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Evasion Differential Game of Multiple Pursuers and a Single Evader with Geometric Constraints in ℓ^2

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Abstract: We investigate a differential evasion game with multiple pursuers and an evader for the infinite systems of differential equations in ℓ^2. The control functions of the players are subject to geometric constraints. The pursuers' goal is to bring the state of at least one of the controlled systems to the origin of ℓ^2, while the evader's goal is to prevent this from happening in a finite interval of time. We derive a sufficient condition for evasion from any initial state and construct an evasion strategy for the evader.

Keywords: differential game; control; strategy; infinite system of differential equations; geometric constraint

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1. Introduction

The research on differential games started with the pioneering works of Isaacs and Pontryagin. Since then the field has been a topic for extensive research studies. Numerous monographs and collections, such as [1–12], have compiled the results in this field. In recent years, there has been a growing interest in differential games, although most of the research has focused on games in finite-dimensional spaces.

Many real-world problems can be modeled as control problems for partial differential equations (PDEs), which is a very active field that remains dynamic. The study of control problems for PDEs began with [13] on the time-optimal control problem for the parabolic equation; an up-to-date account of the theory can be found, for example, in [14]. Differential game problems for processes described by PDEs have been considered, for example, in refs. [15,16]. It turns out that studying differential game problems for PDE is convenient, if it is reduced to an infinite system of ordinary differential equations. This approach is called the decomposition method and has been used extensively, see for example, [17–26].

Concerning evasion problems in the finite-dimensional case, the works [27,28] have shown that one evader can avoid multiple slow pursuers in a finite-dimensional Euclidean space. These results have been extended to cases where the pursuers' control set is a subset of the interior of the evader's control set. In [29], a multi-pursuer–single-evader simple motion differential game in $\mathbb{R}^n$ was studied. It was proved that if the evader's initial position lay in the interior of the convex hull of the pursuers' initial positions, the evader could be captured; otherwise, evasion was possible. Since then, several approaches have been employed in the research of multi-pursuer–single-evader problems. For instance, in [30], the authors assumed that the evader possessed complete information about the pursuers' positions and strategies, and they utilized optimal control theory to analyze the problem. In [31], the authors considered a differential game involving an evader and multiple pursuers moving in an external dynamic flow field, and they derived a simplified capturing condition when the evader's maximum speed was lower than that of each pursuer. In a different context, [32] focused on high-speed-pursuit–evasion games with multiple pursuers and a single evader in an open domain with holonomic constraints and proposed an escape strategy for the evader based on the concept of the Apollonius circle. Problems with integral constraints on the control functions are considerably difficult. In [33], the authors considered such a problem and proved that evasion was possible from multiple pursuers regardless of the initial positions of players, when the evader had an advantage in energy over the pursuers. These works naturally introduced the question of evasion problem studies in infinite dimensions. However, a simple motion pursuit differential game in ℓ^2 with a finite number of pursuers, when players have identical capabilities has not been studied, and no initial position has been identified from which the pursuit can be completed. The pursuit differential game of one pursuer and one evader has been studied in [34] for an infinite system of binary differential equations in the Hilbert space ℓ^2. The general case of this problem was studied in [35]. Moreover, the papers [36–38] relate to differential games with an infinite system.

In this paper, we investigate an evasion differential game involving multiple pursuers and one evader in an infinite system of differential equations with a special operator (refer to (3)), which gives a coupled system. In the this case, in contrast to [34,35] the dynamics cannot be reduced to finite dimensional subsystems. We establish that if the evader's control set contains or overlaps with the control set of any pursuer, then evasion is feasible regardless of the number of pursuers and the initial position of the game. Additionally, we construct an evasion strategy for the evader. Our main result highlights that the infinite dimensionality of the space ℓ^2 confers an advantage to the evader, enabling evasion from any finite number of pursuers.

2. Statement of the Problem

It is remembered that ℓ^2 is the set of all sequences of real numbers

$$\left\{ x = (x_1, x_2, \dots), \mid \sum_{k=1}^{\infty} x_k^2 < \infty \right\}$$

with the inner product and norm given by

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k, \quad ||x|| = \sqrt{\langle x, x \rangle}, \quad x, y \in \ell^2.$$

We consider a differential game described by the following infinite system of differential equations:

$$\begin{array}{ll} \dot{x}_{ik} = \lambda x_{ik} + a_1 x_{i,k+1} + a_2 x_{i,k+2} + \cdots + a_M x_{i,k+M} + u_{ik}, & x_{ik}(0) = x_{ik}^0, \\ \dot{y}_k = \lambda y_k + a_1 y_{k+1} + a_2 y_{k+2} + \cdots + a_M y_{k+M} + v_k, & y_k(0) = y_k^0, \end{array} \tag{1}$$

where $x_{ik}, y_k \in \mathbb{R}$, $i = 1, 2, \dots, m$, $k = 1, 2, \dots$, $\lambda \in \mathbb{R}$ and $M \in \mathbb{N}$ are given constants, $\sum_{j=1}^{M} a_j = 1$ for $a_j \geq 0$, and u_{ik}, v_k are the control parameters of the i-th pursuer and the evader, respectively. Moreover, it is considered that

$$x_i^0 = (x_{i1}^0, x_{i2}^0, ...) \in \ell^2, \;\; y^0 = (y_1^0, y_2^0, ...) \in \ell^2, \;\; i = 1, 2, \dots, m.$$

Further, it is assumed that $x_i^0 \neq y^0$ for all $i = 1, 2, \dots, m$.

With the notation $\eta_{ik} = x_{ik} - y_k$ for $i = 1, 2, \dots, m$, $k = 1, 2 \dots$, we can write (1) in the form

$$\dot{\eta}_i = Q\eta_i + u_i - v, \quad \eta_i(0) = \eta_i^0, \;\; i = 1, 2, ..., m, \tag{2}$$

where $\eta_i = (\eta_{i1}, \eta_{i2}, \dots)$, $u_i = (u_{i1}, u_{i2}, \dots)$, $v = (v_1, v_2, \dots)$, and $\eta_i^0 = x_i^0 - y^0$ for $i = 1, 2, \dots m$, and $\eta_i^0 \neq 0$ is assumed as well. The linear operator $Q : \ell^2 \to \ell^2$ is defined by the following equation:

$$Q\eta = \{\lambda \eta_k + a_1 \eta_{k+1} + a_2 \eta_{k+2} + ... + a_M \eta_{k+M}\}_{k \in \mathbb{N}}. \tag{3}$$

Further, we continue as

$$Q = \lambda I + \sum_{j=1}^{M} a_j E^j,$$

where I is the identity map, and $E : \ell^2 \to \ell^2$ is the shift map whose action is defined as $[Ex]_k = x_{k+1}$. It is easy to see that $\|E^j\| = 1$ for all $j \in \mathbb{N}$; hence, Q is a bounded linear operator. Indeed,

$$\|Q\| = |\lambda| + \sum_{j=1}^{M} a_j \|E^j\| \le |\lambda| + 1.$$

This implies that e^{tQ} is a continuous semi-group, and the solution of (2) can be written as

$$\eta_i(t) = e^{tQ}\eta_i^0 + \int_0^t e^{(t-s)Q}(u_i(s) - v(s))ds. \tag{4}$$

Definition 1. *A function $u_i(t) = (u_{i1}(t), u_{i2}(t), \dots)$, $i \in \{1, 2, \dots, m\}$ with measurable coordinates $u_{ik}(t)$, $k = 1, 2, \dots$, which satisfy the constraint $||u_i(t)|| \le \rho_i$, $0 \le t \le T$, is called the admissible control of the i-th pursuer, where $\rho_i > 0$ and $T > 0$ are given numbers.*

Definition 2. *A function $v(t) = (v_1(t), v_2(t), \dots)$ with measurable coordinates $v_k(t)$, $k = 1, 2, \dots$, which satisfy the constraint $||v(t)|| \le \sigma$, $0 \le t \le T$, is called the admissible control of the evader, where $\sigma > 0$ and $T > 0$ are given numbers.*

Assume that $\sigma \ge \rho_i$ for all $i = 1, \dots, m$.

Definition 3. *If there exists a control $\overline{v}(t)$ of the evader, such that $\eta_i(t) \neq 0$, $i = 1, \dots, m$, $0 \le t \le T$, for any admissible controls $u_i(t)$, $i = 1, \dots, m$, $0 \le t \le T$ of the pursuers, then we say that evasion is possible on the time interval $[0, T]$.*

The problem here is that to find a strategy for the evader, such that evasion is possible in game (2).

3. The Main Result

In this section, we prove the following theorem.

Theorem 1. *In game (2), evasion is possible for any initial states η_i^0, $i = 1, 2, \dots, m$.*

Proof. We can write (4) in the form

$$\eta_i(t) = e^{tQ}\xi_i(t), \;\; \xi_i(t) = \eta_i^0 + \int_0^t e^{-sQ}(u_i(s) - v(s))ds. \tag{5}$$

For any $i = 1, 2, \dots, m$, the equation $\eta_i(t) = 0$ is equivalent to $\xi_i(t) = 0$ since the matrix e^{tQ} is not singular. Furthermore, proving the theorem only requires constructing an admissible control $v(t)$ for the evader during the time interval of $0 \le t \le T$, which means that $\xi_i(t) = (\xi_{i1}(t), \xi_{i2}(t), \dots) \neq 0$ holds for every $i = 1, 2, \dots, m$ and $0 \le t \le T$.

Since $\eta_i^0 \neq 0$, we can conclude that there exists a unit vector $\alpha = (\alpha_1, \alpha_2, ...) \in \ell^2$, where $\|\alpha\| = 1$, such that the inner product $\langle \eta_i^0, \alpha \rangle \ge 0$ for all $i = 1, \dots, m$. We can choose the vector α to be an orthonormal vector to the hyperplane that passes through the points η_i^0, where $i = 1, 2, \ldots, m$.

Now, we set the evader's control as follows

$$v(t) = -\frac{e^{-tQ^*}\alpha\sigma}{\|e^{-tQ^*}\alpha\|}, \tag{6}$$

where Q^* stands for the transpose of the matrix Q. We show that evasion is possible by using the control (6) of the evader on the time interval $[0, T]$. Note that since $\alpha = (\alpha_1, \alpha_2, ...)$ is a unit vector, the denominator in (6) is not equal to zero, i.e.,

$$\|e^{-tQ^*}\alpha\| \neq 0, \quad t \geq 0.$$

To obtain a contradiction, suppose that there exist admissible controls of pursuers, such that $\xi_p(\theta) = 0$ for some $p \in \{1, \ldots, m\}$ and $\theta > 0$, while the evader applies the control (6). From $\langle \eta_i^0, \alpha \rangle \geq 0$, $i = 1, 2, \ldots, m$, we then have

$$\langle \eta_p^0, \alpha \rangle \geq 0, \quad i = 1, 2, \ldots, m. \tag{7}$$

Thus,

$$\begin{aligned}
\langle \xi_p(\theta), \alpha \rangle &= \left\langle \eta_p^0 + \int_0^t e^{-sQ}(u_i(s) - v(s))ds, \alpha \right\rangle && (8)\\
&= \langle \eta_p^0, \alpha \rangle + \int_0^\theta \left\langle e^{-sQ}(u_p(s) - v(s)), \alpha \right\rangle ds && (9)\\
&\geq \int_0^\theta \left\langle e^{-sQ}u_p(s), \alpha \right\rangle ds - \int_0^\theta \left\langle e^{-sQ}v(s), \alpha \right\rangle ds. && (10)
\end{aligned}$$

Using the Cauchy–Schwartz inequality, the first term of the right side of Equation (8) can be bounded as follows

$$\begin{aligned}
\int_0^\theta \left\langle e^{-sQ}u_p(s), \alpha \right\rangle ds &= \int_0^\theta \left\langle u_p(s), e^{-sQ^*}\alpha \right\rangle ds && (11)\\
&\geq -\rho_p \int_0^\theta \left\| e^{-sQ^*}\alpha \right\| ds. && (12)
\end{aligned}$$

In (11), equality holds at

$$u_p(t) = -\frac{e^{-tQ^*}\alpha\rho_p}{\|e^{-sQ^*}\alpha\|}, \quad 0 \leq t \leq \theta. \tag{13}$$

From (6) and (11), we have $\langle \xi_p(\theta), \alpha \rangle \geq 0$ for $\sigma \geq \rho_p$. Indeed,

$$\langle \xi_p(\theta), \alpha \rangle \geq \int_0^\theta \left\langle u_p(s), e^{-sQ^*}\alpha \right\rangle ds - \int_0^\theta \left\langle v(s), e^{-sQ^*}\alpha \right\rangle ds \tag{14}$$

$$\geq -\rho_p \int_0^\theta \left\| e^{-sQ^*}\alpha \right\| ds - \int_0^\theta \left\langle -\frac{e^{-tQ^*}\alpha\sigma}{\|e^{-tQ^*}\alpha\|}, e^{-sQ^*}\alpha \right\rangle ds \tag{15}$$

$$= -\rho_p \int_0^\theta \left\| e^{-sQ^*}\alpha \right\| ds + \sigma \int_0^\theta \left\| e^{-sQ^*}\alpha \right\| ds \tag{16}$$

$$= (\sigma - \rho_p) \int_0^\theta \left\| e^{-sQ^*}\alpha \right\| ds \geq 0. \tag{17}$$

The assumption $\xi_p(\theta) = 0$ shows that $\langle \xi_p(\theta), \alpha \rangle = 0$. However, in (14), the equality sign is only true for $\rho_p = \sigma$. By comparing (6) and (13), we can conclude that $u_p(t) = v(t)$, for $0 \leq t \leq \theta$. By substituting this into (4), we have

$$\xi_p(\theta) = \eta_p^0.$$

This implies that

$$\xi_p(\theta) = (\xi_{p1}(\theta), \xi_{p2}(\theta), ...) = (\eta_{p1}^0, \eta_{p2}^0, ...) \neq 0.$$

This contradicts our assumption $\xi_p(\theta) = 0$. From this, we conclude that $\xi_i(t) \neq 0$, for all $t \in [0, T]$, and $i = 1, 2, \ldots, m$. This completes the proof. □

4. Conclusions

In the present paper work, we studied a multi-pursuer evasion differential game problem for an infinite system of differential equations with a special operator $Q = \lambda I + \sum_{i=1}^M a_i E^i$ on the right hand side, where $a_i \geq 0$ and $\sum_{i=1}^M a_i = 1$.

We considered a differential game with geometric constraints and showed that evasion is possible from any initial position under natural conditions in ℓ^2. In the construction of the evasion strategy, the fact that a finite number of points lies on a hyperplane played the key role.

A similar result can be obtained in the case of an infinite (countable) number of pursuers, if the initial states of all objects lie on a hyperplane. In general, for the countable number of pursuers, this problem is open.

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