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Guaranteed Pursuit and Evasion Times in a Differential Game for an Infinite System in Hilbert Space l_2

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Abstract: The present paper is devoted to studying a pursuit differential game described by an infinite system of binary differential equations in Hilbert space l_2. The control parameters of the players are subject to geometric constraints. The pursuer tries to bring the state of the system to the origin of the Hilbert space l_2, and oppositely, the evader tries to avoid it. Our aim is to construct a strategy for the pursuer to complete a differential game and an evasion control. We obtain an equation for the guaranteed pursuit and evasion times.

Keywords: differential game; pursuit; control; strategy; infinite system of differential equations; geometric constraint

MSC: 91A23; 49N75

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1. Introduction

Control and differential game problems described by partial differential equations (PDE) attracted most researchers' attention. The time-optimal control problem for the parabolic type equation was first concerned by Fattorini [1]. The works of Lions [2], and Osipov [3] were the first papers related to differential games governed by PDE. The parabolic Dirichlet boundary optimal control problem on complex connected domains was studied by Liu et al. [4]. For the optimal control problems where an appropriate quadratic functional subject to a linear PDE is minimized, the paper of Geshkovski and Zuazua [5] presents the methodology for proving turnpike.

Using the decomposition method various control problems for PDE were studied by many researchers such as Butkovskiy [6], Chernous'ko [7], Avdonin and Ivanov [8], Satimov and Tukhtasinov [9], Chaves-Silva et al. [10], Alimov and Albeverio [11], and Philippe Martin et al. [12].

Moreover, using this method, some differential game problems governed by PDE were analyzed (see, for example, [9,13–17]). Though this method results in a differential game described by a system of ordinary differential equations, the main difficulty in solving differential game problems arises because of the infinity of the number of equations in the system.

Thus, differential game problems described by PDE are closely related to those described by an infinite system of differential equations. For instance, the works [9,15,16] studied differential games described by a PDE of the form

$$\frac{\partial z}{\partial t} = Az + u - v, \ \ Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_i}\right). \tag{1}$$

These differential game problems described by (1) were reduced to a differential game described by the following infinite system of differential equations

$$\dot{z}_k + \lambda_k z_k = u_k - v_k, \ k = 1, 2, \dots, \tag{2}$$

where u_k and v_k are control parameters of players, $z_k, u_k, v_k \in \mathbb{R}$, and the coefficients λ_k, $k = 1, 2, \dots$, satisfy the condition $0 < \lambda_1 \le \lambda_2 \le \dots \to \infty$.

Therefore, assuming that $\lambda_1, \lambda_2, \dots$ are any numbers, differential games for the infinite system of differential equations (2) can be studied separately from PDE (1). In its turn, an infinite system of differential equations (2) can be extended by replacing the coefficients λ_k with $d \times d$ matrices A_k, and z_k, u_k, v_k with the d-dimensional vectors. For example, the work [18] studies the problem of optimal pursuit for the following infinite system

$$\begin{aligned} \dot{x}_i &= -\alpha_i x_i - \beta_i y_i + u_{i1} - v_{i1}, \quad & x_i(0) = x_{i0}, \\ \dot{y}_i &= \beta_i x_i - \alpha_i y_i + u_{i2} - v_{i2}, \quad & y_i(0) = y_{i0}, \end{aligned} \tag{3}$$

in Hilbert space l_2, where α_i, β_i are real numbers, and $\alpha_i \ge 0$. In fact, system (3) can be derived from system (2) when λ_k are complex numbers and z_k, u_k, v_k are complex variables. Moreover, by replacing $\dot{x}_i$, $\dot{y}_i$ with the second derivatives $\ddot{x}_i$, $\ddot{y}_i$ in the system (3) we can study a differential game described by the infinite system of second order binary equations (see, for example, [19]). Similar to finite-dimensional differential games, one can consider differential games with geometric or integral constraints (see, for example, [20]), multi pursuer differential games (see, for example, [21]) and so on for the infinite systems of differential equations.

In the work [22], a differential game with geometric constraints for Equation (2) was considered and an equation for the guaranteed pursuit time was obtained. The paper [23] is devoted to a differential game for system (3) with geometric constraints, and an equation for the guaranteed pursuit time was obtained.

Infinite systems of Equations (2) and (3) can be written as an infinite system of 2-systems as follows:

$$\dot{z}_i = A_i z_i + u_i - v_i, \ \ A_i = \begin{bmatrix} -\lambda_{i1} & 0 \\ 0 & -\lambda_{i2} \end{bmatrix}, \ \ \dot{z}_i = B_i z_i + u_i - v_i, \ \ B_i = \begin{bmatrix} -\alpha_i & -\beta_i \\ \beta_i & -\alpha_i \end{bmatrix},$$

where λ_{i1}, λ_{i2}, α_i, β_i are some real numbers, $z_i = (x_i, y_i)$, $u_i = (u_{i1}, u_{i2})$, $v_i = (v_{i1}, v_{i2})$. Clearly, the matrices A_i and B_i are the Jordan matrices. Any 2×2 real matrix is similar to one of the Jordan matrices: A_i, B_i or $C_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}$, with λ_i being a real number. For completeness, Equations (2) and (3) suggest considering an infinite system of 2-equations corresponding to matrices C_i. Therefore, in the present paper, we study a differential game problem described by the following infinite system of binary differential equations

$$\dot{z}_i = C_i z_i + u_i - v_i, \ i = 1, 2, \dots$$

For this system, neither a differential game with integral constraints nor a differential game with geometric constraints was studied in the literature. We study a differential game with geometric constraints, i.e., the values of the control functions of pursuer and evader belong to the balls of radii ρ and σ, respectively. We obtain an equation for the guaranteed pursuit time. Moreover, we construct an admissible strategy for the pursuer, and we obtain a formula for a guaranteed evasion time.

2. Statement of Problem

We recall that the Hilbert space l_2 is a linear space of all sequences of real numbers

$$l_2 = \left\{ \xi = (\xi_1, \xi_2, \dots, \xi_n, \dots) \mid \sum_{n=1}^{\infty} \xi_n^2 < \infty \right\}$$

where the inner product and norm are defined as follows

$$(\xi, \eta) = \sum_{n=1}^{\infty} \xi_n \eta_n, \ \ ||\xi|| = \sqrt{(\xi, \xi)}.$$

A controlled object is described by the following infinite system of differential equations

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + y_i + u_{i1} - v_{i1}, \quad x_i(0) = x_{i0}, \\ \dot{y}_i &= -\lambda_i y_i + u_{i2} - v_{i2}, \qquad\ \ y_i(0) = y_{i0}, \quad i = 1, 2, \dots, \end{aligned} \tag{4}$$

in Hilbert space l_2, where $\lambda_i \geq \lambda_0$, and λ_0 is a given positive number, $x_0 = (x_{10}, x_{20}, \dots) \in l_2$, $y_0 = (y_{10}, y_{20}, \dots) \in l_2$, $u = (u_{11}, u_{12}, u_{21}, u_{22}, \dots)$ and $v = (v_{11}, v_{12}, v_{21}, v_{22}, \dots)$ are the control parameters of pursuer and evader, respectively. We suppose that $0 \leq t \leq T$, where T is a sufficiently large number, and that $\eta_0 = (\eta_{10}, \eta_{20}, \dots) = (x_{10}, y_{10}, x_{20}, y_{20}, \dots) \neq 0$, where $\eta_{i0} = (x_{i0}, y_{i0})$, $i = 1, 2, \dots$.

Let ρ and σ, $\rho > \sigma$, be given positive numbers.

Definition 1. *A function $u(t) = (u_1(t), u_2(t), \dots)$, $t \in [0, T]$, with measurable coordinates $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $i = 1, 2, \dots$, subject to the condition*

$$\sum_{i=1}^{\infty} (u_{i1}^2(t) + u_{i2}^2(t)) \leq \rho^2, \ \ 0 \leq t \leq T, \tag{5}$$

is referred to as the admissible control of the pursuer.

Definition 2. *A function $v(t) = (v_1(t), v_2(t), \dots)$, $t \in [0, T]$, with measurable coordinates $v_i(t) = (v_{i1}(t), v_{i2}(t))$, $i = 1, 2, \dots$, subject to the condition*

$$\sum_{i=1}^{\infty} (v_{i1}^2(t) + v_{i2}^2(t)) \leq \sigma^2, \ \ 0 \leq t \leq T, \tag{6}$$

is referred to as the admissible control of evader.

Let $\eta_i(t) = (x_i(t), y_i(t))$, $U_i = (U_{i1}, U_{i2})$, $v_i = (v_{i1}, v_{i2})$.

Definition 3. *A strategy of the pursuer is defined as a function of the form*

$$U(t, v) = U^0(t) + v = (U_1^0(t) + v_1, U_2^0(t) + v_2, \dots), \tag{7}$$

where $U^0(t) = (U_1^0(t), U_2^0(t), \dots)$, $U_i^0(t) = (U_{i1}^0(t), U_{i2}^0(t))$, has measurable coordinates $U_i^0(t)$, $0 \leq t \leq T$ that satisfy the condition

$$\sum_{i=1}^{\infty} \left(\left(U_{i1}^0(t) \right)^2 + \left(U_{i2}^0(t) \right)^2 \right) \leq (\rho - \sigma)^2, \ \ 0 \leq t \leq T. \tag{8}$$

It is clear that

$$e^{A_i t} = e^{-\lambda_i t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \ \ A_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}, \ \ i = 1, 2, \dots \tag{9}$$

The function $\eta(t) = (\eta_1(t), \eta_2(t), \dots)$ defined by the equation

$$\eta_i(t) = e^{A_i t}\eta_{i0} + \int_0^t e^{A_i(t-s)}(u_i(s) - v_i(s))ds, \;\; i = 1, 2, \dots,$$

is the unique solution of (4) in $C(0, T; l_2)$, which is the class of continuous functions on $[0, T]$ that take their values in l_2. The representation

$$\eta_i(t) = e^{A_i t}\left[\eta_{i0} + \int_0^t e^{-A_i s}(u_i(s) - v_i(s))ds\right], \;\; i = 1, 2, \dots \tag{10}$$

clearly, implies that $\eta(t) = 0$ if and only if $\gamma(t) = 0$, where

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots), \;\; \gamma_i(t) = \eta_{i0} + \int_0^t e^{-A_i s}(u_i(s) - v_i(s))ds, \;\; i = 1, 2, \dots \tag{11}$$

Definition 4. *We call a number θ a guaranteed pursuit time if there exists a strategy $U(t, v)$ of the pursuer such that for any admissible control $v(t)$, $0 \le t \le \theta$, of the evader, we have $\eta(t') = 0$ at some t', $0 \le t' \le \theta$, where $\eta(t) = (\eta_1(t), \eta_2(t), \dots)$ is the solution of the initial value problem (4) at $u(t) = U(t, v(t))$ and $v = v(t)$, $0 \le t \le T$.*

The pursuer is interested in minimizing the guaranteed pursuit time θ by choosing his strategy $U(t, v)$ and the evader tries to maximize θ by choosing his control $v = v(t)$, $0 \le t \le T$.

The research questions are (1) Can the pursuit be completed for any numbers λ_i? (2) Is it possible to obtain an equation for a guaranteed pursuit time? (3) Is it possible to construct an admissible strategy for the pursuer in explicit form to complete the pursuit?

3. Pursuit Differential Game

Let, for $i = 1, 2, \dots$,

$$\begin{aligned} W_i(\theta) &= \int_0^\theta e^{-A_i s}e^{-A_i^* s}ds = \int_0^\theta \begin{bmatrix} e^{\lambda_i s} & -e^{\lambda_i s}s \\ 0 & e^{\lambda_i s}\end{bmatrix}\begin{bmatrix} e^{\lambda_i s} & 0 \\ -e^{\lambda_i s}s & e^{\lambda_i s}\end{bmatrix}ds \\ &= \begin{bmatrix} \int_0^\theta e^{2\lambda_i s}(1+s^2)ds & -\int_0^\theta e^{2\lambda_i s}sds \\ -\int_0^\theta e^{2\lambda_i s}sds & \int_0^\theta e^{2\lambda_i s}ds \end{bmatrix} = \begin{bmatrix} \psi_{11}(\theta) & \psi_{12}(\theta) \\ \psi_{21}(\theta) & \psi_{22}(\theta)\end{bmatrix}, \end{aligned}$$

where A^* is the transpose of the matrix A, and

$$\begin{aligned} \psi_{11}(\theta) &= \int_0^\theta e^{2\lambda_i s}(1+s^2)ds, & \psi_{12}(\theta) &= -\int_0^\theta e^{2\lambda_i s}sds, \\ \psi_{21}(\theta) &= -\int_0^\theta e^{2\lambda_i s}sds, & \psi_{22}(\theta) &= \int_0^\theta e^{2\lambda_i s}ds. \end{aligned}$$

For the matrix $W_i(t)$ and $\eta_{i0} = (x_{i0}, y_{i0})$ the following relation holds

$$\eta_{i0}W_i(t)\eta_{i0} = \int_0^t \left|e^{-A_i^* s}\eta_{i0}\right|^2 ds.$$

Obviously, if $\eta_{i0} \neq 0$, then this integral is positive. Thus, the matrix $W_i(t)$ is positively defined. Thereby, its determinant is positive

$$|W_i(t)| = \psi_{11}(t)\psi_{22}(t) - \psi_{12}^2(t) > 0. \tag{12}$$

Hence, the matrix $W_i(t)$ is invertible and it is not difficult to see that

$$W_i^{-1}(t) = \begin{bmatrix} \frac{\psi_{22}(t)}{|W_i(t)|} & -\frac{\psi_{12}(t)}{|W_i(t)|} \\ -\frac{\psi_{21}(t)}{|W_i(t)|} & \frac{\psi_{11}(t)}{|W_i(t)|} \end{bmatrix}.$$

In this section, we present the main result of our paper, which is the following statement about the guaranteed pursuit time.

Theorem 1. *Let a positive number $t = \theta$ satisfy the inequality*

$$\sum_{i=1}^{\infty} \frac{4e^{2\lambda_i t}\psi_{11}^2(t)(2+t^2)}{|W_i(t)|^2}|\eta_{i0}|^2 \leq (\rho - \sigma)^2. \tag{13}$$

Then, θ is a guaranteed pursuit time in game (4).

Proof. First of all, we have to establish that there is a number θ to satisfy (13). This fact follows from the following statement. □

Lemma 1. *The following is true*

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} \frac{4e^{2\lambda_i t}\psi_{11}^2(t)(2+t^2)}{|W_i(t)|^2}|\eta_{i0}|^2 = 0.$$

Proof. We estimate $|W_i(t)|^2$ from below. By the Cauchy–Schwartz inequality, we have

$$\int_0^t e^{2\lambda_i s}s^2 ds \int_0^t e^{2\lambda_i s} ds \geq \left(\int_0^t e^{2\lambda_i s} s ds \right)^2,$$

and so

$$\begin{aligned} |W_i(t)| &= \int_0^t e^{2\lambda_i s}(1+s^2)ds \int_0^t e^{2\lambda_i s}ds - \left(\int_0^t e^{2\lambda_i s} s ds \right)^2 \\ &= \left(\int_0^t e^{2\lambda_i s} ds \right)^2 + \int_0^t e^{2\lambda_i s}s^2 ds \int_0^t e^{2\lambda_i s} ds - \left(\int_0^t e^{2\lambda_i s} s ds \right)^2 \\ &\geq \left(\int_0^t e^{2\lambda_i s} ds \right)^2 = \psi_{22}^2(t). \end{aligned} \tag{14}$$

Therefore, we obtain

$$f_i(t) \triangleq \frac{4e^{2\lambda_i t}\psi_{11}^2(t)(2+t^2)}{|W_i(t)|^2}|\eta_{i0}|^2 \leq \frac{4e^{2\lambda_i t}(2+t^2)\psi_{11}^2(t)}{\psi_{22}^4(t)}|\eta_{i0}|^2. \tag{15}$$

Since $\lambda_i \geq \lambda_0$, we have

$$\begin{aligned} \psi_{11}(t) &= \int_0^t e^{2\lambda_i s}(1+s^2)ds = \frac{(1+t^2)e^{2\lambda_i t}}{2\lambda_i} + \frac{e^{2\lambda_i t}}{4\lambda_i^3} - \frac{1}{2\lambda_i} - \frac{te^{2\lambda_i t}}{2\lambda_i^2} - \frac{1}{4\lambda_i^3} \\ &\leq \frac{(1+t^2)e^{2\lambda_i t}}{2\lambda_i} + \frac{e^{2\lambda_i t}}{4\lambda_i^3} = \frac{e^{2\lambda_i t}t^2}{2\lambda_i}\left(1 + \frac{1}{t^2} + \frac{1}{2\lambda_i^2 t^2}\right) \\ &\leq \frac{t^2 e^{2\lambda_i t}}{2\lambda_i}(1+\alpha_1(t)),\ \ \alpha_1(t) = \frac{1}{t^2} + \frac{1}{2\lambda_0^2 t^2}, \end{aligned} \tag{16}$$

and

$$\psi_{22}(t) = \int_0^t e^{2\lambda_i s}ds = \frac{e^{2\lambda_i t}-1}{2\lambda_i} \geq \frac{e^{2\lambda_i t}}{2\lambda_i}(1-\alpha_2(t)),\ \ \alpha_2(t) = e^{-2\lambda_0 t}. \tag{17}$$

Since by (16) and (17)

$$\begin{aligned} \frac{4e^{2\lambda_i t}(2+t^2)\psi_{11}^2(t)}{\psi_{22}^4(t)} &\leq \frac{4e^{2\lambda_i t}(2+t^2)\cdot \frac{t^4 e^{4\lambda_i t}}{4\lambda_i^2}(1+\alpha_1(t))^2}{\frac{e^{8\lambda_i t}}{16\lambda_i^4}(1-\alpha_2(t))^4} \\ &= \frac{16\lambda_i^2(2+t^2)t^4(1+\alpha_1(t))^2}{e^{2\lambda_i t}(1-\alpha_2(t))^4}. \end{aligned} \tag{18}$$

In view of the inequality $e^{2\lambda_i t} > (2\lambda_i t)^7/7!$, inequality (18) implies that

$$\begin{aligned} \frac{4e^{2\lambda_i t}(2+t^2)\psi_{11}^2(t)}{\psi_{22}^4(t)} &\leq \frac{16\lambda_i^2(2+t^2)t^4}{(2\lambda_i t)^7/7!}\cdot\frac{(1+\alpha_1(t))^2}{(1-\alpha_2(t))^4} \\ &\leq \frac{630}{\lambda_0^5}\cdot\frac{2+t^2}{t^3}\cdot\frac{(1+\alpha_1(t))^2}{(1-\alpha_2(t))^4}. \end{aligned} \tag{19}$$

From (15) it follows that

$$\sum_{i=1}^{\infty} f_i(t) \leq \frac{630}{\lambda_0^5}\cdot\frac{2+t^2}{t^3}\cdot\frac{(1+\alpha_1(t))^2}{(1-\alpha_2(t))^4}\sum_{i=1}^{\infty}|\eta_{i0}|^2.$$

The right-hand side of this inequality approaches 0 as $t \to +\infty$ since $\lim\limits_{t\to\infty}\alpha_i(t) = 0$, $i = 1,2$, and $\sum\limits_{i=1}^{\infty}|\eta_{i0}|^2 = ||\eta_0||^2$. This is the desired conclusion. Thus, there is a number θ that satisfies inequality (13). The proof of the lemma is complete.

To prove the theorem, we construct the strategy for the pursuer as follows

$$U_i(t) = \begin{cases} -e^{-tA_i^*}W_i^{-1}(\theta)\eta_{i0} + v_i(t), & 0 \leq t \leq \theta, \\ v_i(t), & t > \theta. \end{cases} \tag{20}$$

The functions $U_i^0(t)$, $i = 1,2,\ldots$, (see (26)) in Definition 3 are defined as follows

$$U_i^0(t) = \begin{cases} -e^{-tA_i^*}W_i^{-1}(\theta)\eta_{i0}, & 0 \leq t \leq \theta \\ 0, & t > \theta \end{cases},\ \ i = 1,2,\ldots \tag{21}$$

To show the admissibility of the constructed strategy, first, we prove inequality (8), that is

$$||U^0(t)||^2 = \sum_{i=1}^{\infty}|U_i^0(t)|^2 \leq (\rho-\sigma)^2.$$

Indeed, since

$$\begin{aligned} g_i(t) &\triangleq \left|e^{-tA_i^*}W_i^{-1}(\theta)\eta_{i0}\right| = \frac{e^{\lambda_i t}}{|W_i(\theta)|}\left|\begin{bmatrix}1 & 0\\ -t & 1\end{bmatrix}\begin{bmatrix}\psi_{22}(\theta) & \psi_{12}(\theta)\\ \psi_{21}(\theta) & \psi_{11}(\theta)\end{bmatrix}\begin{bmatrix}x_{i0}\\ y_{i0}\end{bmatrix}\right| \\ &= \frac{e^{\lambda_i t}}{|W_i(\theta)|}\left|\begin{bmatrix}x_{i0}\psi_{22}(\theta)+y_{i0}\psi_{21}(\theta)\\ -tx_{i0}\psi_{22}(\theta)+x_{i0}\psi_{12}(\theta)-ty_{i0}\psi_{21}(\theta)+y_{i0}\psi_{11}(\theta)\end{bmatrix}\right|, \end{aligned}$$

then utilizing the obvious inequalities $|a+b+c+d|^2 \le 4(a^2+b^2+c^2+d^2)$ for the vectors a, b, c, d, and $\psi_{22} \le \psi_{11}$, $|\psi_{21}| \le \psi_{11}$ yields

$$\begin{aligned} g_i^2(t) &= \frac{e^{2\lambda_i t}}{|W_i(\theta)|^2}\left|\begin{bmatrix}x_{i0}\psi_{22}(\theta)\\ tx_{i0}\psi_{22}(\theta)\end{bmatrix}+\begin{bmatrix}-y_{i0}\psi_{21}(\theta)\\ ty_{i0}\psi_{21}(\theta)\end{bmatrix}+\begin{bmatrix}0\\ -x_{i0}\psi_{12}(\theta)\end{bmatrix}+\begin{bmatrix}0\\ y_{i0}\psi_{11}(\theta)\end{bmatrix}\right|^2 \\ &\le \frac{4e^{2\lambda_i t}}{|W_i(\theta)|^2}\left(x_{i0}^2\psi_{22}^2(\theta)(1+t^2)+y_{i0}^2\psi_{21}^2(\theta)(1+t^2)+x_{i0}^2\psi_{21}^2(\theta)+y_{i0}^2\psi_{11}^2(\theta)\right) \\ &\le \frac{4e^{2\lambda_i t}\psi_{11}^2(\theta)(2+t^2)}{|W_i(\theta)|^2}|\eta_{i0}|^2 \le \frac{4e^{2\lambda_i\theta}\psi_{11}^2(\theta)(2+\theta^2)}{|W_i(\theta)|^2}|\eta_{i0}|^2. \end{aligned} \tag{22}$$

From this using the definition of θ we conclude that

$$\begin{aligned} \sum_{i=1}^{\infty}|U_i^0(t)|^2 &= \sum_{i=1}^{\infty}|-e^{-tA_i^*}W_i^{-1}(\theta)\eta_{i0}|^2 = \sum_{i=1}^{\infty}g_i^2(t) \\ &\le \sum_{i=1}^{\infty}\frac{4e^{2\lambda_i\theta}\psi_{11}^2(\theta)(2+\theta^2)}{|W_i(\theta)|^2}|\eta_{i0}|^2 \le (\rho-\sigma)^2, \end{aligned} \tag{23}$$

which is our assertion.

We are now in a position to prove the admissibility of strategy (20). By the Minkowski inequality and the definition of θ we have, for $0 \le t \le \theta$,

$$\begin{aligned} \|U(t)\| &= \|U^0(t)+v(t)\| \le \|U^0(t)\|+\|v(t)\| \\ &= \left(\sum_{i=1}^{\infty}|U_i^0(t)|^2\right)^{1/2}+\left(\sum_{i=1}^{\infty}|v_i(t)|^2\right)^{1/2} \le (\rho-\sigma)+\sigma=\rho. \end{aligned}$$

The proof of admissibility of strategy $U(t)$ is complete.

To show that θ is a guaranteed pursuit time, we show that $\eta(\theta)=0$. To this end, we show that

$$\gamma_i(\theta)=0, \;\; i=1,2,\ldots.$$

Indeed, by (20)

$$\begin{aligned} \gamma_i(\theta) &= \eta_{i0}+\int_0^{\theta}e^{-sA_i}(U_i(s)-v_i(s))ds \\ &= \eta_{i0}+\int_0^{\theta}e^{-sA_i}\left(e^{-sA_i^*}W_i^{-1}(\theta)\eta_{i0}\right)ds = \eta_{i0}-\eta_{i0}=0, \end{aligned}$$

and so $\gamma_i(\theta)=0$; hence, $\eta(\theta)=0$. Thus, the pursuit is completed exactly at the time θ. This completes the proof of the theorem. □

4. Evasion Differential Game

In this section, we study an evasion differential game.

Definition 5. *We call number τ a guaranteed evasion time if, for any number τ', $0 \le \tau' < \tau$, one can construct an admissible control $v_0(t)$ of evader such that, for any admissible control of the pursuer, we have $\eta(t) \neq 0$ for all $0 \le t \le \tau'$ and $i = 1, 2, \ldots$.*

The evader tries to maximize the guaranteed evasion time. The research questions are (1) Is it possible to find a guaranteed evasion time τ in the game for any initial state $\eta_0 = (\eta_{10}, \eta_{20}, \cdots) \neq 0$? (2) Can we construct an admissible control for the evader that ensures the guaranteed evasion time?

Theorem 2. *The number*

$$\tau = \sup_{j=1,2,\ldots} \tau_j,$$

where τ_j is a non-negative root of the equation

$$\int_0^t e^{\lambda_j s}\left(\rho(1+s) - \frac{\sigma}{1+s}\right)ds = |\eta_{j0}|, \tag{24}$$

is a guaranteed evasion time in game (4)

Proof. First, we show that Equation (24) has a unique non-negative root. Clearly, if $\eta_{i0} = 0$, then Equation (24) has the unique root $\tau_j = 0$.

Let $\eta_{j0} \neq 0$. Since $\rho > \sigma$ and $\lambda_j > 0$, we obtain

$$\begin{aligned} |\eta_{j0}| &= \int_0^{\tau_j} e^{\lambda_j s}\left(\rho(1+s) - \frac{\sigma}{1+s}\right)ds > \int_0^{\tau_j} e^{\lambda_j s}(\rho(1+s) - \sigma)ds \\ &> \int_0^{\tau_j} e^{\lambda_j s}\rho s ds > \rho\int_0^{\tau_j} s ds = \rho\frac{\tau_j^2}{2}. \end{aligned} \tag{25}$$

Hence, the root of equation (24) satisfies the condition $\tau_j^2 < 2|\eta_{j0}|/\rho$. Consequently, $\tau = \sup\limits_{j=1,2,\ldots} \tau_j < \infty$.

Next, we construct a control for the evader that ensures $\eta(t) \neq 0$ on $[0, \tau']$ for the arbitrary control $u = u(t)$ of the pursuer, where τ' is an arbitrary time that satisfies the condition $0 < \tau' < \tau$.

By definition of τ, we can find $j \in \{1, 2, \cdots\}$ such that $\tau' < \tau_j$. We show that game (4) is never completed on $[0, \tau_j)$. Observe $\eta_{j0} \neq 0$ since otherwise $\tau_j = 0$, contradicting the positivity of τ_j.

Let the evader apply the following control

$$v_j(t) = -\sigma\frac{1}{1+t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e_j, \;\; v_i(t) = 0, \; i = 1, 2, \cdots, \; i \neq j, \;\; t \in [0, \tau_j). \tag{26}$$

where $e_j = \frac{\eta_{j0}}{|\eta_{j0}|}$ is unit 2-vector, i.e., $e_j = (e_{j1}, e_{j2})$, $|e_j| = 1$.

Next, we establish that control (26) is admissible. Using (6), we obtain

$$\begin{aligned} |v_j(t)| &= \left| -\sigma\frac{1}{1+t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} e_j \right| = \frac{\sigma}{1+t}\left|\begin{bmatrix} e_{j1} + te_{j2} \\ e_{j2} \end{bmatrix}\right| \\ &\le \frac{\sigma}{1+t}\left(\left|\begin{bmatrix} e_{j1} \\ e_{j2} \end{bmatrix}\right| + \left|\begin{bmatrix} e_{j2}t \\ 0 \end{bmatrix}\right|\right) = \frac{\sigma}{1+t}(1 + t|e_{j2}|) \\ &\le \frac{\sigma}{1+t}(1+t) = \sigma. \end{aligned}$$

Consequently, control (26) is admissible.

According to (10) and (11), $\eta(t) \neq 0$ if and only if $\gamma(t) \neq 0$. It suffices, therefore, to show that, for any admissible control of the pursuer $u(\cdot)$, $\gamma(t) \neq 0$, $t \in [0, \tau_j)$. Indeed,

$$\begin{aligned} \gamma_j(t) &= \eta_{j0} + \int_0^t e^{-A_j s} u_j(s) ds - \int_0^t e^{-A_j s} v_j(s) ds \\ &= \eta_{j0} + \int_0^t e^{-A_j s} u_j(s) ds + \sigma e_j \int_0^t \frac{1}{1+s} e^{-\lambda_j s} ds. \end{aligned} \tag{27}$$

Then, by (26) we have

$$\begin{aligned} |(e^{-A_j s} u_j(s), e_j)| &\leq |e^{-A_j s} u_j(s)||e_j| = |(e^{-A_j s} u_j(s)| \\ &= e^{\lambda_j s} \left| \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{j1} \\ u_{j2} \end{bmatrix} \right| = e^{\lambda_j s} \left| \begin{bmatrix} u_{j1} - u_{j2} s \\ u_{j2} \end{bmatrix} \right| \\ &\leq e^{\lambda_j s} \left| \begin{bmatrix} u_{j1} \\ u_{j2} \end{bmatrix} + \begin{bmatrix} -u_{j2} s \\ 0 \end{bmatrix} \right| \leq \rho e^{\lambda_j s}(1+s) \end{aligned}$$

Consequently, using (27) and the fact that τ_j is the root of equation (24), we obtain for $t \in [0, \tau_j)$ that

$$\begin{aligned} (\gamma_j(t), e_j) &= |\eta_{j0}| + \int_0^t \left(e^{-A_j s} u_j(s), e_j \right) ds + \sigma \int_0^t \frac{1}{1+s} e^{\lambda_j s} ds \\ &\geq |\eta_{j0}| - \int_0^t \rho(1+s) e^{\lambda_j s} ds + \int_0^t \frac{\sigma}{1+s} e^{\lambda_j s} ds \\ &> |\eta_{j0}| - \int_0^{\tau_j} e^{\lambda_j s} \left(\rho(1+s) - \frac{\sigma}{1+s} \right) ds = 0. \end{aligned}$$

As the result, $\gamma_j(t) \neq 0$, $t \in [0, \tau_j)$, and hence $\eta_j(t) \neq 0$ by (11), therefore, $\eta(t) \neq 0$ for $t \in [0, \tau_j)$. In particular, $\eta(t) \neq 0$ on the interval $[0, \tau']$. The proof of Theorem 2 is complete. □

5. Conclusions

In the present paper, we have studied a pursuit differential game for an infinite system of binary differential equations (4)

$$\dot{z}_i = C_i z_i + u_i - v_i, \ i = 1, 2, \ldots,$$

which correspond to the Jordan matrices $C_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}$, where λ_i, $i = 1, 2, \ldots$, are real numbers. Previous research only studies differential games described by infinite systems

$$\dot{z}_i = A_i z_i + u_i - v_i, \ \ A_i = \begin{bmatrix} -\lambda_{i1} & 0 \\ 0 & -\lambda_{i2} \end{bmatrix}$$

and

$$\dot{z}_i = B_i z_i + u_i - v_i, \ \ B_i = \begin{bmatrix} -\alpha_i & -\beta_i \\ \beta_i & -\alpha_i \end{bmatrix}.$$

It is well known that any 2×2-matrix can be reduced to one of the Jordan matrices A_i, B_i, and C_i. Therefore, the infinite system we have studied in the present paper fills the gap between the infinite 2-systems corresponding to the 2×2 Jordan matrices. Pursuit and evasion differential game problems for the infinite system of differential equations corresponding to the 2×2 matrices C_i were considered in the present paper for the first time.

We obtained an equation for the guaranteed pursuit time. Moreover, we constructed an explicit strategy for the pursuer that ensures the completion of the game by the guaranteed pursuit time. The main difficulties were obtaining the equation for the guaranteed pursuit time and proving Lemma 1, which plays a key role in establishing the admissibility of the constructed strategy of the pursuer. Moreover, we studied an evasion differential game and obtained a formula for the guaranteed evasion time.

For future work, we recommend studying pursuit or/and evasion differential games described by an infinite system of differential equations of the general form

$$\dot{z}_i = D_i z_i + u_i - v_i,\ \ D_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix},\ i = 1, 2, \dots,$$

where a_i, b_i, c_i, and d_i are any given real numbers.

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