+m)\omega \end{matrix} \qquad (6)$$

Define

$$D(\omega) = \begin{vmatrix} \omega^2 - 1 + m - n & -4\omega \\ (m-2)\omega & -2(\omega^2 - 1 + n) \end{vmatrix}$$

$$E(\omega) = \begin{vmatrix} \omega^2 - 1 + m - n & -4\omega \\ -(2+m)\omega & -2(\omega^2 + 1 - n) \end{vmatrix}$$

and

$$F(\omega) = \begin{vmatrix} \omega^2 - 1 + m - n & \omega^2 - 1 - m - n \\ (m-2)\omega & -(2+m)\omega \end{vmatrix}$$

We find that ω satisfies

$$D(\omega)E(\omega) = [F(\omega)]^2$$

which is a polynomial equation for ω with degree 8:

$$p(\omega) = \omega^8 + s_3\omega^6 + s_2\omega^4 + s_1\omega^2 + s_0 = 0 \qquad (7)$$

with

$$s_3 = 4 - m^2, \qquad s_2 = 6 - 2n^2 + (2n-3)m^2$$

$$s_1 = 4 - 4n^2 + 3(n-1)m^2, \qquad s_0 = (1-n)^2[-m^2 + (1+n)^2]$$

Moreover, it is easy to see that ω^2 is a positive root of the following equation:

$$z^4 + s_3z^3 + s_2z^2 + s_1z + s_0 = 0 \qquad (8)$$

If, instead $(\omega t)/2 = (\pi/2) + j\pi, j \in \mathbb{Z}$, then $D(\omega) = F(\omega) = 0$, and thus ω^2 is still a positive root of (8).

Lemma 1.

1. *If $\pm i\omega(\omega > 0)$ is a pair of purely imaginary roots of the characteristic equation, then ω^2 is a positive root of the above quartic polynomial equation.*
2. *There exists at least one positive solution to* (7) *provided that $n \neq 1$ and $|1+n| < |m|$.*

Proof. The first statement follows from the previous analysis. Since the leading term of the polynomial (7) is ω^8, the polynomial $p(\omega)$ tends to infinity as $\omega \to \infty$. Moreover, $p(0) = s_0 < 0$. Therefore, there is some $\omega > 0$ such that $p(\omega) = 0$. □

The previous condition is only sufficient for $p(\omega)$ to have positive solutions. Even if the parameters do not satisfy it, we can always compute all positive solutions of $p(\omega) = 0$ since $p(\omega)$ is an 8-degree polynomial. Additionally, there are no more than eight positive solutions.

Proposition 1. *If the quartic Equation* (8) *has a positive root* ω_*^2, $\omega_* > 0$, *and* $D(\omega_*) \neq 0$, *then Equation* (6) *has a unique real root* $\theta_* = F(\omega_*)/D(\omega_*)$. *Hence, Equation* (4) *has a pair of purely imaginary roots* $\pm i\omega_*$ *when*

$$\tau = \tau_*^j = \frac{2\tan^{-1}(\theta_*) + 2j\pi}{\omega_*}, \qquad j \in \mathbb{Z} \qquad (9)$$

Proof. From (6), we find $\theta_*^2 = E(\omega_*) \,/\, D(\omega_*)$ and $\theta_* = F(\omega_*)/D(\omega_*)$.
Thus, $D(\omega_*)E(\omega_*) = [F(\omega_*)]^2$. Consequently, (3) has a pair of purely imaginary roots $\pm i\omega_*$ when τ is defined as in (9). This completes the proof of the Proposition. □
The next result establishes that a curve of simple root $\lambda(\tau)$ occurs and travels transversally around the imaginary axis under some conditions of transversality.

Proposition 2. $\lambda = i\omega_*(\omega_* > 0)$ *is a simple root of* (3) *at* $\tau = \tau_*^j$, $j \in \mathbb{Z}$. *The crossing direction through the imaginary axis is determined by the sign of*

$$G(\omega_*, \theta_*) = \left\{\left[(m-2n)\tau_*^j - m + 2\right]\theta_*^2 + 4\omega_*\theta_* + (m+2n)\tau_*^j - m - 2\right\}[m\omega_*\theta_*^2 - 4n\theta_* + m\omega_*]$$
$$- \left[\left(m\tau_*^j + 2\right)\omega_*\theta_*^2 - 4\left(n\tau_*^j + 1\right)\theta_* + \left(m\tau_*^j - 2\right)\omega_*\right][(m-2n)\theta_*^2 + m + 2n]$$

Proof. First, we differentiate (3) with respect to τ and obtain

$$P(\lambda, \tau)\frac{d\lambda}{d\tau} = \lambda Q(\lambda, \tau) \qquad (10)$$

where we denote

$$P(\lambda, \tau) = 2\lambda + 2 + me^{-\lambda\tau} - (m\lambda + m)\tau e^{-\lambda\tau} - 2n\tau e^{-2\lambda\tau}$$

$$Q(\lambda, \tau) = (m\lambda + m)e^{-\lambda\tau} + 2ne^{-2\lambda\tau}$$

The fact that $\lambda = i\omega_*$ is a simple root follows from (10). We argue by contradiction. If not, then $Q(i\omega_*, \tau_*^j) = 0$ holds true. Using (5), this yields

$$mi\omega_* + m + 2n(1 - i\theta_*)(1 + i\theta_*) = 0$$

Thus, $m\omega_* - 2n\theta_* = 0$ and $-m\omega_*\theta_* + 2n = 0$, which leads to $\pm 1 = \theta_* = \tan\left[\left(\omega\tau_*^j\right) / 2\right]$, i.e., the absurd $\left(\omega\tau_*^j\right)/2 = (\pi \,/\, 4) + j\pi$.

We are now left to verify the transversality condition. For simplicity, we consider $d\tau/d\lambda$ instead of $d\lambda/d\tau$. In light of (10), we can find

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_*} = \frac{P\left(i\omega_*, \tau_*^j\right)}{i\omega_* Q\left(i\omega_*, \tau_*^j\right)} = -\frac{iP\left(i\omega_*, \tau_*^j\right)\overline{Q\left(\imath\omega_*, \tau_*^j\right)}}{\omega_* Q\left(i\omega_*, \tau_*^j\right)\overline{Q\left(\imath\omega_*, \tau_*^j\right)}}$$

By straightforward calculations, we can find

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_*} = \frac{G(\omega_*, \theta_*)}{(1+\theta_*^2)^2\omega_*\left|Q\left(i\omega_*, \tau_*^j\right)\right|^2}$$

Finally, we can conclude

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\bigg|_{\lambda=i\omega_*}\right\} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_*}\right\} = sign\{G(\omega_*, \theta_*)\}$$

Therefore, we complete our proof. □

Remark 1. *A positive sign of $G(\omega_*, \theta_*)$ means that each crossing of the real part of characteristic roots at τ_*^j must be from left to right, whereas a negative sign indicates that the real part of a pair of conjugate roots of Equation* (3) *changes from positive value to negative value when τ_*^j is crossed.*

Due to the discussions and facts above, we have the following conclusions.

Theorem 1.

1. *If Equation* (7) *has no positive root, then the equilibrium E_* of system* (2) *is locally asymptotically stable for $\tau \geq 0$.*
2. *If Equation* (7) *has a unique positive root ω_*, then there exists a $\tau_* > 0$, where $\tau_* = \min\{\tau_*^j, j \in \mathbb{Z}\}$, such that the equilibrium E_* of system* (2) *is locally asymptotically stable when $\tau \in [0, \tau_*)$. As τ increases, the system dynamic may switch from stable to unstable, a Hopf Bifurcation occurs, and then back to stable, and so on, according to $sign[G(\omega_*, \theta_*)]$.*
3. *If Equation* (7) *has at least two positive roots, then there may exist many stability switches, with the occurrence of a Hopf bifurcation at each switch.*

4. Concluding Remarks

In this work, we considered a continuous-time version with delays of a discrete model with R&D competition. Our modification of the discrete model was suggested by real economic situations

where there are always time delays between the times when information is obtained and when the decisions are implemented. Delay Differential Equations were used to model the resulting system.

Employing results on the distribution of the zeros of transcendental functions, we found a set of conditions to determine the stability of the Nash equilibrium point and the existence of Hopf bifurcations. More precisely, as the delay parameter increased, stability loss and gain emerged (which could repeat alternatively), and complicated, chaotic behavior dynamics emerged.

This scenario did not occur in the discrete version model. Chaotic behavior is undesirable in an economic system since chaos means market confusion or irregularity. When the market runs irregularly, product prices, firm earnings, and R&D spending all suffer. Therefore, all market players expect a stable market and hope to bring the market back to equilibrium when it is unbalanced. Hence, time delay has the important dual role of destabilizer and stabilizer.

Hopf Bifurcation Analysis in a Modified R&D Model with Delay

We consider a duopoly game model of R&D (research and development) rivalry between two high-tech firms with time delay, in which a monopoly sector with R&D spillover is studied using a mixture of game theory and nonlinear dynamics theory. The local asymptotic stability of the equilibrium point is studied by analyzing the corresponding characteristic equation. It is found that the delay can lead the system dynamic behavior to exhibit stability switches and Hopf bifurcations appear.

1. Introduction

With the exponential advancement of economics, business rivalry has been increasingly fierce. If companies, who are an essential component of the industry, are to survive and stay invincible in the market, they must develop a major competition that is distinct from that of other firms. Research and development (or R&D for short) has emerged as the primary driver of corporate growth and is, therefore, an essential means for businesses to achieve core competitiveness. In reality, enterprises' R&D activities will lower manufacturing costs, enhance product quality, increase market share, and boost their market competitiveness. However, in the modern world, R&D spillovers often occur while R&D operations are carried out. Because of the knowledge sharing of R&D among firms and the flow of human capital, R&D spillovers are unavoidable.

Entrepreneurs and economists have been paying attention to competitiveness and collaboration in R&D investment in recent years. The classic research dates back to the 1980s when D'Aspremont and Jacquemin [10] developed a popular duopoly model of technology spillover. Ziss [25] created a two-stage duopoly game model with R&D spillovers based on the AJ and KMZ models. Amir [2] looked into the consequences of R&D spillover and found that as the volume of technology outflow increased, so did the expense of R&D for businesses. Katsoulacos and Ulph [15] have studied endogenous spillovers in R&D collaboration. Yin [21] looked at the asymmetric R&D cooperation system and found that increasing market share is one of the main reasons for delivering collaborative creativity. Under asymmetrical R&D spillover, Atallah [4] investigated the impact of spillover levels on cooperative motivation and shared benefit. Chi *et al.* [9] built a two-stage game theoretic model to explore the cooperative and noncooperative R&D activities in alternative fuel vehicles with spillovers. Wei-wen *et al.* [20] discussed the impact of mixed spillovers and product differentiation on duopoly enterprises equilibrium output and profits in different cooperative states using the AJ model.

Bischi *et al.* [5,6] developed a two-stage competitive game based on industry information share and R&D competitiveness and examined the model's stability. Zhang *et al.* [23] looked at a two-stage duopoly model with semi-collusion in development, and they investigated the model's dynamic disorderly behavior. The multi-stability of a three-dimensional dynamical oligopoly model was investigated by Agiza *et al.* [1] In their article, they looked at the paths to complex attractors and the formation of basins with complex structures. Cavalli and Naimzada [7] created an oligopoly model with several rational players and then investigated the complex dynamical behaviors caused by the coexistence of multiple attractors in the model. It is not difficult to conclude from the abovementioned research findings that R&D rivalry between duopoly firms that manufacture complementary goods is rarely examined. However, R&D rivalry between companies that make complementary goods is prevalent in the real world. Li and Ma [16] studied a bounded rational dual-channel game, simulating the model's complicated dynamical behavior. Many academics have looked at the complicated dynamical behaviors of nonlinear oligopolies from a variety of perspectives, including differentiated products [3-22], heterogeneous companies [14,19], and delayed choices [11,17].

Zhou and Wang [24] produced a two-stage duopoly game model with R&D spillover, complementary goods, and joint benefit maximization as the backdrop. According to their study, the firm learns its marginal profitability at time t and uses that knowledge to produce at time $t+1$. In other words, the monopolist decides to begin production at time t with a product that will be effective in the period $[t, t+1)$. At time $t+1$, this production process will deliver the quantity $q(t+1)$ to the market. With this type of modeling approach, markets are open at discrete-time intervals, and no trading happens in the interval of time $(t, t+1)$. One of the benefits of working with a continuous-time arrangement is that it allows markets to remain open at all times, allowing for the consideration of some latencies related to the firm's information set or production technology through time delays (gestation lags). Pansera *et al.* [18] converted their model into a continuous-time model by introducing a time delay. It was found that the Nash equilibrium was destabilized via Hopf bifurcation. Similarly, Ferrara *et al.* [13] transformed the discrete model of Zhang *et al.* [23] into a continuous delayed time model and found the model to be capable of exhibiting extremely complicated dynamic behaviors.

This paper considers a different modeling approach to study a duopoly game model of R&D rivalry based on the model by Zhou and Wang [24]. In Pansera *et al.* [18], they assumed that production is immediately available, but there exists a time lag between the time at which the firm computes its own marginal profit $(t-\tau)$ and the time at which such a marginal profit is used to produce the final output (t). In contrast, the instantaneous variation of production is now based on the differential

existing between the target (based on past information with a delay τ) and current production. If such a differential is positive (resp. negative), production will tend to increase (resp. reduce). Unlike the mechanism detailed previously, the explicit knowledge of production at time t is now necessary to adjust production. We demonstrate that depending on how a discrete-time model is transformed into a continuous-time model, various conclusions about the local stability of the Nash equilibrium can be obtained. It is then seen how equilibrium can be switched to chaos and instability can be shifted back to stability. As a result, the time delay has both a destabilizing and a stabilizing impact on the system.

The paper develops as follows. Section 2 presents the model. Section 3 studies its dynamical properties and analyses the bifurcation. Section 4 gives the numerical example. The final section outlines the conclusions.

2. The Model

In this section, we transform the discrete duopoly game model of R&D competition between two high-tech enterprises of Zhou and Wang [24] using a different approach than the one presented in Pansera *et al.* [18]. To this purpose, we note that their system $x_m(t+1) = \phi(x_m(t))$, with x_m the R&D effort of firm $m(m = 1, 2)$, is also equivalent

to $[x_m(t+1) - x_m(t)] = -x_m(t) + \phi(x_m(t))$. Hence, by assuming that the term in brackets represents an approximation of $\partial x_m(t)/\partial t$, and also the existence of a time delay about the knowledge of the expression on the right-hand side, we derive the following time-delayed model

$$\dot{x}_1(t) = -x_1(t) + x_1(t-\tau)\left\{\frac{4(\beta+1)^2 b}{9}[x_1(t-\tau) + x_2(t-\tau)] + \frac{4(a-2bc)(\beta+1)}{9} - \gamma x_1(t-\tau)\right\}$$

$$\dot{x}_2(t) = -x_2(t) + x_2(t-\tau) + \alpha_2(t-\tau)\left\{\frac{4(\beta+1)^2 b}{9}[x_1(t-\tau) + x_2(t-\tau)] + \frac{4(a-2bc)(\beta+1)}{9} - \gamma x_2(t-\tau)\right\}$$

where the parameters $a > 0$, $b > 0$ and $c > 0$, respectively, reflect the market size, commodity price sensitivity, and unit cost of manufactured products without R&D efforts; $\alpha_m > 0$ denotes the speed of adjustment of a firm $m, \beta \in (0, 1)$ is equivalent to R&D spillover, and $\gamma > 0$ is the expense parameter of a firm's technical advancement. As shown in Zhou and Wang [24], the previous system has a unique Nash-Cournot equilibrium point

$$E_3 = (x_*, x_*), \text{with } x_* = \frac{4(a-2bc)(\beta+1)}{9\gamma - 8(\beta+1)^2 b}$$

which is stable in the absence of delays. In order to guarantee the economic meaningfulness of E_3, we assume the conditions

$$a > 2bc, \quad 9\gamma > 8(\beta+1)^2 b$$

The local stability of the positive equilibrium point is governed by the roots of the corresponding characteristic equation for our system. By linearizing it at E_3, it follows that the associated characteristic equation is the following two degree exponential polynomial equation where namely

$$\lambda^2 + 2\lambda + 1 + (p\lambda + p)^{e-\lambda\tau} + qe^{-2\lambda\tau} \qquad (1)$$

where

$$p = -2 + \left[\gamma - \frac{4(\beta+1)^2 b}{9}\right](\alpha_1 + \alpha_2)x_*$$

and

$$q = 1 - \left[\gamma - \frac{4(\beta+1)^2 b}{9}\right](\alpha_1 + \alpha_2)x_* + \left[\gamma - \frac{8(\beta+1)^2 b}{9}\gamma\alpha_1\alpha_2 x_*^2\right]$$

3. Local Stability and Bifurcation Analysis

We follow the method proposed in Chen *et al.* [8] to investigate the distribution of characteristic roots of [1]. First, we observe that $\lambda = 0$ is not an eigenvalue since
$1 + p + q = [\gamma - 8(\beta+1)^2 b/9]\gamma\alpha_1\alpha_2 x^2 > 0$. Assuming that $\lambda = i\omega(\omega > 0)$ is a purely imaginary root of [1], we substitute it in [1] and obtain

$$-\omega^2 + 2i\omega + 1 + (pi\omega + p)e^{-i\omega\tau} + qe^{-2i\omega} = 0 \qquad (2)$$

Case $(\omega\tau)/2 \neq (\pi/2) + k\pi,\ k \in \mathbb{Z}$. Letting $\theta = \tan[(\omega\tau)/2]$, we get

$$e^{-i\omega\tau} = \frac{1 - i\theta}{1 + i\theta} \qquad (3)$$

Separating real and imaginary parts in [25] leads to

$$\begin{cases} (\omega^2 - 1 + p - q)\theta^2 - 4\omega\theta = \omega^2 - 1 + p - q \\ (p-2)\omega\theta^2 + 2(-\omega^2 + 1 - q)\theta = -(2+p)\omega \end{cases} \qquad (4)$$

Denote

$$D(\omega) = \begin{vmatrix} \omega^2 - 1 + p - q & -4\omega \\ (p-2)\omega & 2(-\omega^2 + 1 - q) \end{vmatrix} = -2[\omega^4 + (-p+2)\omega^2 - p + pq - q^2 + 1]$$

$$E(\omega) = \begin{vmatrix} \omega^2 - 1 - p - q & -4\omega \\ -(2+p)\omega & -2(\omega^2 + 1 - q) \end{vmatrix} = -2[\omega^4 + (p - 2q + 4)\omega^2 - p + pq + q^2 - 1]$$

and

$$F(\omega) = \begin{vmatrix} \omega^2 - 1 + p - q & \omega^2 - 1 - p - q \\ (p-2)\omega & -(2+p)\omega \end{vmatrix} = (-2p)\omega(\omega^2 + 1 - q)$$

In case $D(\omega) = 0$, solving Equation (4) for θ, we must have $E(\omega) = F(\omega) = 0$. This means that ω satisfies Equation (4). If $D(\omega) \neq 0$, Cramer's rule implies

$$\theta^2 = \frac{E(\omega)}{D(\omega)} \quad \text{and} \quad \theta = \frac{F(\omega)}{D(\omega)}$$

which give rise to the follow identity $D(\omega)E(\omega) = [F(\omega)]^2$. By a direct calculation, we get the following polynomial equation in $\omega^2 = z$,

$$z^4 + s_3 z^3 + s_2 z^2 + s_1 z + s_0 = 0 \qquad (5)$$

where

$$s_3 = 4 - p^2, \quad s_2 = 6 - 2q^2 - 3p^2 + 2p^2 q$$

$$s_1 = -3p^2 + 4 + 3p^2 q - 4q^2, \quad s_0 = (1-q)^2[-p^2 + (1+q)^2]$$

Since the leading term of the polynomial is z^4, it tends to infinity as $z \to +\infty$. If $s_0 < 0$, i.e., $q \neq 1$ and $(1+q)^2 < p^2$, the existence of positive roots of (5) is guaranteed.

Case $\frac{\omega\tau}{2} = \left(\frac{\pi}{2}\right) + k\pi, k \in \mathbb{Z}$. Since $e^{-i\omega\tau} = -1$, according to (2), we obtain $\omega^2 - 1 + p - q = 0$ and $(2 - p)\omega = 0$. Hence, $D(\omega) = F(\omega) = 0$. Under these circumstances, ω^2 is still a positive root of (5).

Based on the previous analysis, we have the following result.

Lemma 1. If (5) has a positive root ω_*^2, then (1) has a purely imaginary root $i\omega_*$ at $\tau = \tau_*^k (k = 0, 1, 2, \dots)$, with

$$\tau_*^k = \begin{cases} \dfrac{2\arctan(\theta_* + 2k\pi)}{\omega_*}, & \text{if} \quad D(\omega_*) \neq 0 \\ \dfrac{\pi + 2k\pi}{\omega_*}, & \text{if} \quad D(\omega_*) = 0 \end{cases}$$

The next result establishes that a basic root curve occurs and travels transversally around the imaginary axis under some conditions of transversality.

Lemma 2. Let $\lambda(\tau) = u(\tau) + i\omega(\tau)$ be a root of the characteristic Equation (1) satisfying $u(\tau_*^k) = 0$ and $\omega(\tau_*^k) = \omega_*$. Then the root is simple and the root crossing criterion is derived as

$$sign\left[\frac{d(Re\lambda)}{d\tau}\right]_{\tau=\tau_*^k} = sign[G(\omega_*, \theta_*)]$$

where

$$\begin{aligned} G(\omega_*, \theta_*) = \{&[(p-2q)\tau_*^k - p + 2]\theta_*^2 + 4\omega_*\theta_* + (p-2q)\tau_*^k - p - 2\}[p\omega_*\theta_*^2 - 4q\theta_* + p\omega_*] \\ &- [(p\tau_*^k + 2)\omega_*\theta_*^2 - 4(q\tau_*^k + 1)\theta_* + (p\tau_*^k - 2)\omega_*][(p-2q)\theta_*^2 + p + 2q] \end{aligned}$$

Proof. Taking the derivatives in Equation (1) with respect to τ, we get

$$M(\lambda, \tau)\frac{d\lambda}{d\tau} = \lambda N(\lambda, \tau) \qquad (6)$$

with

$$M(\lambda, \tau) = 2\lambda + 2 + pe^{-\lambda\tau} - (p\lambda + p)\tau e^{-\lambda\tau} - 2q\tau e^{-2\lambda\tau}$$

and

$$N(\lambda, \tau) = (p\lambda + p)e^{-\lambda\tau} + 2qe^{-2\lambda\tau}$$

From (6) it follows

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_*} = \frac{M(i\omega_*, \tau_*^k)}{i\omega_* N(i\omega_*, \tau_*^k)} = \frac{iM(i\omega_*, \tau_*^k)\overline{N\left((\iota\omega_*, \tau_*^k)\right)}}{\omega_* N(i\omega_*, \tau_*^k)N(i\omega_*, \tau_*^k)} \qquad (7)$$

Using (3) in (7) yields

$$sign\left\{\frac{d(Re\lambda)}{d\tau}\bigg|_{\lambda=i\omega_*}\right\} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega_*}\right\} = sign\left\{\frac{G(\omega_*, \theta_*)}{(1+\theta_*^2)^2\omega_*|N(i\omega_*, \tau_*^k)|^2}\right\}$$

It remains to show the simplicity of the root $\lambda = i\omega*$. Suppose it is not simple. Then (6) would imply

$$pi\omega_* + p + 2qe^{-i\omega_*\tau_*^k} = 0$$

If $\frac{(\omega\tau_*^k)}{2} = \left(\frac{\pi}{2}\right) + k\pi, k \in \mathbb{Z},$ then $e^{-i\omega_*\tau_*^k} = -1$, so that

$$pi\omega_* + p - 2q = 0$$

leads to the absurd $\omega_* = 0$.

If $\frac{(\omega\tau_*^k)}{2} \neq \left(\frac{\pi}{2}\right) + k\pi, k \in \mathbb{Z},$ then

$$pi\omega_* + p + 2q\frac{1 - i\theta_*}{1 + i\theta_*} = 0$$

Consequently, one has $p\omega_* - 2q\theta_* = 0$ and $-p\omega_*\theta_* + 2q = 0$. Thus, $\theta_* = \pm 1$, i.e., $\tan\left[\frac{(\omega\tau_*^k)}{2}\right] = \pm 1$. In conclusion, we get $\frac{(\omega\tau_*^k)}{2} = \left(\frac{\pi}{4}\right) + k\pi, k \in \mathbb{Z}$, which is a contradiction. The statement follows. □

A positive sign of $G(\omega_*, \theta_*)$ corresponds to crossings of the imaginary axis at $\tau = \tau_*^k$ as τ increases from right to left, while a negative sign means a crossing from left to right. In view of the above Lemmas, we derive the following results.

Theorem 1

1) If Equation (5) has no solutions, then no stability switches exist.
2) If Equation (5) has a unique solution ω_*^2, then our system is locally asymptotically stable for $\tau \in [0, \tau_*)$. If $sign[G(\omega_*, \theta_*)] < 0$, the equilibrium E_3 remains stable, while in case one has $sign[G(\omega_*, \theta_*)] > 0$ it loses its stability via a Hopf bifurcation at $\tau = \tau_*$.
3) If Equation (5) has at least two positive roots, then, according to the sign of $G(\omega_*, \theta_*)$, a finite number of stability switches may occur as the time delay τ increases from zero to the positive infinity, with the occurrence of a Hopf Bifurcation at each switch.

4. Numerical Example

In this section, a numerical example is given to illustrate the validity of Theorem 1 obtained in Section 3. Consider the system with parameters $a = 1.5, b = -2.2, c = 1, \alpha_1 = \alpha_2 = 1, \gamma = 1.2,$ and $\beta = 0.8$, it can be calculated that the equilibrium point E_3 is equal to 0.6263 and therefore has a unique solution ω_*^2 based on Equation (5) with $\omega_* = 3.4937$. Then one has $\theta_* = 1.3309$ and $\tau_* = \tau_*^0 = 0.5303$, which implies $G(\omega_*, \theta_*) = 387.6839 > 0$. Based on Condition (2) in Theorem 1, if $sign[G(\omega_*, \theta_*)] > 0$, the system loses its stability via a Hopf Bifurcation at $\tau = \tau_*$. Therefore, the waveform diagrams are illustrated in Figures 1 and 2. From Figures 1 and 2, one can find that trajectories converge to the equilibrium point when $0.4 = \tau < \tau_* = 0.5303$, that is, $x_1(t)$ and $x_2(t)$ are stable, and they lose their stability at $\tau = \tau_*$, then there exists Hopf bifurcation with $0.6 = \tau > \tau_* = 0.5303$. Define the initial conditions $x_1(0) = 0.5$ and $x_2(0) = 1$, for the discrete duopoly game model of R&D competition between two high-tech enterprises, Figures 3 and 4 show that the system state $x_1(t)$ is asymptotically stable at $0.4 = \tau < \tau_* = 0.5303$, and there exists a Hopf bifurcation when $0.6 = \tau > \tau_* = 0.5303$, the state $x_1(t)$ still exhibits periodic changes over time in the case of competition with enterprise $x_2(t)$. Similarly, Figures 5 and 6 depict the dynamic trajectory phase diagrams of state $x_2(t)$ under competition with enterprise $x_1(t)$. Therefore, the two competing states trade off each other, showing periodic changes over time. In addition, simulations for other cases can be given similarly.

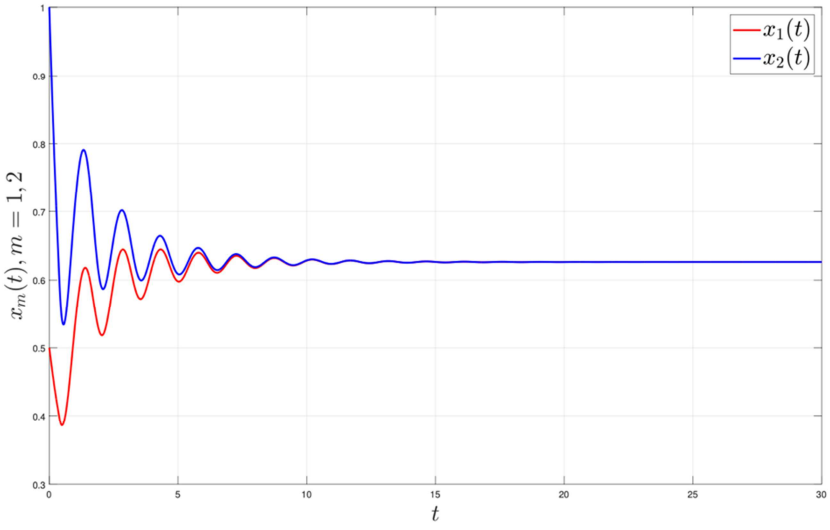


Figure 1. Waveform diagram with $0.4 = \tau < \tau_* = 0.5303$, the system is stable

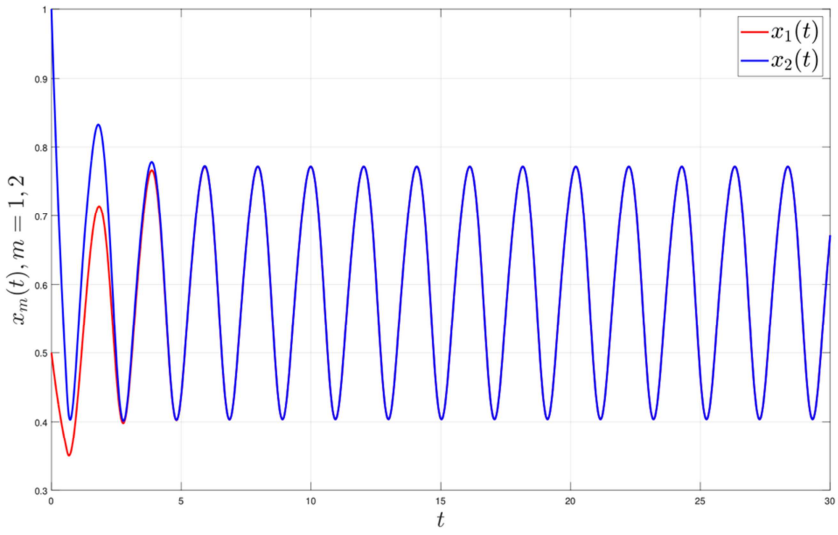


Figure 2. Waveform diagram with $0.6 = \tau > \tau_* = 0.5303$, the system with a Hopf Bifurcation

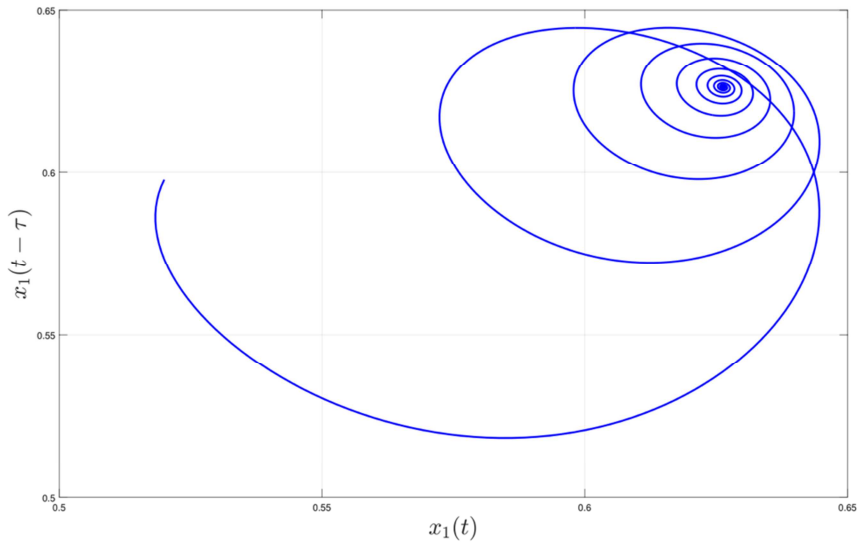


Figure 3. Phase diagram of $x_1(t)$ with $0.4 = \tau < \tau_* = 0.5303$, the system is stable

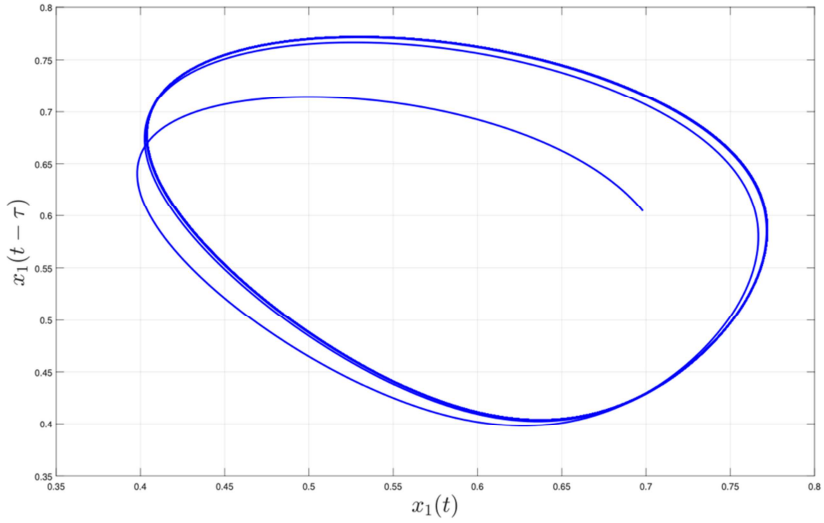


Figure 4. Phase diagram of $x_1(t)$ with $0.6 = \tau > \tau_* = 0.5303$, the system with a Hopf Bifurcation

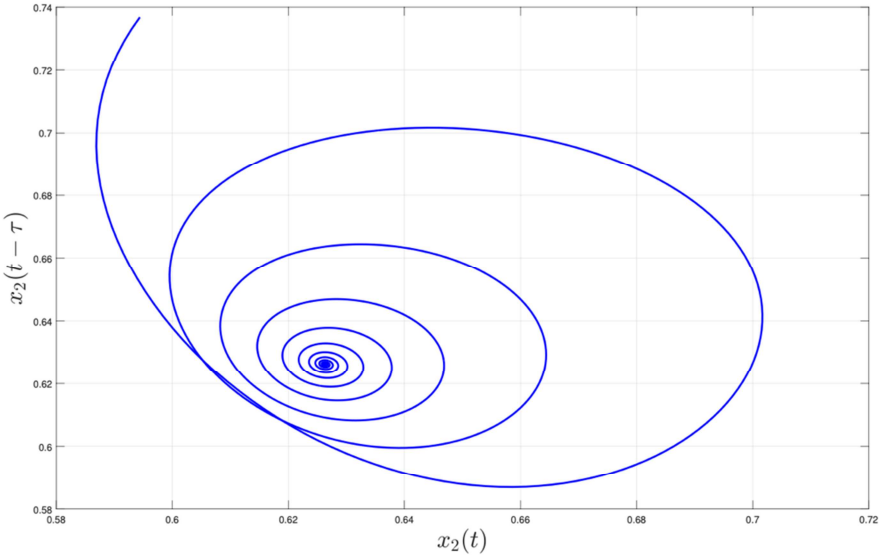


Figure 5. Phase diagram of $x_2(t)$ with $0.4 = \tau < \tau_* = 0.5303$, the system is stable

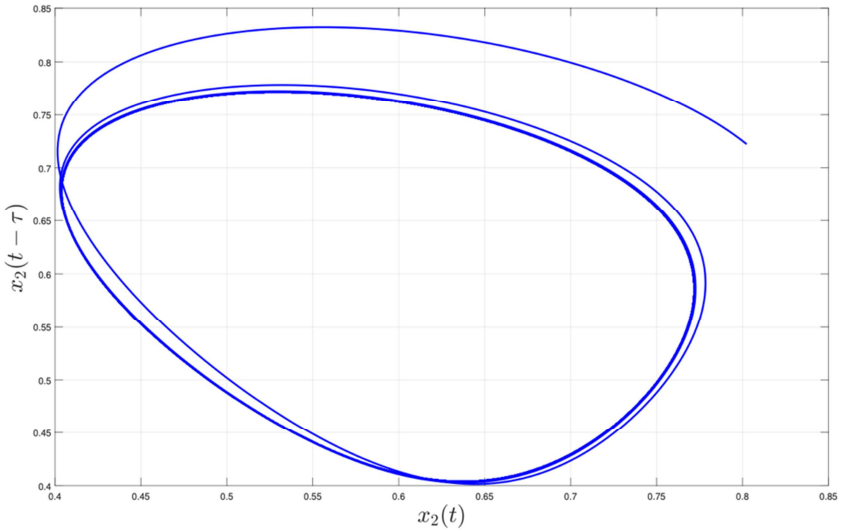


Figure 6. Phase diagram of $x_2(t)$ with $0.6 = \tau > \tau_* = 0.5303$, the system with a Hopf Bifurcation

5. Conclusions

By taking the discrete-time nonlinear monopoly model with R&D spillover of Zhou and Wang [24] as a starting point, Pansera *et al.* [18] proposed a continuous-time version of it with discrete delays. This paper has considered a different way of transforming a discrete-time dynamic set up into a continuous-time model with delays. The resulting R&D model presents Hopf bifurcation phenomena at the positive equilibrium. In addition, the dynamic behavior of the system around the positive equilibrium may change from local asymptotic stability to instability or vice versa. Contrary to Pansera *et al.* [18], this new scenario exhibits more complex dynamics that may dramatically change the long-term findings of an economy. A final comment: studying this kind of model in a stochastic (i.e., non-deterministic) context could be fruitful for a possible future research agenda. We did not develop a stochastic model here because we wanted to concentrate on deterministic chaos. In addition, further development of the topic would be to use a model with distributed delays.

PART III

COVID-19 SARS-2

Beginning from early 2020, a new highly contagious form of flu, supported by an unknown pathogen, classified as a Coronavirus type 2 strain, discovered at the end of the previous year, whence the COVID-19 (CoronaVirus Disease 2019) acronym, spread in the human population with virulence huge, claiming millions of lives worldwide due to lung complications responsible for a potentially fatal Severe Acute Respiratory Syndrome (SARS-2) (life-threatening one).

The *epidemic*, which rapidly took on the distinctive characteristics of a *pandemic*, to slowly transform into a common *endemic*, aggravated by the *syndemic* with several morbid cofactors, found above all in the weakest strata of the populations, has attracted the attention of scholars of numerous branches of knowledge, who examined the salient aspects from every point of view attempt to assess its global impact on the anthropic community.

In particular, due to the exceptional scientific (biomedical) interest and the extraordinary sociological (economic) relevance of the phenomenon, the pathology has been studied by microbiology, infectivology and epidemiology, with the typical sectorial systematic methodologies, to clarify the nosological, etiological and pathogenetic aspects, defining morbidity, morbosity and mortality in view of the application of appropriate hygiene and health regulations, of the adoption of adequate preventive measures and of the prescription of suitable prophylaxis provisions, but it was also analyzed, by demographic statistics, with suitable mathematical tools, to evaluate the rate of penetration into the populations in relation to the initial immune responses (production of specific antibodies), to the intermediate clinical symptoms (overt manifestations of disease) and the terminal, favorable (convalescence with healings) or unfavorable (worsening with deaths), outcomes.

By appropriately (and also opportunistically) exploiting the stringent scientific relevance of the topic, the opportunity is taken to mathematically analyze some phenomenologies related (or that can be related) in the framework of an extensive algorithms implementation already identified in modeling including parametric evaluations that can be correlated with the reference systems, further demonstrating their extraordinary operational flexibility on a quantitative level and their exceptional

functional versatility from a qualitative point of view in terms of technical use (critical applications of powerful direct instrumentation during the methodical inspection procedures of the materials examined) and responses concrete (practical acquisition of valid probative information during the systematic processing of the data obtained).

By statistically integrating the information collected demographically with the implementations deriving from the definition of convenient exponential development models, it is possible to calculate the course of the disease in absence of targeted containment expedients and in presence of appropriate restrictive measures imposition, in emergency conditions, by studying frequency and distribution based on certain parameters to obtain the elaboration of strategies capable of validly controlling the scope of phenomenology, even though it cannot selectively prevent it (in addition to compliance with the protocols established by international health regulations, which impose limitation of movements, disinfection of environments, sanitation of contaminated places, use of individual protection devices, social distancing, preventive quarantine in case of suspected incubation and sicks isolation to try to interrupt the viral transmission chain).

Nevertheless, a valid prevention can be extrapolated: by the attuation of analytical decomposition procedures of the significant elements of complex transformation kinetics including the detailed description of the temporal evolution of pandemic phenomena, in terms of incidence (relevance), distribution (penetration), expansion (diffusion), prevalence (preponderance), recrudescence (exacerbation), attenuation (alleviation) and remission (extinction), within the framework of deterministic models, formulated using differential equations; by the implementation of procedures of synthetic consideration of the crucial probabilistic factors concerning the random variability of morbid events in relation to incidental risk factors (susceptibility), such as the danger of infection (exposure to contagion), immune reactions (specific seropositivity), or health conditions (previous pathologies), inserted in the context of stochastic models; and by the development of mathematical simulation processes based on hypothetical assumptions, concerning potentially homogeneous parametric characteristics, which allow to establish, in a fairly reliable way qualitatively and in a sufficiently precise quantitative way, the elective propagation modalities, the expansion peak time and the disease total duration in the context of behavioral and compartmental models.

The pragmatic application of the general principles to the specific question includes the identification of the main characteristics of the living organisms involved in the phenomena, with the construction of suitable mathematical structures to include the descriptive particularities, the validation of the models obtained with the determination of the interactions between the subjects involved in the system and the study of emerging behaviors both for the clarification of the relatively predictable (foreseeable or unexpected) reciprocal influences and for the search for

absolutely unpredictable events (remembering that, based on contingent needs, epidemiological - descriptive, analytical and clinical - screenings can be observational or experimental, without forgetting that the two operating modes are not mutually exclusive). Specifically, the discussion analyzes Multi-Attribute Decision Making Based on Interval-Valued Trapezoidal Neutrosophic Number, the problems inherent in additional environmental stresses resulting from the disposal of personal protective equipment, and the difficulties in developing occupational choices based on uncertain parameters during pandemic due to multifactorial implications.

Multi-Attribute Decision Making Based on Interval-Valued Trapezoidal Neutrosophic Number and its Application in the Diagnosis of Viral Flu

Decision-Making Technique (DMT) is mostly “used in artificial intelligence and cognitive sciences to” elaborate individual and social perception. So, one of the most important strategies in DMT evolved in medical diagnosis scrutiny regarding the connection of symptoms and diagnosis of diseases due to uncertainty and fuzziness in the relevant information. The focus of this article is to develop a diagnostic decision-making strategy for the diagnosis of Viral diseases with close related symptoms using the Interval-Valued Trapezoidal Neutrosophic Numbers (IVTrNN) w.r.t Multiple Attribute Decision Making (MADM) strategy where, the attribute value is evolved to Interval-Valued Trapezoidal Neutrosophic Number and the attribute weight is unknown and can be related to the GRA (Grey Relational Analysis Projection) technique. In this research several operational laws are developed as well as the expected value and the hamming distance between two IVTrNs are introduced. Moreover, the information entropy method is used to determine the attributes weights and the grey relational analysis as well as the projection method are involved too in the proposed framework. The ranks of the alternative decisions are evaluated by their relative closeness to PIS (Positive Ideal Solutions), which combine the grey relational projection values from positive and negative ideal solutions associated with each alternative. Finally, a Viral disease example is “given to verify the developed approach and to demonstrate its practicality and effectiveness”.

1. Introduction

Multi-Attribute Decision Making Technique (MADMT) is the best tool to tackle such real-life situation problems. In the field of medical sciences, different Viral diseases have many common symptoms due to which it is often difficult to successfully diagnose a certain disease. Zadeh [37] has put forward the fuzzy set theory, which has proved to be important in tackling uncertain situations and fuzziness in real world problems. In 1989 Atanassov and Gargov [3] gave extension to the idea of fuzzy theory by putting forward the theory of Intuitionistic Fuzzy Set (IFS) and Interval Valued Intuitionistic Fuzzy Set (IVIFS). Adlassing [2] used the fuzzy set theory to discuss the connection among medical and the fuzzy logic and explain computerized diagnosing strategy. Touqeer *et al.* [28,29,30] also presented different techniques regarding MCDM using different types of fuzzy sets. De *et al*. [5] put forward the tool of medical diagnosis using intuitionistic fuzzy sets.

Abdullah *et al.* [1] as well as Garg *et al.* [10,11,12,13,32] discussed several decision-making problems employing fuzzy and intuitionistic fuzzy sets environment.

In 1995 Florentin Smarandache [25] brought into attention the introductory research material for the new concept of Neutrosophy. “Neutrosophic” set is an important tool through which we can perceive and explain the hierarchy of the “classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set” etc. In the field of Medicare with passage of time the symptoms of different diseases change according to the different individuals infected. Therefore, there is need to develop and adapt sophisticated tools for proper patient diagnosis.

Ye *et al.* [33,35,36] brought into attention the different multi-procedure techniques based on the Neurotrophic equation. In Neutrosophy the employment of different fuzzy sets introduces helpful solutions for the real world demanding medical diagnosis situation [33,36,35,27,22]. In the herein research the GRA (Grey Relational Analysis) introduced by Deng [6,7] and widely used for MCDM is involved in the proposed framework. Zhang [38] presented GRA method for MCDM with interval numbers. This method has a practical and theoretical appeal in estimating under uncertainty the correlation degree of factors. Accordingly, this theory is based on and, also, leads to supporting the idea that the more the similarity between factors the more these factors correlation. In the previously mentioned literature as well as in the relevant literature about the GRA methodology, as for instance [6,7,39,38], the different researchers have involved only the IFS (Intuitionistic Fuzzy Sets), the ITFS (Intuitionistic Trapezoidal Fuzzy Sets) [39] and the IVIFS (Interval Valued Intuitionistic Fuzzy Sets) [19]. IFS, ITFS and the IVIFS can tackle efficiently incomplete only information but not indeterminate and inconsistent information. The Neutrosophic fuzzy set concept is able to generalize the above GRA schemes in decision making. Neutrosophic set theory, integrating the three major degrees of uncertainty in the analysis of information, that is the degree of truth, the degree of indeterminacy and the degree of falsity describes efficiently all the aspects involved and needed in any successful decision-making technique applied under uncertainty.

In the herein proposed framework, it is attempted to fill in this research gap in literature. That is, to investigate the use of Neutrosophic rich information representation framework in GRA (Grey Relational Analysis) based Decision Making, with emphasis to Medical Decision Making, aiming at improving GRA methodology. More specifically, we are using the Interval-Valued Trapezoidal Neutrosophic numbers (IVTrNN) in a (MADMT) model. The difficulties in the decision-making process, due to single-valued uncertainty and membership functions of IFS, ITFS can be overcome with the vector valued (T, F, I), standing for truth, falsity, indeterminacy. membership function in Neutrosophic situation. It is much more convenient to express fuzziness in (T, F, I) vector valued memberships using interval numbers than involving real crisp numbers, standing for the single

valued membership functions. In the IVTrNN representation the values will be given in the form of intervals rather than in the form of exact numbers.

Therefore, it is herein considered IVTrNN Neutrosophic membership representation, rather than the single valued Neutrosophic number representation. Thus, the herein proposed research has investigated the GRA methodology within the newly developed scope of "Neurotrophic set" theory using the IVTrNN representation. In Neutrosophic set theory, indeterminacy is "quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are modelled as independent. Neutrosophic logic clearly distinguishes between relative truth and absolute truth while fuzzy logic does not" provide such rich representations capabilities. Involving the IVTrNN representational model more robust inference capabilities are provided, rather than the application of rival previous techniques. It is well known that in medical decision-making field large amount of complex knowledge is employed and, therefore, the "whole process involves low intra and inter person consistency. So, contradictions, inconsistency, indeterminacy and fuzziness should be accepted as unavoidable as it is integrated in the behavior of biological systems as well as in their characterization and decision processes.

The paper outline is as follow. Next section reviews the most important basics of Neutrosophic Sets. Section 2, also, introduces IVTrNNs and explains their relevant arithmetic operations. Section 3 presents a novel strategy for solving Multi-Attribute Decision Making (MADM) problems involving the novel GRA methodology based on the IVTrNN representation. Section 4 illustrates a relevant real world numerical medical decision-making problem and the associated application of the proposed framework, providing an efficient solution, as well as its explanation and practicality. Section 5 provides a qualitative comparative methodological discussion on the proposed and rival approaches and finally, section 6 provides conclusive remarks and prospects.

2. Basic Definition

Neutrosophic Sets

Neutrosophic set theory is a section of neutrosophy, which studies the origin, nature and the scope of neutralities, as well as the interaction with different ideational spectra [25] and consists a powerful formal framework.

Definition 2.1. [31] Assume the X be the sets of points and $x \in X$. Then, the neutrosophic set A in X is defined as $\dot{A} = \{< x, \dot{T}(x), \dot{I}(x), \dot{F}(x) >: x \in X\}$ where, $\dot{T}(x) + \dot{I}(x) + \dot{F}(x) \in [0,1]$ and the

$\dot{T}(x)$ is the truth-membership grade, while $\dot{I}(x)$ is the indeterminacy grade and the $\dot{F}(x)$ is the falsity grade.

Definition 2.2. [21] An interval $A(x) = [A^U, A^L]$ represents the (IVFS) defined on universe X where, A^L is the lower fuzzy set $A^L: \mathrm{X} \to [0,1]$ and A^U is the upper fuzzy set
$A^U: \mathrm{X} \to [0,1]$ $A = \{(x, [A^U, A^L]): x \in X\}$ $0 \leq A^L \leq A^U \leq 1$.

Definition 2.3. An Interval Valued Trapezoidal Neutrosophic Number (IVTrNN) is an Interval Valued Neutrosophic Set (IVTrNS) on X defined by $N(x) = \{N^L, N^U\}$ where, N^U and N^L are the lower and upper trapezoidal Neutrosophic sets of N such that $N^L \subseteq N^U$.

Definition 2.4. If N^U is considered to be an Upper (IVTrNN) then, we have these following three Trapezoidal Neutrosophic Numbers as follows, $\tilde{T}^U = \left(\bar{\dot{l}}, \bar{\dot{m}}, \bar{\dot{n}}, \bar{\dot{p}}\right) : X \to [0,1]$, $\tilde{I}^U = \left(\bar{\dot{q}}, \bar{\dot{r}}, \bar{\dot{s}}, \bar{\dot{t}}\right) : X \to [0,1]$ and $\tilde{F}^U = \left(\bar{\dot{u}}, \bar{\dot{v}}, \bar{\dot{w}}, \bar{\dot{x}}\right): X \to [0,1]$. If $\bar{\dot{m}} = \bar{\dot{n}}$, $\bar{\dot{r}} = \bar{\dot{s}}$ and $\bar{\dot{v}} = \bar{\dot{w}}$ these IVTrNN are reduced to a Triangular Neutrosophic Number to be defined as

$$\tilde{T}^U = \begin{cases} \dfrac{x - \bar{\dot{l}}}{\bar{\dot{m}} - \bar{\dot{l}}} & for\ \bar{\dot{l}} \leq x < \bar{\dot{m}} \\ \tilde{T}^U, & for\ \bar{\dot{m}} \leq x \leq \bar{\dot{n}} \\ \dfrac{\bar{\dot{p}} - x}{\bar{\dot{p}} - \bar{\dot{n}}}, & for\ \bar{\dot{n}} < x \leq \bar{\dot{p}} \\ 0, & for\ otherwise \end{cases}$$

$$\tilde{I}^U = \begin{cases} \dfrac{\bar{\dot{r}} - x + \tilde{I}^U\left(x - \bar{\dot{q}}\right)}{\bar{\dot{m}} - \bar{\dot{l}}}, & for\ \bar{\dot{q}} \leq x < \bar{\dot{r}} \\ \tilde{I}^U, & for\ \bar{\dot{r}} \leq x \leq \bar{\dot{s}} \\ \dfrac{x - \bar{\dot{s}} + \tilde{I}^U\left(\bar{\dot{t}} - x\right)}{\bar{\dot{t}} - \bar{\dot{s}}}, & for\ \bar{\dot{s}} < x \leq \bar{\dot{t}} \\ 1, & for\ otherwise \end{cases}$$

$$\tilde{F}^U = \begin{cases} \dfrac{\bar{\dot{v}} - x + \tilde{F}^U\left(x - \bar{\dot{u}}\right)}{\bar{\dot{m}} - \bar{\dot{l}}}, & for\ \bar{\dot{u}} \le x < \bar{\dot{v}} \\ \tilde{F}^U, & for\ \bar{\dot{v}} \le x \le \bar{\dot{w}} \\ \dfrac{x - \bar{\dot{w}} + \tilde{F}^U\left(\bar{\dot{x}} - x\right)}{\bar{\dot{x}} - \bar{\dot{w}}}, & for\ \bar{\dot{w}} < x \le \bar{\dot{x}} \\ 1, & for\ otherwise \end{cases}$$

Definition 2.5. Let $\dot{N}^L$ be a lower trapezoidal Neutrosophic number then, we have these following three trapezoidal Neutrosophic numbers

$\tilde{T}^L = \left(\underline{\dot{l}}, \underline{\dot{m}}, \underline{\dot{n}}, \underline{\dot{p}}\right) : X \to [0,1]$, $\tilde{I}^L = \left(\underline{\dot{q}}, \underline{\dot{r}}, \underline{\dot{s}}, \underline{\dot{t}}\right) : X \to [0,1]$ and $\tilde{F}^L = \left(\underline{\dot{u}}, \underline{\dot{v}}, \underline{\dot{w}}, \underline{\dot{x}}\right) : X \to [0,1]$.

If $\underline{\dot{m}} = \underline{\dot{n}}, \underline{\dot{r}} = \underline{\dot{s}}$ and $\underline{\dot{v}} = \underline{\dot{w}}$ these Trapezoidal Neutrosophic Numbers are reduced to a Triangular Neutrosophic Numbers defined as

$$\tilde{T}^L = \begin{cases} \dfrac{x - \underline{\dot{l}}}{\underline{\dot{m}} - \underline{\dot{l}}}\tilde{T}^L, & for\ \underline{\dot{l}} \le x < \underline{\dot{m}} \\ \tilde{T}^L, & for\ \underline{\dot{m}} \le x \le \underline{\dot{n}} \\ \dfrac{\underline{\dot{p}} - x}{\underline{\dot{p}} - \underline{\dot{n}}}, & for\ \underline{\dot{n}} < x \le \underline{\dot{p}} \\ 0, & for\ otherwise \end{cases}$$

$$\tilde{I}^L = \begin{cases} \dfrac{\underline{\dot{r}} - x + \tilde{I}^L\left(x - \underline{\dot{q}}\right)}{\underline{\dot{r}} - \underline{\dot{q}}}, & for\ \underline{\dot{q}} \le x < \underline{\dot{r}} \\ \tilde{I}^L, & for\ \underline{\dot{r}} \le x \le \underline{\dot{s}} \\ \dfrac{x - \underline{\dot{s}} + \tilde{I}^L\left(\underline{\dot{t}} - x\right)}{\underline{\dot{t}} - \underline{\dot{s}}}, & for\ \underline{\dot{s}} < x \le \underline{\dot{t}} \\ 1, & for\ otherwise \end{cases}$$

$$\tilde{F}^L = \begin{cases} \dfrac{\underline{\dot{v}} - x + \tilde{F}^L\left(x - \underline{\dot{u}}\right)}{\underline{\dot{r}} - \underline{\dot{q}}}, & for\ \underline{\dot{u}} \le x < \underline{\dot{v}} \\ \tilde{F}^L, & for\ \underline{\dot{v}} \le x \le \underline{\dot{w}} \\ \dfrac{x - \underline{\dot{w}} + \tilde{F}^L\left(\underline{\dot{x}} - x\right)}{\underline{\dot{x}} - \underline{\dot{w}}}, & for\ \underline{\dot{w}} < x \le \underline{\dot{x}} \\ 1, & for\ otherwise \end{cases}$$

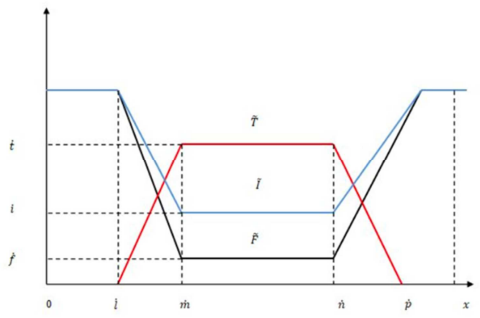


Figure 1. A Trapezoidal Neutrosophic Number

Definition 2.6. [38] Let $n_1 = \left(\bar{\dot{l}}, \bar{\dot{m}}, \bar{\dot{n}}, \bar{\dot{p}} \ : \ \tilde{T}^U, \tilde{I}^U, \tilde{F}^U\right)$ and $n_2 = \left(\bar{\dot{l}}, \bar{\dot{m}}, \bar{\dot{n}}, \bar{\dot{p}} \ : \ \tilde{T}^L, \tilde{I}^L, \tilde{F}^L\right)$ are the two IVTrNN. Therefore, the corresponding Hamming distance between n_1 and n_2 could be reasonably defined by

$$\begin{aligned}\widetilde{D}(n_1, n_2) = \frac{1}{12}\Big[&\Big|\bar{\dot{l}}_1(2 + \bar{\tau} - \bar{\imath} - \bar{F}) - \underline{\dot{l}}_2(2 + \bar{\tau} - \bar{\imath} - \bar{F})\Big| \\ &+ \Big|\bar{\dot{m}}_1(2 + \bar{\tau} - \bar{\imath} - \bar{F}) - \underline{\dot{m}}_2(2 + \bar{\tau} - \bar{\imath} - \bar{F})\Big| \\ &+ \Big|\bar{\dot{n}}_1(2 + \bar{\tau} - \bar{\imath} - \bar{F}) - \underline{\dot{n}}_2(2 + \bar{\tau} - \bar{\imath} - \bar{F})\Big| \\ &+ \Big|\bar{\dot{p}}_1(2 + \bar{\tau} - \bar{\imath} - \bar{F}) - \underline{\dot{p}}_2(2 + \bar{\tau} - \bar{\imath} - \bar{F})\Big|\Big] \qquad (1)\end{aligned}$$

Definition 2.7. The expected value I_n of an IVTrNN can be defined as

$n = \left(\left[\bar{\dot{l}}, \bar{\dot{m}}, \bar{\dot{n}}, \bar{\dot{p}}\right], \left[\underline{\dot{l}}, \underline{\dot{m}}, \underline{\dot{n}}, \underline{\dot{p}}\right]\right)\left(\tilde{T}(x), \tilde{I}(x), \tilde{F}(x)\right)$ and is shown as follows

$$\begin{aligned}I_n = \frac{1}{12}\Big[&\Big|\bar{\dot{l}} - \underline{\dot{l}}\Big| + \Big|\bar{\dot{m}} - \underline{\dot{m}}\Big| + \Big|\bar{\dot{n}} - \underline{\dot{n}}\Big| + \Big|\bar{\dot{p}} - \underline{\dot{p}}\Big|\Big] \\ &\times \left[2 + \left|\tilde{T}_U - \tilde{T}_L\right| - \left|\tilde{I}_U - \tilde{I}_L\right| + \left|\tilde{F}_U - \tilde{F}_L\right|\right] \qquad (2)\end{aligned}$$

Definition 2.8. Let n_1 and n_2 be two IVTrNNs

$$n_1 = \left(\bar{l}_1, \bar{m}_1, \bar{n}_1, \bar{p}_1\right), \left(\underline{l}_1, \underline{m}_1, \underline{n}_1, \underline{p}_1 : \tilde{T}_1\right), \left(\bar{q}_1, \bar{r}_1, \bar{s}_1, \bar{t}_1\right), \left(\underline{q}_1, \underline{r}_1, \underline{s}_1, \underline{t}_1 : \tilde{I}_1\right)\left(\bar{u}_1, \bar{v}_1, \bar{x}_1, \bar{z}_1\right), \left(\underline{u}_1, \underline{v}_1, \underline{x}_1, \underline{z}_1 : \tilde{F}_1\right) \quad (3)$$

$$n_2 = \left(\bar{l}_2, \bar{m}_2, \bar{n}_2, \bar{p}_2\right), \left(\underline{l}_2, \underline{m}_2, \underline{n}_2, \underline{p}_2 : \tilde{T}_2\right), \left(\bar{q}_2, \bar{r}_2, \bar{s}_2, \bar{t}_2\right), \left(\underline{q}_2, \underline{r}_2, \underline{s}_2, \underline{t}_2 : \tilde{I}_2\right)\left(\bar{u}_2, \bar{v}_2, \bar{x}_2, \bar{z}_2\right), \left(\underline{u}_2, \underline{v}_2, \underline{x}_2, \underline{z}_2 : \tilde{F}_2\right) \quad (4)$$

Definition 2.9. [19,34] Therefore, the basic operations on IVTrNNs could be defined as follows

Addition

$$n_1 \oplus n_2$$
$$= \left\{\left(\underline{1_1} + \underline{1_2} - \underline{1_1 1_2}, \underline{m_1} + \underline{m_2} - \underline{m_1 m_2}, \underline{n_1} + \underline{n_2} - \underline{n_1 n_2}, \underline{p_1} + \underline{p_2} - \underline{p_1 p_2}\right), (\bar{l}_1 + \bar{l}_2 - \bar{l}_1 \bar{l}_2, \bar{m}_1 + \bar{m}_2 - \bar{m}_1 \bar{m}_2, \bar{n}_1 + \bar{n}_2 - \bar{n}_1 \bar{n}_2, \bar{p}_1 + \bar{p}_2 - \bar{p}_1 \bar{p}_2)\right\}, \left\{\left(\underline{q_1 q_2}, \underline{q_1 q_2}, \underline{r_1 r_2}, \underline{r_1 r_2}\right), (\bar{s}_1 \bar{s}_2, \bar{s}_1 \bar{s}_2, \bar{t}_1 \bar{t}_2, \bar{t}_1 \bar{t}_2), \left(\underline{u_1 u_2}, \underline{u_1 u_2}, \underline{v_1 v_2}, \underline{v_1 v_2}\right), (\bar{w}_1 \bar{w}_2, \bar{w}_1 \bar{w}_2, \bar{x}_1 \bar{x}_2, \bar{x}_1 \bar{x}_2)\right\}$$

Multiplication

$$n_1 \otimes n_2 = \left[\left(\underline{l_1 l_2}, \underline{m_1 m_2}, \underline{n_1 n_2}, \underline{p_1 p_2}\right), (\bar{l}_1 \bar{l}_2, \bar{m}_1 \bar{m}_2, \bar{n}_1 \bar{n}_2, \bar{p}_1 \bar{p}_2)\right] \left[\left(\underline{q_1} + \underline{q_2} - \underline{q_1 q_2}, \underline{r_1} + \underline{r_2} - \underline{r_1 r_2}, \underline{s_1 s_2}, \underline{t_1} + \underline{t_2} - \underline{t_1 t_2}\right)\right] \left[\left(\underline{u_1} + \underline{u_2}, -\underline{u_1 u_2}, \underline{v_1 v_2}, \underline{w_1} + \underline{w_2} - \underline{w_1 w_2}, \underline{w_1} + \underline{x_2} - \underline{x_1 x_2}\right)\right] [(\bar{u}_1 + \bar{u}_2 - \bar{u}_1 \bar{u}_2, \bar{v}_1 + \bar{v}_2 - \bar{v}_1 \bar{v}_2)][\bar{w}_1 + \bar{w}_2 - \bar{w}_1 \bar{w}_2, \bar{x}_1 + x_2 - \bar{x}_1 \bar{x}_2]$$

Scalar Multiplication

$$\lambda n_1$$
$$= \left\{\left(1 - \left(1 - \underline{l_1}\right)^{\lambda}, 1 - \left(1 - \underline{m_1}\right)^{\lambda}, 1 - \left(1 - \underline{n_1}\right)^{\lambda}, 1 - \left(1 - \underline{p_1}\right)^{\lambda}\right), \left(1 - (1 - \bar{l}_1)^{\lambda}, 1 - (1 - \bar{m}_1)^{\lambda}, 1 - (1 - \bar{n}_1)^{\lambda}, 1 - (1 - \bar{p}_1)^{\lambda}\right)\right\}, \left[\left(\underline{q_1}^{\lambda}, \underline{r_1}^{\lambda}, \underline{s_1}^{\lambda}, \underline{t_1}^{\lambda}\right), \left(\bar{q}_1^{\lambda}, \bar{r}_1^{\lambda}, \bar{s}_1^{\lambda}, \bar{t}_1^{\lambda}\right)\right] \left[\left(\underline{u_1}^{\lambda}, \underline{v_1}^{\lambda}, \underline{w_1}^{\lambda}, \underline{x_1}^{\lambda}\right), \left(\bar{u}_1^{\lambda}, \bar{v}_1^{\lambda}, \bar{w}_1^{\lambda}, \bar{x}_1^{\lambda}\right)\right] \quad (5)$$

Exponentiation

$$n_1^{\lambda} = \left[\left(\underline{l_1}^{\lambda}, \underline{m_1}^{\lambda}, \underline{n_1}^{\lambda}, \underline{p_1}^{\lambda}\right), \left(\bar{l}_1^{\lambda}, \bar{m}_1^{\lambda}, \bar{n}_1^{\lambda}, \bar{p}_1^{\lambda}\right)\right], \left\{\left(1 - \left(1 - \underline{q_1}\right)^{\lambda}, 1 - \left(1 - \underline{r_1}\right)^{\lambda}, 1 - \left(1 - \underline{s_1}\right)^{\lambda} 1 - \left(1 - \underline{t_1}\right)^{\lambda}, 1 - (1 - \bar{q}_1)^{\lambda}, 1 - (1 - \bar{r}_1)^{\lambda}, 1 - (1 - \bar{s}_1)^{\lambda}, 1 - (1 - \bar{t}_1)^{\lambda}\right)\right\} \left\{1 - \left(1 - \underline{u_1}\right)^{\lambda}, 1 - \left(1 - \underline{v_1}\right)^{\lambda}, 1 - \left(1 - \underline{w_1}\right)^{\lambda}, 1 - \left(1 - \underline{x_1}\right)^{\lambda}, 1 - (1 - \bar{u}_1)^{\lambda}, 1 - (1 - \bar{v}_1)^{\lambda}, 1 - (1 - \bar{w}_1)^{\lambda}, 1 - (1 - \bar{x}_1)^{\lambda}\right\}$$

3. Multi-Criteria Decision Making (MCDM) strategy based upon the Interval-Valued Trapezoidal Neutrosophic Numbers

Regarding the multi-attribute decision making technique (MADMT) problem m alternatives are considered $A = (A_1, A_2, \dots, A_3)$, n decision criteria $Q = Q_1, Q_2, \dots, Q_n$ and $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the corresponding weight vector of an attribute with $\omega_j \in [0, 1]$ and their specific values ω_j not defined. The attribute number of an alternative A_i on the criteria Q_j is the Interval-Valued Trapezoidal Neutrosophic Number $n_{ij} = ([a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4], [\tau(x), \iota(x), F(x)]$ where, the $\tau(x)$ tells us about the extent the alternative A_j belongs to IVTrNN $[a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4]$ and the $\iota(x)$ and $F(x)$

tell us about the extent the alternative A_j does not belong to IVTrNN $[a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4]$ on the criteria Q_j $0 \leq \tau(x) + \iota(x) + F(x) \leq 1$ and the decision matrix (DM) denoted $D = [a_{ij}]$ constructs the Positioning of alternative required.

The sequence of computation steps in the Decision Making (DM) methodology of the (IVTrNN) are defined as follows:

3.1. Strategy and the algorithm to standardize the Decision Making (DM) process

To eliminate the difference in the scaling of various physical dimensions in the computations defined in the DM process, at first normalization is performed. Suppose $N = ([n_{ij}]_{m\times n})$ denotes the decision matrix where $n_{ij}^z = [n_{ij}^1, n_{ij}^2, n_{ij}^3, n_{ij}^4]$, then, the methodology for normalization of two distinct kind of criteria is given below:

3.2. Cost type criteria

$$n_{ij}^p = \frac{\max(a_{ij}^1) - a_{ij}^p}{\max(a_{ij}^4) - \min(a_{ij}^1)} \qquad z = 1,2,3,4 \qquad (6)$$

3.3. Benefit type criteria

$$n_{ij}^p = \frac{a_{ij}^p - \min(a_{ij}^1)}{\max(a_{ij}^4) - \min(a_{ij}^1)} \qquad z = 1,2,3,4 \qquad (7)$$

3.4. Calculating the attribute weight

There are many techniques in the literature to calculate the attributes weights in Decision Making. In the herein research the information entropy technique is employed and discussed as the most suitable and computationally effective for managing uncertainty in weights calculation. Entropy is an important concept of the thermodynamics originated by Shannon [24] who has, also, introduced the method of the entropy as equivalent to the uncertainty.

Entropy is a concept easily understandable and related to fuzziness, so that the highest the fuzziness characterizes a system, then, the greatest its entropy value (EP) would be. Therefore, we know that if the attribute values of all alternatives have small differences regarding the specific attribute then, it is associated with the highest EP. It provides adequate knowledge to define the position of a specific alternative, by assuming it possesses a little value in the prioritizing technique. On the other hand, if the attribute values of all alternatives have obvious differences then, this attribute presents little entropy and similarly is helping to determine the best alternative. So, in order to figure out the

alternatives, if one attribute has the highest entropy it is assigned with the lowest weight and if it has the lowest entropy then, it is assigned with the highest weight.

3.5. A Technique to calculate the entropy value of attribute

To verify the entropy value of an attribute the formulae given below are involved:

$$E_j = -Z \times \sum_{x=1}^{n} f_{ij} \, ln \, f_{xy} (1 \le i \le p, 1 \le j \le q) \qquad (8)$$

where $Z = \frac{1}{lpq}$ and if $f_{ij} = 0$, then $0 \times \ln 0 = 0$

$$f_{ij} = \frac{I(r_{ij})}{\sum_{x=1}^{q} I(r_{ij})} (1 \le i \le p), (1 \le j \le q) \qquad (9)$$

The entropy weight is calculated as:

$$\omega_j = \frac{(1 - H_j)}{\sum_{x=1}^{q} (1 - H_j)} \qquad (10)$$

The equation mentioned above is used to find the weight. But a research effort. namely, that of Zhou *et al*. [40] comes with the refined formula for measuring the weight as follows:

$$\omega_j = \frac{\sum_{j=1}^{n} H_j + 1 - 2 \times H_j}{\sum_{j=1}^{q} \left(\sum_{j=1}^{q} H_y + 1 - 2 \times H_j \right)} \; (1 \le y \le q) \qquad (11)$$

3.6. Positioning the alternatives based on Grey Relational Projection Technique (GRPT)

Deng [6,7] introduced and discussed the concepts of the Grey Relational theory and its computational method dealing with the system scrutiny under incomplete knowledge. Grey Relational Analysis recommended the uncertain relation among things, complex systems of elements and their behaviors. Grey Relational theory is the process of quantitative analysis scrutiny of alternatives. Grey Relational theory is used in many real-world applications spanning recruitment and hiring processes, employment, power distribution systems operation and decision making processes.

The main procedure of GRPT is converting the attribute values of all alternatives into a comparability sequence by reducing the effects from different physical parameters. Based on these sequences, the methodology determines a reference sequence, the so-called ideal target sequence or negative ideal target sequence. Then, this approach calculates the grey relational coefficient

between all such comparability sequences and the reference sequence. Finally, based on these grey relational coefficients, the grey relational degree between the reference sequence and every such considered comparability sequence is completely determined. If a comparability sequence that is converted from an alternative decision has the highest grey relational degree with the reference sequence, in case it is the ideal target sequence, or the lowest grey relational degree with the reference sequence, in case it is the negative ideal target sequence, that alternative solution will be the best choice. Suppose then, the decision-making matrix (DM) has been normalized as previously outlined, the steps of grey relational projection technique are illustrated as follows:

3.7. Optimistic ideal solution (OIS) and Pessimistic ideal solution (PIS)

If a normalized IVTrNN decision matrix has been determined, then, an IVTrNN Optimistic ideal solution (OIS) and the corresponding IVTrNN Pessimistic ideal solution (PIS) could be defined as follows. Regarding the OIS definition:

$$n^{+} = \left(n_1^{+}, n_2^{+}, \dots, n_p^{+}\right) \qquad (12)$$

where n_p^{+} is evaluated as

$$n_j^{+} = \left[\left(\left[n_j^1 + n_j^2 + n_j^3 + n_j^4 +\right]\right); T(N_x), I(N_x), F(N_x)\right]$$
$$= \left[\left(\underbrace{\max_i}\left(n_j^1\right), \underbrace{\max_i}\left(n_j^2\right), \underbrace{\max_i}\left(n_j^3\right), \underbrace{\max_i}\left(n_j^4\right)\right)\right]; \left[\left(\underbrace{\max_i}\,\tau(N_x), \underbrace{\min_i}\,\iota(N_x), \underbrace{\min_i}\,F(N_x)\right)\right]$$

Moreover, the Fuzzy trapezoidal Pessimistic Ideal Solution (PIS) can be evaluated using:

$$n^{-} = \left(n_1^{-}, n_2^{-}, \dots, n_p^{-}\right) \qquad (13)$$

where, each of n_q^{-} can be calculated using the following relation

$$n_j^{-} = \left[\left(\left[n_j^1 + n_j^2 + n_j^3 + n_j^4 -\right]\right); T(N_x), I(N_x), F(N_x)\right]$$
$$= \left[\left(\underbrace{\max_i}\left(n_{ij}^1\right), \underbrace{\max_i}\left(n_{ij}^2\right), \underbrace{\max_i}\left(n_{ij}^3\right), \underbrace{\max_i}\left(n_{ij}^4\right)\right)\right]; \left[\left(\underbrace{\min_i}\,\tau(N_x), \underbrace{\max_i}\,\iota(N_x), \underbrace{\max_i}\,F(N_x)\right)\right]$$

Based on the above and similarly to [39], another direct definition of the corresponding Fuzzy Optimistic Ideal Solution (OIS) could be:

$$n_j^{+} = ([1,1,1,1]; 1,1,1)$$

and the relevant definition of the Fuzzy Pessimistic Ideal Solution (PIS) is

$$n_j^- = ([0,0,0,0];0,0,0)$$

3.8. Calculation of the Grey relational coefficient (GRC)

The GRC of each alternative from (OIS) and (PIS) can be calculated using the following equations respectively. The GRC of each alternative from (OIS) is given as:

$$\varsigma_{ij}^+ = \frac{S^+ + \varrho E^+}{d_{ij}^+ + \varrho P^+} \qquad (14)$$

where, the Normalized Hamming Distance

$$\begin{aligned} D(n_1, n_2^+) = \frac{1}{12}\big[&\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^1 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^1 +\big| \\ &+\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^2 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^2 +\big| \\ &+\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^3 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^3 +\big| \\ &+\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^4 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^4 +\big|\big] \end{aligned} \qquad (15)$$

$$S^+ = \underbrace{\min_i}\,\underbrace{\min_j} d_{ij}^+ , P^- = \underbrace{\max_i}\,\underbrace{\max_j} d_{ij}^+ \qquad (16)$$

ϱ is the resolution coefficient and $\varrho \in (0,1)$ and its value could be considered 0.5 in case of incomplete and uncertain knowledge.

So, the GRC of each alternative in (OIS) is given as

$$\varsigma_{ij}^- = \frac{S^- + \varrho E^-}{d_{ij}^- + \varrho P^-} \qquad (16)$$

where, the normalized hamming distance is

$$\begin{aligned} D(n_1, n_2^-) = \frac{1}{12}\big[&\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^1 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^1 -\big| \\ &+\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^2 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^2 -\big| \\ &+\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^3 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^3 -\big| \\ &+\big|(2+\bar{\tau}-\bar{\iota}-\bar{F})n_{ij}^4 - (2+\bar{\tau}-\bar{\iota}-\bar{F})n_j^4 -\big|\big] \end{aligned} \qquad (18)$$

$$S^- = \underbrace{\min_i}\,\underbrace{\min_j} d_{ij}^- , P^- = \underbrace{\max_i}\,\underbrace{\max_j} d_{ij}^- \qquad (19)$$

ϱ is the resolution coefficient and $\varrho \in (0,1)$ and its value is 0.5 again in case of complete ignorance and uncertainty.

Similarly, the GRC of each alternative in (PIS) is given as

3.9. Calculation of Grey Relational Grade GRG

We can find the GRG of every alternative from (OIS) and (PIS) by using the formulae given below.

$$\varsigma_{ij}^{+} = \sum_{j=1}^{q} \omega_j \varsigma_{ij}^{+} \qquad (20)$$

$$\varsigma_{ij}^{-} = \sum_{j=1}^{q} \omega_j \varsigma_{ij}^{-} \qquad (21)$$

3.10. Positioning the alternative

The positioning process in the GRA method is the procedure of selecting the alternative with the "greatest degree of grey relation" from (OIS) and the "smallest degree of grey relation" from (PIS).

3.11. Projection

The weight associated with the GPC (Grey relational Projection coefficient) will be

$$\overline{\omega}_j = \frac{\omega_j^2}{\sqrt{\sum_{j=1}^{q} \omega_j^2}} \qquad (22)$$

In the sequel the weighted grey correlation projection technique is applied for the alternative from the (OIS) solution and the (PIS) as follows:

$$\wp_i^{+} = \sum_{j}^{q} \left(\overline{\omega}_j \times \varsigma_{ij}^{+}\right) \qquad (23)$$

$$\wp_i^{-} = \sum_{j}^{q} \left(\overline{\omega}_j \times \varsigma_{ij}^{-}\right) \qquad (24)$$

3.12. Similarity approximation to OIS and Positioning an alternative

The alternative under consideration can now be positioned based on the GPC projection coefficient of each and every alternative on the basis of the OIS and the PIS solutions. The greatest the projected number on the OIS solution is, then, the closest to OIS and the best the alternative would be, while on the other hand, the smallest the projection onto PIS is, the more away the alternative to PIS should be and the better the considered alternative would be. Therefore, the similarity closeness can be defined as

$$RC_i = \frac{\sum_j^q \left(\bar{\omega}_j \times \varsigma_{ij}^-\right)}{\sum_j^q \left(\bar{\omega}_j \times \wp_i^-\right) \sum_j^q \left(\bar{\omega}_j \times \varsigma_{ij}^-\right)} \qquad (25)$$

4. An Illustrative Numerical Example for the Application of the Proposed GRPT Methodology based on IVTrNN Representation Framework

From the beginning of the mankind a multitude of different viral diseases have been arose with the passage of time. In the herein numerical example three viral diseases with common symptoms are discussed and the responsible medical staff should be able to efficiently make a decision about the viral disease under consideration on the basis of the symptoms provided. Let A_1, A_2, A_3 are the three viral diseases named as Spanish flu, COVID-19 and Swine Flu having the following four different symptoms Q_1, Q_2, Q_3, Q_4 given below.

- Abdominal pain, lethargy, sneezing, weight loss – high fever, tiredness, dry cough, breathing issues fever.
- Cough, sore throat, joint pain, body aches.
- Earache, petechial rashes, hoarseness, nasal congestion.

The assumption of the alternatives A_1, A_2, A_3 would be to be represented in the form of IVTrNN w.r.t four attributes Q_1, Q_2, Q_3, Q_4. The Decision Matrix (DM) will be in the form:

Alternative	Q_1	Q_2	Q_3	Q_4
A_1	$(1,2,3,4)$ $(0.1,0.2,0.4)$ $(2,4,8,9)$ $(0.2,0.3,0.5)$	$(2,5,7,8)$ $(0.2,0.3,0.7)$ $(3,4,5,10)$ $(0.4,0.5,0.6)$	$(3,7,8,9)$ $(0.3,0.4,0.8)$ $(7,8,9,10)$ $(0.5,0.6,0.9)$	$(1,5,7,9)$ $(0.1,0.3,0.2)$ $(2,4,5,7)$ $(0.4,0.5,0.9)$
A_2	$(1,5,6,9)$ $(0.1,0.3,0.5)$ $(2,3,5,7)$ $(0.2,0.5,0.8)$	$(1,3,7,8)$ $(0.1,0.2,0.6)$ $(1,2,7,5)$ $(0.5,0.3,0.7)$	$(1,2,8,9)$ $(0.1,0.2,0.3)$ $(2,4,8,9)$ $(0.4,0.7,0.9)$	$(5,6,8,9)$ $(0.3,0.4,0.5)$ $(2,5,8,6)$ $(0.5,0.6,0.9)$
A_3	$(3,6,4,9)$ $(0.1,0.2,0.4)$ $(1,2,4,7)$ $(0.3,0.6,0.7)$	$(1,5,8,10)$ $(0.1,0.3,0.5)$ $(2,5,8,10)$ $(0.5,0.2,0.9)$	$(4,5,6,7)$ $(0.7,0.8,0.9)$ $(1,2,8,9)$ $(0.8,0.9,0.7)$	$(1,3,5,7)$ $(0.3,0.5,0.7)$ $(2,5,7,9)$ $(0.4,0.8,0.9)$

Table 1 Attributive values of alternatives

The computational steps of the application of the herein proposed methodology for the problem at hand are as follows:

Step I.

Standardize the matrix

$$N = \left(\tilde{n}^{ij}\right)_{m\times n}$$

using the equation 6 and 7.

Step II.

The notion of expected value arises and it could be defined as the central value of the corresponding interval. The expected value is calculated from the entropy weight as previously analyzed and could be defined by using the equation 2.7 and 11.

$$I_n = \begin{bmatrix} 2.05 & 0.9 & 1.108 & 0.7 \\ 0.8 & 0.733 & 0.5833 & 0.933 \\ 1 & 0.316 & 1.5 & 0.933 \end{bmatrix}$$

Alternative	Q_1	Q_2	Q_3	Q_4
A_1	$(0, 0.12, 0.2, 0.37)$ $(0.1, 0.2, 0.4)$ $(0, 0.25, 0.75, 0.87$ $(0.2, 0.3, 0.5)$	$(0.2, 0.6, 0.7, 0.8)$ $(0.2, 0.3, 0.7)$ $(0.6, 0.7, 0.8, 1)$ $(0.4, 0.5, 0.6)$	$(0, 0.5, 0.6, 1)$ $(0.3, 0.4, 0.8)$ $(0.12, 0.25, 0.5, 0,75)$ $(0.5, 0.6, 0.9)$	$(0, 0.2, 0.6, 0.7)$ $(0.1, 0.3, 0.2)$ $(0, 0.1, 0.6, 0.4)$ $(0.4, 0.5, 0.9)$
A_2	$(0, 0.12, 0.8, 1)$ $(0.1, 0.3, 0.5)$ $(0, 0.25, 0.5, 0.8)$ $(0.2, 0.5, 0.8)$	$(0.5, 0.62, 0.87, 1)$ $(0.1, 0.2, 0.6)$ $(0.12, 0.5, 0.87, 0.6)$ $(0.5, 0.3, 0.7)$	$(0.5, 0.62, 0.87, 1)$ $(0.1, 0.2, 0.3)$ $(0.12, 0.5, 0.87, 0.6)$ $(0.4, 0.7, 0.9)$	$(0.25, 0.62, 0.37, 1)$ $(0.3, 0.4, 0.5)$ $(0, 0.12, 0.37, 0.75)$ $(0.5, 0.6, 0.9)$
A_3	$(0.25, 0.62, 0.37, 1$ $(0.1, 0.2, 0.4)$ $(0, 0.12, 0.37, 0.75$ $(0.3, 0.6, 0.7)$	$(0, 0.44, 0.7, 0.8)$ $(0.1, 0.3, 0.5)$ $(0.1, 0.4, 0.7, 1)$ $(0.5, 0.2, 0.9)$	$(0.37, 0.5, 0.62, 0.75)$ $(0.7, 0.8, 0.9)$ $(0, 0.12, 0.87, 0.17)$ $(0.8, 0.9, 0.7)$	$(0, 0.25, 0.5, 0.75)$ $(0.3, 0.5, 0.7)$ $(0.125, 0.5, 0.75, 1)$ $(0.4, 0.8, 0.9)$

Table 2 Standardize values of alternatives

The attribute weights could be calculated as

$$\omega_1 = 0{,}247, \omega_2 = 0{,}433, \omega_3 = 0{,}3193$$

Step III.

As it has been already analyzed the optimistic ideal solution (OIS) maximizes the benefits criterion and minimizes the cost criterion. On the other hand, the pessimistic ideal solution (PIS) minimizes the benefits criterion and maximizes the cost criterion of the IVTrNN. Such ideal alternatives could be verified with the help of the equations 14 and 17.

$$c_j^+ = \{([0.25, 0.62, 0.75, 0.87](0.3, 0.3, 0.4)[0.6, 0.62, 0.87, 0.87](0.5, 0.2, 0.5)[0.25, 0.62, 0.75, 0.75](0.5, 0.3, 0.2))\} \quad (26)$$

$$c_j^- = \{([0, 0.12, 0.2, 0.37](0.1, 0.6, 0.8)[0,4, 0.7, 0.6\,](0.1, 0.5, 0.9)[0.25, 0.5, 0.67](0.1, 0.9, 0.9)[0.25, 0.37, 0.4](0.1, 0.8, 0.9))\} \quad (27)$$

Step IV.

GRC is verified with respect to the relationship between the ideal and the actual values. The GRC for the OIS optimistic solution and the PIS, the negative ideal solution, are herein verified with the help of equations 20 and 21.

$$\left(\varsigma_{ij}^+\right)_{3\times4} = \begin{bmatrix} 0.668 & 0.7186 & 0.619 & 0.6234 \\ 0.958 & 1.000 & 0.661 & 0.5965 \\ 0.6753 & 0.707 & 0.8033 & 0.788 \end{bmatrix}$$

$$\left(\varsigma_{ij}^-\right)_{3\times4} = \begin{bmatrix} 0.810 & 0.8961 & 0.610 & 0.7909 \\ 0.893 & 1.000 & 0.824 & 0.752 \\ 0.827 & 0.8591 & 0.956 & 0.952 \end{bmatrix}$$

Step V.

After calculating the GRC the Grey Weighted Correlation Projection Coefficient (GWCP) is calculated for every alternative A_i on the OIS and PIS solutions by using equations 23 and 24:

$$P_1^+ = 0.23866, \quad P_2^+ = 0.3133, \quad P_3^+ = 0.25573$$

$$P_1^- = 0.27966, \quad P_2^- = 0.7948, \quad P_3^- = 0.30904$$

Step VI.

Then, the relative closeness is calculated by using 25:

$$RC_1 = 0{,}46045, \;\; RC_2 = 0{,}28144, \;\; RC_3 = 0{,}45280$$

Step VII.

Finally, the process of positioning the alternative according to its similarity approximation to each OIS solution is taking place

$$N_2 > N_3 > N_1 \qquad (28)$$

So herein the three different types of Flu with their common symptoms are ordered and explained according to the proposed generalized GRA methodology. The proposed framework provides, therefore, a tool helpful for the Viral diagnostic team in the diagnosis of the Flu type.

5. Conclusion and prospects

In all the previous researches regarding the application of the Grey Relational Analysis (GRA) methodology most of the researches evolve and involve the representation framework of the IFS and the ITFS theory which does not provide adequate information about the indeterminate and the inconsistent character of the alternative strategies inherent in real world complex Multi-Criteria (MCDM) and Multi-Attribute decision making (MADM) problems that exist in every real life situation. The IVTrNN representation framework provides the required additional information for successfully solving such problems that is, suitable measures of uncertainty, impreciseness, incompleteness and inconsistency regarding the information involved in such decision-making processes. The IVTrNN framework is much more suitable to be utilized in defining the indeterminate and inconsistent parameters in Decision Making Techniques (DMT). In the IFS representation the fuzzy set is defined w.r.t to Truth (T) as well as its associated memberships.

Therefore, the indeterminate and the inconsistent state of information is nullified and not considered. Moreover, even the involvement of the Interval Valued Fuzzy Sets (IVFS) and the Interval Valued Intuitionistic Fuzzy Sets (IVIFS) theory cannot provide all required means to effectively manage uncertainty of information in MCDM and MADM problems, since the parameters using IVFS or IVIFS are defined w.r.t to the membership and the non-membership only, regarding Truth of Information (T). On the other hand, in the IVTrNN representation the relevant parameters are defined w.r.t to the (T, I, F) vector of Truth, Indeterminacy and Falsity, which constructs a much more general and proper framework to explain the ambiguity, uncertainty and fuzziness that arises in the integration of decision-making processes and strategies information. In the herein presented study w.r.t the MADM situation, the proposed framework and methodology has been used to cope up with the attributes values involving the Interval-Valued Trapezoidal Neutrosophic Numbers (IVTrNN) representation with null knowledge about the weight of the attribute. After the introduction of basic arithmetic laws, the calculation of expected values and the corresponding normalized hamming distances between two IVTrNFN numbers are illustrated. In the sequel, the information Entropy Technique is used to estimate the weight value of the MADM attributes in order that the Grey Relational Analysis (GRA) could be developed. In the proposed generalized GRA methodology the alternative decision-making strategies have been ranked by introducing the similarity closeness with respect to the Optimistic Ideal Solution (OIS) and merging the GRA values from the OIS and the Pessimistic Ideal Solution (PIS) for each alternative. Finally, we illustrate the application of the proposed methodology in deriving the numerical solution of a Medical Decision-Making example in order to explain the detailed concepts involved and to demonstrate its computational applicability and efficiency. Based on this research effort a quantitative comparative study is planned by the authors regarding the applicability, computational efficiency and representational power of the different GRA approaches discussed in solving complex MCDM and MADM problems.

Analysis on Additional Environmental Stress - PPE Kit Disposal during Pandemic, a Dual Hesitant q-Rung Orthopair MARCOS Methodology under Uncertainty

Plastics are a major hazard to the environment; they have nearly taken on a new dimension in terms of contaminating ecology, especially when pollution levels rise and pandemics such as COVID-19 emerge, causing a global problem and raising concerns about polymeric waste management. So, this study focused on the disposal of enhanced personal protective equipment (PPE) (face masks, gloves, and other protective equipment) and other single-use plastic medical equipment that were used extensively and cent percent transferred into waste during the crisis, adding to the burden on the waste management authority to dispose with the available methods, which may help in reducing the increasing quantity of dumped debris but are ineffective. For this, we utilized the fuzzy Measurement Alternatives and Ranking according to the COmpromise Solution (MARCOS) approach, which uses an enhanced MARCOS method with a dual hesitant q-rung orthopair fuzzy set to analyze the PPE waste disposal problem. We are encouraged to carry PPE kits and dispose of them as soon as possible in order to prevent plastic waste and make the environment more sanitary in the event of a pandemic. In the example of picking a PPE kit disposal problem, the applicability and efficacy of the proposed strategy is illustrated. A sensitivity and comparison analysis is used to verify the suggested approach's stability and validity.

1. Introduction

Plastic materials have become an essential part of everyday life, resulting in huge plastic pollution on a global scale. When plastic is thrown away after its use, it becomes a plastic waste. This waste never degrades, and it can be found in the earth for many years. Plastic is compact, adaptable, efficient, liquid impervious, formidable, and reasonably priced. These are the appealing qualities that drive our great appreciation and excessive consumption of plastic goods and products. However, because plastic materials are long-lasting and take a long time to degrade, they subsequently end up as waste. Our incredible fascination with plastic, combined with an indisputable behavioral proclivity for increasingly over-consuming, disregarding, improper waste disposal, and thus contaminating, has become a dangerous mix. Plastics are composed of synthetic chemicals and are utilized in a variety of aspects of water cans, clothing, packaging materials, medical equipment, electrical items, building products, and so on. Plastic was initially thought to be innocuous and odorless. However, many years of plastic disposal into the environment has resulted

in a variety of associated issues. Plastic pollution is now widely recognized as a major environmental burden that also has an impact on human health. Despite the fact that PW is reusable, recycled products are more harmful to the environment due to added chemicals and skin tones.

COVID-19 is a corona virus infection caused by a newly discovered virus SARS-CoV-2, which can cause a respiratory tract infection and is mainly spread from person to person. During a pandemic, hospital beds, masks, and other safety medical equipment are used more frequently. Personal protective equipment (PPE) is also increasingly being used for this purpose in corona treatment hospitals. All of these are made of plastics. The coronavirus crisis has highlighted the importance of plastics in our daily lives. In this scenario, plastics made a significant contribution to the health-care sector and public health safety. In addition to the implementation of a countrywide lockdown, social distancing, and regulations on travel and social demonstrations, regular use of sanitizers, and mostly plastic-based PPE for ordinary citizens, such as surgical masks and hand gloves, as well as preventative medical outfit, protective clothing, face masks, and other PPE for health workers has contributed to the sudden increase in the amount of plastic waste [18]. Plastics provide exceptional stability, lifespan, and flexibility. Because of these qualities, plastics are vital in the healthcare system, with widespread usage in single-use medical equipment, industrial processes, and a variety of other surgical operations and transplants. Therefore, COVID-19 creates additional challenges for PWM all over the countries.

Prominently, in the pandemic time, PPEs and other plastic-based health-care equipment have started to emerge as a lifesaver for ensuring the health and safety of healthcare professionals and ordinary people. However, a fair assessment is required to compare all of the benefits and drawbacks of plastics, their strategic planning or mishandling, and their destiny in the ecosystem during the COVID-19 pandemic. In this situation, questions arise as to whether plastics act as peoples health protectors or polluters of the environment. The effects on worldwide plastic production and utilization are highlighted, including PPE production and consumption, online shopping and takeaway service providers, and the temporary inversion or remain of single-use-plastics restrictions. The impact of the pandemic on plastic consumption, as well as the essential problems and challenges of PWM, are also important.

Coronavirus has created a huge impact on all aspects of our ecosystem, including waste management, as the used PPE kits are not disposed of according to typical standards in emerging regions, which has resulted in an increase in the quantity of contaminated surface waste, necessitating increased proper processing and treatment methods for PPE kits in PWM. One of the main adverse facets of the COVID-19 pandemic is the disregard for environmental protection and the resulting disasters of increased plastic pollution all over the world. The used PPE kit disposal is

one of the trickiest and most serious issues of plastic's environmental effects in this period. The abundance of used PPE kits will add to our earth's plastic burden. This paper discusses how to properly dispose of used personal protective equipment (PPE) and single-use plastics, and also discusses the various guidelines and advisories issued by various international agencies and countries for PWM in this current situation. Here, we address several disposal methods for PPEs that can determine the advantages of the system of disposal when applying the alternatives to the proposed method. There are no impacts and drawbacks to the alternatives that would minimize used PPE waste and enhance waste disposal in the future.

As a part of the investigation of these impacts, it was decided to fill a research gap in the research of PPE kit disposal problem. Hence, the authors propose a fuzzy Measurement Alternatives and Ranking according to the COmpromise Solution (MARCOS) method with Best-Worst Method (BWM) based on Dual Hesitant Q-Rung Orthopair Fuzzy Set (DHq-ROFS). The Hesitant Fuzzy Set (HFS) [35] and the Q-Rung Orthopair Fuzzy Set (q-ROFS) [43] have recently received a lot of attention and are widely used in decision making. In this paper, we combine the HFS and q-ROFS to form a DHq-ROFS. The dual hesitant q-rung orthopair fuzzy set [13] can characterize fuzziness and uncertainty better than the PFS and IFS. In comparison to fuzzy sets, IFSs, HFSs, and q-ROFs, the concept of DHq-ROFS is more suitable in dealing with uncertainty and fuzziness. This realization raises awareness of the DHq-ROFS and motivates us to develop new techniques under the hesitant and q-ROFS. Because of the advantages of the DHq-ROF set, researchers have recently proceeded to develop the MCDM method for the DHq-ROFS environment.

The novelty of this research is to determine a suitable method for the disposal of PPE kits, which will help in minimizing the amount of plastic waste in this pandemic situation. To evaluate the disposal method for PPEs, a new fuzzy linguistic scale with accreditation in DHq-ROFSs and a new fuzzy MARCOS method were developed in this paper. The following are the advantages of the proposed method: early evaluation of fuzzy reference points using fuzzy ideal and anti-ideal solutions; more accurate dedication of the utility degree with respect from both; proposal for a new method of determining utility functions and aggregating them; and the possibility of taking into account a large number of criteria and alternatives. Here, the BWM is a hybrid with the MARCOS method, which obtains the criteria weights. Finally, the proposed method is to reduce the PW and create new sources from the used PPEs, and the research methodology is shown in Figure 1.

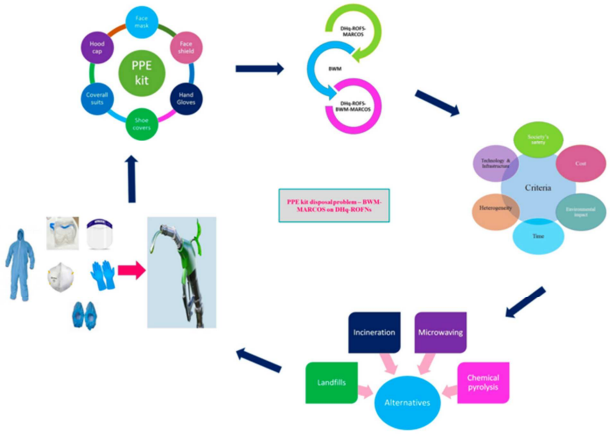


Fig. 1: The Methodology

The nomenclature with their abbreviations are given in Table 1.

DM	Decision Making
NDM	Normalized Decision Matrix
MCDM	Multi-Criteria Decision Making
COVID-19	COronaVIrus Disease-2019
MARCOS	Measurement Alternatives and Ranking according to the COmpromise Solution
BWM	Best-Worst Method
PPE	Personal Protective Equipment
BMW	Bio Medical Waste
q-ROFS	q-Rung Orthopair Fuzzy Set
DHq-ROFS	Dual Hesitant q-Rung Orthopair Fuzzy Set
WM	Waste Management
PW	Plastic Waste
PWM	Plastic Waste Management

Table 1: Nomenclature

2. Review of the Literature

This section includes several study results on MCDM, MARCOS, BWM, plastic waste, PPEs, and some relevant studies on used PPE kit disposal methods under MCDM methods. Several studies providing the effects of bio-medical waste and their disposal methods are addressed. Mrinalini Goswami *et al*. [10] examined BMW management in India during the COVID-19 pandemic. The

significance of the investments required in these projects in order to provide a safe and environmentally friendly method of disposing of these waste products. Due to the pandemic, there is an increase in the use of single-use plastics in the health care sector, which could pose an environmental risk. Parashar *et al.* [24] discussed the importance of using innovative methods for recycling plastics and raising awareness in society about the importance of maintaining a green environment. Kumar *et al.* [21] executed a life cycle assessment of PPE kits using two disposal methods, landfill and incineration, based on six environmental impacts, and decentralized incineration was declared to be a safe disposal method for the PPE kits. The various plastic waste treatment methods have been discussed, and thermo-chemical pyrolysis is preferred as an efficient method because it is cost-effective, environmentally friendly, requires less landfill space, produces less pollution, and aids in the production of energy products such as oil and gas, which have properties similar to fossil fuels. Selvakumar Dharmaraj *et al.* [7] proposed this method. Since there is a possibility of SARS-CoV-2 viruses in bio-medical waste collected from various sources, researchers have been frustrated about the importance of disinfecting these wastes before disposal. Ilyas *et al.* [14] discovered that the "identify, isolate, disinfect, and safe treatment practices" strategy works well for safer COVID-19 waste management. Benson *et al.* [4] described the importance of implementing dynamic WM strategies aimed at minimizing environmental pollution caused by plastics generated during the COVID-19 pandemic. Klemes *et al.* [19] discussed the future of plastic waste reduction in relation to COVID-19. Aragaw [3] proposed a three-layered surgical mask that protects against COVID-19, which has become a main cause of micro pollution. The authors emphasize the importance of raising awareness and its connection to the environment. Torres *et al.* [36] reviewed the recent development of sustainable face mask alternatives, as well as reprocessing and reusing routes, during the COVID-19 pandemic in Peru. Kumar Das *et al.* [20] provide an overview of various waste management strategies used for disposing of hazardous health care waste, as well as the importance of using an effective disposal method that helps to recycle waste or convert it into valuable products such as energy. Prashantkumar Kudli *et al.* [27] discussed the various methods for disposing of BMWs that are used in India, as well as the innovations that can be implemented in these processes. Vanapalli *et al.* [37] has suggested the sources of plastic in India, and also the recycling of certain plastics, and the effects of using single-use plastics and the importance of recycling these plastics. Geetha *et al.* [9] proposed a healthcare waste disposal method using MULTIMOORA method under intuitionistic hesitant fuzzy set. Hantoko *et al.* [12] presented the MCDM method for selecting an efficient health care waste disposal method during and after COVID-19. In the method, incineration has been found to be an efficient disposal method during a pandemic situation. Praharaj [26] explained the research technologies developed to allow

PPEs to be disinfected and reused up to 5 times during their life cycle. This paper also discusses the importance of using recycled PPE. The use of macroplastics, but also the awareness and importance of using bio-based plastics, must be created among plastic producers and politicians, as proposed by Patricio Silva *et al.* [25], in order to promote sustainable growth and stimulate both green and blue economies. The use of large amounts of single-use plastics, which are becoming a major source of pollution, is a threat to the environment. As Zhang *et al.* [44] have discussed, authorities and non-authorities' management must plan and implement policies to minimize the use of plastics and develop alternative materials. Shammi *et al.* [30] have focused on the disposal of personal protective equipment (PPE) in Bangladesh. They have also been discussed in terms of improper disposal and their side effects. The significance of collaborating with various organizations to improve hazardous PPE waste management capacity has also been highlighted. Aragaw *et al.* [2] identify the polymers used in the manufacture of face masks and gloves, and the results of FTIR and TGA/DTA analysis show that these polymers can be recycled into fuel energy using the pyrolysis method. According to Jain *et al.* [15], thermal pyrolysis is a more efficient method of disposing of PPE kits because the polymers in them can be recycled into liquid fuel with properties similar to fossil fuel. The shortcomings and inefficiencies of India's current waste management systems may exacerbate the mismanagement of these plastics and their leakage into the environment. Narayanamoorthy *et al.* [23] examined the bio-medical waste disposal using MOOSRA method under hesitant fuzzy approach.

The review by Hantoko *et al.* [12] of the various waste management techniques used in different countries has been presented, and the importance of following proper guidelines in disposing of these waste products has been emphasized. This paper summarizes COVID-19's impact on waste management. The various waste management technologies used in various parts of the world have been reviewed. Fan *et al.* [8] have advocated a long-term systematic assessment of waste management. WHO has issued guidelines for waste management during the COVID-19 pandemic. The role of PPE kits in the prevention and control of COVID-19 is discussed. According to Mahmood *et al.* [22], the importance of optimizing the use of protective gear and following the re-use of PPE kits may reduce demand to some extent. da Costa [6] examines the regulatory instruments that have been halted, paused, cancelled, or delayed. The importance of enforcing existing and new, stricter regulations is emphasized. Chew *et al.* [5] proposed to recycle hazardous waste through pyrolysis and co-pyrolysis, which converts it into energy. In these methods, the reaction mechanism and optimization of the criteria have been investigated. When the Pyrolysis method is compared to other disposal methods, it is discovered to be more capable of preserving environmental sustainability and lowering health risks. Su *et al.* [34] also discusses the pyrolysis of

medical waste and how the recycling method aids in the production of bio-oil and biochar. Jedruchniewicz *et al.* [16] has focused on and studied the various types of gloves used by people, such as latex, vinyl, and nitrile, as well as their improper disposal, which has an impact on the environment. Ammendolia *et al.* [1] recommended that municipal efforts to improve PPE collection methods in metropolitan cities be developed, must be developed and the impact of using single-use plastics have to be studied.

The need to narrow the gap in research for used PPE kit disposal methods, which should be followed by a thorough discussion and motivation. As a result, we implemented the fuzzy MARCOS method. To find the best method for used PPE disposal, we use the main criteria of society's safety, cost, environmental impact, time, heterogeneity, technology, and infrastructure aspects. Decision making is the process of choosing the best solution based on a combination of criteria. Several researchers have provided a wide range of methods based on MCDM to solve problems and make decisions easier. The decision-making process is enhanced with proficiency, logic, and clarity when the MCDM methods are used. The system is highly rewarded for increasing proficiency. The process is exonerated from its indecisiveness by employing better logic. Finally, by improving clarity, the decision-making process becomes more appealing, even to novices in the field. The MARCOS method [32] is the most realistic MCDM models, and it is used to solve complex decision-making problems in various types of research. Stevic *et al.* [33] established a novel MARCOS approach for sustainable supplier selection in the healthcare industry in Bosnia and Herzegovina. Stankovic *et al.* [31] proposed a fuzzy MARCOS model for traffic risk assessment. Puska *et al.* [28] developed a novel MARCOS model for SSS decision-making. The fuzzy approach was used in response to the linguistic values provided by decision-makers. It is an innovative procedure that allows a high-quality response to stakeholder and societal demands. In 2015, Rezaei [29] proposed the best-worst method (BWM), that can find the criteria weights and alternatives with regard to multiple criteria based on pairwise comparisons with less compared data. The BWM, on the other hand, can proficiently correct inconsistencies exacerbated by pairwise comparisons. In contrast to AHP, the BWM performs pairwise comparisons using a 1-9 scale. Because it does not use supplementary comparisons, this procedure appears to be simpler, extra precise, and much less superfluous [11]. However, experts' subjective judgments usually have characteristics of uncertainty and complexity, and the information criteria in the adult situation are unclear. As a result, for some practical problems, BWM comparisons can be executed by using fuzzy numbers rather than crisp values. This will be more relevant to the current situation and can produce quite persuasive ranking results. A fuzzy-based BWM was proposed, with comparisons carried out using fuzzy comparing decisions.

Many researchers showed their keen interest in the generalizations of fuzzy sets such as IFSs, HFSs, PFSs, IHFSs and PHFSs due to their strong points of view to tackle the vagueness and uncertainty. Since these assumptions have gained some traction among researchers in the HF environment, they are only valid in cases where their corresponding sum of existing members (θ) and non-existing (ϕ) grade is less than or equal to 1. Moreover, if the experts can express their priority for the element in the form of a discrete set, that is $(\theta)^2 + (\phi)^2 > 1$. So the ordinary IHFSs and PHFSs failed to handle such situations and is unable to classify the decision-making approaches [13]. Therefore some more comprehensive model is required for such situations. To cope this situation, in this manuscript we introduced the concept of DHq-ROFSs, which is the generalized form of IHFSs and PHFSs. In DHq-ROFSs the sum of qth power of membership grade and qth power of non-membership grade belongs to [0,1]. The remaining portions of the paper is designed as [42]. The present study developed the fuzzy MARCOS with BWM weighting method, which is aimed at disposing of the used PPE kits during the COVID-19 pandemic with different kinds of alternatives and solving the problem of a great way to dispose of plastic waste. We compared our proposed method to those of TOPSIS, VIKOR, and MULTIMOORA. We also performed a sensitivity analysis.

2.1 The research motivation

The motivation of this study is to find the best PPEs disposal method that will reduce the plastic waste during the pandemic. To address this issue, we applied the fuzzy BWM-MARCOS method under dual hesitant q-rung orthopair fuzzy environment.

- Obtaining a suitable disposal method for used PPE kits using the MCDM method. To find the best disposal method for this problem, the fuzzy MARCOS is being used to evaluate the alternatives. It will reduce plastic waste during this pandemic and it will help to create new kinds of products from this method.
- The weights of criteria are obtained in the BWM method to expose decision makers preferences, and the criteria weights are used to rank the alternatives for solving this problem. Here, we combine the BWM with the MARCOS method and also use a dual hesitant q-rung orthopair fuzzy set to select the best solution in a good manner.
- Thus, the research suggests a framework for disposing of health-care waste during a pandemic, particularly for PPE kits, on what options are suitable for this problem.

- This type of disposal system will improve plastic waste management and reduce plastic waste quantities during the COVID-19 period. In this case, Fuzzy DHq-ROFS-MARCOS would face a practical and appropriate waste disposal problem.

2.2 The research contribution

This research paper's contribution can be described as follows:

- In this paper, we proposed the BWM and a fuzzy MARCOS method based on dual hesitant q-rung orthopair fuzzy number. The DHq-ROFS-MARCOS procedure is used to rank the alternatives to disposing of the PPEs.
- The proposed methods, as well as the weights of the criteria, are used in the application of plastic waste management to select the best PPEs kit disposal treatment. The BWM is used to estimate the weight, and the obtained weights are combined with the fuzzy MARCOS method, which provides a juridical way to dispose of the waste.
- The DHq-ROFS-MARCOS-BWM method is proposed, which combines the MARCOS method with a DHq-ROF environment and the BWM weight finding procedure. In this proposed DHq-ROFS-MACOS method, decision makers express their opinions and values in terms of dual hesitant q-rung orthopair fuzzy elements.
- We changed the weight values of the criteria in our proposed method in order to discuss our findings in terms of certain sensitivity analysis modifications. When compared to the proposed method, the ranking values and ranking results have changed.
- To exemplify the dependability of our proposed methods, we compared our results to those of existing methods. We compared our proposed ranking method DHq-ROFS-MARCOS to the TOPSIS, VIKOR, and MULTIMOORA methods in this paper. We hope that our proposed methods will help to provide the best disposal solution for this problem.

Hence, our contribution is regarded as an effective method for choosing the required method of PPE disposal, and we believe that it will contribute to the creation of a healthy ecosystem for our society during the COVID-19 pandemic.

The paper is arranged as follows. Preliminaries are provided in Section 3. Section 4 describes the conventional MARCOS method. Section 5 presents a method that has been proposed. This section describes the MARCOS with DHq-ROFNs and the BWM methods. Section 6 includes a numerical example that demonstrates the effectiveness of the proposed method and how weight values are

calculated. Section 7 examines sensitivity and comparison. Section 8 concludes with a conclusion and future work.

3. Preliminaries

Definition 3.1. *[43]*

Assume that S be a non-empty fix set, then a q-ROFS A on U can be described as follows:

$$A = \{< s, (\theta_A(s), \phi_A(s)) > | s \in S\} \qquad (1)$$

where $\theta_A(s) : S \to [0,1]$ *and* $\phi_A(s) : S \to [0,1]$ *are represent the degree of membership and non-membership of* s *to* A*, respectively, which satisfies* $0 \leq (\theta_A(s))^q + (\phi_A(s))^q \leq 1, (q \geq 1)$*. The indeterminacy degree is given as* $\lambda_A(s) = \sqrt[q]{(\theta_A(s))^q + (\phi_A(s))^q - ((\theta_A(s))^q)((\phi_A(s))^q)}$*,* $< \theta_A(s), \phi_A(s) >$ *is called a q-ROFN, which is represented by* $\Phi = (\theta_A, \phi_A)$*.*

Definition 3.2. *[35] Assume* R *is a reference set, and describe the hesitant fuzzy set* A *on* R *with function* h*, when applied to* R *returns a subset of* $[0,1]$ *which is denoted mathematically as [41]:*

$$A = \{< r, h(r) > | r \in R\} = \{< r, \cup_{\gamma \in h(r)} \{\gamma\} > | r \in R\} \qquad (2)$$

in which $h(r)$ *is called the hesitant fuzzy element (HFE) and it contains all possible membership degrees of* $r \in R$ *to the set* H*.*

Definition 3.3. *[45] Assume* S *is a fixed set, and dual hesitant fuzzy set (DHFS)* T *on* S *is defined as,*

$$T = \{< s, h(s), g(s) > | s \in S\}, \qquad (3)$$

where $h(s)$ *and* $g(s) \in [0,1]$*, representing the possible grade of membership and non-membership element* $s \in S$ *to* T*, respectively, with the conditions:*

$$0 \leq \alpha, \beta \leq 1, \quad 0 \leq \alpha^+ + \beta^+ \leq 1$$

where $\alpha \in h(s), \beta \in g(s), \alpha^+ \in h^+(s) = \cup_{\alpha \in h(s)} \max\{\alpha\}$ *and* $\beta^+ \in g^+(s) = \cup_{\beta \in g(s)} \max\{\beta\} \forall s \in S$*. For, the pair* $t(s) = (h(s), g(s))$ *is called a DHFE denoted by* $t = (h, g)$*, with* $\alpha \in h, \beta \in g, \alpha^+ \in h^+ = \cup_{\alpha \in h(s)} \max\{\alpha\}, \beta^+ \in g^+ = \cup_{\beta \in g} \max\{\beta\}, 0 \leq \alpha, \beta \leq 1$*, and* $0 \leq \alpha^+ + \beta^+ \leq 1$.

Definition 3.4. *[41, 42] Let S is non-empty set. A DHq-ROFS A described on S is*

$$A = \{< s, h_A(s), g_A(s) > | s \in S\}, \qquad (4)$$

in which $h_A(s)$ and $g_A(s) \in [0,1]$ representing the possible grade of membership and non-membership element $s \in S$ to the set A, respectively, with

$$\alpha^q + \beta^q \leq 1 (q \geq 1),$$

where $\alpha \in h_A(s), \beta \in g_A(s) \ \ \forall s \in S$. For, the pair $t_q(s) = (h_A(s), g_A(s))$ is called a DHq-ROF element denoted by $t_q = (h, g)$ with $\alpha \in h, \beta \in g, 0 \leq \alpha, \beta \leq 1, \alpha^q + \beta^q \leq 1$.When $q = 2$, then DHq-ROFS is reduced to DHPFS [39], and when $q = 1$, then DHq-ROFS is reduced DHIFS [45].

Definition 3.5. *[13] For a DHq-ROFN $t_q = (h, g)$ characterized by h and g, the score function of t_q is described as*

$$S(t_q) = \frac{1}{\#h} \sum_{\alpha \in h} \alpha^q - \frac{1}{\#g} \sum_{\beta \in g} \beta^q \qquad (5)$$

where $\#h$ and $\#g$ represents the cardinality of h and g respectively.

4. The Conventional MARCOS Method

Consider m alternatives $\{C_1, C_2, C_3, \ldots, C_m\}$, n criteria $\{B_1, B_2, \ldots, B_n\}$, then the steps of the conventional MARCOS method are given below. **Step 1:** Construct an initial decision matrix. **Step 2**: Developing an extended initial matrix by using the anti-ideal solution (AIS) and ideal solution (IDS)

$$D = \begin{array}{c} \\ AIS \\ C_1 \\ C_2 \\ \vdots \\ C_m \\ IDS \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \ldots & B_n \end{array} \\ \begin{pmatrix} c_{ai1} & c_{ai2} & \ldots & c_{ain} \\ c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mn} \\ c_{id1} & c_{id2} & \ldots & c_{idn} \end{pmatrix} \end{array} \qquad (6)$$

Here, AIS represents the worst alternative and IDS represents the best alternative. Based on criteria, AIS and IDS are described from equations (7) and (8)

$$AIS = \min_i c_{ij} \ \ if \ \ j \in P \qquad \text{and} \qquad \max_i c_{ij} \ \ if \ \ j \in N \qquad (7)$$

$$IDS = \max_i c_{ij} \quad if \quad j \in P \qquad \text{and} \qquad \min_i c_{ij} \quad if \quad j \in N \qquad (8)$$

where P belongs positive criteria and N belongs negative criteria.

Step 3: Normalize the extended initial matrix (D) and the normalized matrix (NDM) denoted as $X = [x_{ij}]_{m\times n}$ are calculated by equations (9) and (10)

$$x_{ij} = \frac{c_{ai}}{c_{ij}} \quad if \quad j \in N \qquad (9)$$

$$x_{ij} = \frac{c_{ij}}{c_{ai}} \quad if \quad j \in P \qquad (10)$$

where c_{ij} and c_{ai} are the elements of D.

Step 4: Compute the weighted normalized matrix $Z = [z_{ij}]_{m\times n}$ using equation (11)

$$z_{ij} = x_{ij} \times w_j \qquad (11)$$

Step 5: Calculate the utility degree of the alternative U_i by equation (12) and (13)

$$U_i^- = \frac{S_i}{S_{ais}} \qquad (12)$$

$$U_i^+ = \frac{S_i}{S_{ids}} \qquad (13)$$

where $S_i(i = 1,2,\dots,m)$ denotes the sum of the elements of the matrix Z, which is given in equation (14)

$$S_i = \sum_{i=1}^{n} z_{ij} \qquad (14)$$

Step 6: Compute the utility function of alternative $f(U_i)$ described by (15)

$$f(U_i) = \frac{U_i^+ + U_i^-}{1 + \frac{1 - f(U_i^+)}{f(U_i)^+} + \frac{1 - f(U_i^-)}{f(U_i)^-}} \qquad (15)$$

where $f(U_i^-)$ denotes the utility function is to (AIS), and $f(U_i^+)$ denotes the utility function is to (IDS). Both function is obtained by equation (16) and (17)

$$f(U_i^-) = \frac{U_i^+}{U_i^+ + U_i^-} \qquad (16)$$

$$f(U_i^+) = \frac{U_i^-}{U_i^+ + U_i^-} \qquad (17)$$

Step 7: Finally, ranking the alternatives.

5. Proposed Method

5.1 The DHq-ROFS-MARCOS Method

Consider m alternatives $\{C_1, C_2, C_3, \ldots, C_m\}$, n criteria $\{B_1, B_2, \ldots, B_n\}$, then the steps of the DHq-ROFS-MARCOS method are given below and shown in Figure 2

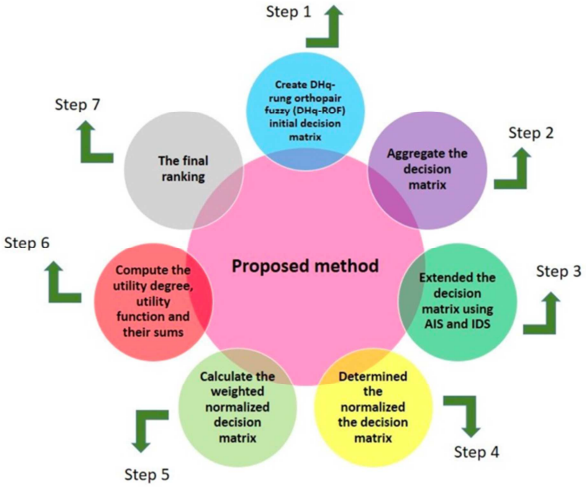


Figure 2: DHq-ROFS-MARCOS Method

Step 1: Create an initial decision matrix using DHq-ROFNs according to Table 2.

Linguistic term	DHq-ROF membership values	DHq-ROF non-membership values
Certainly high (CH)	0,95	0,15
Very high (VH)	0,85	0,25
High (H)	0,75	0,35
Above average (AA)	0,65	0,45
Average (A)	0,55	0,55
Under average (UA)	0,45	0,65
Low (L)	0,35	0,75
Very low (VL)	0,25	0,85
Certainly low (CL)	0,15	0,95

Table 2: Fuzzy Linguistic Variables

The initial matrix D is given in equation (18)

$$D = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{pmatrix} (\theta_{11}^1, \theta_{11}^2, \dots, \theta_{11}^l), (\phi_{11}^1, \phi_{11}^2, \dots, \phi_{11}^l) & (\theta_{12}^1, \theta_{12}^2, \dots, \theta_{12}^l), (\phi_{12}^1, \phi_{12}^2, \dots, \phi_{12}^l) & \dots & (\theta_{1n}^1, \theta_{1n}^2, \dots, \theta_{1n}^l), (\phi_{1n}^1, \phi_{1n}^2, \dots, \phi_{1n}^l) \\ (\theta_{21}^1, \theta_{21}^2, \dots, \theta_{21}^l), (\phi_{21}^1, \phi_{21}^2, \dots, \phi_{21}^l) & (\theta_{22}^1, \theta_{22}^2, \dots, \theta_{22}^l), (\phi_{22}^1, \phi_{22}^2, \dots, \phi_{22}^l) & \dots & (\theta_{2n}^1, \theta_{2n}^2, \dots, \theta_{2n}^l), (\phi_{2n}^1, \phi_{2n}^2, \dots, \phi_{2n}^l) \\ \vdots & \vdots & \ddots & \vdots \\ (\theta_{m1}^1, \theta_{m1}^2, \dots, \theta_{m1}^l), (\phi_{m1}^1, \phi_{m1}^2, \dots, \phi_{m1}^l) & (\theta_{m2}^1, \theta_{m2}^2, \dots, \theta_{m2}^l), (\phi_{m2}^1, \phi_{m2}^2, \dots, \phi_{m2}^l) & \dots & (\theta_{mn}^1, \theta_{mn}^2, \dots, \theta_{mn}^l), (\phi_{mn}^1, \phi_{mn}^2, \dots, \phi_{mn}^l) \end{pmatrix} \end{array} \quad (18)$$

where $D_{cb}^l = (\theta_{cb}^l, \phi_{cb}^l)(c = 1, 2, \dots, m, b = 1, 2, \dots, n)$ denotes the elements of the alternatives $C_c(c = 1, 2, \dots, m)$ based on attribute $B_b(b = 1, 2, \dots, n)$ given by the experts e^l. Then aggregate the experts opinion is:

$$c_{ij} = \frac{1}{\#h} \sum_{\alpha \in h} (\alpha)^q - \frac{1}{\#g} \sum_{\beta \in g} (\beta)^q \qquad (19)$$

The aggregated decision matrix D is in equation (20),

$$D = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \dots & B_n \end{array} \\ \begin{pmatrix} (\theta_{11}, \phi_{11}) & (\theta_{12}, \phi_{12}) & \dots & (\theta_{1n}, \phi_{1n}) \\ (\theta_{21}, \phi_{21}) & (\theta_{22}, \phi_{22}) & \dots & (\theta_{2n}, \phi_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\theta_{m1}, \phi_{m1}) & (\theta_{m2}, \phi_{m2}) & \dots & (\theta_{mn}, \phi_{mn}) \end{pmatrix} \end{array} \qquad (20)$$

Step 2: Construct the extended initial matrix by using the ideal and anti-ideal solutions, as shown in equation (21). The anti-ideal solution (AIS) represents the worst alternative whereas an alternative with best characteristic defines an ideal solution (IDS).

$$D = \begin{array}{c} \\ AIS \\ C_1 \\ C_2 \\ \vdots \\ C_m \\ IDS \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \dots & B_n \end{array} \\ \begin{pmatrix} (\theta_{ai1}, \phi_{ai1}) & (\theta_{ai2}, \phi_{ai2}) & \dots & (\theta_{ain}, \phi_{ain}) \\ (\theta_{11}, \phi_{11}) & (\theta_{12}, \phi_{12}) & \dots & (\theta_{1n}, \phi_{1n}) \\ (\theta_{21}, \phi_{21}) & (\theta_{22}, \phi_{22}) & \dots & (\theta_{2n}, \phi_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\theta_{m1}, \phi_{m1}) & (\theta_{m2}, \phi_{m2}) & \dots & (\theta_{mn}, \phi_{mn}) \\ (\theta_{id1}, \phi_{id1}) & (\theta_{id2}, \phi_{id2}) & \dots & (\theta_{idn}, \phi_{idn}) \end{pmatrix} \end{array} \qquad (21)$$

Based on criteria, AIS and IDS are obtained by (22) and (23).

$$AIS = \left(min_i \theta_{ij}, min_i \phi_{ij}\right) \;\; if \;\; j \in P \quad and \quad \left(max_i \theta_{ij}, max_i \phi_{ij}\right) \;\; if \;\; j \in N \qquad (22)$$

$$IDS = \left(max_i \theta_{ij}, max_i \phi_{ij}\right) \;\; if \;\; j \in P \quad and \quad \left(min_i \theta_{ij}, min_i \phi_{ij}\right) \;\; if \;\; j \in N \qquad (23)$$

Where P denotes the positive criteria and N denotes the negative criteria.

Step 3: By using equation (24) and (25) normalize the extended initial matrix, which is denoted as $X = [x_{ij}]_{m\times n}$.

$$x_{ij} = \left(\frac{\theta_{ai}}{c_{ij}}, \frac{\phi_{ai}}{c_{ij}}\right) \quad if \quad j \in N \qquad (24)$$

$$x_{ij} = \left(\frac{c_{ij}}{\theta_{ai}}, \frac{c_{ij}}{\phi_{ai}}\right) \quad if \quad j \in P \qquad (25)$$

where (θ_{ij}, ϕ_{ij}) and c_{ai} are the elements of D.

Step 4: Now, compute the WND $Z = [z_{ij}]_{m\times n}$ using equation (26)

$$z_{ij} = (\theta_{ij} \times w_j, \phi_{ij} \times w_j) \qquad (26)$$

Step 5: Calculate the utility degree of the alternative U_i based on equation (27) and (28)

$$U_i^- = \frac{S_i}{S_{ais}} \qquad (27)$$

$$U_i^+ = \frac{S_i}{S_{ids}} \qquad (28)$$

where $S_i(i = 1,2,\dots,m)$ denotes the sum of the elements of the matrix Z, which is given in equation (29)

$$S_i = \sum_{i=1}^{n} z_{ij} \qquad (29)$$

Step 6: Compute the utility function of alternative $f(U_i)$ as described in equation (30)

$$f(U_i) = \frac{U_i^+ + U_i^-}{1 + \frac{1 - f(U_i^+)}{f(U_i)^+} + \frac{1 - f(U_i^-)}{f(U_i)^-}} \qquad (30)$$

where $f(U_i^-)$ denotes the utility function related to (AIS), and $f(U_i^+)$ denotes the utility function with respect to (IDS) which are calculated using equations (31) and (32).

$$f(U_i^-) = \frac{U_i^+}{U_i^+ + U_i^-} \qquad (31)$$

$$f(U_i^+) = \frac{U_i^-}{U_i^- + U_i^+} \qquad (32)$$

Step 7: Finally, rank the alternatives based on the final utility function values. The alternative with the highest value is selected as the best option.

5.2 The Fuzzy BWM Method

Here, we describe the steps of BWM for calculating the criteria weights in PPE disposal problem. The steps involved are shown in Figure 3

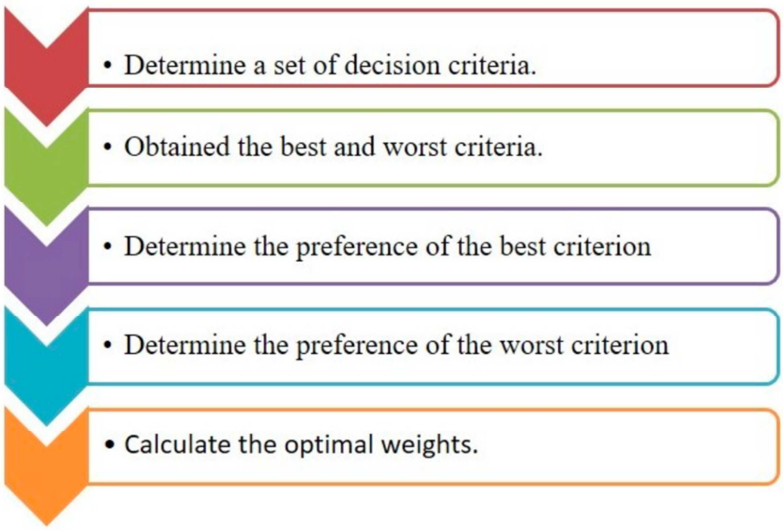


Figure 3: BWM Method

Step 1: Create a list of decision criteria.

Here, we look at the criteria $(b_1, b_2, \dots, b_n)$ which are used to make a decision. The PPEs disposing criterion are B_1-society's safety, B_2-cost, B_3-environmental impact, B_4-time, B_5-heterogeneity, and B_6-technology and infrastructure.

Step 2: Identify the best criteria (e.g., most beneficial and valuable) and the worst criteria (e.g., least beneficial and valuable).

Now, the decision-maker determines the overall best and worst criteria. At this point, no comparisons are made. Then, (b_3)-environmental impact and b_4-time are the best and the worst criteria for the disposal of PPEs problem.

Step 3: Evaluate their preference for the best overall criterion among the other criteria by assigning values between 1 and 9. The Best-to-Others vector result is:

$$B_G = b_{G1}, b_{G2}, \dots, b_{Gn}$$

where B_{Gj} denotes a preference for the best criterion G over criterion j (i.e) $B_{GG} = 1$. In our case, the vector demonstrates a preference for B_3-environmental impact over the other criteria.

Step 4: Using a number between 1 and 9, rank all of the criteria in order of preference over the worst criterion. The resulting others-to-worst vector is:

$$B_P = b_{1P}, b_{2P}, \dots, b_{nP}$$

where b_{jP} denotes a preference for criterion j over the worst criterion P. It is obvious that $B_{PP} = 1$. In this case, the vector depicts preferences for all criteria over the criteria time-b_4.

Step 5: Obtain the optimal weights $(\omega_1^*, \omega_2^*, \dots, \omega_n^*)$.

The optimal weight for the criteria is the one where, for each pair of $\frac{\omega_G}{\omega_j}$ and $\frac{\omega_j}{\omega_P}, \frac{\omega_G}{\omega_j} = b_{Gj}$ and $\frac{\omega_j}{\omega_P} = b_{jP}$. To satisfy these conditions for all j, we should find a solution where $\left|\frac{\omega_G}{\omega_j} - b_{Gj}\right|$ and $\left|\frac{\omega_j}{\omega_P} - b_{jP}\right|$ for all j is minimized. Consider the non-negativity and sum conditions for the weights as:

$$\min \max_j \left\{\left|\frac{\omega_G}{\omega_j} - b_{Gj}\right| - \left|\frac{\omega_j}{\omega_P} - b_{jP}\right|\right\}$$

s.t.

$$\sum_j \omega_j = 1, \quad \omega_j \geq 1 \quad \forall j \qquad (33)$$

Equation (33) can be transferred to the following problem:

$$\begin{gathered} \min \chi \\ s.t. \\ \left|\frac{\omega_G}{\omega_j} - b_{Gj}\right| \leq \chi, \quad \forall j \\ \left|\frac{\omega_j}{\omega_P} - b_{jP}\right| \leq \chi, \quad \forall j \qquad (34) \\ \sum_{j=1}^{t} \omega_j = 1, \quad \forall j \\ \omega_j \geq 0, \quad \forall j \end{gathered}$$

The optimal weights $(\omega_1^*, \omega_2^*, \dots, \omega_n^*)$ and χ^* are obtained by solving equation (34). Then, using χ^*, we present a consistency ratio (CR). It is larger the χ^*, the higher the CR and the less reliable the comparison become.

5.2.1 Consistency Ratio

We propose a consistency ratio for the proposed best-worst method in this section.

A comparison is fully consistent when $b_{Gj} \times b_{jP} = b_{GP}\ \forall j$, where b_{Gj} is preference of the best criteria over j, b_{jP} is preference of the worst criteria j, and b_{GP} is the preference of the best criteria over the worst criteria [29].

Moreover, some j may be inconsistent, for this we suggest a CR to demonstrate how consistent a comparison is. To that end, we begin by computing the minimum consistency of a comparison, which is as follows:

As previously stated, where $b_{ij} \in \{1, 2, \dots, b_{GP}\}$ the largest possible value of b_{GP} is 9. Consistency decreases when $b_{Gj} \times b_{jP}$ is smaller or larger than b_{GP} or equivalently $b_{Gj} \times b_{jP} \neq 1$, and the largest inequality occurs when b_{Gj} and b_{jP} have the maximum value, which will result in χ.

We also know that $\left(\frac{\omega_G}{\omega_j}\right) \times \left(\frac{\omega_j}{\omega_P}\right) = \frac{\omega_G}{\omega_P}$ and given the largest in equality as a results by b_{Gj} and b_{jP}, χ is a value that must be subtracted from b_{Gj} and b_{jP} , then added to b_{GP} or:

$$\left(b_{Gj} - \chi\right) \times \left(b_{jP} - \chi\right) = (b_{GP} + \chi) \qquad (35)$$

As for the minimum consistency $b_{Gj} = b_{jP} = b_{GP}$ we have

$$(b_{GP} - \chi) \times (b_{GP} - \chi) = (b_{GP} + \chi) \Rightarrow \chi^2 - (1 + 2b_{GP})\chi + (b_{GP}^2 - b_{GP}) = 0 \qquad (36)$$

Solving for different values of $b_{GP} \in \{1, 2, \dots, 9\}$, we can find the maximum possible χ (max χ). These values are used as the consistency index in Table 3.

b_{GP}	1	2	3	4	5	6	7	8	9
CI	0,0	0,52	1,00	1,72	2,19	3,01	3,97	4,81	5,50

Table 3: Consistency Index (CI)

The CR is then calculated using χ^* and the corresponding consistency index, as follows:

$$Consistency\ Ratio = \frac{\chi^*}{Consistency\ index} \qquad (37)$$

6. Numerical Example

Natural creatures have a tough time breaking down the manmade chemical bonds in plastic, making the material's persistence a major concern. Only a small fraction of plastic manufacturing gets recycled; the remainder is disposed of in landfills or incinerated. In landfills, it will be non-degradable for many years. Incineration releases harmful chemicals into the atmosphere, contributing to increased air pollution. A pure plastic can only be reused two to three times because it deteriorates due to thermal pressure, and it has a shorter life span after each reprocessing. As a result of inadequate collection and segregation methods, single-use plastic (SUP) trash disposal has been highlighted as a serious concern. Before COVID-19, only 60 percent of plastic was recycled; the remaining 40 percent of plastic was left unused, causing water, land, and air pollution. 70

percent of plastic products are converted into plastic waste during this pandemic. The COVID-19 issue has resulted in an increase in the volume of plastic garbage, which has had a detrimental impact on plastic management. SUP, in particular, adds to our planet's plastic weight, as do a plethora of abandoned PPE kits. The corona virus has resulted in an increase in the use and disposal of plastic-based items for medical and other purposes. During the pandemic, the increased use of SUP and plastic-based materials, together with greater availability to healthcare items and packaging, caused in a significant increase in global plastic waste generation. ([17, 38, 40]). Hence, the pandemic has posed a significant environmental challenge in terms of plastic waste management.

Many cities report a considerable rise in medical waste in the form of hospital-generated personal protection equipment. As a result, it is critical to increase the capacity to handle and treat this health-care waste as early as possible. In this situation, we must assess the most viable or acceptable waste disposal solutions for the current plastic waste management system that are compatible with the environmental system and human health consequences of waste management. This paper discusses how to properly dispose the infected personal protective equipment (PPE) and single-use plastics. Here, we address several disposal methods for PPEs from which an efficient disposal technique is chosen based on the proposed method. There are no impacts and drawbacks to the alternatives that would minimize single use plastics, used PPE kits and enhance waste disposal in the future. Here, we consider the four alternatives based on six criteria to dispose these wastes. The criteria are explained in Table 4 and the alternatives are explained as follows.

Criteria	**Description**
Society's safety (B_1)	Method must be safe for the public and workers to use, and it must not have a negative effect on society.
Cost (B_2)	Operating, transportation, technical, and other costs must be low.
Environmental impact (B_3)	Method adopted must not pollute the environment.
Time (B_4)	The disposal method must be capable of disposing a large amount of waste in a short period of time.
Heterogeneity (B_5)	The disposal method must be appropriate for the disposal of various types of plastic waste.
Technology and infrastructure (B_6)	Technology and land requirements for disposing and handling of waste.

Table 4: Selected Criterion for Ranking the Alternative

Landfills: The Landfills are facilities for the final disposal of single-use plastic waste on land that are designed and built with the goal of minimizing environmental impacts. The landfill site is the oldest type of waste treatment. Landfills have been the most prevalent means of storing and disposing of organized trash, and they are still in practice in many parts of the world. Plastic containers can take ten to a hundred years to degrade in landfills. Other plastic items may take the same number of years or longer to decay in such an environment, which lacks sunlight, air, and moisture (three essential components for facilitating bio-degradation). Plastic buried deep in landfills can leach dangerous chemicals into groundwater. The most serious environmental problem caused by landfills is groundwater contamination. A modern landfill that addresses these issues is a complex structure outfitted with a variety of environmentally friendly equipment.

Incineration: Incineration is an extremely high, dry combustion process that converts organic and combustible waste into inorganic, non-flammable material, resulting in a significant reduction in waste volume and weight. Incineration is the rapid oxidation of trash at high temperatures of 870-1200°C. When plastics are burned, toxic gases such as dioxins, fluorinated gases, mercury, and PCBs are released into the air, endangering both plants and people's lives. Once plastic is combusted, black carbon (soot) is released, which is a greenhouse gas that contributes to climate change and pollution, and a class of flame retardants known as halogens is also formed. Cancer, neurological damage, birth defects, child developmental disorders, asthma, and multiple organ damage are all known to be caused by these harmful chemicals.

Microwaving: The microwave device is essentially a steam-based technique that uses the action of soaking and heat created by microwave radiation. The contaminated water is reheated by microwave energy at a frequency of approximately 2,450 MHz and a wavelength of 12.24 cm. Microwaving is a technique for treating bio-medical waste made of plastic and glass. At the full design capacity of each microwave device, microwave should kill bacteria and other harmful organisms as determined by an approved biological indicator. In the absence of hazardous waste, it has the advantage of producing no liquid effluents and little emissions. The downsides are that they have high initial costs, have odor issues, and are susceptible to energy loss.

Chemical pyrolysis: Pyrolysis is the thermal degradation of polymers into smaller molecules in the presence of a catalyst (such as aluminium oxides, fly ash, red mud, and calcium hydroxide) in an inert atmosphere at temperatures between 300°C and 400°C. It is a thermo-chemical plastic disposal method that involves the oxidative decomposition of lengthy polymer structures into simpler, smaller units under high temperature and pressure in the lack of oxygen for a short period of time.

The oil produced during the pyrolysis process resembles normal diesel. Chemical pyrolysis products can be refined in the same way that as oil, using conventional refining technologies to produce polymer building blocks. They can also be used directly as a fuel.

In this section, the proposed methodology is used to evaluate the disposal method for used PPEs. To obtain the best disposal method that has a low environmental impact and is useful for all requirements in society, we have chosen six criteria to evaluate the four alternatives. Those criteria are society's safety (B_1), cost (B_2), environmental impact (B_3), time (B_4), heterogeneity (B_5), and technology & infrastructure (B_6). Here, we consider three decision-makers to evaluate the alternatives based on those selected criteria. Then, the alternatives are landfills (C_1), incineration (C_2), microwaving (C_3), and chemical pyrolysis (C_4). The selected alternatives and criteria are shown in Figure 4.

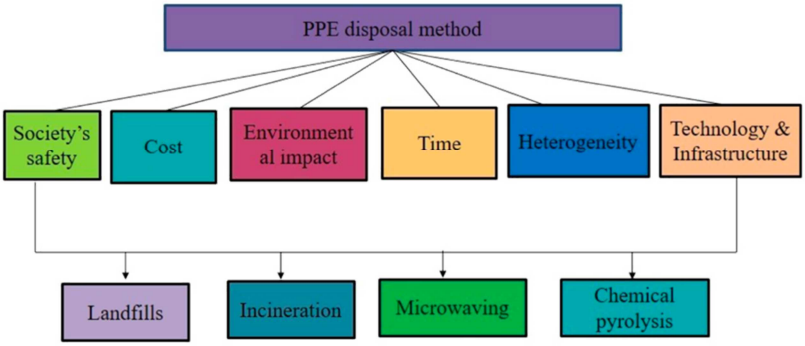


Figure 4: The used PPEs Disposal Problem

6.1 The Fuzzy BWM Method

Step 1: Create a list of decision criteria.

Here, we look at the criteria $(B_1, B_2, \ldots, B_n)$ that is used to make a decision. The PPEs disposing criterion are B_1-society's safety, B_2-cost, B_3-environmental impact, B_4-time, B_5-heterogeneity, and B_6-technology and infrastructure.

Step 2: Obtain the best and the worst criteria, (B_3)-environmental impact is best and B_4-time is the worst criteria.

Step 3: Here, B_3 is the best criteria and the pairwise comparison vector for the best criterion values are given in Table 5.

Criteria	B_1	B_2	B_3	B_4	B_5	B_6
$\boldsymbol{B_3}$	8	4	1	6	3	5

Table 5: Pairwise Comparison Vector for the Best Criterion

Step 4: Here, B_4 is the worst criteria and the pairwise comparison vector for the best criterion values are given in Table 6.

	B_4
B_1	5
B_2	7
B_3	6
B_4	1
B_5	2
B_6	4

Table 6: Pairwise Comparison Vector for the Worst Criterion

Step 5: From Table 5 and Table 6 results in equation (34) for this problem, as follows:

$$\begin{gathered}\min \chi, \\ s.t. \\ \left|\frac{\omega_3}{\omega_1}-b_{31}\right| \leq \chi, \left|\frac{\omega_3}{\omega_2}-b_{32}\right| \leq \chi, \left|\frac{\omega_3}{\omega_4}-b_{34}\right| \leq \chi, \left|\frac{\omega_3}{\omega_5}-b_{35}\right| \leq \chi, \left|\frac{\omega_3}{\omega_6}-b_{36}\right| \leq \chi, for\ all\ j \\ \left|\frac{\omega_1}{\omega_4}-b_{14}\right| \leq \chi, \left|\frac{\omega_2}{\omega_4}-b_{24}\right| \leq \chi, \left|\frac{\omega_3}{\omega_4}-b_{34}\right| \leq \chi, \left|\frac{\omega_5}{\omega_4}-b_{54}\right| \leq \chi, \left|\frac{\omega_6}{\omega_4}-b_{64}\right| \leq \chi, for\ all\ j \\ \omega_1+\omega_2+\omega_3+\omega_4+\omega_5+\omega_6=1 \\ \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \geq 0, \ \ \forall j\end{gathered} \tag{38}$$

Solving this equation (38), we found the optimal weights $(\omega_1^*, \omega_2^*, \ldots, \omega_n^*)$ are $\omega_1 = 0.0751$, $\omega_2 = 0.1503$, $\omega_3 = 0.4154$, $\omega_4 = 0.0382$, $\omega_5 = 0.2004$, $\omega_6 = 0.1202$ which are shown in Figure 5 and $\chi^* = 0.1858$.

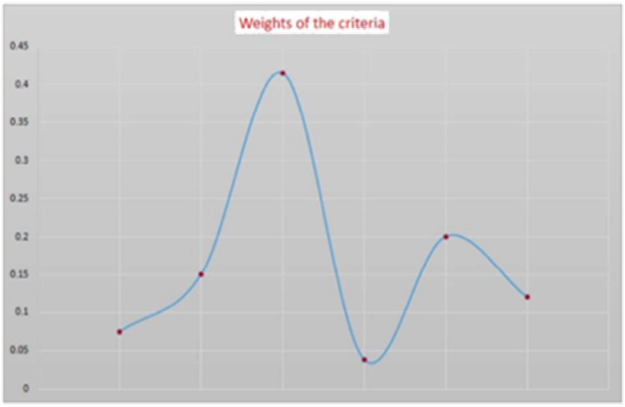


Figure 5: Sensitivity Analysis Results

For the consistency ratio, as $b_{GP} = b_{34} = 6$, the consistency index is 3.01 (Table 3), and the CR is $\frac{0.1858}{3,01}$, which implies a very good consistency.

6.2 The DH-q-ROFNs-MARCOS Method

In this section, experts evaluate the disposal methods under the selected criteria are represented as a DHq-ROFNs. The linguistic scale (Table 2) helps the experts to assist their opinions in evaluating the alternatives.

Step 1: Create the initial decision matrix using linguistic variables, which is shown in Table 7.

		$\boldsymbol{B_1}$	$\boldsymbol{B_2}$	$\boldsymbol{B_3}$	$\boldsymbol{B_4}$	$\boldsymbol{B_5}$	$\boldsymbol{B_6}$
$\boldsymbol{DM_1}$	C_1	(AA, A)	(H, A)	(VH, L)	(VH, VL)	(H, VL)	(CH, A)
	C_2	(L, H)	(A, H)	(CH, UA)	(VH, UA)	(A, L)	(H, A)
	C_3	(H, VH)	(H, VH)	(UA, A)	(A, VH)	(VH, A)	(AA, A)
	C_4	(CH, H)	(VH, H)	(CH, A)	(AA, VH)	(VH, AA)	(VH, A)
$\boldsymbol{DM_2}$	C_1	(A, AA)	(L, AA)	(H, A)	(H, L)	(VH, L)	(VH, UA)
	C_2	(L, A)	(H, A)	(CH, A)	(VH, A)	(UA, H)	(VH, UA)
	C_3	(L, VL)	(VH, AA)	(A, H)	(AA, H)	(L, A)	(A, UA)
	C_4	(VH, VH)	(VH, AA)	(CH, UA)	(VH, VH)	(H, A)	(VH, VH)
$\boldsymbol{DM_3}$	C_1	(L, H)	(A, H)	(VH, H)	(VH, L)	(AA, H)	(AA, H)
	C_2	(VH, A)	(A, AA)	(H, UA)	(CH, UA)	(H, A)	(VH, H)
	C_3	(A, AA)	(H, VH)	(AA, VH)	(H, VH)	(A, H)	(L, VH)
	C_4	(H, UA)	(H, AA)	(VH, AA)	(A, AA)	(H, VH)	(A, H)

Table 7: Decision Matrix based on Linguistic Scale for DM_1, DM_2, DM_3

Then, aggregating the initial matrix with $q = 3$ and which is given in Table 8.

	B_1	B_2	B_3	B_4	B_5	B_6
C_1	0.0611	0.1102	0.3397	0.0641	0.0772	0.4208
C_2	0.108	0.1514	0.4737	0.4567	0.0161	0.3888
C_3	0.0222	0.4452	0.1024	0.1275	0.1492	0.009
C_4	0.5201	0.475	0.5989	0.311	0.3949	0.3899

Table 8: Aggregate Decision Matrix

Step 2: The extended initial matrix is shown in Table 9 by applying the equation (22) and (23).

	B_1	B_2	B_3	B_4	B_5	B_6
AAI	0.0222	0.475	0.1024	0.0641	0.0772	0.4208
C_1	0.0611	0.1102	0.3397	0.0641	0.0772	0.4208
C_2	0.108	0.1514	0.4737	0.4567	0.0161	0.3888
C_3	0.0222	0.4452	0.1024	0.1275	0.1492	0.009
C_4	0.5201	0.475	0.5989	0.311	0.3949	0.3899
ID	0.5201	0.1102	0.5989	0.4567	0.3949	0.009

Table 9: Extended Aggregate Matrix

Step 3: By using equation (24) and (25) to NDM, then multiplied the weights values $w_1 = 0.0751, w_2 = 0.1503, w_3 = 0.4154, w_4 = 0.0382, w_5 = 0.2004, w_6 = 0.1202$ with NDM, we get the weighted NDM and which is shown in Table 10.

	B_1	B_2	B_3	B_4	B_5	B_6
AAI	0.0031	0.0348	0.0709	0.0053	0.0391	0.0025
C_1	0.0088	0.1503	0.2356	0.0053	0.0391	0.0025
C_2	0.0155	0.1093	0.3285	0.0382	0.0081	0.0027
C_3	0.0031	0.0371	0.0709	0.0106	0.0757	0.1202
C_4	0.0751	0.0348	0.4154	0.0260	1.2004	0.0027
ID	0.0751	0.1503	0.4154	0.0382	0.2004	0.1202

Table 10: Weighted Normalized Matrix

Step 4: Calculate the utility degree of the alternatives using equation (27) and (28), and utility function and final ranking results are obtained using equation (29), (30), (31), (32) are shown in Table 11.

	$f(U_i^-)$	$f(U_i^+)$	U_-	U_+	$\boldsymbol{f(U_i)}$	**Rank**
C_1	0.1347	0.8652	2.8362	0.4417	0.4324	**3**
C_2	0.1347	0.8652	3.2260	0.5025	0.4919	**2**
C_3	0.1347	0.8652	2.0398	0.3177	0.3110	**4**
C_4	0.1347	0.8652	4.8452	0.7547	0.7388	**1**

Table 11: The Final Ranking Values for Proposed Method

Step 5: The final ranking results are given in Table 11 and graphical representation is shown in Figure 6.

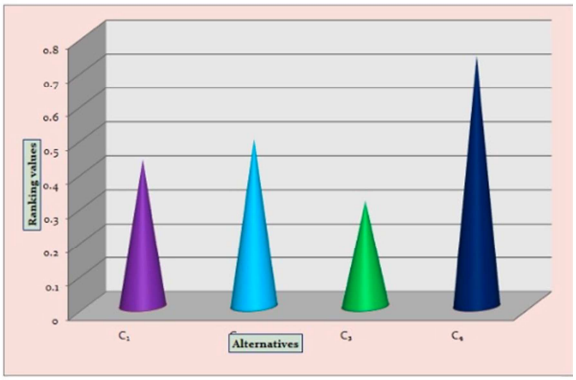


Figure 6: The Final Ranking Result

From Table 11, the best disposal method for used PPE kits is C_4-chemical pyrolysis. The manufacturing of liquid fuel from disposal of PPEs could address both the challenges of PPE waste management and rising energy demand. Plastic-derived liquid fuel is environmentally friendly and has similar fuel properties to fossil fuels. In this situation, converting plastic waste into a liquid biofuel is a viable option for protecting environmental resources. The process of converting any material at extremely high temperatures is known as pyrolysis. All PPEs currently disposed of in landfills, oceans, and other places that harm aquatic species can be relocated to a combustor and exposed to high temperatures to decompose the propane bonds and form a liquid. The liquid that results can be used as a biofuel. We need the fuel to power various mechanical devices. This fuel

can be reprocessed, which helps to protect the environment. Because of the growing population and their need for energy, this would be a healthy alternative that would also aid in the preservation of our natural environment.

7. Analysis of Sensibility and Comparison

7.1 Sensitivity analysis

This section depicts the analysis of sensitivity for used PPE kit disposal alternatives using the proposed method, based on the three weight cases shown in Table 12.

Criteria	Case 1	Case 2	Case 3
B_1	0.0751	0.4120	0.1639
B_2	0.1503	0.1128	0.3538
B_3	0.4154	0.0705	0.0431
B_4	0.0382	0.2265	0.0703
B_5	0.2004	0.0371	0.2459
B_6	0.1202	0.1411	0.1230

Table 12: Weight values for Sensibility Analysis

The results of three cases are compared for this model's sensitivity analysis, and the pictorial representation is shown in Figure 7.

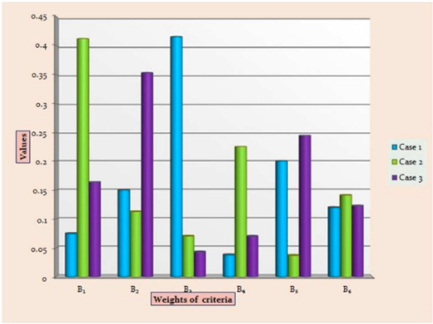


Figure 7: Weight of the Criteria for Sensitivity Analysis

Case 1 is the proposed result, and cases 2 and 3 are the other results explored using the different kinds of criteria weights obtained from the BWM weight finding method. The PPE kit disposal method used in this paper is based on six criteria: B_1-society's safety, B_2-cost, B_3-environmental impact, B_4-time, B_5-heterogeneity, and B_6-technology and infrastructure. The fuzzy MARCOS method is used to find the alternative ranks. The final ranks are determined by the weights of each criterion, which are determined using the BWM weight finding method. The best alternative to the used PPE kit disposal method can be found. When criteria are important, they can lead to a better compromise option.

Case 2 B_1-society's safety is the best criteria and B_5-heterogeneity is the worst criteria. In BWM method, we found the weights of the criteria are $\omega_1 = 0.4120$, $\omega_2 = 0.1128$, $\omega_3 = 0.0705$, $\omega_4 = 0.2264$, $\omega_5 = 0.0371$, $\omega_6 = 0.1411$, and $\chi^* = 0.1522$ and 0.0408. Which implies a good consistency.

The ranking order in this case is as follows:

$$C_4 > C_2 > C_3 > C_1$$

when B_1 is best criterion and B_5 is worst criterion.

Case 3 B_2-cost is the best criteria and B_3-environmental impact is the worst criteria. In BWM method, we found the weights of the criteria are $\omega_1 = 0.1639$, $\omega_2 = 0.3538$, $\omega_3 = 0.0431$, $\omega_4 = 0.0703$, $\omega_5 = 0.2459$, $\omega_6 = 0.1230$, and $\chi^* = 0.1381$ and consistency ratio is 0.0600. Which implies a good consistency.

The ranking order in this case is as follows:

$$C_4 > C_1 > C_2 > C_3$$

when B_2 is best criterion and B_3 is worst criterion.

Table 13:

Alternatives	**Case 1**	**Rank**	**Case 2**	**Rank**	**Case 3**	**Rank**
C_1	0.4324	3	0.2407	4	0.4479	2
C_2	0.4919	2	0.4502	2	0.3997	3
C_3	0.3110	4	0.2731	3	0.3299	4
C_4	0.7388	1	0.6966	1	0.5729	1

Table 13: Ranking Results for Sensitivity Analysis

shows that alternative C_4 has the same rank in all three cases (1). In three cases, alternatives C_1, C_2, and C_3 have different rank positions and rank values. The ranking results are shown in Figure 8.

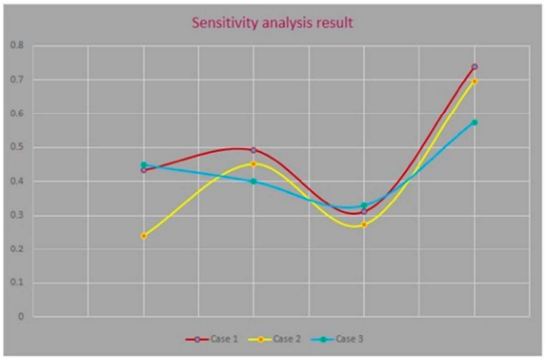


Figure 8: Sensitivity Analysis Results

Furthermore, criterion weight determines its significance, implying that the ranking sequential keeps changing as the weighting factor of the criteria changes. So, in order to test the effectiveness, we keep hoping that if a weight detection system is available, we can select the best alternative for disposing of used PPE kits depending on the priorities assigned in such cases.

7.2 Comparison Analysis

To investigate the robustness of the method, we compared the results obtained from the proposed method with other relevant MCDM methods like TOPSIS, VIKOR, and MULTIMOORA for similar data considered in the fuzzy MARCOS method. The preference order obtained using VIKOR, and TOPSIS methods coincides with that of the proposed method, whereas in the MULTIMOORA method, but the alternatives C_1, C_2, C_3 and C_4 have different ranking values when compared to the proposed approach. Table 14

Methods	Ranking values	Ranking order	Rank result
TOPSIS	C_1-0.3980, C_2-0.4902, C_3-0.2467, C_4-0.7476	$C_4 > C_2 > C_1 > C_3$	C_4
VIKOR	C_1-0.4528, C_2-0.3076, C_3-1.0000, C_4-0.0000	$C_4 < C_2 < C_1 < C_3$	C_4
MULTIMOORA	C_1-0.1446, C_2-0.3310, C_3-0.1341, C_4-0.3310	$C_4 > C_2 > C_1 > C_3$	C_4
Proposed Method	C_1-0.4324, C_2-0.4919, C_3-0.3110, C_4-0.7388	$C_4 > C_2 > C_1 > C_3$	C_4

Table 14: Comparison Analysis Results

shows the obtained results and are represented graphically in the Figure 9

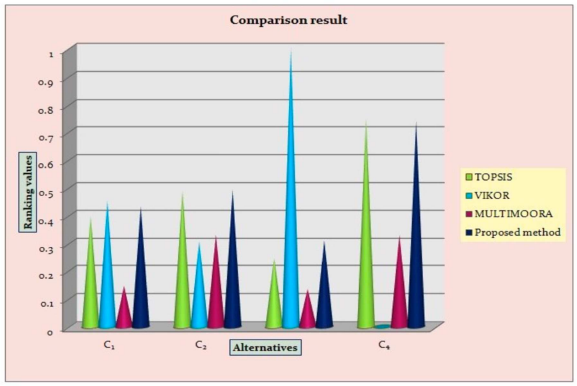


Figure 8: Comparison Analysis Results

8. Conclusion

As a consequence of the research, a billion tonnes of extra plastic items have been created throughout the world to combat the spread of COVID-19 in this specific pandemic. This has an influence on plastic waste management, and the amount of polymeric debris in our ecosystem has doubled. The proposed procedure develops a new fuzzy linguistic scale with accreditation in DHq-ROFNs and a new fuzzy MARCOS technique to establish an appropriate approach for the disposal

of PPE kits, which will contribute to reducing the amount of plastic waste from unconditioned dumping in this pandemic crisis. To create the criteria weights for this problem, the hybrid approach BWM was used with the MARCOS technique, resulting in a more flexible and acceptable solution to the plastic waste problem. Micro-plastic litter has been discovered as a substantial source of single-use disposable plastics in the landscape. Recent research has found that worn PPE kits are important sources of microplastic contamination in water sources, suggesting an environmental danger that is much greater than the COVID-19 scenario. As a result, a large number of secondhand personal protective equipment (PPEs) are now readily available in a variety of forms and uses all over the world. The presented findings are favourable for national authorities to handle the correct disposal of personal protective equipment (PPE) according to WHO recommendations. This study presents a way for turning personal protective equipment kits into alternative fuel to dispose of them. In the future, we may consider appropriately disposing of old PPE kits and other single-use medical equipment at a suitable location. Using different forms of MCDM techniques, we choose a site using a sophisticated q-rung orthopair fuzzy approach.

EXPLICIT

According to balanced concinnitas betwixt λόγοι e τόποι, just like dense threads between warps and wefts, the dissertation intends to contribute to drawing, with the help of the most powerful mathematical tools made available to sectorial scholars by contemporary science, a path based on the modulated elaboration of current operational strategies, founded on the rational application of modern functional tactics and developed on the critical experimentation of recent survey techniques capable of leading to new aims and higher goals in the field of economic research. Obviously the work does not at all pretend to be exhaustive (nemo omnia novit), but at least aspires to appear comprehensive (quod est in votis), humbly proposing (si parva licet componere magnis) some systematic methodologies inspired by the great scholars of the past, to address the serious problems of the present, perhaps solving the cogent unknowns of the future, because “Nos, quasi nani super gigantum humeros sumus, quorum beneficio longius quam ipsi speculamus”, “We are like dwarves on the shoulders of giants, thanks to which we see farther than them” (John of Blois, *Epistulae*, passim).

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