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Design sensitivity analysis of structural systems with damping devices subjected to fully non-stationary stochastic seismic excitations

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A B S T R A C T

In the framework of dynamic excitations to be considered during the design phase of structures, the most crucial one is the ground motion acceleration. To increase the structural performances against seismic actions, one of the most effective design criteria is to introduce damping devices.

An efficient approach to define the optimal parameters of the damping system is based on the *design sensitivity analysis*, which provides a quantitative estimate of desirable design change, by relating the available design variables.

In this paper a method to evaluate the sensitivities of stochastic response characteristics of structural systems with damping devices subjected to seismic excitations, modelled as fully non-stationary Gaussian stochastic processes, is proposed. The main steps are: i) to define the *time–frequency varying response* (*TFR*) function for non-classically damped systems; ii) to evaluate closed form solutions for the first-order derivatives of the *TFR* function as well as of the one-sided *evolutionary power spectral density* function of the structural response, with respect to damping parameters of devices; iii) to perform a *design sensitivity analysis* selecting as performance measure function the non-geometric spectral moments of nodal displacements.

A numerical application demonstrates how the proposed approach is suitable to cope with practical problems of engineering interest.

1. Introduction

One of more effective design criteria that has been recently suggested and applied to protect structures against earthquake effects is to introduce damping devices inside the building or between adjacent buildings to increase their performances against seismic actions [1–4]. These devices, by absorbing or dissipating part of the energy transmitted to the main structure from an earthquake, significantly mitigate the motion amplitude of: interstory drifts, absolute accelerations induced by earthquake actions and so on [5–10]. Even though these devices have often a non-linear behaviour (they can dissipate energy by mechanisms that involve alternatively yielding of metallic elements, sliding friction, motion of a piston within a viscous fluid, deformation of viscoelastic materials) the linearized viscous damping model is an attractive idealization for its mathematical simplicity, especially when the parameters of the structural system have to be changed for design reasons. A recent review of viscous dampers and viscoelastic dampers design strategies for seismic protection of structures can be found in the papers by De Domenico et al. [11], and Alhasan et al. [12], respectively.

Sensitivity analysis (*SA*) simply refers to a process that investigates how uncertainty in the model input parameters modifies a given quantity of interest of the output. Therefore, it is a suitable vehicle to evaluate the variation of structural responses under the influence of changes of structural parameters [13–15]. The *SA* can be divided into two main categories, *local sensitivity analysis* (*LSA*) and *global sensitivity analysis* (*GSA*). The *LSA* is defined as the partial derivative of the output with respect to the input parameters at the nominal values. The *GSA* attempts to provide a "global" representation of how different uncertain quantities interact to influence some function of the output [16]. An overview of the state of the art on *SA* for deterministic input and uncertain parameters can be found in a recent paper by Razavi et al. [17].

In structural engineering, *SA* aims to identify those factors, which are often only a small subset, that have a significant influence on a specific system output. It follows that the *SA* plays a significant role in structural design. Moreover, in the framework of

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LSA, the *design sensitivity analysis* (*DSA*), that concerns with the relationship between design variables available and the structural response, provides a quantitative estimate of desirable design change, even if a systematic design optimization method is not used. Therefore, the *DSA* results help engineers to decide on the direction and amount of design change needed to improve the performance measures [18,19]. Furthermore, for seismic excitations, *DSA* can be used to evaluate the effective performance of structures equipped with seismic devices, often used in seismic engineering. In fact, the effectiveness of these devices can be significantly affected by manufacturing tolerances. It follows that the effective response of structural systems equipped with devices, whose properties differ from the nominal ones, can present performances very different from those expected [see e.g. [20]]. Therefore, papers have recently been devoted to the evaluation of the *DSA* of the structural response, in the presence of viscous and viscoelastic damping devices, for deterministic excitations [see e.g. [21]].

It is well known that the accelerations induced by strong motion earthquakes have a stochastic nature. Furthermore, the analysis of recorded accelerograms have shown that ground motion accelerations change in time both their amplitude and frequency content [22,23]. Recently, Muscolino et al. [24,25] and Genovese et al. [26], analysing several recorded accelerograms, have shown that amplitude changes are strictly related to the time-variation of the energy of the accelerogram, while the frequency change depends on the time-variation of both zero-level up-crossings as well as the number of peaks. The stochastic processes involving both the intensity and the spectral variation in time are referred in the literature as fully non-stationary (or non-separable) stochastic processes [27,28]. It follows that a recorded accelerogram can be considered as a sample of zero-mean Gaussian fully non-stationary stochastic process. Then, in order to reproduce the real characteristics of recorded accelerograms, the fully non-stationary random processes should be introduced. These processes can be obtained by modulating in both frequency and amplitude a stationary zero-mean Gaussian random process through a deterministic time–frequency modulating function. Recently, Muscolino et al. [24] proposed a new model that use the *evolutionary power spectral density* (*EPSD*) function to generate samples of a fully non stationary zero mean Gaussian process, having a target acceleration time-history as one of its own samples. In [24], the *EPSD* function of the fully non-stationary process is evaluated as the sum of uniformly modulated processes, each one given by the product of a deterministic modulating function per a stationary zero-mean Gaussian sub-process, whose unimodal *PSD* function is filtered by high pass and low pass Butterworth filters. The stochastic process proposed by Muscolino et al. [24] is able to capture simultaneously the time-varying intensity and the time-varying frequency content of a target accelerogram.

On the contrary to *SA* for deterministic actions on structural systems, for which many approaches are now well established [18,19], the stochastic sensitivity, that is the variation of statistics of structural stochastic responses as a consequence of structural parameters modifications, still needs further investigation, especially for fully non-stationary stochastic excitations. In the framework of stochastic sensitivity, after the pioneering studies by Szopa [29] and Socha [30] several papers have been devoted to this topic. As an example, Benfratello et al. [31] proposed a procedure, in the time domain, to evaluate the sensitivity of the statistical moments of the response for stationary Gaussian and non-Gaussian white input processes. Yan et al [32], utilizing the pseudo-excitation method [33], implemented a procedure to evaluate the sensitivity of first and second order of response *PSD* functions once the derivatives of eigenpair are evaluated. Ding et al [34] presented two numerical methods to capture the sensitivity and Hessian matrix of the *PSD* function for non-classically damped systems subject to stationary stochastic excitations.

In the framework of uniformly modulated non-stationary stochastic excitations, Chaudhuri and Chakraborty [35] developed the formulation in double frequency domain for obtaining the analytical sensitivity statistics of various dynamic response quantities with respect to structural parameters. Cacciola et al. [36] proposed a numerical procedure for the determination of the evolution of the response statistics sensitivity for both classically and non-classically damped structural systems subjected to non-stationary non-white input processes, by solving set of differential equations once the Kronecker algebra was applied. Marano et al. [37] performed a parametric *SA* of the spectral response of a single-degree-of-freedom system with respect to uncertain soil parameters. Liu [38,39] proposed numerical methods for the calculation of the sensitivity and Hessian matrix of the response *power spectral density* matrix function of structural systems. The methods were formulated by accompanying the pseudo-excitation method with the Gauss precise time step method or the Newmark method. Tombari et al [40] proposed a method for the evaluation of the sensitivity of the stochastic response of structures coupled with Vibrating Barrier devices. Hu et al. [41] proposed an explicit time-domain method for *SA* of variances of responses of structures under uniformly modulated non-stationary random excitations.

In this paper, *DSA* of structures with viscous damping devices subjected to seismic excitations modeled by zero-mean fully-non-stationary Gaussian stochastic processes is performed. The main purpose of the proposed approach is to describe a procedure evaluating closed form solutions of the sensitivity of the *evolutionary power spectral density* (*EPSD*) function of the stochastic response. To do this, since the structural systems with damping devices are non-classically damped, first, according to the formulation recently proposed by Alderucci and Muscolino [42], the *time–frequency varying response* (*TFR*) function for non-classically damped systems is evaluated in explicit form. Then, closed form solutions for the first-order derivatives of the *TFR* function as well as of the one-sided *EPSD* function of the structural response, with respect to damping parameters of devices, are evaluated. Finally, the non-geometric spectral moments [43–47] of both nodal displacements and interstory drifts are selected as performance measure functions. Numerical applications show the computational efficiency of the proposed approach which is very suitable to cope with practical problems of engineering interest.

2. Dynamic structural response sensitivities in time domain for deterministic seismic loads

2.1. Equations of motion

Let us consider a structural system subjected to seismic excitations whose configuration could be modified for design reasons introducing viscous dampers having linear behavior. It follows that the equations of motion of a n-degree of freedom (n-DOF) structural linear system, quiescent at time $t = t_0$, can be written in the form:

$$\begin{aligned}&\mathbf{M}\,\ddot{\mathbf{U}}(t,\boldsymbol{\alpha}) + \mathbf{C}(\boldsymbol{\alpha}_C)\dot{\mathbf{U}}(t,\boldsymbol{\alpha}) + \mathbf{K}(\boldsymbol{\alpha}_K)\,\mathbf{U}(t,\boldsymbol{\alpha}) = -\mathbf{M}\,\boldsymbol{\tau}\,\ddot{U}_g(t);\\ &\mathbf{U}(t_0,\boldsymbol{\alpha}) = \mathbf{0};\quad \dot{\mathbf{U}}(t_0,\boldsymbol{\alpha}) = \mathbf{0}\end{aligned} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}(\boldsymbol{\alpha}_C)$, and $\mathbf{K}(\boldsymbol{\alpha}_K)$ are the $n \times n$ mass, damping, and stiffness matrices of the structure, $\mathbf{U}(t,\boldsymbol{\alpha})$ is the n-dimensional vector of nodal displacements relative to the ground; $\boldsymbol{\tau}$ is the n-dimensional array listing the influence coefficients of the ground shaking; $\ddot{U}_g(t)$ is the seismic acceleration; a dot over a variable denotes differentiation with respect to time. In Eq.(1), the dependence of the damping and stiffness matrices of the structure, as well as of the response vector, on the r-order design variable vector $\boldsymbol{\alpha}$, characterizing sizing of viscous device parameters, is stressed. The vector $\boldsymbol{\alpha}^T = \left[\boldsymbol{\alpha}_C^T \quad \boldsymbol{\alpha}_K^T \right]$, of order $r\,(r = r_c + r_k)$, collects device design parameters, which must be evaluated by the design procedure. It can be split as:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \Delta\boldsymbol{\alpha} \tag{2}$$

where $\Delta\boldsymbol{\alpha}^T = \left[\Delta\boldsymbol{\alpha}_C^T \quad \Delta\boldsymbol{\alpha}_K^T \right]$ is assumed to be a vector collecting small parameter variations with respect to the nominal parameter vector $\boldsymbol{\alpha}_0^T = \left[\boldsymbol{\alpha}_{C,0}^T \quad \boldsymbol{\alpha}_{K,0}^T \right]$. It follows that the $n \times n$ damping, and stiffness matrices of the structure, defined in Eq.(1), can be split as follows:

$$\mathbf{K}(\boldsymbol{\alpha}_K) = \mathbf{K}_\mathrm{S} + \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K); \quad \mathbf{C}(\boldsymbol{\alpha}_C) = \mathbf{C}_\mathrm{S} + \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) \tag{3}$$

in which $\mathbf{K}_\mathrm{S}$ and $\mathbf{C}_\mathrm{S}$ are the stiffness and damping matrices of the structure without devices, respectively; $\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K) = \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_{K,0}) + \Delta\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K)$ and $\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) = \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_{C,0}) + \Delta\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C)$ are the additional stiffness and damping matrices due to the installation of devices. They are composed by $\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_{K,0})$ and $\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_{C,0})$, evaluated in correspondence of the nominal parameter vector $\boldsymbol{\alpha}_0$ of seismic devices, and by $\Delta\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) = \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) - \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_{C,0})$ and $\Delta\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K) = \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K) - \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_{K,0})$, their deviations with respect to the additional stiffness and damping matrices evaluated at nominal seismic device parameters.

Due to the presence of seismic devices, the structural system generally could become non-classically damped, it follows that to evaluate the structural response, the equations of motion (1) have to be written in state-variables:

$$\dot{\mathbf{Z}}(t,\boldsymbol{\alpha}) = \mathbf{D}(\boldsymbol{\alpha})\,\mathbf{Z}(t,\boldsymbol{\alpha}) + \mathbf{w}\,\ddot{U}_\mathrm{g}(t); \quad \mathbf{Z}(t_0,\boldsymbol{\alpha}) = \mathbf{0} \tag{4}$$

where $\mathbf{Z}(t,\boldsymbol{\alpha})$ is the $2n$-state-variable vector while the matrix $\mathbf{D}(\boldsymbol{\alpha})$, of order $2n \times 2n$, and the vector $\mathbf{w}$, of order $2n$, are defined, respectively, as:

$$\mathbf{Z}(t,\boldsymbol{\alpha}) = \begin{bmatrix} \mathbf{U}(t,\boldsymbol{\alpha}) \\ \dot{\mathbf{U}}(t,\boldsymbol{\alpha}) \end{bmatrix}; \quad \mathbf{D}(\boldsymbol{\alpha}) = \begin{bmatrix} \mathbf{O}_{n,n} & \mathbf{I}_n \\ -\mathbf{M}^{-1}\mathbf{K}(\boldsymbol{\alpha}_K) & -\mathbf{M}^{-1}\mathbf{C}(\boldsymbol{\alpha}_C) \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} \mathbf{O}_{n,1} \\ -\boldsymbol{\tau} \end{bmatrix} \tag{5}$$

with $\mathbf{I}_n$ the n-order identity matrix and $\mathbf{O}_{n,s}$ the zero matrix of order $n \times s$.

Once the equations of motion are written in state-variables, the solution of Eq.(4), for quiescent structural systems, can be formally written as [48,49]:

$$\mathbf{Z}(t,\boldsymbol{\alpha}) = \int_{t_0}^{t} \boldsymbol{\Theta}(t-\tau,\boldsymbol{\alpha})\mathbf{w}\,\ddot{U}_\mathrm{g}(\tau)\,\mathrm{d}\tau \tag{6}$$

where $\boldsymbol{\Theta}(t,\boldsymbol{\alpha})$ is the transition matrix which can be evaluated once the following eigenproblem is solved:

$$\mathbf{D}^{-1}(\boldsymbol{\alpha})\boldsymbol{\Psi}(\boldsymbol{\alpha}) = \boldsymbol{\Psi}(\boldsymbol{\alpha})\boldsymbol{\Lambda}^{-1}(\boldsymbol{\alpha}); \quad \boldsymbol{\Psi}^T(\boldsymbol{\alpha})\mathbf{A}(\boldsymbol{\alpha}_C)\boldsymbol{\Psi}(\boldsymbol{\alpha}) = \mathbf{I}_{2m} \tag{7}$$

where the superscript T denotes the transpose operator, $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ is a diagonal matrix collecting the first $2m$ complex eigenvalues ($m \leqslant n$ is the number of *complex modes* selected for the analysis), and $\boldsymbol{\Psi}(\boldsymbol{\alpha})$ is a complex matrix, of order $(2n \times 2m)$, collecting the corresponding 2 m complex eigenvectors. In Eq.(7) the following matrix has been introduced:

$$\mathbf{A}(\boldsymbol{\alpha}_C) = \begin{bmatrix} \mathbf{C}(\boldsymbol{\alpha}_C) & \mathbf{M} \\ \mathbf{M} & \mathbf{O}_{n,n} \end{bmatrix}. \tag{8}$$

Once the eigenproblem (7) is solved, the transition matrix can be evaluated as follows:

$$\begin{aligned} \boldsymbol{\Theta}(t,\boldsymbol{\alpha}) &= \exp\left[t\,\mathbf{D}(\boldsymbol{\alpha})\right] = \boldsymbol{\Psi}(\boldsymbol{\alpha})\exp\left[t\,\boldsymbol{\Lambda}(\boldsymbol{\alpha})\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha})\mathbf{A}(\boldsymbol{\alpha}_C) \\ &\equiv \boldsymbol{\Psi}^*(\boldsymbol{\alpha})\exp\left[t\,\boldsymbol{\Lambda}^*(\boldsymbol{\alpha})\right]\boldsymbol{\Psi}^{*\,T}(\boldsymbol{\alpha})\mathbf{A}(\boldsymbol{\alpha}_C) \end{aligned} \tag{9}$$

where the asterisk * denotes the complex conjugate matrix.

2.2. Deterministic local sensitivity analysis

The local sensitivity analysis (*LSA*) consists in the evaluation of the change in the system response due to system parameter variations in the neighborhood of prefixed values, called "nominal parameters". In state-variables the first-order sensitivity vector of the structural response, $\mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0)$, with respect to i-th parameter α_i, i-th element of the r-order parameter vector $\boldsymbol{\alpha}$, is defined as follows:

$$\mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0) = \left.\frac{\partial \mathbf{Z}(t,\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \tag{10}$$

By performing the differentiation of Eq.(4), with respect to i-th parameter α_i, and setting $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$, the following differential equation governing the evolution of state-variable sensitivity vector is obtained [36]:

$$\dot{\mathbf{s}}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0) = \mathbf{D}(\boldsymbol{\alpha}_0)\,\mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0) + \bar{\mathbf{F}}(t,\boldsymbol{\alpha}_0); \quad \mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0) = \mathbf{0} \tag{11}$$

where $\bar{\mathbf{F}}(t,\boldsymbol{\alpha}_0)$ is the pseudo-force vector given as:

$$\bar{\mathbf{F}}(t,\boldsymbol{\alpha}_0) = \mathbf{D}'_i(\boldsymbol{\alpha}_0)\mathbf{Z}(t,\boldsymbol{\alpha}_0) \tag{12}$$

in which the matrix $\mathbf{D}'_i(\boldsymbol{\alpha}_0)$ can be readily determined differentiating the matrix $\mathbf{D}(\boldsymbol{\alpha})$ with respect to i-th parameter, α_i, that is,

$$\mathbf{D}'_i(\boldsymbol{\alpha}_0) = \left.\frac{\partial}{\partial \alpha_i}\mathbf{D}(\boldsymbol{\alpha})\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} = \begin{bmatrix} \mathbf{O}_{n,n} & \mathbf{O}_{n,n} \\ -\mathbf{M}^{-1}\mathbf{K}'_i(\boldsymbol{\alpha}_{K,0}) & -\mathbf{M}^{-1}\mathbf{C}'_i(\boldsymbol{\alpha}_{C,0}) \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} \mathbf{K}'_i(\boldsymbol{\alpha}_{K,0}) &= \left.\tfrac{\partial}{\partial \alpha_i}\mathbf{K}(\boldsymbol{\alpha}_K)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_{K,0}} \equiv \left.\tfrac{\partial}{\partial \alpha_i}\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_{K,0}}, \quad \alpha_i \in \boldsymbol{\alpha}_K; \\ \mathbf{C}'_i(\boldsymbol{\alpha}_{C,0}) &= \left.\tfrac{\partial}{\partial \alpha_i}\mathbf{C}(\boldsymbol{\alpha}_C)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_{C,0}} \equiv \left.\tfrac{\partial}{\partial \alpha_i}\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C)\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_{C,0}}, \quad \alpha_i \in \boldsymbol{\alpha}_C \end{aligned} \tag{14}$$

Nothing that the set of first-order ordinary differential in Eq.(11) is formally similar to Eq.(4). It follows, according to Eq.(6), that the i-th state-variable sensitivity vector of nodal response, for quiescent structural systems, can be calculated as:

$$\begin{aligned} \mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0) &= \int_{t_0}^{t} \boldsymbol{\Theta}(t-\tau,\boldsymbol{\alpha}_0)\bar{\mathbf{F}}(\tau,\boldsymbol{\alpha}_0)\,\mathrm{d}\tau \\ &\equiv \int_{t_0}^{t} \boldsymbol{\Theta}(t-\tau,\boldsymbol{\alpha}_0)\mathbf{D}'_i(\boldsymbol{\alpha}_0)\mathbf{Z}(\tau,\boldsymbol{\alpha}_0)\,\mathrm{d}\tau \end{aligned} \tag{15}$$

where

$$\mathbf{Z}(t,\boldsymbol{\alpha}_0) = \int_{t_0}^{t} \boldsymbol{\Theta}(t-\tau,\boldsymbol{\alpha}_0)\mathbf{w}\,\ddot{U}_\mathrm{g}(\tau)\,\mathrm{d}\tau = \boldsymbol{\Psi}(\boldsymbol{\alpha}_0)\,\mathbf{X}(t,\boldsymbol{\alpha}_0) \tag{16}$$

with $\mathbf{X}(t,\boldsymbol{\alpha}_0)$ the complex modal response

$$\mathbf{X}(t,\boldsymbol{\alpha}_0) = \int_{t_0}^{t} \exp\left[(t-\rho)\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\right]\mathbf{v}(\boldsymbol{\alpha}_0)\,\ddot{U}_\mathrm{g}(\rho)\,\mathrm{d}\rho \tag{17}$$

Alternatively, the state-variable sensitivity vector (15), with respect to the i-th parameter, can be evaluated as:

$$\mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0) = \boldsymbol{\Psi}(\boldsymbol{\alpha}_0)\,\mathbf{Y}_i(t,\boldsymbol{\alpha}_0) \tag{18}$$

where $\mathbf{Y}_i(t,\boldsymbol{\alpha}_0)$ is the sensitivity vector of the response, with respect to the parameter α_i, into the complex modal subspace, given as:

$$\mathbf{Y}_i(t,\boldsymbol{\alpha}_0) = \int_{t_0}^{t} \exp\left[(t-\tau)\,\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\right]\mathbf{B}_i(\boldsymbol{\alpha}_0)\mathbf{X}(\tau,\boldsymbol{\alpha}_0)\mathrm{d}\tau \tag{19}$$

In the previous equations the following positions have been made:

$$\mathbf{B}_i(\boldsymbol{\alpha}_0) = \boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0)\mathbf{A}(\boldsymbol{\alpha}_{C,0})\mathbf{D}'_i(\boldsymbol{\alpha}_0)\boldsymbol{\Psi}(\boldsymbol{\alpha}_0);$$
$$\mathbf{v}(\boldsymbol{\alpha}_0) = \boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0)\mathbf{A}(\boldsymbol{\alpha}_{C,0})\mathbf{w} \tag{20}$$

Notice that for deterministic excitation the state-variable sensitivity vectors (15) and (19), with respect to the i-th design parameter, can be easily evaluated by step-by-step procedures [36].

3. Explicit form of dynamic structural response sensitivities in the mixed time–frequency domain

3.1. Definition of seismic accelerations as fully non-stationary random processes

In this section it is assumed that the ground motion acceleration, $\ddot{U}_g(t)$, is a zero-mean Gaussian fully non-stationary random process. In order to define this process, here the Priestley spectral representation of non-stationary processes is adopted [50,51]. Moreover, in the stochastic analysis the one-sided *Power Spectral Density* (*PSD*) function is generally used to characterize the input process. It has been demonstrated that, because the one-sided *PSD* function is not symmetric [43–45], the corresponding autocorrelation function is a complex function having real part coincident with the autocorrelation function corresponding to the two-sided *PSD* function [44]. This implies that, from a mathematical point of view, the zero-mean Gaussian fully non-stationary random process is a complex process. It can be defined by means of the following *Fourier-Stieltjes integral* [46]:

$$\ddot{U}_g(t) = \sqrt{2}\int_0^{\infty} \exp(\mathrm{i}\ \omega\ t)\, a(\omega,t)\, \mathrm{d}N(\omega) \tag{21}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit; $a(\omega,t)$ is a slowly varying complex deterministic time–frequency modulating function which has to satisfy the condition: $a(\omega,t) \equiv a^*(-\omega,t)$; $N(\omega)$ is a zero-mean process with orthogonal increments satisfying the condition:

$$\mathrm{E}\langle \mathrm{d}N(\omega_1)\ \mathrm{d}N^*(\omega_2)\rangle = \delta(\omega_1-\omega_2)\, G_0(\omega_1)\ \mathrm{d}\omega_1\ \mathrm{d}\omega_2 \tag{22}$$

where the operator $\mathrm{E}\langle\cdot\rangle$ denotes the stochastic average; $\delta(\cdot)$ is the Dirac delta, and $G_0(\omega)$ is the one-sided *PSD* function of the "embedded" stationary counterpart process [47], which is a real function for $\omega \geqslant 0$, while $G_0(\omega) = 0$ for $\omega < 0$.

Notice that, because of the *PSD* function has been assumed one-sided, the zero-mean Gaussian non-stationary random process $\ddot{U}_g(t)$ is a complex one [43–45]. This process can be completely defined, in the time domain, by the knowledge of its complex autocorrelation function:

$$R_{\ddot{U}_g\ddot{U}_g}(t_1,t_2) \equiv \mathrm{E}\left\langle \ddot{U}_g(t_1)\ \ \ddot{U}_g(t_2)\right\rangle = \int_0^{\infty} \exp[\mathrm{i}\omega(t_1-t_2)]\, G_{\ddot{U}_g\ddot{U}_g}(\omega,t_1,t_2)\mathrm{d}\omega \tag{23}$$

where

$$G_{\ddot{U}_g\ddot{U}_g}(\omega,t_1,t_2) = a(\omega,t_1)\ \ a^*(\omega,t_2)G_0(\omega) \tag{24}$$

The complex process, $\ddot{U}_g(t)$, which generates the complex autocorrelation function (Eq. (23)) has been called *pre-envelope process* by Di Paola [44]. In the Priestley evolutionary process model, the function

$$G_{\ddot{U}_g\ddot{U}_g}(\omega,t) = |a(\omega,t)|^2 G_0(\omega) \tag{25}$$

is called one-sided *evolutionary power spectral density* (*EPSD*) function of the non-stationary process $\ddot{U}_g(t)$. This process is called *fully non-stationary* random process since both time and frequency content change. If the modulating function is a time dependent function, $a(\omega,t) \equiv a(t)$, the non-stationary process is called *uniformly modulated* (or *quasi-stationary*) random process. In the latter case the *EPSD* function assumes the expression: $G_{\ddot{U}_g\ddot{U}_g}(\omega,t) = a(t)^2 G_0(\omega)$.

3.2. Closed form solution for the time–frequency varying response vector function

It has been shown that the *time–frequency varying response* (*TFR*) vector function of the response plays a central role in the evaluation of the statistics of the response for both classically and non-classically damped structural systems subjected to fully non-stationary stochastic input [27,28,42,46]. In the presence of the unknown r-order parameter vector $\boldsymbol{\alpha}$, the *TFR* vector function of nodal response, $\mathbf{Z}(\omega,t,\boldsymbol{\alpha})$, according to Eq.(16), can be evaluated as follows

$$\mathbf{Z}(\omega,t,\boldsymbol{\alpha}) = \boldsymbol{\Psi}(\boldsymbol{\alpha})\,\mathbf{X}(\omega,t,\boldsymbol{\alpha}) \tag{26}$$

where $\mathbf{X}(\omega,t,\boldsymbol{\alpha})$ is the *TFR* vector function of the modal complex response, given by:

$$\mathbf{X}(\omega,t,\boldsymbol{\alpha}) = \int_{t_0}^{t} \exp[(t-\tau)\boldsymbol{\Lambda}(\boldsymbol{\alpha})]\mathbf{v}(\boldsymbol{\alpha})\exp(\mathrm{i}\omega\tau)a(\omega,\tau)\mathrm{d}\tau \tag{27}$$

This vector function, in the following denoted by the acronym *MTFR* (*modal time–frequency varying response*), can be evaluated as the solution of a set of 2 m first order uncoupled differential equations. Indeed, the following relationship holds [42,46]:

$$\dot{\mathbf{X}}(\omega,t,\boldsymbol{\alpha}) = \boldsymbol{\Lambda}(\boldsymbol{\alpha})\,\mathbf{X}(\omega,t,\boldsymbol{\alpha}) + \mathbf{v}(\boldsymbol{\alpha})\ \exp(\mathrm{i}\omega\, t)a(\omega,t)\mathcal{U}(t-t_0);$$
$$\mathbf{X}(\omega,t_0,\boldsymbol{\alpha}) = \mathbf{X}_0(\omega,\boldsymbol{\alpha}) \tag{28}$$

where $\mathbf{X}(\omega,t_0,\boldsymbol{\alpha}) \equiv \mathbf{X}_0(\omega,\boldsymbol{\alpha})$ is the vector of the initial condition at time $t = t_0$ and $\mathcal{U}(t)$ is the *unit step function.* When the particular solution of Eq.(28), $\mathbf{X}_p(\omega,t,\boldsymbol{\alpha})$, can be determined in explicit form, the *MTFR* vector function, according to the dynamics of non-classically damped systems, can be written as [42]:

$$\mathbf{X}(\omega,t,\boldsymbol{\alpha}) = \{\mathbf{X}_p(\omega,t,\boldsymbol{\alpha}) + \exp[t\ \boldsymbol{\Lambda}(\boldsymbol{\alpha})] \times [\mathbf{X}_0(\omega,\boldsymbol{\alpha}) - \mathbf{X}_p(\omega,t_0,\boldsymbol{\alpha})]\}\mathcal{U}(t-t_0). \tag{29}$$

It has been recently shown that the analytical expression of the particular solution vector $\mathbf{X}_p(\omega,t,\boldsymbol{\alpha})$, can be easily obtained in closed form for the most common models of modulating function $a(\omega,t)$ proposed in literature [46]. In particular, here the Spanos and Solomos [52] model for the fully non-stationary seismic excitation is adopted. It is well known that this model is very useful in the framework of seismic engineering. Obviously, the proposed formulation can be easily particularized for other simpler models. In the Spanos and Solomos [52] model the time–frequency functions can be written as:

$$a(\omega,t) = \varepsilon(\omega)\ (t-t_0)\ \exp[-\alpha_a(\omega)\,(t-t_0)]\mathcal{U}(t-t_0)\,; \tag{30}$$

where $\varepsilon(\omega)$ and $\alpha_a(\omega)$ could be complex functions which have to be chosen to satisfy the condition: $a(\omega,t) \equiv a^*(-\omega,t)$. Moreover, for quiescent structural systems at time $t_0 = 0$, $\mathbf{X}_0(\omega,\boldsymbol{\alpha}) = \mathbf{0}$, and for the modulating function, defined in Eq.(30), the vector $\mathbf{X}(\omega,t,\boldsymbol{\alpha})$, defined in Eq.(29), can be evaluated in explicit form as [42]:

$$\mathbf{X}(\omega,t,\boldsymbol{\alpha}) = -\varepsilon(\omega)\{\exp(-\beta(\omega)\,t)[\boldsymbol{\Gamma}^2(\omega,\boldsymbol{\alpha}) + t\,\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha})] - \exp[t\ \boldsymbol{\Lambda}(\boldsymbol{\alpha})]\ \boldsymbol{\Gamma}^2(\omega,\boldsymbol{\alpha})\} \times\ \mathbf{v}(\boldsymbol{\alpha})\,\mathcal{U}(t) \tag{31}$$

where $\beta(\omega) = \alpha_a(\omega) - \mathrm{i}\omega$ and $\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha})$ is a diagonal matrix defined as:

$$\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha}) = [\boldsymbol{\Lambda}(\boldsymbol{\alpha}) + \beta(\omega)\mathbf{I}_{2m}]^{-1}. \tag{32}$$

3.3. Closed form solution for the sensitivity time–frequency varying response vector function

According to Eqs.(18) and (26), the *sensitivity* of the *TFR* vector function, with respect to *i*-th parameter, can be also evaluated as

$$\mathbf{s}_{\mathbf{Z},i}(\omega,t,\boldsymbol{\alpha}_0)=\boldsymbol{\Psi}(\boldsymbol{\alpha}_0)\,\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0) \tag{33}$$

where $\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0)$ is the *sensitivity* of the *TFR* vector function, with respect to the parameter α_i, projected into the complex modal subspace. It can be evaluated as follows:

$$\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0)=\int_0^t \exp\left[(t-\tau)\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\right]\mathbf{B}_i(\boldsymbol{\alpha}_0)\mathbf{X}(\omega,\tau,\boldsymbol{\alpha}_0)\,\mathrm{d}\tau. \tag{34}$$

In the following this vector will be synthetically denoted by the acronym *MSTFR* (*modal sensitivity time–frequency response)* vector function.

The main problem is now to evaluate in explicit form the *MSTFR* vector function, $\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0)$, taking into account Eq.(31). After very simple algebra it can be shown that this vector function can be evaluated as solution of the following differential equation, with zero start conditions at time $t_0=0$:

$$\begin{aligned}&\dot{\mathbf{Y}}_i(\omega,t,\boldsymbol{\alpha}_0)=\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\,\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0)\;+\mathbf{B}_i(\boldsymbol{\alpha}_0)\mathbf{X}(\omega,\tau,\boldsymbol{\alpha}_0)\,\mathcal{U}(t);\\&\mathbf{Y}_i(\omega,0,\boldsymbol{\alpha}_0)=\mathbf{0}.\end{aligned} \tag{35}$$

To perform the solution of this set of differential equations the *MTFR* vector function, defined in Eq.(31), is rewritten as:

$$\mathbf{X}(\omega,t,\boldsymbol{\alpha}_0)\;=\;\mathbf{X}_1(\omega,t,\boldsymbol{\alpha}_0)+\mathbf{X}_2(\omega,t,\boldsymbol{\alpha}_0) \tag{36}$$

where

$$\begin{aligned}&\mathbf{X}_1(\omega,t,\boldsymbol{\alpha}_0)\;=\;-\varepsilon(\omega)\,\exp\left(-\beta(\omega)\,t\right)\,\left[\boldsymbol{\Gamma}_0^2(\omega)+\,t\,\boldsymbol{\Gamma}_0(\omega)\right]\mathbf{v}_0\,\mathcal{U}(t);\\&\mathbf{X}_2(\omega,t,\boldsymbol{\alpha}_0)\;=\;\varepsilon(\omega)\exp\left(t\,\boldsymbol{\Lambda}_0\right)\,\boldsymbol{\Gamma}_0^2(\omega)\mathbf{v}_0\,\mathcal{U}(t).\end{aligned} \tag{37}$$

Notice that for simplicity's sake in the previous equations the following position have been made:

$$\boldsymbol{\Lambda}_0\;=\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\,;\quad \boldsymbol{\Gamma}_0(\omega)=\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha}_0);\quad \mathbf{v}_0=\mathbf{v}(\boldsymbol{\alpha}_0). \tag{38}$$

Since the *MTFR* vector function has been split as the sum of two contributions, the *MSTFR* vector function, solution of Eq.(35), can be split as the sum of two vectors too, solutions of the following two sets of differential equations, with zero start initial conditions at time $t_0=0$:

$$\begin{aligned}&\dot{\mathbf{Y}}_{i,1}(\omega,t,\boldsymbol{\alpha}_0)=\boldsymbol{\Lambda}_0\,\mathbf{Y}_{i,1}(\omega,t,\boldsymbol{\alpha}_0)\;+\mathbf{B}_i(\boldsymbol{\alpha}_0)\mathbf{X}_1(\omega,t,\boldsymbol{\alpha}_0)\,;\\&\mathbf{Y}_{i,1}\,(\omega,0,\boldsymbol{\alpha}_0)=\;\mathbf{0}\\&\dot{\mathbf{Y}}_{i,2}(\omega,t,\boldsymbol{\alpha}_0)=\boldsymbol{\Lambda}_0\,\mathbf{Y}_{i,2}(\omega,t,\boldsymbol{\alpha}_0)\;+\mathbf{B}_i(\boldsymbol{\alpha}_0)\mathbf{X}_2(\omega,t,\boldsymbol{\alpha}_0)\,;\\&\mathbf{Y}_{i,2}(\omega,0,\boldsymbol{\alpha}_0)=\mathbf{0}\end{aligned} \tag{39}$$

It follows that the *MSTFR* vector function can be evaluated as the sum of the two terms:

$$\begin{aligned}&\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0)=\mathbf{Y}_{i,1}(\omega,t,\boldsymbol{\alpha}_0)+\,\mathbf{Y}_{i,2}(\omega,t,\boldsymbol{\alpha}_0)\\&=\;\big\{\mathbf{Y}_{i,1,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0)+\mathbf{Y}_{i,2,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0)\\&\quad-\exp\left(t\,\boldsymbol{\Lambda}_0\right)\left[\mathbf{Y}_{i,1,\mathrm{p}}(\omega,0,\boldsymbol{\alpha}_0)+\mathbf{Y}_{i,2,\mathrm{p}}(\omega,0,\boldsymbol{\alpha}_0)\right]\big\}\mathcal{U}(t)\end{aligned} \tag{40}$$

where the particular solution vectors of Eqs.(39), can be evaluated, after some algebra, as follows:

$$\begin{aligned}&\mathbf{Y}_{i,1,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0)\;=\;\varepsilon(\omega)\,\exp\left(-\beta(\omega)\,t\right)\\&\times\boldsymbol{\Gamma}_0(\omega)[\boldsymbol{\Gamma}_0(\omega)\,\mathbf{B}_i(\boldsymbol{\alpha}_0)+\,\mathbf{B}_i(\boldsymbol{\alpha}_0)\,\boldsymbol{\Gamma}_0(\omega)+t\,\mathbf{B}_i(\boldsymbol{\alpha}_0)]\boldsymbol{\Gamma}_0(\omega)\mathbf{v}_0;\\&\mathbf{Y}_{i,2,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0)\;=\;\varepsilon(\omega)\mathbf{P}_i(t,\boldsymbol{\alpha}_0)\exp\left(t\,\boldsymbol{\Lambda}_0\right)\boldsymbol{\Gamma}_0^2(\omega)\mathbf{v}_0;\end{aligned} \tag{41}$$

where $\mathbf{P}_i(t,\boldsymbol{\alpha}_0)$ is a matrix of order $(2m\times 2m)$ whose elements, $P_{i,jk}(t,\boldsymbol{\alpha}_0)$, are defined as:

$$P_{i,jj}(t,\boldsymbol{\alpha}_0)=t\;B_{i,jj}(\boldsymbol{\alpha}_0);\quad P_{i,jk}(t,\boldsymbol{\alpha}_0)=\frac{B_{i,jk}(\boldsymbol{\alpha}_0)}{\lambda_k-\lambda_j},\;j\neq k \tag{42}$$

with $B_{i,jk}(\boldsymbol{\alpha}_0)$ elements of the matrix $\mathbf{B}_i(\boldsymbol{\alpha}_0)$ introduced in Eq.(20).

4. Local sensitivities of stochastic structural response for fully non-stationary seismic input processes

4.1. Closed form solutions for the sensitivities of the EPSD response matrix function

Since it has been assumed that the ground motion acceleration, $\ddot{U}_g(t)$, is a zero-mean Gaussian fully non-stationary random process, with one-sided *EPSD* function, $G_{\ddot{U}_g\ddot{U}_g}(\omega,t)$, the stochastic response is a zero-mean fully non-stationary stochastic vector process, whose one-sided *EPSD* matrix function, $\mathbf{G}_{\mathbf{ZZ}}(\omega,t,\boldsymbol{\alpha})$, can be evaluated as follows [46]:

$$\begin{aligned}\mathbf{G}_{\mathbf{ZZ}}(\omega,t,\boldsymbol{\alpha})&=G_0(\omega)\,\mathbf{Z}^*(\omega,t,\boldsymbol{\alpha})\mathbf{Z}^T(\omega,t,\boldsymbol{\alpha})\\&=G_0(\omega)\,\boldsymbol{\Psi}^*\,(\boldsymbol{\alpha})\mathbf{X}^*(\omega,t,\boldsymbol{\alpha})\mathbf{X}^T(\omega,t,\boldsymbol{\alpha})\boldsymbol{\Psi}^T(\boldsymbol{\alpha})\end{aligned} \tag{43}$$

where $G_0(\omega)$ is the one-sided *PSD* function of the stationary counterpart of the input process, while $\mathbf{Z}(\omega,t,\boldsymbol{\alpha})$ and $\mathbf{X}(\omega,t,\boldsymbol{\alpha})$ are the *TFR* vector responses, in state variables, into nodal and modal complex spaces, respectively. Once the one-sided *EPSD* matrix function, $\mathbf{G}_{\mathbf{ZZ}}(\omega,t)$, is defined, it is possible to evaluate in compact form the statistics of the response as follows:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(t,\boldsymbol{\alpha})=\boldsymbol{\Psi}^*(\boldsymbol{\alpha})\left[\int_0^\infty G_0(\omega)\,\mathbf{X}^*(\omega,t,\boldsymbol{\alpha})\,\mathbf{X}^T(\omega,t,\boldsymbol{\alpha})\mathrm{d}\omega\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha}) \tag{44}$$

This matrix is the so-called *pre-envelope covariance* (*PEC*) matrix function, in nodal space, it is a $2n\times 2n$ Hermitian matrix, whose real part coincides with the classical covariance matrix. It can be evaluated formally as [43,44]:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(t,\boldsymbol{\alpha})=\mathrm{E}\Big\langle\mathbf{Z}(t,\boldsymbol{\alpha})\mathbf{Z}^{*T}(t,\boldsymbol{\alpha})\Big\rangle=\begin{bmatrix}\boldsymbol{\Lambda}_{0,\mathbf{UU}}(t,\boldsymbol{\alpha})&\mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}(t,\boldsymbol{\alpha})\\-\mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}^{*T}(t,\boldsymbol{\alpha})&\boldsymbol{\Lambda}_{2,\mathbf{UU}}(t,\boldsymbol{\alpha})\end{bmatrix} \tag{45}$$

where the matrices $\boldsymbol{\Lambda}_{i,\mathbf{UU}}(t,\boldsymbol{\alpha})$ collect the so-called *i*-th order *non-geometric spectral moments* (*NGSM*) [44,47] of the stochastic response.

As stated before the *sensitivity analysis* consists in the evaluation of the change in the system response due to system parameter variations in the neighbourhood of nominal parameters, $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$. By differentiating the *PEC* matrix function, defined in Eq.(45), it is possible to evaluate its *sensitivity* function, with respect to the *i*-th parameter, as follows:

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathbf{s}_{\mathbf{Z},i}}(t,\boldsymbol{\alpha}_0)&=\frac{\partial\boldsymbol{\Sigma}_{\mathbf{ZZ}}(t,\boldsymbol{\alpha})}{\partial\alpha_i}\bigg|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}=\frac{\partial}{\partial\alpha_i}\begin{bmatrix}\boldsymbol{\Lambda}_{0,\mathbf{UU}}(t,\boldsymbol{\alpha})&\mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}(t,\boldsymbol{\alpha})\\-\mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}^{T}(t,\boldsymbol{\alpha})&\boldsymbol{\Lambda}_{2,\mathbf{UU}}(t,\boldsymbol{\alpha})\end{bmatrix}\Bigg|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}\\&=\;\mathrm{E}\Big\langle\mathbf{Z}(t,\boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*\mathrm{T}}(t,\boldsymbol{\alpha}_0)\Big\rangle+\mathrm{E}\Big\langle\mathbf{Z}(t,\boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*\mathrm{T}}(t,\boldsymbol{\alpha}_0)\Big\rangle^{*T}\end{aligned} \tag{46}$$

whose elements are the *sensitivity* of first three spectral moments with respect to the parameter α_i. In the previous equation the sensitivity vector $\mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0)$ which has been defined in Eq.(15) appears. It follows that the following relationship holds:

$$\begin{aligned}&\mathrm{E}\Big\langle\mathbf{Z}(t,\boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*T}(t,\boldsymbol{\alpha}_0)\Big\rangle=\boldsymbol{\Psi}^*(\boldsymbol{\alpha}_0)\\&\times\left\{\int_0^\infty\mathbf{X}^*(\omega,t,\boldsymbol{\alpha}_0)\mathbf{Y}_{\mathrm{i}}^T(\omega,t,\boldsymbol{\alpha}_0)\,G_0(\omega)\mathrm{d}\omega\right\}\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0)\end{aligned} \tag{47}$$

where the vector $\mathbf{Y}_i(\omega, t, \boldsymbol{\alpha}_0)$ is the *MSTFR* vector function introduced in Eq.(34). Substituting Eq.(47) into Eq.(46) the so-called *sensitivity* of the *PEC* matrix function can be rewritten as:

$$\boldsymbol{\Sigma}_{\mathbf{s}_{\mathbf{Z},i}}(t, \boldsymbol{\alpha}_0) \equiv \int_0^\infty \mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega, t, \boldsymbol{\alpha}_0)\mathrm{d}\omega \tag{48}$$

where the matrix

$$\mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega, t, \boldsymbol{\alpha}_0) = G_0(\omega)\ \boldsymbol{\Psi}^*(\boldsymbol{\alpha}_0) \times \left[\mathbf{X}^*(\omega, t, \boldsymbol{\alpha}_0)\,\mathbf{Y}_i^T(\omega, t, \boldsymbol{\alpha}_0) + \mathbf{Y}_i^*(\omega, t, \boldsymbol{\alpha}_0)\,\mathbf{X}^T(\omega, t, \boldsymbol{\alpha}_0)\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0). \tag{49}$$

can be interpreted as the *sensitivity* of the *EPSD* matrix function. This equation shows that, since the *MTFR* vector functions in Eq.(31) as well as the *MSTFR* vector function in Eq. (40) are evaluated by explicit relationships, the so-called *sensitivity* of the *EPSD* matrix function can be evaluated by means of closed form solutions too.

4.2. Closed form solutions for the sensitivities of the PSD response matrix function

For zero-mean stationary Gaussian excitation stochastic process, the sensitivities of (geometric) spectral moments of structural response can be evaluated by differentiating *PEC* matrix, that in this case is not a time-dependent one. In fact, assuming the modulating function equal to *unit step function* (Heaviside function, $a(\omega, t) = 1;\ t > 0$), and taking the limit as $t \to \infty$, it is possible to evaluate *PEC* matrix for stationary excitation as follows:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha}) = \int_0^\infty \mathbf{G}_{\mathbf{ZZ}}(\omega, t)\mathrm{d}\omega = \boldsymbol{\Psi}^*(\boldsymbol{\alpha})\left[\int_0^\infty G_0(\omega)\ \mathbf{H}_{\mathrm{m}}^*(\omega)\ \mathbf{v}_0^*\mathbf{v}_0^T\ \mathbf{H}_{\mathrm{m}}(\omega)\ \mathrm{d}\omega\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha}) \tag{50}$$

This matrix collects the geometric spectral moments:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha}) = \begin{bmatrix} \boldsymbol{\Lambda}_{0,\mathbf{UU}}(\boldsymbol{\alpha}) & \mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}(\boldsymbol{\alpha}) \\ -\mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}^{*T}(\boldsymbol{\alpha}) & \boldsymbol{\Lambda}_{2,\mathbf{UU}}(\boldsymbol{\alpha}) \end{bmatrix} \tag{51}$$

The *sensitivity* of the *PEC* matrix for stationary excitations is then obtained by differentiating Eq.(50) with respect to the *i*-th parameter obtaining:

$$\boldsymbol{\Sigma}_{\mathbf{s}_{\mathbf{Z},i}}(\boldsymbol{\alpha}_0) = \left.\frac{\partial \boldsymbol{\Sigma}_{\mathbf{ZZ}}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} = \frac{\partial}{\partial \alpha_i}\left.\begin{bmatrix} \boldsymbol{\Lambda}_{0,\mathbf{UU}}(\boldsymbol{\alpha}) & \mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}(\boldsymbol{\alpha}) \\ -\mathrm{i}\boldsymbol{\Lambda}_{1,\mathbf{UU}}^{*T}(\boldsymbol{\alpha}) & \boldsymbol{\Lambda}_{2,\mathbf{UU}}(\boldsymbol{\alpha}) \end{bmatrix}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} = \mathrm{E}\left\langle \mathbf{Z}(t, \boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*T}(t, \boldsymbol{\alpha}_0)\right\rangle + \mathrm{E}\left\langle \mathbf{Z}(t, \boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*T}(t, \boldsymbol{\alpha}_0)\right\rangle^{*T} \tag{52}$$

where

$$\mathrm{E}\left\langle \mathbf{Z}(t, \boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*T}(t, \boldsymbol{\alpha}_0)\right\rangle = \boldsymbol{\Psi}^*(\boldsymbol{\alpha}_0) \times \left[\int_0^\infty G_0(\omega)\ \mathbf{H}_{\mathrm{m}}^*(\omega)\ \mathbf{v}_0^*\mathbf{v}_0^T\ \mathbf{H}_{\mathrm{m}}(\omega)\ \mathbf{B}_i^T(\boldsymbol{\alpha}_0)\ \mathbf{H}_{\mathrm{m}}(\omega)\mathrm{d}\omega\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0) \tag{53}$$

In this equation $\mathbf{H}_{\mathrm{m}}(\omega) = [\mathrm{i}\,\omega\,\mathbf{I}_{2m} - \boldsymbol{\Lambda}_0]^{-1}$ is the transfer function matrix in the complex modal subspace. It follows that the *sensitivity* of the *PSD* matrix function for stationary excitations, $\mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega, \boldsymbol{\alpha}_0)$, can be evaluated as:

$$\mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega, \boldsymbol{\alpha}_0) = G_0(\omega)\ \boldsymbol{\Psi}^*(\boldsymbol{\alpha}_0)\mathbf{H}_{\mathrm{m}}^*(\omega) \times \left[\mathbf{v}_0^*\mathbf{v}_0^T\,\mathbf{H}_{\mathrm{m}}(\omega)\mathbf{B}_i^T(\boldsymbol{\alpha}_0) + \mathbf{B}_i^*(\boldsymbol{\alpha}_0)\mathbf{H}_{\mathrm{m}}^*(\omega)\mathbf{v}_0^*\mathbf{v}_0^T\right]\mathbf{H}_{\mathrm{m}}(\omega)\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0). \tag{54}$$

Obviously, also this matrix can be evaluated by means of closed form solutions.

5. Design sensitivity analysis

The main purpose of *design sensitivity analysis* (*DSA*) is to find information on the structural behaviour of a structural system by analysing the sensitivities, with respect to design variables, of a performance measure quantity, opportunely selected. To do this, it is herein assumed that the selected *performance measure function*, $\varphi(t, \boldsymbol{\alpha})$, of a structural response quantity of interest, depends on device parameters, collected in the vector $\boldsymbol{\alpha}$, as well as on the state-variable response $\mathbf{Z}(t, \boldsymbol{\alpha})$. For deterministic excitations, this function can be defined as:

$$\varphi(t, \boldsymbol{\alpha}) = \mathbf{q}^T\,\mathbf{Z}(t, \boldsymbol{\alpha}) \tag{55}$$

where $\mathbf{q}$ is a vector collecting the combination coefficients relating the response quantity of interest with the structural response in state-variables $\mathbf{Z}(t, \boldsymbol{\alpha})$. In the case of fully non-stationary stochastic excitations, a very useful performance measure could be a generic *non-geometric spectral moment* of nodal displacements or interstory drifts. It follows that the performance measure function can be evaluated as a function of *PEC* matrix, that, by taking into account Eqs. (44) and (45), can be evaluated as follows:

$$\varphi(t, \boldsymbol{\alpha}) = \mathbf{q}^T\boldsymbol{\Sigma}_{\mathbf{ZZ}}(t, \boldsymbol{\alpha})\ \mathbf{q} \equiv \mathbf{q}^T\left[\int_0^\infty \mathbf{G}_{\mathbf{ZZ}}(\omega, t, \boldsymbol{\alpha})\mathrm{d}\omega\right]\mathbf{q} \tag{56}$$

It follows that its first-order derivative with respect to *i*-th parameter leads to:

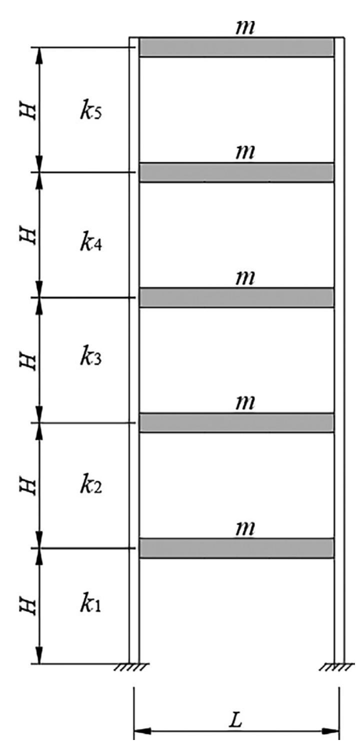

Fig. 1. Five-storey plane frame.

Table 1
Modal information of the analysed undamped building.

mode	ω [rad/s]	T [s]
1	14.331	0.438
2	41.832	0.150
3	65.944	0.095
4	84.713	0.074
5	96.620	0.065

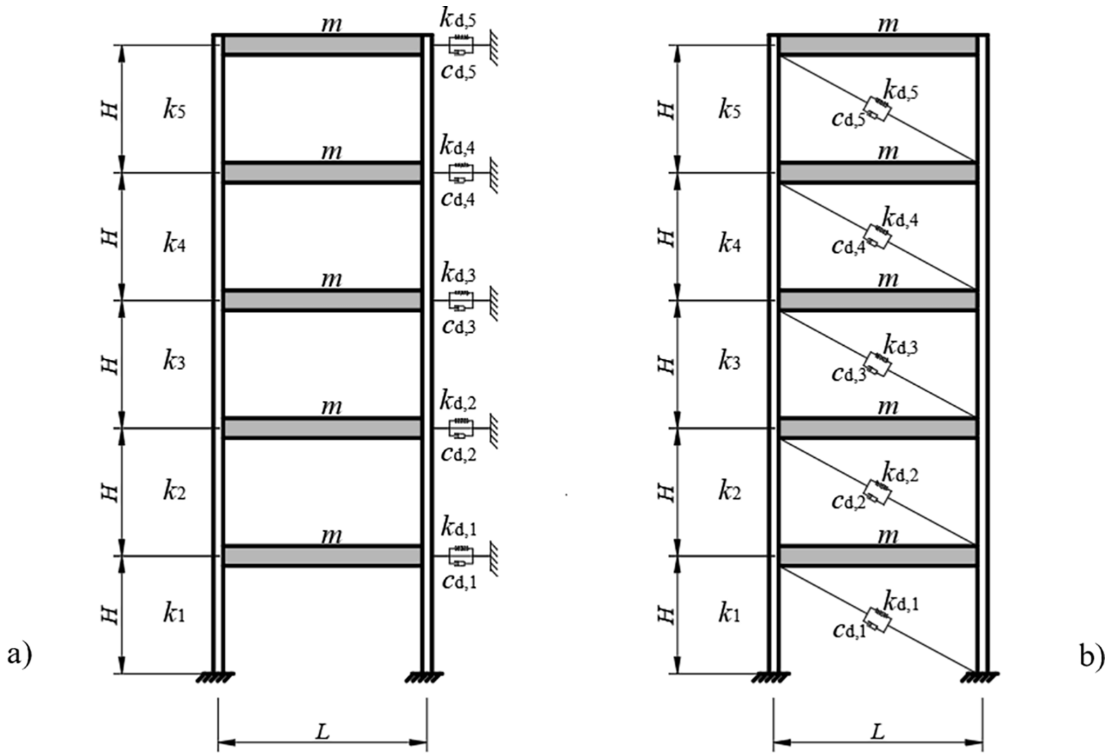

Fig. 2. Structure with viscous damper devices: a) external dampers; b) internal dampers.

Table 2
Percentage performance measure sensitivities for the structure with external dampers.

Floor	$\varepsilon_{c,i}$ (%)	$\varepsilon_{k,i}$ (%)
1	−3.22	-2.4×10^{-5}
2	−11.48	-8.3×10^{-5}
3	−22.24	-1.7×10^{-4}
4	−32.68	-2.9×10^{-4}
5	−39.61	-4.3×10^{-4}

Table 3
Percentage of performance measure sensitivities for the structure with internal dampers.

Drifts	$\varepsilon_{c,i}$ (%)	$\varepsilon_{k,i}$(%)
1	−26.66	$+1.2 \times 10^{-5}$
2–1	−22.23	$+1.2 \times 10^{-5}$
3–2	−15.62	$+6.1 \times 10^{-6}$
4–3	−8.56	-1.2×10^{-6}
5–4	−5.27	-1.2×10^{-4}

$$s_{\varphi_{\alpha_i}}(t,\boldsymbol{\alpha}_0) = \left.\frac{\partial\varphi(t,\boldsymbol{\alpha})}{\partial\alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} = \mathbf{q}^T\boldsymbol{\Sigma}_{\mathbf{s}_{\mathbf{Z},i}}(\boldsymbol{\alpha}_0)\,\mathbf{q} = \mathbf{q}^T\left[\int_0^\infty \mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega,t,\boldsymbol{\alpha}_0)\mathrm{d}\omega\right]\mathbf{q} \quad (57)$$

where $\mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega,t,\boldsymbol{\alpha}_0)$ is the matrix function evaluated in explicit closed form in Eq.(49).

Finally, for stationary excitation the derivatives of the performance measure are not time-dependent and can be particularized as:

$$s_{\varphi_{\alpha_i}}(\boldsymbol{\alpha}_0) = \mathbf{q}^T\left[\int_0^\infty \mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega,\boldsymbol{\alpha}_0)\mathrm{d}\omega\right]\mathbf{q} \quad (58)$$

where $\mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega,\boldsymbol{\alpha}_0)$ is the matrix function evaluated in explicit closed form in Eq.(54).

For design purposes, it is very useful to identify the most influential design parameters of the viscous devices on the response quantity of interest. To this aim, a *percentage function of performance measure sensitivities* is herein introduced as a percentage measure of the influence of the generic parameter on the selected performance measure, i.e.:

$$\varepsilon_i(t,\boldsymbol{\alpha}_0)(\%) = \frac{s_{\varphi_{\alpha_i}}(t,\boldsymbol{\alpha}_0)}{\varphi(t,\boldsymbol{\alpha}_0)} \times 100 \quad (59)$$

Obviously for stationary excitation this quantity is not time-dependent.

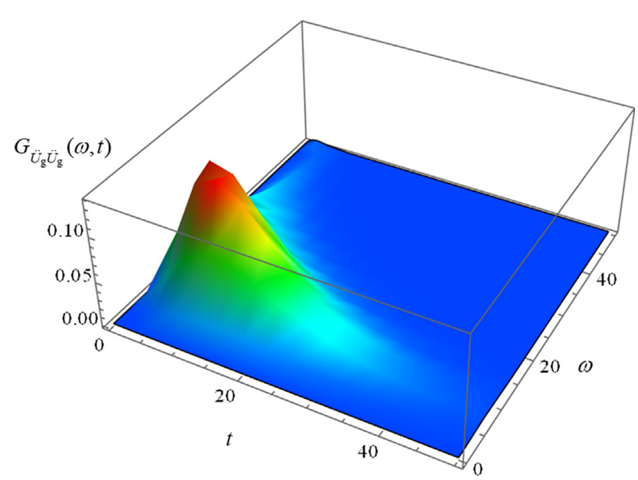

Fig. 3. *EPSD* function of the input process according to Spanos and Solomos model [52].

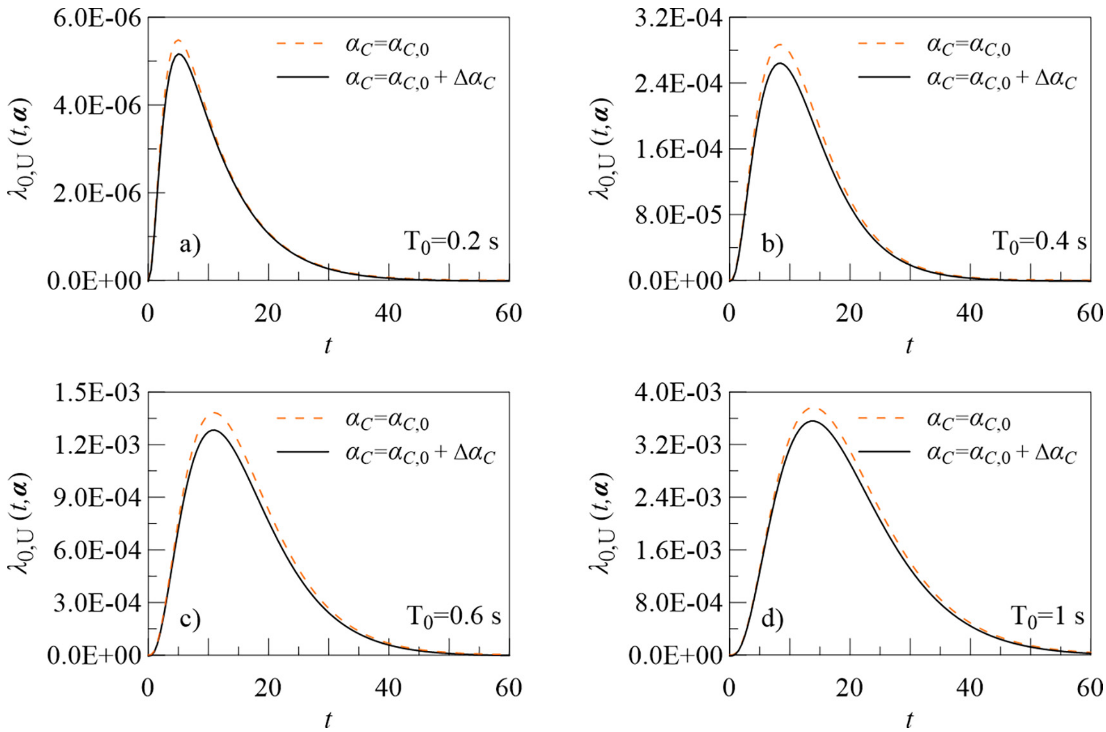

Fig. 4. Comparison between the first *NGSMs* $\lambda_{0,U}(t,\boldsymbol{\alpha})$ $[m^2]$ of four different oscillators having two different values of the damping coefficient α_C.

In order to better expose the proposed procedure, let consider a five-storey one-bay shear-type steel frame (see Fig. 1), whose story height is H = 3.2 m and bay width is L = 6 m. The steel columns are HE340A wide flange beams, with Young's modulus E = 200 GPa and inertia along the strong axis $I = 2.769 \times 10^{-4}$ m^4. The tributary mass per-storey is $m = 16 \times 10^3$ kg while the lateral stiffness of the frame is $k_i = 4.05 \times 10^7$ N/m, $i = 1, \ldots, 5$. The modal characteristics of the undamped structure are summarized in Table 1. The damping ratio of the structure is assumed equal to $\zeta_0 = 2\%$ for all the vibration modes.

In order to reduce the displacement of the structure, commercial fluid damper devices are connected to the system depicted in Fig. 1; each device is modelled as a combination of a linear spring, having nominal stiffness $K_D(\alpha_{K,0,i}) = k_{d,i} = 30$ N/m, and a linear dashpot, having damping coefficient $C_D(\alpha_{C,0,i}) = c_{d,i} = 3.1\times 10^5$ N s/m, with $i = 1, \ldots, 5$.

The analysed structure is subjected to a ground motion acceleration modelled as a stationary stochastic process, whose *PSD* function is evaluated according to the Tajimi-Kanai model:

$$G_0(\omega) = G_W \frac{4\,\zeta_K^2\,\omega_K^2\,\omega^2 + \omega_K^4}{(\omega_K^2 - \omega^2)^2 + 4\,\zeta_K^2\,\omega_K^2\,\omega^2} \tag{60}$$

where $G_W = 0.1$ m^2/s^3, $\omega_K = 12.56$ rad/s is the filter frequency that determines the dominant input frequency and $\zeta_K = 0.6$ is the filter damping coefficient that indicates the sharpness of the *PSD* function.

Two different configurations of the vibration control system are considered: in the first model external damper devices, considered fixed to a rigid support, are connected to the structure at each floor (see Fig. 2a); the second configuration presents viscous dampers across each storey (see Fig. 2b).

Since the main aim of this study is to reduce the displacement of the top floor of the two selected systems (see Fig. 2) under ground motion excitation, the selected performance measure function, which in this case is not time depending, is herein assumed equal to the *first spectral moment*, $\varphi(t,\boldsymbol{\alpha}_0) = \lambda_0(\boldsymbol{\alpha}_0)$, being $\boldsymbol{\alpha}_0^T = [\boldsymbol{\alpha}_{C,0}^T \quad \boldsymbol{\alpha}_{K,0}^T]$. Thus, the *first spectral moment* of the structural response of the top floor $\lambda_0(\boldsymbol{\alpha}_0) = \lambda_{0,U5}(\boldsymbol{\alpha}_0)$ and its sensitivity with respect to the i-th dissipation $s_{\lambda_0,\alpha_{c,i}}(\boldsymbol{\alpha}_0)$ and stiffness $s_{\lambda_0,\alpha_{k,i}}(\boldsymbol{\alpha}_0)$ parameters associated to each damper device located at each floor of the structure, have been evaluated. The influence of the parameters of the damper devices in the reduction of the response of the top floor of the two systems, has been evaluated by the introduction of the *percentage of performance measure sensitivities* associated to the damping, $\varepsilon_{c,i}(\%)$, and to the stiffness, $\varepsilon_{k,i}(\%)$ parameters, defined as follows, respectively:

$$\varepsilon_{c,i}(\%) = \frac{s_{\lambda_0,\alpha_{c,i}}(\boldsymbol{\alpha}_0)}{\lambda_0(\boldsymbol{\alpha}_0)} \times 100 \tag{61}$$

$$\varepsilon_{k,i}(\%) = \frac{s_{\lambda_0,\alpha_{k,i}}(\boldsymbol{\alpha}_0)}{\lambda_0(\boldsymbol{\alpha}_0)} \times 100 \tag{62}$$

In Tables 2 and 3 are summarized the aforementioned quantities evaluated for the systems with external and internal damper devices, respectively. Notice that for the structure with external dampers the results are shown in term of absolute displacements, while for the other structure the analysis is conducted in term of interstorey-drift.

Analyzing Tables 2 and 3 it can be immediately noticed that the main contribution to the reduction of the structural response of the

Table 4
Information about the maximum values of *NGSMs* $\lambda_{0,U}(t_{max},\boldsymbol{\alpha}_0)$ and $\lambda_{0,U}(t_{max},\boldsymbol{\alpha})$, for the four analysed cases.

T_0 [s]	t_{max} [s]	$\lambda_{0,U}(t_{max},\boldsymbol{\alpha}_0)$ [m^2]	$\lambda_{0,U}(t_{max},\boldsymbol{\alpha})$ [m^2]	$R(\%)$
0.2	5	5.48×10^{-6}	5.18×10^{-7}	5.48
0.4	8.5	2.87×10^{-4}	2.65×10^{-4}	7.55
0.6	11	1.38×10^{-3}	1.29×10^{-3}	6.83
1	14	3.76×10^{-3}	3.57×10^{-3}	5.14

top floor is given by the damping coefficients $c_{d,i}$, while the stiffness of the damper devices $k_{d,i}$ is not influent $\left(\left|\varepsilon_{k,i}(\%)\right| \ll \left|\varepsilon_{c,i}\ (\%)\right|\right)$.

Moreover, by analyzing the percentage function of the *performance measure sensitivities* in the structure with external dampers $\varepsilon_{c,i}\ (\%)$, it can be observed that the main contribution to the reduction of the displacement of the top floor is obtained by devices placed at third to fifth floor given that the absolute values of the corresponding *percentage of performance measure sensitivities* are higher than 20% $(|\varepsilon_{c,3}| = 22.24\%;\ |\varepsilon_{c,4}| = 32.68\%;\ |\varepsilon_{c,5}| = 39.61\%)$. Instead, the same quantity $\left|\varepsilon_{c,i}\ (\%)\right|$ evaluated for the devices located at the first two floors of the system reachs values lower than 12% $(|\varepsilon_{c,1}| = 3.22\%;\ |\varepsilon_{c,2}| = 11.48\%)$.

On the contrary, when using internal dampers, the devices located at first to second floor contribute more to the reduction of the response $(|\varepsilon_{c,1}| = 26.66\%;\ |\varepsilon_{c,2}| = 22.23\%)$ than the dampers located on the upper three floors.

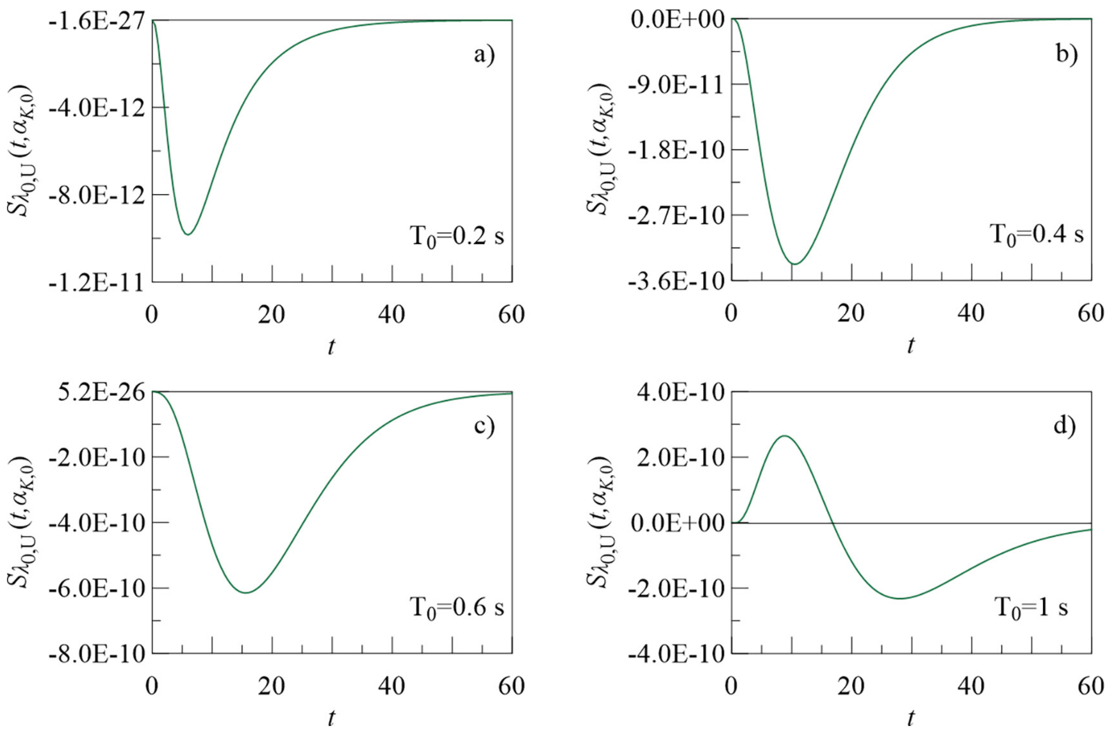

Fig. 5. Time-variant histories of the sensitivity functions of first *NGSM* $S_{\lambda_0,U}(t,\alpha_{K,0})$ with respect to damper stiffness coefficient of four different oscillators.

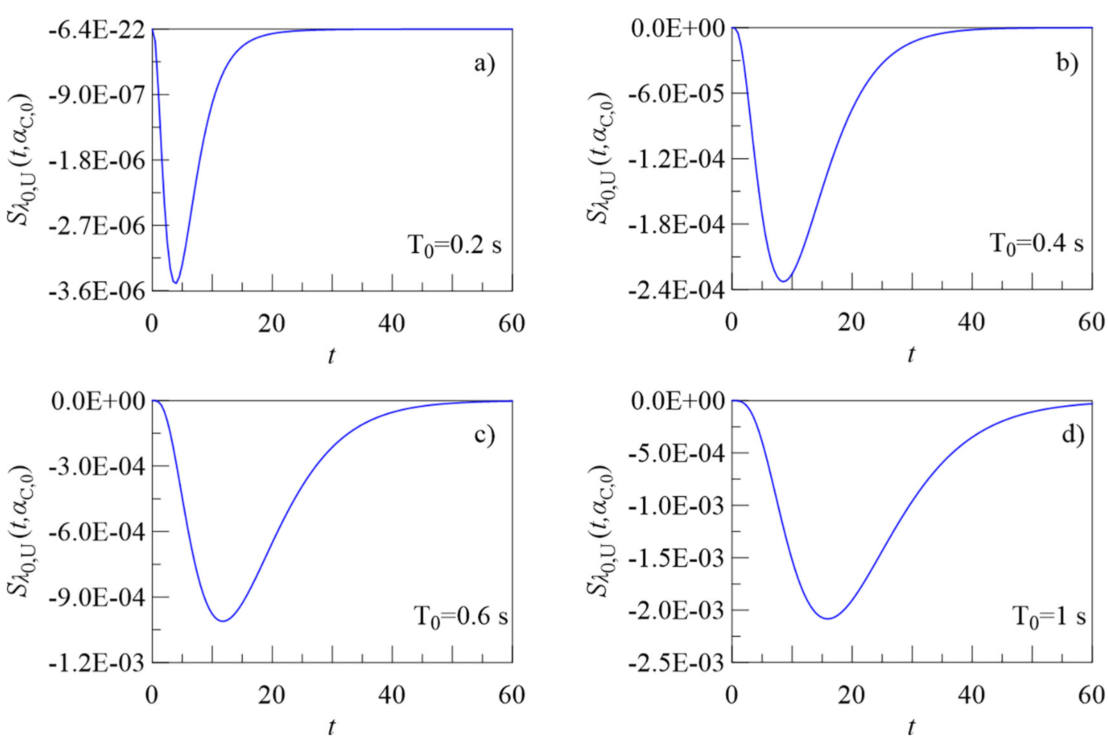

Fig. 6. Time-variant histories of the sensitivity functions of first *NGSM* $S_{\lambda_0,U}(t,\alpha_{C,0})$ with respect to damper damping coefficient of four different oscillators.

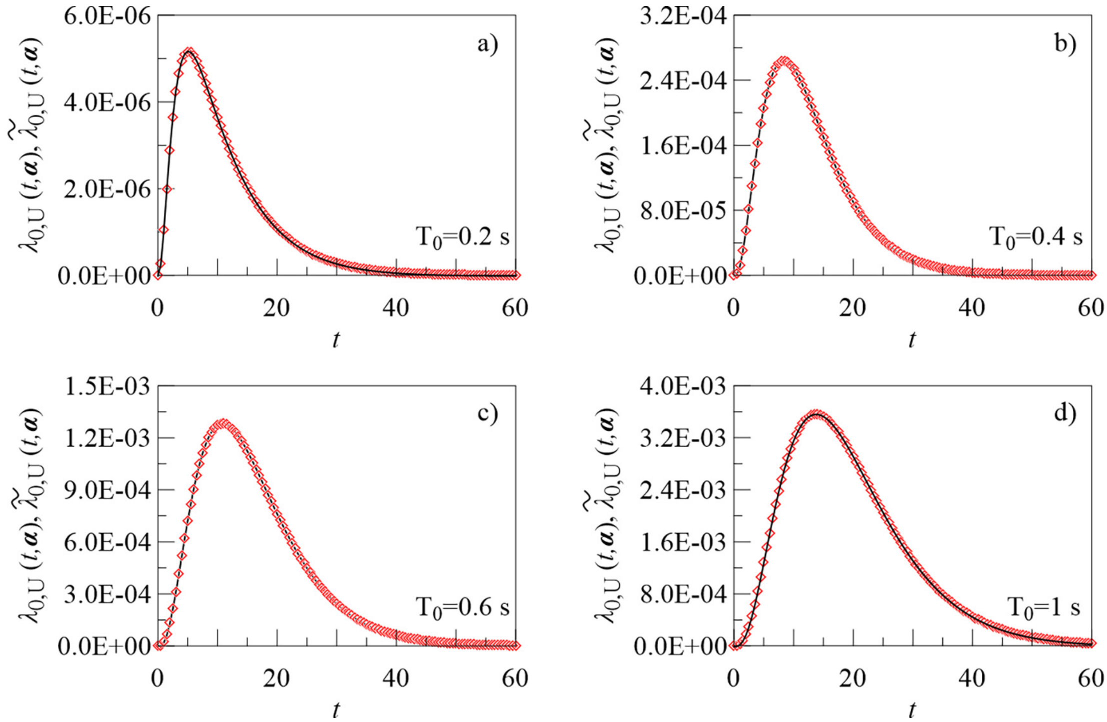

Fig. 7. Comparison between the first *NGSMs* $\lambda_{0,\mathrm{U}}(t,\boldsymbol{\alpha})$ (black solid line) and $\tilde{\lambda}_{0,\mathrm{U}}(t,\boldsymbol{\alpha})$ (dotted line) of four different oscillators.

6. Numerical application

In this section, in order to show the effectiveness of the proposed method, two different numerical applications will be conducted. The first one is based on the analysis of four Single-Degree-of-Freedom (SDOF) systems, in order to quantify the differ-

Table 5
Information about the sensitivity of *NGSMs* $\lambda_{0,\mathrm{U}}(t,\boldsymbol{\alpha}_0)$ evaluated with respect to the damping coefficient for the four analysed cases.

T_0 [s]	$t_{\min}$[s]	$S_{\lambda_0,\mathrm{U}}(t_{\min},\alpha_{C,0})$	$\varepsilon_c(t_{\min},\alpha_{C,0})(\%)$
0.2	4.0	- 3.50×10^{-6}	−6.62
0.4	8.5	- 2.33×10^{-4}	−8.12
0.6	11.5	- 1.00×10^{-3}	−7.34
1	16.0	- 2.08×10^{-3}	−5.68

Table 6
Geometric configuration of the 2-D frame.

Columns: *a, q* [cm]	**Columns**: *b, d, e, f, g h, j, k, l, m, n, o, p* [cm]	**Column**: *i* [cm]
20×50	60×60	20×40

Table 7
Tributary mass per story.

Floor	Mass [kg]
1	217,505
2	242,634
3	239,372
4	230,454
5	105,312

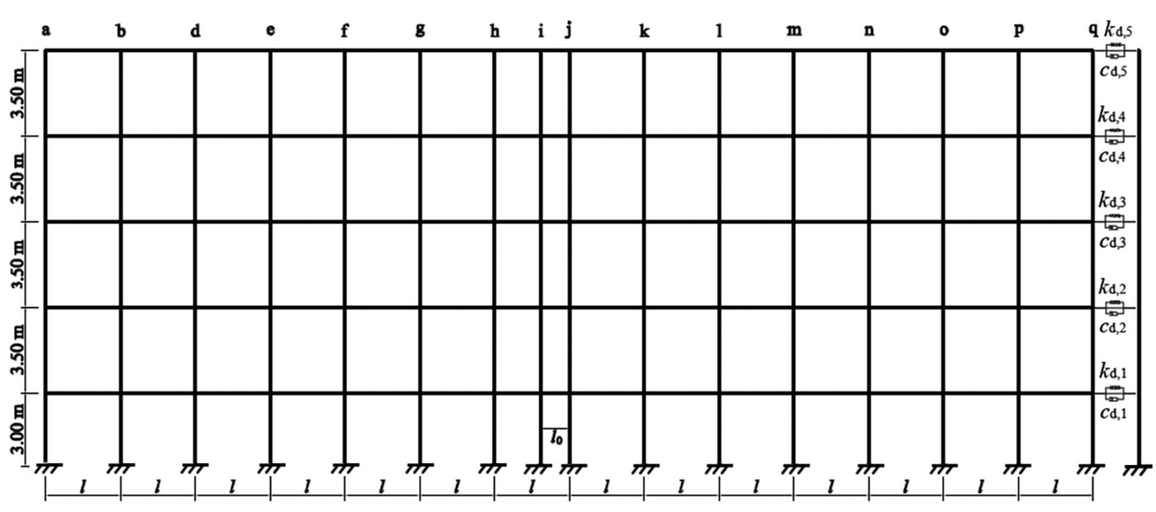

Fig. 8. Five-story plane frame structure, $l = 3.07\,\mathrm{m}$, $l_0 = 1.18\,\mathrm{m}$.

ence in the *percentage functions of performance measure sensitivities* between non-stationary responses. Then the proposed procedure will be applied to a Multi-Degree-of-Freedom (MDOF) system, connected at each floor to external fluid damper devices in order to reduce the absolute displacements of the system.

For both applications the *PSD* function of the stationary seismic input has been modelled according to Eq.(60). The selected time–frequency modulating function for the fully non stationary process is the one proposed by Spanos and Solomos [52], defined in Eq.(30), whose parameters herein selected are:

$$\alpha_a(\omega) = \frac{1}{2}\left(0.15 + \frac{\omega^2}{225\pi^2}\right); \quad \varepsilon(\omega) = \frac{\sqrt{2}}{15\,\pi\, a_{\max}}\omega; \quad t_0 = 0 \tag{63}$$

In order to normalize to one the modulating function, the parameter $a_{\max}$ is herein set equal to $1.34\ \mathrm{m/s^2}$. In Fig. 3 is depicted the *EPSD* function, having as stationary counterpart the *PSD* function introduced in Eq.(60).

6.1. SDOF systems

Four different SDOF systems connected to an external damper device with stiffness and damping coefficients respectively $K_D(\alpha_K) = K_D(\alpha_{K,0}) + \Delta K_D(\alpha_K)$ and $C_D(\alpha_C) = C_D(\alpha_{C,0}) + \Delta C_D(\alpha_C)$, have been herein studied. The natural periods of the systems without the installation of the damping devices have been assumed as: $T_0 = 0.2$ s, $T_0 = 0.4$ s, $T_0 = 0.6$ s and $T_0 = 1.0$ s; while, a damping ratio equal to $\zeta_0 = 0.02$ has been set for the evaluation of the damping $C_S = 2\zeta_0\omega_0$ of each oscillator, being $\omega_0 = 2\pi/T_0$ the circular frequency.

The nominal parameters of the external damper devices have been assumed as $C_D(\alpha_{C,0}) = 3.1\times 10^5$ N s/m and $K_D(\alpha_{K,0}) = 30$ N /m [3], for each SDOF system. The additional stiffness and damping parameters due the installation of the seismic devices, have been set as: $\Delta K_D(\alpha_K) = 0$ N /m and $\Delta C_D(\alpha_C) = 3.4\times 10^5$ N s/m because the small parameter deviations $\Delta\boldsymbol{\alpha}^T = [\Delta\alpha_C \quad \Delta\alpha_K]$ with respect to the nominal parameter vector $\boldsymbol{\alpha}_0^T = [\alpha_{C,0} \quad \alpha_{K,0}]$ have been considered as $\Delta\alpha_K = 0\%$ and $\Delta\alpha_C = 10\%$; as a consequence, $\boldsymbol{\alpha}^T = [\alpha_C \quad \alpha_{K,0}]$ being $\alpha_K = \alpha_{K,0}$ and $\alpha_C = \alpha_{C,0} + \Delta\alpha_C$.

For the SDOF systems the *PEC* matrix function is a 2×2 Hermitian matrix, whose real part coincides with the classical covariance matrix. It can be evaluated as:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(t,\boldsymbol{\alpha}) = \begin{bmatrix} \lambda_{0,U}(t,\boldsymbol{\alpha}) & \mathrm{i}\,\lambda_{1,U}(t,\boldsymbol{\alpha}) \\ -\mathrm{i}\,\lambda^{*T}_{1,U}(t,\boldsymbol{\alpha}) & \lambda_{2,U}(t,\boldsymbol{\alpha}) \end{bmatrix} \tag{64}$$

where the function $\lambda_{i,U}(t,\boldsymbol{\alpha})$ is the so-called *non-geometric spectral moment* (*NGSM*) of *i*-th order of stochastic response [47]. The same quantities $\lambda_{i,U}(t,\boldsymbol{\alpha}_0)$ can be evaluated for the four nominal systems assuming $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$ in Eq. (64).

In Fig. 4 the time histories of the first *NGSMs* $\lambda_{0,U}(t,\boldsymbol{\alpha}_0)$ and $\lambda_{0,U}(t,\boldsymbol{\alpha})$ are depicted for each of the four analysed systems. Increasing dissipation by $\Delta\alpha_C = 10\%$ causes a reduction of the spectral moment function $\lambda_{0,U}(t,\boldsymbol{\alpha})$ (black lines) with respect to the trend of the nominal one $\lambda_{0,U}(t,\boldsymbol{\alpha}_0)$ (orange dashed lines).

The maximum values of the non-geometrical spectral moments and the temporal instants $t_{\max}$ in which the two functions reach their maximum values have been reported in Table 4. The percentage reduction $R(\%)$ of the peak of the non-geometrical spectral moment $\lambda_{0,U}(t_{\max},\boldsymbol{\alpha})$ with respect to the nominal one $\lambda_{0,U}(t_{\max},\boldsymbol{\alpha}_0)$, has been also reported in Table 4:

$$R(\%) = \left|\frac{(\lambda_{0,U}(t_{\max},\boldsymbol{\alpha}_0) - \lambda_{0,U}(t_{\max},\boldsymbol{\alpha}))}{\lambda_{0,U}(t_{\max},\boldsymbol{\alpha}_0)}\right| \times 100 \tag{65}$$

For each of the four analysed systems, in Figs. 5 and 6 are plotted the sensitivity functions of the first *NGSM* with respect to the stiffness $S_{\lambda_{0,U}}(t,\alpha_{K,0}) = (\partial\lambda_{0,U}(t,\boldsymbol{\alpha})/\partial\alpha_K)|_{\alpha_K=\alpha_{K,0}}$ and damping coefficients $S_{\lambda_{0,U}}(t,\alpha_{C,0}) = (\partial\lambda_{0,U}(t,\boldsymbol{\alpha})/\partial\alpha_C)|_{\alpha_C=\alpha_{C,0}}$ of the damper device, respectively.

For a small variation of a parameter α with respect to the nominal one α_0, it's possibile to predict with good accuracy the variation of the response spectral moment by the knwoledge of its sensitivity. Infact, *NGSMs* $\lambda_{0,U}(t,\boldsymbol{\alpha})$ of Eq. (64) can be evaluated more easily $\tilde{\lambda}_{0,U}(t,\boldsymbol{\alpha})$ by the sum of the nominal *NGSMs* $\lambda_{0,U}(t,\boldsymbol{\alpha}_0) = \lambda_{0,U}(t,\alpha_{K,0},\alpha_{C,0})$ and the corresponding sensitivity functions calculated with respect both the stiffness and damping coefficients of the damper device. The two sensitivity functions must be multiplied by the small parameter deviations $\Delta\alpha_K$ and $\Delta\alpha_C$ obtaining:

$$\begin{aligned}\tilde{\lambda}_{0,U}(t,\boldsymbol{\alpha}) &= \tilde{\lambda}_{0,U}(t,\alpha_K,\alpha_C) \\ &= \lambda_{0,U}(t,\alpha_{K,0},\alpha_{C,0}) + S_{\lambda_{0,U}}(t,\alpha_{K,0})\,\Delta\alpha_K \\ &\quad + S_{\lambda_{0,U}}(t,\alpha_{C,0})\Delta\alpha_C\end{aligned} \tag{66}$$

In this numerical application, only the sensitivity with respect to the damper device $S_{\lambda_{0,U}}(t,\alpha_{C,0})$ has been considered in Eq.(66). As highlighted by Fig. 6, for the four analysed systems, the sensitivity functions have a negative trend thus, according to Eq. (66) $\lambda_{0,U}(t,\alpha_K,\alpha_C) \cong \tilde{\lambda}_{0,U}(t,\alpha_K,\alpha_C) < \lambda_{0,U}(t,\alpha_{K,0},\alpha_{C,0})$. As a consequence, a negative sensitivity means that the *NGSM* decreases when the parameter α changes.

A comparison between the *NGSMs* $\lambda_{0,U}(t,\boldsymbol{\alpha})$ (black solid line) and $\tilde{\lambda}_{0,U}(t,\boldsymbol{\alpha})$(dotted line) has been reported in Fig. 7. It can be noticed that for the four analysed cases, the two trends are coincident.

Table 8
Natural circular frequencies and natural periods of vibration of the 2-D frame.

Mode	ω [rad/s]	T [s]
1	10.273	0.612
2	29.447	0.213
3	45.395	0.138
4	56.087	0.112
5	59.866	0.105

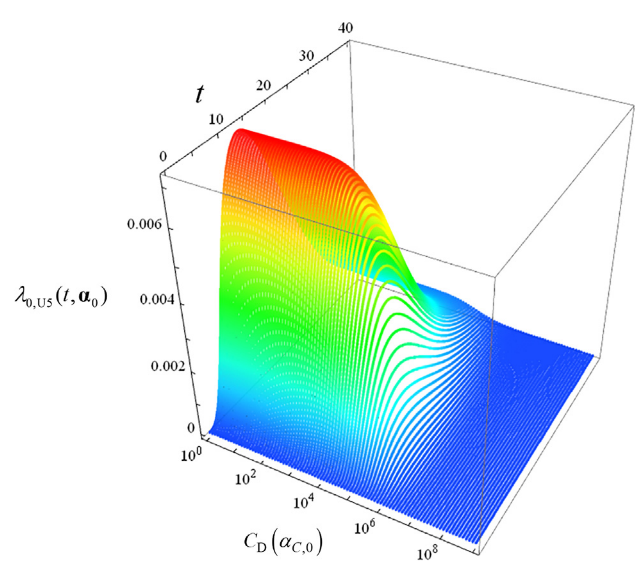

Fig. 9. Time-varying first *NGSM*, $\lambda_{0,U5}(t,\boldsymbol{\alpha}_0)$ [m^2], of the fifth floor versus damping coefficient of external devices equal for all floors, $C_D(\alpha_{C,0})$[N s/m].

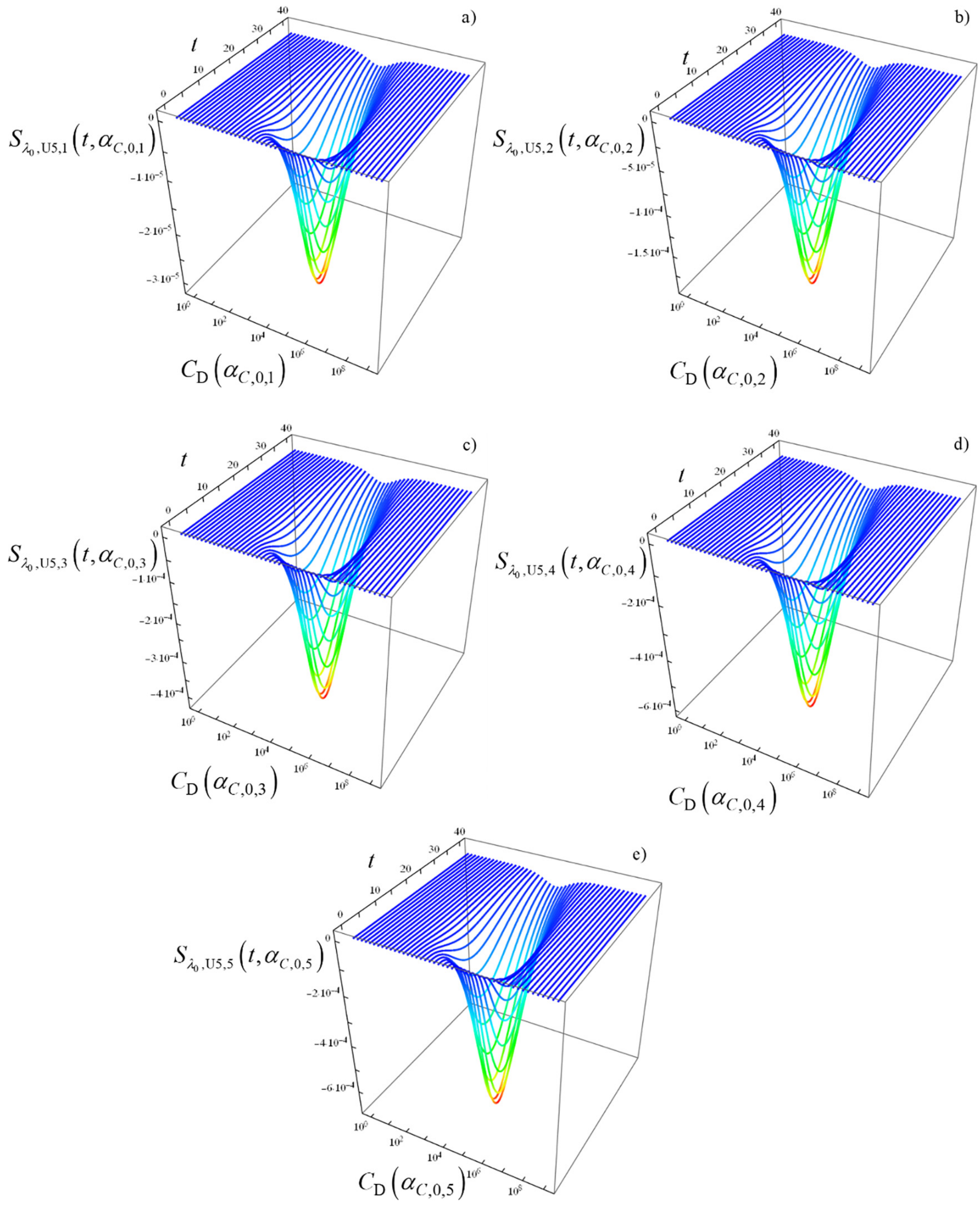

Fig. 10. Time-varying sensitivity of the first *NGSM* of the top floor, $\lambda_{0,\mathrm{U5}}(t,\boldsymbol{\alpha}_0)$ [m^2], with respect to the dissipation parameter of *i*-th floor, $S_{\lambda_0,\mathrm{U5},i}(t,\alpha_{C,0,i})$, versus damping coefficients $C_\mathrm{D}(\alpha_{C,0,i})$, for $K_\mathrm{D}(\alpha_{K,0,i}) = 30$ N/m.

Table 9
Information about the minimum value of the sensitivity of the NGSMs $S_{\lambda_0,\mathrm{U5},i}(t_{\min,i},\alpha_{C,0,i})$ evaluated with respect to the dissipation parameters $C_\mathrm{D}(\alpha_{C,0,i})$.

Floor	$S_{\lambda_0,\mathrm{U5},i}(t_{\min,i},\alpha_{C,0,i})$	$t_{\min,\mathrm{i}}$ [s]	$C_\mathrm{D}(\alpha_{C,0,i})$ [Ns/m]	$\varepsilon_C(t_{\min,i},\alpha_{C,0})(\%)$
1	-3.1×10^{-5}	12	1.0×10^5	1.0
2	-1.9×10^{-4}	12	1.0×10^5	6.1
3	-4.1×10^{-4}	12	1.0×10^5	13.2
4	-5.9×10^{-4}	12	1.0×10^5	19.0
5	-6.6×10^{-4}	12	1.0×10^5	21.3

In Table 5 are reported the sensitivity functions and the *percentage of performance measure sensitivities* associated to damping values $\varepsilon_c(t_{\min}, \alpha_{C,0})(\%)$:

$$\varepsilon_c(t_{\min}, \alpha_{C,0})(\%) = \frac{S_{\lambda_0,\mathrm{U}}(t_{\min}, \alpha_{C,0})}{\lambda_{0,\mathrm{U}}(t_{\min}, \alpha_{K,0}, \alpha_{C,0})} \Delta\alpha_C \times 100 \quad (67)$$

where $t_{\min}$ is the time instant in which the sensitivity functions reach their minimum values.

It can be immediately observed that *percentage of performance measure sensitivities* gives value near the percentage reduction $R(\%)$ of the peak of the *NSGM* $\lambda_{0,\mathrm{U}}(t, \boldsymbol{\alpha})$ with respect to the nominal one $\lambda_{0,\mathrm{U}}(t, \boldsymbol{\alpha}_0)$ as a consequence, the sensitivity represents a useful tool in the design phase.

6.2. MDOF system

The MDOF system herein analyzed is a five-story planar shear frame shown in Fig. 8. The columns, whose properties are reported in Table 6, repeated at each floor, are made of concrete, modelled as linear elastic with Young's modulus $E = 28000$ MPa; furthermore the beams are considered rigid in order to model a typical shear building behaviour. The tributary mass per story is obtained considering the planar frame as a part of a spatial frame, with a transversal distance between frames equal to $L_0 = 835$ cm, and it is summarized in Table 7. The circular frequencies ω and periods T of the analysed frame are reported in Table 8.

In order to reduce the absolute displacements of the top floor U5 of the analysed frame, external fluid damper devices, considered fixed to a rigid support, are connected each floor to the studied structure. The nominal stiffness of each external damper located at i-th floor has been set as $K_\mathrm{D}(\alpha_{K,0,i}) = k_{\mathrm{d},i} = 30$ N/m [3], $i = 1, \ldots, 5$. The damping parameter of each external device $C_\mathrm{D}(\alpha_{C,0,i}) = c_{\mathrm{d},i}$ has been chosen through a parametric analysis assuming $1\,\mathrm{Ns/m} \leqslant C_\mathrm{D}(\alpha_{C,0,i}) \leqslant 10^{10}\,\mathrm{Ns/m}$. In order to define the best damping coefficient value $C_\mathrm{D}(\alpha_{C,0,i})$, both first response *NGSM* of the top floor of the structure $\lambda_{0,\mathrm{U5}}(t, \boldsymbol{\alpha}_0)\,[\mathrm{m}^2]$ (see Fig. 9) and its sensitivity with respect to the i-th dissipation parameter $S_{\lambda_0,\mathrm{U5},i}(t, \alpha_{C,0,i})$ associeted to each damper device located at each floor of the structure, have been analysed.

The trend of the time-varying first *NGSM* of the fifth floor, assuming the same dissipation value in all the devices located on each floor, that is $C_\mathrm{D}(\alpha_{C,0}) \equiv C_\mathrm{D}(\alpha_{C,0,i}),\ i = 1, \ldots, 5$, is depicted in Fig. 9.

In Fig. 10 a-e, are reported the time-varying sensitivity of the first *NGSM* of the top floor, $S_{\lambda_0,\mathrm{U5},i}(t, \alpha_{C,0,i})$, with respect to the dissipation parameter of the device located at i-th floor, versus damping coefficients $C_\mathrm{D}(\alpha_{C,0,i})$.

For each damper device, in Table 9 are reported, the minimum value of the $S_{\lambda_0,\mathrm{U5},i}(t, \alpha_{C,0,i})$ together with the time instant $t_{\min,}$ and the dissipation value $C_\mathrm{D}(\alpha_{C,0,i})$ corresponding to the minimum of the sensitivity functions.

The best parameter to choose in design phase is the one corresponding to the minimum value of the sensitivity functions. As evidenced from Fig. 10 a-e, the minimum value of the function $S_{\lambda_0,\mathrm{U5},i}(t, \alpha_{C,0,i})$ has been reached at $t = t_{min,i} = 12$ s assuming $C_\mathrm{D}(\alpha_{C,0,i}) = 10^5$ Ns/m. Consequently, the obtained damping value can be assumed as the optimal one in the deisgn phase for all the damper devices.

The first *NGSMs* $\lambda_{0,\mathrm{U5}}(t, \boldsymbol{\alpha}_0)\,[\mathrm{m}^2]$, evaluated in $t = 12$ s and assuming $C_\mathrm{D}(\alpha_{C,0}) = 10^5$ N s/m, is equal to $\lambda_{0,\mathrm{U5}} = 3.1 \times 10^{-3}$ $[\mathrm{m}^2]$. By analyzing the values of the percentage of *performance measure sensitivities*, $\varepsilon_c(t_{\min,i}, \alpha_{C,0})(\%)$, reported in the last column of Table 9:

$$\varepsilon_c(t_{\min,i}, \alpha_{C,0})(\%) = \left[S_{\lambda_0,\mathrm{U5},i}(t_{\min,i}, \alpha_{C,0,i})/\lambda_{0,\mathrm{U5}}\right] \times 100 \quad i = 1, \ldots, 5 \quad (68)$$

it can be observed that the main contribution in reduction of the displacement is obtained by the device placed at last floor of the structure.

7. Conclusions

Among the possible solutions to increase the mechanical performances of those structures subjected to ground motion acceleration, one of the most effective design criteria is to introduce damping devices. In order to optimize the parameters of the damping system during the design phase, the design sensitivity analysis, which provides a quantitative estimate of desirable design change, by relating the available design variables and the structural response, represents an efficient approach.

The present work aimed to define a new method to evaluate sensitivities of stochastic response characteristics of structural systems with damping devices subjected to seismic excitations; the ground motion acceleration was herein modelled as fully non-stationary Gaussian stochastic process.

Once closed form solutions for the first-order derivatives of the *TFR* function as well as of the one-sided *evolutionary PSD* (*EPSD*) function of the structural response, with respect to damping parameters of devices, are evaluated, the proposed approach allows to perform a design sensitivity analysis selecting as performance measure function the non-geometric spectral moments of both nodal displacements and interstory drifts.

Several numerical applications showed the applicability of the proposed method in practical problems of engineering interest.

The proposed procedure, to perform the design sensitivity analysis of structural systems with damping devices subjected to fully non-stationary stochastic seismic excitations, can also be extended to the case of damping devices with non-linear behaviour once the equivalent linear motion equations are determined by applying the statistical linearization technique.

Data availability

Data will be made available on request.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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