DOI: 10.1478/AAPP.1022A4

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 102, No. 2, A4 (2024)

A LAPLACE-TYPE PROBLEM FOR A LATTICE WITH CELL COMPOSED BY REGULAR POLYGONS WITH OBSTACLES

MASSIMILIANO FERRARA *

ABSTRACT. In this paper we consider a lattice with a fundamental cell composed of two triangles and two trapezoids in the presence of some obstacles and we determine the probability p that a random segment with uniform and random distribution of constant length intersects a side of the lattice. By this note we solve a Stochastic Geometry's open problem as well and we start to connect this field to some aspects related Artificial Intelligence issues.

1. Introduction

Starting from some previous papers of Stoka (1975, 2015), authentic sources of inspiration, in this note we solve a Laplace-type problem for a lattice of the Euclidean plane with a cell composed of two triangles and two trapezoids (both regular polygons) in the presence of some obstacles (Poincaré 1912; Blaschke 1995). We compute the probability that a segment of uniformly distributed random position and constant length also intersects one side of the lattice.

2. Results

Let $\Re\ (a,b,\alpha,m)$ a lattice with fundamental cell C_0 as in Fig. 1, where α is an angle with $\alpha \in [0, \frac{\bar{x}}{4}]$, $a > b\tan\alpha$ and $0 < m < min(a,b)$. Starting from Fig. 1 we can define:

$$|BC| = b, |AB| = a, |AE| = |DE| = |BF| = |CF| = \frac{b}{2\cos\alpha}, |EF| = a - \tan\alpha; \qquad (1)$$

$$\widehat{F_1,F_2,F_3} = \widehat{F,F_3,F_2} = \widehat{E,E_2,E_3} = \widehat{E,E_3,E_2} = \alpha, \widehat{E_1,E,E_3} = \widehat{F_2,F,F_3} = \pi - 2a,$$

$$\widehat{E_1,E,E_2} = \widehat{E_1,E,E_3} = \widehat{F_1,F,F_2} = \widehat{F_1,F,F_3} = \frac{\pi}{2} + a, \widehat{E,E_2,E_1} = \widehat{E,E_3,E_1} = \widehat{F,E_3,F_1} = \frac{\bar{x}}{4} - \frac{\alpha}{2}$$

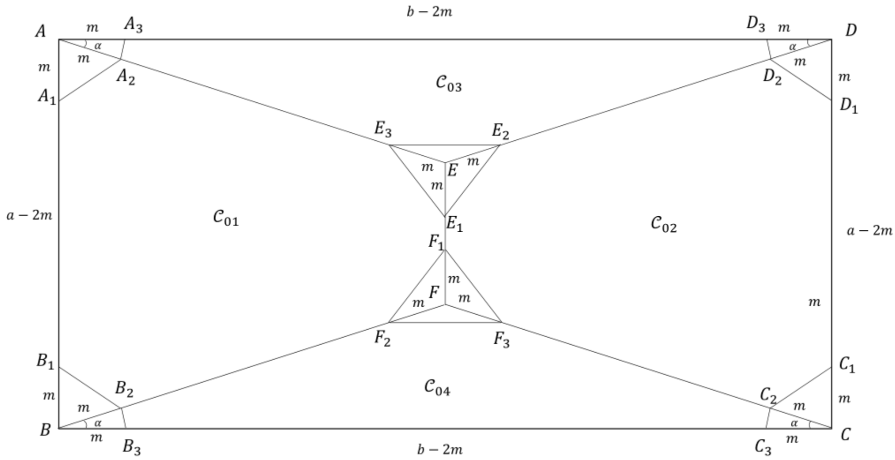

FIGURE 1. Fundamental cell

$$\widehat{A,A_1,A_2} = \widehat{A,A_2,A_1} = \widehat{B,B_1,B_2} = \widehat{B,B_2,B1} = \widehat{C,C_1,C_2} = \widehat{C,C_2,C_1} =$$

$$= \widehat{D,D_1,D_2} = \widehat{D,D_2,D_1} = \frac{\bar{x}}{4} + \frac{\alpha}{2}; \tag{2}$$

$$area\ AA_1A_2 = area\ BB_1B_2 = area\ CC_1C_2 = area\ DD_1D_2 = area\ EE_1E_2 = area\ EE_1E_3 =$$

$$= area\ FF_1F_2 = FF_1F_3 = \frac{m^2}{\alpha}\cos\alpha, area\ AA_2A_3 = area\ BB_2B_3 = area\ CC_2C_3 =$$

$$= area\ DD_2D_3 = \frac{m^2}{2}\sin\alpha, area\ EE_2E_1 = area\ FF_2F_3 = \frac{m^2}{2}\sin 2\alpha; \tag{3}$$

$$|A_1A_2| = |B_1B_2| = |C_1C_2| = |D_1D_2| = m\sqrt{2}(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}), |E_2E_3| = |F_2F_3| = 2m\cos\alpha,$$

$$|E_1E_2| = |E_1E_3| = |F_1F_2| = |F_1F_3| = m\sqrt{2}(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2},$$

$$|A_2A_3| = |B_2B_3| = |C_2C_3| = |D_2D_3| = 2m\sin\frac{\alpha}{2}; \tag{4}$$

$$area\ C_0 = ab - 2m^2[2\cos\alpha + \sin\alpha(1+\cos\alpha)]. \tag{5}$$

We want to compute the probability that a segment s of uniformly distributed random position and constant length $l < \frac{1}{2}min(a,b) - m$ intersects a side of the lattice $\Re$, that is, the probability P_{int} that s intersects a side of the fundamental cell C_o. If we denote as ϕ the angle between segment S and the line BC (or AD), then the position of segment s is determined by its midpoint and angle φ. To calculate the probability P_{int}, we consider the limit positions of segment s for a fixed value of angle φ, located within cell C_0. This results in the following Fig. 2:

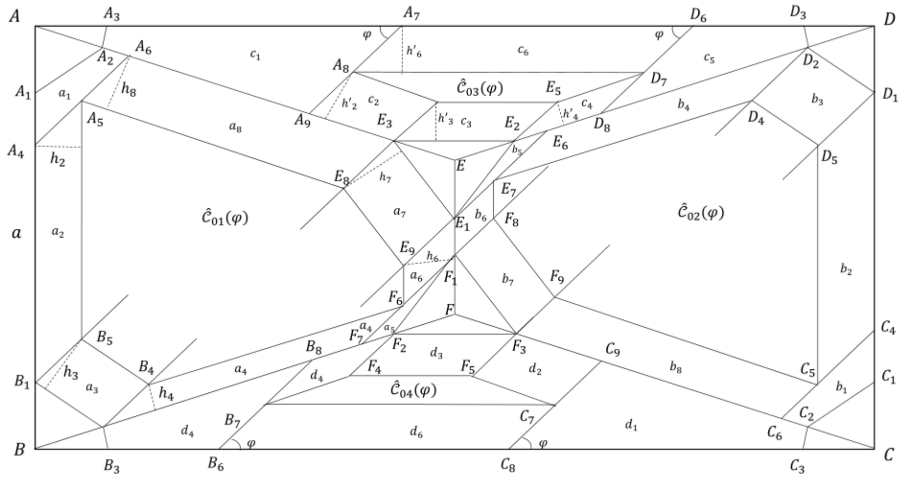

FIGURE 2. Lattice result

and the formulas:

$$area\,\widehat{C_{01}}(\varphi) = area\,\widehat{C_{02}}(\varphi) = area\,\widehat{C_{01}} - \sum_{i=1}^{8} area\;\; a_i(\varphi), \tag{6}$$

$$area\,\widehat{C_{03}}(\varphi) = area\,\widehat{C_{04}}(\phi) = area\,\widehat{C_{03}} - \sum_{i=1}^{6} area\;\; \alpha_i(\varphi). \tag{7}$$

First of all, we consider the following Fig. (3). We have:

$$\widehat{AA_4A_6} = \frac{\pi}{2} - \varphi, \widehat{A_4AA_6} = \frac{\pi}{2} - \alpha, \widehat{AA_6A_4} = \varphi + \alpha \tag{8}$$

as a consequence

$$|AA_6| = \frac{l\sin(\varphi+\alpha)}{\cos\alpha}, |AA_6| = \frac{l\cos\varphi}{\cos\alpha}. \tag{9}$$

Then

$$area\,AA_4A_6 = \frac{l^2\cos\varphi\sin(\varphi+\alpha)}{2\cos\alpha}$$

Thus, by considering Eq. 3

$$area\;a_1(\varphi) = \frac{l^2\cos\varphi\sin(\varphi+\alpha)}{2\cos\alpha} - \frac{m^2}{2}\cos\alpha, \tag{10}$$

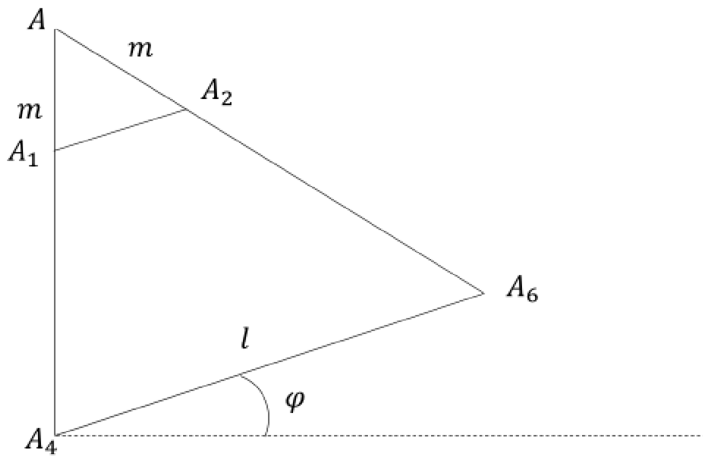

FIGURE 3. Lattice section

from Fig. 2 and Eqs. 8 and 9 we obtain

$$h_2 = \frac{l}{2}\cos\varphi, |A_4B_1| = a - |AA_4| - m = a - m - \frac{l\sin(\varphi+\alpha)}{\cos\alpha},$$

and then

$$area\ a_2(\varphi) = \frac{al}{2}\cos\varphi - \frac{ml}{2}\cos\varphi - \frac{l^2\cos\varphi\sin(\varphi+\alpha)}{2\cos\alpha}. \tag{11}$$

Figure 2 and the first part of Eq. 8 give us

$$\widehat{A_4B_1B_5} = \widehat{AA_4A_6} = \frac{\bar{x}}{2} - \varphi, \widehat{BB_1B_2} = \frac{\pi}{4} + \frac{\alpha}{2}, \widehat{B_5B_1B_2} = \frac{\pi}{4} + \varphi - \frac{\alpha}{2}, \widehat{B_1B_2B_4} = \frac{3\pi}{4} - \varphi + \frac{\alpha}{2},$$

$$h_3 = \frac{l}{2}\sin(\varphi + \frac{\pi}{4} - \frac{\alpha}{2}). \tag{12}$$

So, taking in consideration Eq. 4, we have

$$area\ \ a_3(\varphi) = \frac{ml}{2}[\cos\alpha\sin\varphi - (1-\sin\alpha)\cos\varphi]$$

Figure 2 and Eqs. 2 and 12 give us

$$\widehat{F_1F_7F_2} = \widehat{B_1B_2F_7} = \varphi - \alpha, \widehat{F_1F_2F_7} = \frac{3\pi}{4} + \frac{\alpha}{2}, \widehat{F_2F_1F_7} = \frac{\pi}{4} + \frac{\alpha}{2} - \varphi \tag{13}$$

With these angles and from triangle $\widehat{F_1F_2F_7}$ we obtain

$$\frac{|F_2F_7|}{\sin(\frac{\pi}{4}+\frac{\alpha}{2}-\varphi)} - \frac{l}{\sin(\frac{\pi}{4}-\frac{\alpha}{2})} = \frac{|F_1F_2|}{\sin(\varphi-\alpha)}$$

$$|F_2F_7| = \frac{l\sin(\frac{\pi}{4}+\frac{\alpha}{2}-\varphi)}{\sin(\frac{\pi}{4}-\frac{\alpha}{2})}. \quad (14)$$

If we consider that $|F_1F_2| = m\cos\alpha \times \left[2\sin(\frac{\pi}{4}-\frac{\alpha}{2}\right]^{-1}$, we obtain the follow condition:

$$2l\sin(\varphi-\alpha) = m\cos\alpha, (m \neq 0) \quad (15)$$

As a consequence

$$area\ a_5(\varphi) = \frac{l^2\sin(\varphi-\alpha)\sin(\frac{\pi}{4}-\frac{\alpha}{2})}{2\sin(\frac{\pi}{4}-\frac{\alpha}{2})}. \quad (16)$$

Now, from Eq. 14 we have obtained that $\varphi \geq \alpha$, hence

$$\varphi \in [\alpha, \frac{\pi}{2}]. \quad (17)$$

From Fig. 2 and from Eqs. 1, 4, 14, 15, and 16 we obtain

$$h_4 = \frac{l}{2}\sin(\varphi-\alpha), |B_2F_7| = \frac{b}{2\cos\alpha} - 2m - \frac{l\sin(\frac{\pi}{4}+\frac{\alpha}{2}-\varphi)}{\sin(\frac{\pi}{4}-\frac{\alpha}{2})},$$

Then

$$area\ a_4(\varphi) = \frac{bl}{4\cos\alpha}\sin(\varphi-\alpha) - \frac{m^2\cos\alpha}{2} - \frac{l^2\sin(\varphi-\alpha)\sin(\frac{\pi}{4}+\frac{\alpha}{2}-\varphi)}{2\sin(\frac{\pi}{4}-\frac{\alpha}{2})}. \quad (18)$$

Figure 2 and Eqs. 1, 2, and 14 give us

$$\widehat{F_1E_1E_9} = \frac{\pi}{2} - \varphi, h_6 = \frac{l}{2}\cos\varphi, |E_1F_1| = a - b\tan\alpha - 2m,$$

and

$$area\ a_6(\varphi) = \frac{a-b(ga-2m)l}{2}\cos\varphi. \quad (19)$$

From Fig. 2 and Eq. 4 we have

$$\widehat{E_1E_3E_8} = \frac{\pi}{2} - \varphi, h_7 = \frac{l}{2}\cos\varphi, |E_1E_3| = m\sqrt{2}(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}),$$

hence

$$area\ a_1(\varphi) = \frac{ml\sqrt{2}(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2})}{2} \quad (20)$$

Finally, Fig. 2 and Eqs. 1, 8, and 9 will give us

$$\widehat{A_6A_5E_8} = \varphi+\alpha, h_8 = \frac{l}{2}\sin(\varphi+\alpha), |A_6E_3| = \frac{b}{\cos\alpha} - m - \frac{l\cos\varphi}{\cos\alpha}.$$

Then

$$area\ a_8(\varphi) = (\frac{b}{\cos\alpha} - m)\frac{l}{2}\sin\varphi - \alpha - \frac{l^2\cos\varphi\sin(\varphi-\alpha)}{2\cos\alpha}, \tag{21}$$

From Eqs. 10, 11, 13, 17, 19, 20, 21, and 22 we have

$$a_9(\varphi) = \sum_{i=1}^{8} area\ \ a_i(\varphi) = al\cos(\varphi) + \frac{bl}{4}(3\sin\varphi - \tan\alpha) + [\sqrt{2}(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}) - 4]\frac{ml}{2}\cos\varphi -$$

$$-\frac{l^2}{4\cos\alpha}(\sin 2\varphi + \sin\alpha\cos 2\varphi + \sin\varphi) - m^2\cos\alpha. \tag{22}$$

By using the previous equation in Eq. 6 it results

$$area\ \widehat{C_{01}}(\varphi) = area\ \widehat{C_{02}}(\varphi) = area\ C_{01} - A_1(\varphi). \tag{23}$$

Figure 2 and Eq. 3 give us

$$\widehat{DD_6D_8} = \pi - \varphi, \widehat{DD_8D_6} = \varphi - \alpha \tag{24}$$

and

$$|DD_6| = \frac{l\sin(\varphi-\alpha)}{\sin\alpha}, |DD_8| = \frac{l\sin\varphi}{\sin\alpha} \tag{25}$$

hence

$$area\ a_5(\varphi) = \frac{l^2\sin\varphi\sin(\varphi-\alpha)}{2\sin\alpha} - \frac{m^2}{2}\sin\alpha \tag{26}$$

Again from Fig. 2 and Eq. 3 it follows

$$\widehat{AA_9A_7} = \pi - (\varphi + \alpha) \tag{27}$$

and

$$|AA_7| = \frac{l\sin(\varphi+\alpha)}{\sin\alpha}, |AA_9| = \frac{l\sin\varphi}{\sin\alpha}. \tag{28}$$

Then

$$area\ C_1(\varphi) = \frac{l^2\sin\varphi\sin(\varphi+\alpha)}{2\sin\alpha} - \frac{m^2}{2}\sin\alpha, \tag{29}$$

Figure 2 and Eqs. 26 and 29 show us

$$h_6' = \frac{l}{2}\sin\varphi$$

and

$$|A_7D_6| = b - |AA_7| - |DD_6| = b - \frac{l\sin(\varphi+\alpha)}{\sin\alpha} - \frac{l\sin(\varphi-\alpha)}{\sin\alpha}.$$

Therefore

$$area\ a_6 = \frac{bl}{2}\sin\varphi - \frac{l^2\sin\varphi\sin(\varphi+\alpha)}{2\sin\alpha} - \frac{l^2\sin\varphi\sin(\varphi-\alpha)}{2\sin\alpha}. \tag{30}$$

From Fig. 2 e from Eqs. 1 and 29 we obtain

$$h_2' = \frac{l}{2}\sin(\varphi+\alpha), |A_9E_3| = |AE| - |AA_9| - m = \frac{b}{2\cos\alpha} - m - \frac{l\sin\varphi}{\sin\alpha},$$

then

$$area\ a_2 = (\varphi) = (\frac{b}{2\cos\alpha} - m)\frac{l}{2}\sin(\varphi+\alpha) - \frac{l^2\sin\varphi\sin(\varphi+\alpha)}{2\sin\alpha}. \tag{31}$$

Figure 2 and Eq. 4 give us

$$h_3' = \frac{l}{2}\sin\varphi, |E_2E_3| = 2m\cos\alpha,$$

hence

$$area\ a_3 = ml\cos\alpha\sin\varphi. \tag{32}$$

Finally, from Fig. 2 and from Eqs. 1, 25, and 26 it results

$$h_4' = \frac{l}{2}\sin(\varphi-\alpha), |E_2D_8| = |DE| - m - |DD_8| = \frac{b}{2\cos\alpha} - m - \frac{l\sin\varphi}{\sin\alpha},$$

then

$$area\ a_4 = (\frac{b}{2\cos\alpha} - m)\frac{l}{2}\sin(\varphi-\alpha) - \frac{l^2\sin(\varphi-\alpha)\sin\varphi}{2\sin\alpha}. \tag{33}$$

From Eqs. 27, 30, 31, 32, 33, and 34 we obtain

$$A_3(\varphi) = \sum_{i=1}^{6} area\ a_i(\varphi) = bl\sin\varphi - \frac{l^2}{2}\cot\alpha(1-\cos 2\varphi) - m^2\sin\alpha. \tag{34}$$

If we consider this equation in Eq. 7 we obtain

$$area\ \bar{C_{01}}(\varphi) = area\ \bar{C_{04}}(\varphi) = area\ C_{03} - A_3(\varphi) \tag{35}$$

Calling with M_i, with $i = 1, 2, 3, ...$, the set of the segment s that have the midpoint in the cell C_{0i} and denoting by N_i the set of the segment s that is totally included in C_{0i}, we obtain (Stoka 2015):

$$P_{int} = 1 - \frac{\sum_{i=1}^{4}\mu(N_i)}{\sum_{i=1}^{4}\mu(M_i)}, \tag{36}$$

where μ is the Lebesgue measure in the Euclidean plan. By computing the measure $\mu\ (M_i)$ and $\mu\ (N_i)$ we use the Poincaré 's kinematic measure (Stoka 1975)

$$dK = dx \wedge dy \wedge dz,$$

where x, y are the coordinates of the midpoint of s and φ is the predefinite angle. Taking into consideration Eq. 18 we have

$$\mu\,(M_i) = \int_{\alpha}^{\frac{\pi}{2}} d\varphi \int\int_{\{(x,y)\varepsilon C_{0i}\}} dxdy = \int_{\alpha}^{\frac{\pi}{2}} (area\ C_{0i})d\varphi = (\frac{\pi}{2} - \alpha)area\ C_{0i}, (i = 1,2,3,4)$$

then

$$\sum_{i=1}^{4}\mu(M_i) = (\frac{\pi}{2} - \alpha)\sum_{i=1}^{4} area\ C_{0i},$$

by using Eq. 5.

$$\sum_{i=1}^{4} = \mu(M_i) = (\frac{\pi}{2} - \alpha)\left\{ab - 2m^2[2\cos\alpha + \sin\alpha(1+\cos\alpha)]\right\}. \tag{37}$$

Taking into consideration Eqs. 24 and 36 we can write:

$$\mu(N_i) = \int_{\alpha}^{\frac{\pi}{2}} d\varphi \int\int_{\left\{(x,y)\varepsilon\widehat{C_{0i}}(\varphi)\right\}} dxdy = \int_{\alpha}^{\frac{\pi}{2}} [area\,\widehat{C_{0i}}(\varphi)]d\varphi = \int_{\alpha}^{\frac{\pi}{2}} [area\, C_{0i} - A_i(\varphi)]d\varphi =$$

$$= (\frac{\pi}{2} - \alpha)area\; C_{0i} - \int_{0}^{\frac{\pi}{2}} [A_i - (\varphi)]d\varphi, \tag{38}$$

then

$$\sum_{i=1}^{4} \mu(N_i) = (\frac{\pi}{2} - \alpha)area\; C_0 - \int_{0}^{\frac{\pi}{2}} [\sum_{i=1}^{4} A_i(\varphi)]d\varphi. \tag{39}$$

Equations 37, 38, and 39 give us

$$P_{int} = \frac{1}{(\frac{\pi}{2})[ab - m^2(4\cos\alpha + 2\sin\alpha + \sin 2\alpha)} \int_{\alpha}^{\frac{\pi}{2}} [\sum_{i=1}^{4} A_i(\varphi)]d\varphi.$$

From Eqs. 23 and 35 we obtain

$$\int_{\alpha}^{\frac{\pi}{2}} [\sum_{i=1}^{4} A_i(\varphi)]d\varphi = al(1-\sin\alpha) - \frac{bl}{4}[7\cos\alpha - \tan\alpha(1-\sin\alpha)] - \frac{l^4}{4}[1+\cos\alpha+\cos^2\alpha+$$

$$+\cot\alpha(\pi+1-2\alpha+\sin 2\alpha)] + (1-\sin\alpha)[\frac{\sqrt{2}}{2}(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}-2]ml - m^2(\sin\alpha+\cos\alpha)(\frac{\pi}{2}-\alpha)$$

and then

$$P_{int} = \frac{1}{(\frac{\pi}{2}-1)[ab - m^2(4\cos\alpha + 2\sin\alpha + \sin 2\alpha)]}\cdot$$

$$\cdot\{ab(1-\sin\alpha) + \frac{bl}{4}[7\cos\alpha - \tan\alpha(1-\sin\alpha)] - \frac{l^2}{4}[1+\cos\alpha+\cos^2\alpha+\cot\alpha(\pi-2\alpha+\sin 2\alpha)]$$

$$+(1-\sin\alpha)[\frac{\sqrt{2}}{2}(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}) - 2]ml - m^2(\sin\alpha+\cos\alpha)(\frac{\pi}{2})\}.$$

3. Conclusions and future developments

In this paper we have solved an open problem in the field of stochastic geometry on the basis of well known Stoka's approach. In the next future we are going to connect this fascinating field of study to some Artificial Intelligence (AI) subfields as criptography and blockchain issues, machine learning techniques related to Big Data selections and so on. The evolution of mathematical studies follows the frontier changes of knowledge.

Dedication

This scientific work represents the solution to an open problem that my master Marius Ion Stoka, a Corresponding Member of the *Accademia Peloritana dei Pericolanti*, left me with a few years before his death in December 2016. I dedicate these results to his memory.

References

Blaschke, W. (1995). *Vorlesungen über Integralgeometrie*. 3rd ed. Berlin: Deutscher Verlag der Wisschenschaften.

Poincaré, M. (1912). *Calcul des Probabilités*. Paris: Gauthier-Villars. URL: https://gallica.bnf.fr/ark:/12148/bpt6k29064s/f1.item.

Stoka, M. I. (1975). "Probabilités géométriques de type "Buffon" dans le plan Euclidien". *Atti dell'Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali* **110**, 53–59.

Stoka, M. I. (2015). "Laplace type problem for a rectangular lattice and non-uniform distribution". *Atti dell'Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali* **149**, 93–98.

* Università Mediterranea di Reggio Calabria,
Dipartimento di Giurisprudenza, Economia e Scienze Umane,
Via dell'Università, 98124 Reggio Calabria, Italy

Email: massimiliano.ferrara@unirc.it

Communicated 6 May 2024; manuscript received 20 June 2024; published online 7 September 2024