

# Regular solutions for nonlinear elliptic equations, with convective terms, in Orlicz spaces

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## Abstract

We establish some existence and regularity results to the Dirichlet problem, for a class of quasilinear elliptic equations involving a partial differential operator, depending on the gradient of the solution. Our results are formulated in the Orlicz Sobolev spaces and under general growth conditions on the convection term. The sub and supersolutions method is a key tool in the proof of the existence results.

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ , with  $C^{1,\alpha}$  boundary. We consider the following quasilinear elliptic problem involving the  $A$ -laplacian operator

$$(1.1) \quad \begin{cases} -\Delta_A u = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $A : [0, \infty) \rightarrow [0, \infty)$  is a convex function, vanishing at 0,  $A \in C^2((0, +\infty))$ , and  $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a Carathéodory function. The  $A$ -laplacian operator is defined by  $\Delta_A u = \operatorname{div} \left( A'(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right)$ . The properties of the function  $A$  guarantee that  $\Delta_A u$  makes sense also when  $|\nabla u| = 0$ .

A wide class of operators can be incorporated in (1.1). The  $p$ -Laplacian and the  $(p, q)$ -Laplacian,

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for which  $A(t) = t^p$  and  $A(t) = t^p + t^q$ ,  $t \geq 0$ , respectively, are the most known, but we can also consider functions like  $A(t) = (\sqrt{1+t^2} - 1)^\gamma$ , for  $t \geq 0$  and  $\gamma > 1$  or  $A(t) = t^p \lg(1+t)$ , for  $t \geq 0$  and  $p > 1$ . All the  $\Delta_A$  corresponding to the functions  $A$  considered above appear in many physical contests, like nonlinear elasticity and plasticity theory.

The presence of the gradient in the nonlinear term, called convection term, makes variational methods not applicable. Among the techniques used to study problems with a convection term, we cite: topological degree method ([BF, R]), theory of pseudomonotone operators ([GW]), fixed point theorems ([BNV, Z]), sub and super solution methods ([FMMT, FM, G, NS]), approximation methods ([T]), or a combination of the techniques above ([BaTo1, FMP, MW]).

We deal with existence, regularity and sign of the solutions to (1.1). Results in this direction can be found in [BF, FMMT, FMP, MW, NS, R, T, Z]. In the papers above there are various growth conditions on  $f$ , with respect to each variable, which make it necessary to use different methods to approach the problem, depending on the behavior of the convective term.

In all the papers cited above, the abstract framework is the classical Sobolev space  $W_0^{1,p}(\Omega)$  and the growth conditions with respect  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$  are of polynomial type. By contrast, we work in Orlicz spaces and take into account a class of operators which, although they depend only on the gradient, cannot be treated in the Sobolev spaces. Furthermore, this allows for  $f$  a wider choice than that seen above. Roughly speaking, for a problem with the  $p$ -Laplacian, a function  $f(x, s, \xi) = -c + \frac{|s|^{p^*-1}}{\lg(1+|s|)} + a(|s|)|\xi|^p$  (see Theorem 3.3) is allowed. This does not happen if we consider standard growths.

In [BF, FMP, R, Z] the authors establish the existence and the regularity of positive solutions for a problem involving the  $p$ -Laplacian. In [BF, R] the convection term is a continuous, nonnegative function with subcritical growth with respect to  $u$  and growth less than  $p$  with respect to  $\nabla u$ . In [FMP] the growth of the convection term is at most  $p-1$  with respect to  $u$  and  $\nabla u$ , while in [Z] the convection term is superlinear for  $(s, \xi) \rightarrow (0, 0)$  and its growth is at most  $p$  with respect to  $u$  and strictly less than  $p$  with respect to  $\nabla u$ . An existence and regularity result for the  $p$ -Laplacian with a convection term that can be singular at 0 can be found in [NS]. The existence of a suitable pair of sub and supersolutions plays a crucial role in their proof. In general, sub and supersolution methods allow to study also the case of a singular convection term, provided the interval of sub and supersolutions does not contain the singular point.

In [MW] the authors give also sign information on the solutions. They use sub and supersolution methods, in combination with variational techniques, for an operator that can be treated as the  $(p, q)$ -laplacian.

Existence and regularity results, for a general operator  $A(x, \nabla u)$  can be found in [NS, T], and in [FMMT] for  $A(x, u, \nabla u)$ . In [T] the convection term is a continuous function with growth less than  $p-1$  with respect to  $u$  and  $|\nabla u|$ , while in [FMMT, NS] the growth is at most  $p$  with respect to  $|\nabla u|$ .

Some more detailed comments on our new existence and regularity results to (1.1) (Theorems 3.5, 3.2 and 3.3). In Theorem 3.5 we assume the existence of an ordered pair of sub and supersolutions  $\underline{u}, \bar{u} \in W^{1,\infty}(\Omega)$  and use Theorem 3.6 in [BaTo1] to prove the existence of a regular solution to (1.1). The growth condition on  $f$ , in Theorem 3.5, is weaker than that used the Theorem 3.6 in [BaTo1]. A limit in the use of the method of sub and supersolutions is due to the fact that establishing their existence may not be easy. So we give two existence results, Theorems 3.2 and 3.3, where a unified hypothesis on  $f$  guarantees the existence of a suitable pair of sub and supersolutions and enable us to apply Theorem 3.5 to obtain the existence of a regular constant sign solution.

The paper is arranged as follows: in Section 2 we give the basic definitions and collect some auxiliary results. Our main theorems are proved in Section 3. Finally, in Section 4, we present some examples in which it is easy to verify the existence of constant sub and supersolutions.

## 2 Preliminaries

In this Section we give the main definitions on Young functions and define the Orlicz Sobolev spaces that we use in the sequel. For a comprehensive treatment of Young functions and Orlicz spaces we refer the reader to [Chl, KrRu, RR1, RR2]. We also collect some auxiliary results for the proof of the main theorems.

**Definition 2.1** *A function  $A : [0, \infty) \rightarrow [0, \infty]$  is called a Young function if it is convex, vanishes at 0, and is neither identically equal to 0, nor to infinity (in  $(0, +\infty)$ ).*

For Young functions

$$(2.1) \quad A(\lambda t) \leq \lambda A(t) \quad \text{for } \lambda \leq 1 \text{ and } t \geq 0.$$

**Definition 2.2** *The Young conjugate of a Young function  $A$  is the Young function  $\tilde{A}$  defined as*

$$\tilde{A}(s) = \sup\{st - A(t) : t \geq 0\} \quad \text{for } s \geq 0.$$

**Definition 2.3** *A Young function  $A$  is said to satisfy the  $\Delta_2$ -condition near infinity (briefly  $A \in \Delta_2$  near infinity) if it is finite valued and there exist two constants  $K \geq 2$  and  $M \geq 0$  such that*

$$(2.2) \quad A(2t) \leq KA(t) \quad \text{for } t \geq M.$$

**Definition 2.4** *The function  $A$  is said to satisfy the  $\nabla_2$ -condition near infinity (briefly  $A \in \nabla_2$  near infinity) if there exist two constants  $K > 2$  and  $M \geq 0$  such that*

$$(2.3) \quad A(2t) \geq KA(t) \quad \text{for } t \geq M.$$

If (2.2) or (2.3) holds with  $M = 0$ , then  $A$  is said to satisfy the  $\Delta_2$ -condition (globally), or the  $\nabla_2$ -condition (globally), respectively. Given a Young function  $A \in C^1([0, +\infty))$ , define the quantities

$$(2.4) \quad p_A = \inf_{t>0} \frac{t \cdot A'(t)}{A(t)}, \quad \text{and} \quad q_A = \sup_{t>0} \frac{t \cdot A'(t)}{A(t)}.$$

The conditions

$$p_A > 1 \quad \text{and} \quad q_A < +\infty$$

are equivalent to the fact that  $A \in \nabla_2 \cap \Delta_2$  (globally).

We give basic definitions and the main properties on the Orlicz spaces. Let  $\Omega$  be a measurable set in  $\mathbb{R}^n$ , with  $n \geq 1$ . Given a Young function  $A$ , the Orlicz space  $L^A(\Omega)$  is the set of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  such that the Luxemburg norm

$$\|u\|_{L^A(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(\frac{1}{\lambda}|u|\right) dx \leq 1 \right\}$$

is finite. The functional  $\|\cdot\|_{L^A(\Omega)}$  is a norm on  $L^A(\Omega)$ , and the latter is a Banach space (see [Ad]). If  $A$  is a Young function, then a generalized Hölder inequality

$$(2.5) \quad \int_{\Omega} |uv| dx \leq 2\|u\|_{L^A(\Omega)}\|v\|_{L^{\tilde{A}}(\Omega)}$$

holds for every  $u \in L^A(\Omega)$  and  $v \in L^{\tilde{A}}(\Omega)$ .

If  $A \in \Delta_2$  globally (or  $A \in \Delta_2$  near infinity and  $\Omega$  has finite measure) then

$$(2.6) \quad \int_{\Omega} A(k|u|)dx < +\infty \quad \text{for all } u \in L^A(\Omega), \quad \text{all } k \geq 0.$$

Let  $\Omega$  be an open set in  $\mathbb{R}^n$  with  $|\Omega| < \infty$ . The isotropic Orlicz-Sobolev spaces  $W_0^{1,A}(\Omega)$  is defined as

$$W_0^{1,A}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \\ \text{is weakly differentiable in } \mathbb{R}^n, |u|, |\nabla u| \in L^A(\Omega)\}.$$

The space  $W_0^{1,A}(\Omega)$  equipped with the norm

$$\|u\|_{W_0^{1,A}(\Omega)} = \| |u|, |\nabla u| \|_{L^A(\Omega)}.$$

is a Banach space. This norm is equivalent to the standard one

$$\|u\|_{W_0^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \| |\nabla u| \|_{L^A(\Omega)}.$$

For the Young function  $A$  in (1.1), we assume:

[ A1 ]  $A \in C^2(]0, +\infty[)$  (this implies  $A' \in C^1(]0, +\infty[)$ );

[ A2 ] there exist two positive constants  $\delta, g_0 > 0$  such that

$$(2.7) \quad \delta \leq \frac{tA''(t)}{A'(t)} \leq g_0 \quad \text{for } t > 0.$$

We point out that, (2.7) guarantees that  $A'(0) = 0$  and  $A \in \nabla_2 \cap \Delta_2$  globally. In fact integrating (2.7),

$$\left(\frac{t}{t_0}\right)^\delta \leq \frac{A'(t)}{A'(t_0)} \leq \left(\frac{t}{t_0}\right)^{g_0} \quad \text{for } t > t_0 > 0.$$

Choosing  $t = 2t_0$

$$2^\delta A'(t_0) \leq A'(2t_0) \leq 2^{g_0} A'(t_0) \quad \text{for } t > t_0 > 0.$$

Thus

$$A(2t) = \int_0^{2t} A'(\tau) d\tau = 2 \int_0^t A'(2s) ds \leq 2^{g_0+1} \int_0^t A'(s) ds = 2^{g_0+1} A(t) \quad \text{for all } t > 0,$$

and

$$A(2t) = \int_0^{2t} A'(\tau) d\tau = 2 \int_0^t A'(2s) ds \geq 2^{\delta+1} \int_0^t A'(s) ds = 2^{\delta+1} A(t) \quad \text{for all } t > 0.$$

We investigate the existence and the regularity of the solutions to problem (1.1). The proof of the existence is based on sub and supersolution methods, while the main tool for the regularity is Theorem 1.7 of [Li1] and the remark immediately after the statement (see also [Li, Theorem 1]), that we recall below.

**Proposition 2.5** (see [Li1, Theorem 1.7]) *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with  $C^{1,\alpha}$  boundary, for some  $0 < \alpha \leq 1$ . Let  $g : [0, +\infty[ \rightarrow [0, +\infty[$  be a  $C^1$ , increasing function, satisfying  $0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0$ , for  $t > 0$ , and let  $G(t) = \int_0^t g(\tau) d\tau$ . Consider the problem*

$$\operatorname{div}(\mathcal{A}(x, u, \nabla u)) + B(x, u, \nabla u) = 0 \text{ in } \Omega.$$

*Suppose  $\mathcal{A}$  and  $B$  satisfy the structure conditions (here  $a_{ij}(x, z, \eta) = \frac{\partial \mathcal{A}^i}{\partial \eta_j}$ )*

- (a)  $\sum_{i,j=1}^n a_{ij}(x, z, \eta) \xi_i \xi_j \geq \frac{g(|\eta|)}{|\eta|} |\xi|^2$
- (b)  $\sum_{i,j=1}^n |a_{ij}(x, z, \xi)| \leq \Lambda \frac{g(|\xi|)}{|\xi|}$
- (c)  $|\mathcal{A}(x, z, \xi) - \mathcal{A}(y, w, \xi)| \leq \Lambda_1(1 + g(|\xi|)(|x - y|^2 + |z - w|^2))$
- (d)  $|B(x, z, \xi)| \leq \Lambda_1(1 + g(|\xi|)|\xi|),$

for some positive constants  $\Lambda, \Lambda_1, M_0$ , for all  $x$  and  $y \in \Omega$ , for all  $z, w \in [-M_0, M_0]$  and for all  $\xi \in \mathbb{R}^n$ . Then, any solution  $u \in W^{1,G}(\Omega)$ , with  $|u| \leq M_0$  in  $\Omega$ , is  $C^{1,\beta}(\bar{\Omega})$  for some positive  $\beta$ . Moreover

$$(2.8) \quad \|u\|_{C^{1,\beta}(\bar{\Omega})} \leq C(\alpha, \Lambda, \delta, g_0, n, \Lambda_1, g(1), \Omega, M_0)$$

**Lemma 2.6** *Let  $A$  be a Young function satisfying [A1] and [A2]. Put  $\Phi(\xi) = A(|\xi|)$ . Then*

$$(2.9) \quad \sum_{i,j=1}^n \partial_{ij} \Phi(\eta) \xi_i \xi_j \geq \min\{\delta, 1\} \frac{A'(|\eta|)}{|\eta|} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n \setminus \{0\},$$

$$(2.10) \quad \sum_{i,j=1}^n |\partial_{ij} \Phi(\eta)| \leq [2 \max\{|\delta - 1|, |g_0 - 1|\} + n] \frac{A'(|\eta|)}{|\eta|} \quad \text{for all } \eta \in \mathbb{R}^n \setminus \{0\},$$

and  $\nabla \Phi = \mathcal{A}$  satisfies conditions a–c in Proposition 2.5, with  $g(t) = \min\{\delta, 1\} A'(t)$ ,  $\Lambda = \frac{\lambda}{\min\{\delta, 1\}}$ , where  $\lambda = 2 \max\{|\delta - 1|, |g_0 - 1|\} + n$ .

**Proof** From (2.7)

$$(\delta - 1) \frac{A'(|\eta|)}{|\eta|} \leq A''(|\eta|) - \frac{A'(|\eta|)}{|\eta|} \leq (g_0 - 1) \frac{A'(|\eta|)}{|\eta|} \quad \text{for all } \eta \in \mathbb{R}^n \setminus \{0\}.$$

Also,  $\partial_i \Phi(\eta) = A'(|\eta|) \frac{\eta_i}{|\eta|}$ , and

$$\partial_{ij} \Phi(\eta) = A''(|\eta|) \frac{\eta_i \eta_j}{|\eta|^2} + A'(|\eta|) \left( \frac{\delta_{ij}}{|\eta|} - \frac{\eta_i \eta_j}{|\eta|^3} \right) \quad \text{for all } \eta \in \mathbb{R}^n \setminus \{0\}.$$

Thus

$$\begin{aligned} \sum_{i,j=1}^n \partial_{ij} \Phi(\eta) \xi_i \xi_j &= \sum_{i,j=1}^n \left( A''(|\eta|) \frac{\xi_i \eta_i \xi_j \eta_j}{|\eta|^2} + A'(|\eta|) \frac{\delta_{ij} \xi_i \xi_j}{|\eta|} - A'(|\eta|) \frac{\xi_i \eta_i \xi_j \eta_j}{|\eta|^3} \right) \\ &= \left( \frac{A''(|\eta|)}{|\eta|^2} - \frac{A'(|\eta|)}{|\eta|^3} \right) (\xi, \eta)^2 + \frac{A'(|\eta|)}{|\eta|} |\xi|^2 \\ &\geq (\delta - 1) \frac{A'(|\eta|)}{|\eta|^3} (\xi, \eta)^2 + \frac{A'(|\eta|)}{|\eta|} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

If  $\delta \geq 1$

$$(2.11) \quad \sum_{i,j=1}^n \partial_{ij} \Phi(\eta) \xi_i \xi_j \geq \frac{A'(|\eta|)}{|\eta|} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n \setminus \{0\}.$$

If  $\delta < 1$

$$(2.12) \quad \begin{aligned} \sum_{i,j=1}^n \partial_{ij} \Phi(\eta) \xi_i \xi_j &\geq (\delta - 1) \frac{A'(|\eta|)}{|\eta|^3} |\xi|^2 |\eta|^2 + \frac{A'(|\eta|)}{|\eta|} |\xi|^2 \\ &= \delta \frac{A'(|\eta|)}{|\eta|} |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

Putting together (2.11) and (2.12), we get (2.9).

Consider

$$|\partial_{ij} \Phi(\eta)| \leq \frac{|\eta_i| |\eta_j|}{|\eta|^2} \left| A''(|\eta|) - \frac{A'(|\eta|)}{|\eta|} \right| + \delta_{ij} \frac{A'(|\eta|)}{|\eta|} \text{ for all } \eta \in \mathbb{R}^n \setminus \{0\}.$$

Thus

$$\begin{aligned} \sum_{i,j=1}^n |\partial_{ij} \Phi(\eta)| &\leq \frac{(\sum_{i=1}^n |\eta_i|)^2}{|\eta|^2} \left| A''(|\eta|) - \frac{A'(|\eta|)}{|\eta|} \right| + n \frac{A'(|\eta|)}{|\eta|} \\ &\leq [2 \max\{|\delta - 1|, |g_0 - 1|\} + n] \frac{A'(|\eta|)}{|\eta|} \text{ for all } \eta \in \mathbb{R}^n \setminus \{0\}. \end{aligned}$$

So (2.10) holds with  $\lambda = 2 \max\{|\delta - 1|, |g_0 - 1|\} + n$ .  $\square$

### 3 Main results

In this section first we give two existence and regularity results (Theorems 3.2 and 3.3), in which we assume a global growth condition on  $f$ , unilateral with respect to  $s \in \mathbb{R}$ . In Theorem 3.2 we require that  $f$  satisfies some conditions for  $(x, s, \xi) \in \Omega \times [0, +\infty) \times \mathbb{R}^n$  and obtain the existence of a nonnegative solution. Similarly, in Theorem 3.3,  $f$  satisfies some conditions for  $(x, s, \xi) \in \Omega \times (-\infty, 0] \times \mathbb{R}^n$  that guarantee the existence of a non-positive solution.

Here is the definition of weak solution to (1.1).

**Definition 3.1** *A function  $u \in W_0^{1,A}(\Omega)$  is a weak solution to problem (1.1) if*

$$\int_{\Omega} A'(|\nabla u|) \cdot \frac{\nabla u}{|\nabla u|} \cdot \nabla v dx = \int_{\Omega} f(x, u, \nabla u) v dx$$

for all  $v \in W_0^{1,A}(\Omega)$ .

For the first two Theorems, we assume

$$(\mathcal{H}) : \begin{cases} a : [0, +\infty[ \rightarrow [0, +\infty[ \text{ is a locally essentially bounded function;} \\ \rho_1, \rho_2 : \Omega \rightarrow [0, +\infty[ \text{ are two measurable functions, } \rho_1, \rho_2 \in L^\infty(\Omega) \text{ and} \\ \rho_2(x) > 0 \text{ on a set of positive measure;} \\ g_1, g_2 : [0, +\infty[ \rightarrow [0, +\infty[ \text{ are two non-decreasing functions such that } g_1(0) = g_2(0) = 0 \\ \text{and there exist } s_0 > 0, k_1 \in \left] 0, \omega_n^{\frac{1}{n}} |\Omega|^{-\frac{1}{n}} \right[ , \text{ such that } g_1(|s|)|s| \leq A(k_1|s|) \text{ for all } |s| \geq s_0. \end{cases}$$

Here  $\omega_n$  is the measure of the unit ball in  $\mathbb{R}^n$ .

**Theorem 3.2** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with  $C^{1,\alpha}$  boundary. Let  $A : [0, +\infty[ \rightarrow [0, +\infty[$  be a Young function, satisfying [A1] and [A2]. Let  $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a Carathéodory function fulfilling*

$$(3.1) \quad \rho_2(x) - g_2(s) - a(s)A'(|\xi|)|\xi| \leq f(x, s, \xi) \leq \rho_1(x) + g_1(s) \quad \text{for a.e. } x \in \Omega, \text{ all } s \geq 0, \text{ all } \xi \in \mathbb{R}^n.$$

*The functions  $a, \rho_1, \rho_2, g_1, g_2$  are as in  $(\mathcal{H})$ . Then problem (1.1) has a nontrivial, nonnegative solution  $u \in C_0^{1,\beta}(\overline{\Omega})$ .*

*If, in addition, there exist  $\bar{\delta} > 0$  and  $k_3 > 0$  such that  $g_2(s)s \leq A(k_3s)$  for every  $s \in (0, \bar{\delta})$ , then  $u > 0$  in  $\Omega$ .*

**Theorem 3.3** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with  $C^{1,\alpha}$  boundary. Let  $A : [0, +\infty[ \rightarrow [0, +\infty[$  be a Young function, satisfying [A1] and [A2]. Let  $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a Carathéodory function fulfilling*

$$(3.2) \quad -\rho_1(x) - g_1(|s|) \leq f(x, s, \xi) \leq -\rho_2(x) + g_2(|s|) + a(s)A'(|\xi|)|\xi| \quad \text{for a.e. } x \in \Omega, \text{ all } s \leq 0, \text{ all } \xi \in \mathbb{R}^n,$$

*where the functions  $a, \rho_1, \rho_2, g_1, g_2$  are as in  $(\mathcal{H})$ . Then problem (1.1) has a nontrivial, non-positive solution  $u \in C_0^{1,\beta}(\overline{\Omega})$ .*

*If, in addition, there exist  $\bar{\delta} > 0$  and  $k_3 > 0$  such that  $g_2(s)s \leq A(k_3s)$  for every  $s \in (0, \bar{\delta})$ , then  $u < 0$  in  $\Omega$ .*

**Remark 3.4** *In [BF], Theorem 1, the authors prove the existence of a positive solution for a problem with the  $p$ -Laplacian, and a convection term  $f$  satisfying the hypotheses of Theorem 3.3.*

For the proof of the Theorems above, we need an abstract existence result, where sub and supersolutions come into play.

The definition of sub and supersolution in general domains, for which a trace theory may not hold, can be found in [BaTo1]. Our hypotheses on  $\Omega$  allow to adopt the classical definition (see [Ci4, Theorem 3.1]).

We say that  $\bar{u} \in W^{1,A}(\Omega)$  is a supersolution to (1.1) if  $\bar{u}|_{\partial\Omega} \geq 0$  (in the sense of traces) and

$$\int_{\Omega} A'(|\nabla \bar{u}|) \cdot \frac{\nabla \bar{u}}{|\nabla \bar{u}|} \cdot \nabla v dx \geq \int_{\Omega} f(x, \bar{u}, \nabla \bar{u}) v dx$$

for all  $v \in W_0^{1,A}(\Omega)$ ,  $v \geq 0$  a.e. in  $\Omega$ .

We say that  $\underline{u} \in W^{1,A}(\Omega)$  is a subsolution to (1.1) if  $\underline{u}|_{\partial\Omega} \leq 0$  (in the sense of traces) and

$$\int_{\Omega} A'(|\nabla \underline{u}|) \cdot \frac{\nabla \underline{u}}{|\nabla \underline{u}|} \cdot \nabla v dx \leq \int_{\Omega} f(x, \underline{u}, \nabla \underline{u}) v dx$$

for all  $v \in W_0^{1,A}(\Omega)$ ,  $v \geq 0$  a.e. in  $\Omega$ .

For the next Theorem we assume that problem (1.1) has a subsolution and supersolution,  $\underline{u}, \bar{u} \in W^{1,\infty}(\Omega)$ , with  $\underline{u}(x) < \bar{u}(x)$  for all  $x \in \Omega$ . Also,  $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a Carathéodory function satisfying the following growth condition:

(H) there exists a function  $\sigma \in L^\infty(\Omega)$  and a constant  $a > 0$ , such that

$$|f(x, s, \xi)| \leq \sigma(x) + aA'(|\xi|)|\xi| \quad \text{for a.e. } x \in \Omega, \text{ all } s \in [\underline{u}(x), \bar{u}(x)], \text{ all } \xi \in \mathbb{R}^n.$$

The local condition on  $f$ , with respect to  $s$ , is sufficient for our purposes. The use of the method of sub and supersolutions requires an a priori analysis of the problem. Only once the existence of sub and supersolutions has been established does one proceed to search for the existence of a solution.

**Theorem 3.5** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with  $C^{1,\alpha}$  boundary. Let  $A : [0, +\infty[ \rightarrow [0, +\infty[$  be a Young function satisfying [A1] and [A2]. Let  $\underline{u}, \bar{u} \in W^{1,\infty}(\Omega)$  be as above and assume that  $f$  satisfies hypothesis (H). Then problem (1.1) admits at least a solution  $u \in C_0^{1,\beta}(\bar{\Omega})$ . Moreover  $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$  a.e in  $\Omega$ .*

**Proof.** Let  $M = \max\{\|\bar{u}\|_\infty, \|\underline{u}\|_\infty\}$  and  $R > \max\{\|\nabla \bar{u}\|_\infty, \|\nabla \underline{u}\|_\infty\}$ . Consider the truncated function  $f_R$  defined by

$$f_R(x, s, \xi) = \begin{cases} f(x, s, \xi) & \text{if } |\xi| \leq R, \\ f(x, s, \xi) \cdot \frac{A'(R)R}{A'(|\xi|)|\xi|} & \text{if } |\xi| > R, \end{cases}$$

and the problem

$$(3.3) \quad \begin{cases} -\Delta_A(u) = f_R(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In view of the choice of  $R$ ,  $\underline{u}$  and  $\bar{u}$  are a subsolution and a supersolution to (3.3) respectively. Using the monotonicity of  $A'$  we deduce that  $|f_R(x, s, \xi)| \leq \sigma(x) + aA'(R)R$ , for a.e.  $x \in \Omega$ , all  $s \in [\underline{u}(x), \bar{u}(x)]$ , all  $\xi \in \mathbb{R}^n$ . From Theorem 3.6 in [BaTo1] problem (3.3) admits a solution  $u \in W_0^{1,A}(\Omega)$  with  $\underline{u}(x) \leq u(x) \leq \bar{u}(x)$  a.e in  $\Omega$ . Thus  $u \in L^\infty(\Omega)$ .

The functions  $A$  and  $f$  satisfy the hypotheses of Proposition 2.5, with  $\Lambda_1 = \max\{\|\sigma\|_\infty, \min\{\delta, 1\}^{-1}a\}$  (see also Lemma 2.6). Since  $|f_R| \leq |f|$  the same holds for  $f_R$  whatever  $R$  is. Due to Proposition 2.5 there exist two positive constants  $0 < \beta \leq 1$  and  $C$ , independent from  $R$ , such that any solution to (3.3) belongs in  $C_0^{1,\beta}(\bar{\Omega})$  and  $\|u\|_{C_0^{1,\beta}(\bar{\Omega})} \leq C$ . Choosing  $R > C$  we deduce that  $u$  is a solution to (1.1).  $\square$

**Remark 3.6** *When the solution  $u$  has constant sign, it should be interest to verify if it is positive (or negative) in  $\Omega$ . The maximum principle by Pucci-Serrin (see [PS, Theorem 3.5]) is a powerful tool, as it ensures that (under some proper conditions on  $A$  and  $f$ ) any nonnegative solution to (1.1) is positive in  $\Omega$ . A quite standard situation occurs when  $f$  is bounded below by suitable monotone functions and  $A \in \Delta_2$  near 0, as the following corollary shows.*

**Corollary 3.7** *Under the hypotheses of Theorem 3.5, assume that  $f$  satisfies*

$$(3.4) \quad f(x, s, \xi) \geq -aA'(|\xi|) - b(s) \text{ for all } x \in \Omega, s > 0, \xi \in \mathbb{R}^n, |\xi| \leq 1,$$

where  $a > 0$ ,  $b : [0, +\infty[ \rightarrow [0, +\infty[$  is a function increasing in  $(0, \bar{\delta})$  (for some  $\bar{\delta} > 0$ ),  $b(0) = 0$ , and  $b(s) = \frac{A(ks)}{s}$  for  $s \in (0, \bar{\delta})$  and some  $k > 0$ . Then any nonnegative, nontrivial solution to (1.1) is positive.

**Proof.** Let  $u \in W_0^{1,A}(\Omega)$  be a nonnegative, nontrivial solution to (1.1). Theorem 3.5 ensures that  $u \in C_0^{1,\beta}(\bar{\Omega})$ . In order to prove that  $u > 0$  in  $\Omega$  we use Theorem 5.3.1 of [PS].

Conditions (A1)' and (A2) of the Theorem cited above hold, because  $A \in C^2((0, +\infty))$ ,  $A'(0) = 0$ , and  $s \mapsto A'(s)$  is strictly increasing. Conditions (F2) and (B1) are satisfied too.



It remains to verify condition (1.1.5) of Theorem 5.3.1 of [PS]. Put  $B(s) = \int_0^s b(t)dt$ . Due to the monotonicity of  $\frac{A(t)}{t}$ , for  $s \in (0, \bar{\delta})$ , it holds

$$B(s) = \int_0^s \frac{A(kt)}{t} dt \leq \int_0^s \frac{A(ks)}{s} dt = A(ks).$$

If  $h \in \mathbb{N}$  is such that  $k < 2^h$  then, in view of (2.2)

$$A(ks) \leq K^h A(s) \quad \text{for all } s \geq 0.$$

Let  $b_1 = \max\{p_A - 1, K^h\}$ . Then, for  $s \in (0, \bar{\delta})$ , using (2.1) and the inequality above

$$\begin{aligned} H(s) &= sA'(s) - A(s) \geq (p_A - 1)A(s) = \frac{p_A - 1}{b_1} b_1 A(s) \\ &\geq b_1 A\left(\frac{(p_A - 1)s}{b_1}\right) \geq K^h A\left(\frac{(p_A - 1)s}{b_1}\right) \geq A\left(\frac{k(p_A - 1)s}{b_1}\right) \geq B\left(\frac{(p_A - 1)s}{b_1}\right), \end{aligned}$$

or equivalently

$$H\left(\frac{b_1 s}{p_A - 1}\right) \geq B(s) \quad \text{for } 0 < s < \frac{b_1 \bar{\delta}}{p_A - 1}.$$

$H$  is increasing, so

$$\frac{b_1 s}{p_A - 1} \geq H^{-1}(B(s)) \quad \text{for } 0 < s < \frac{b_1 \bar{\delta}}{p_A - 1}.$$

Finally

$$(3.5) \quad \frac{1}{H^{-1}(B(s))} \geq \frac{p_A - 1}{b_1 s} \quad \text{for } 0 < s < \frac{b_1 \bar{\delta}}{p_A - 1}.$$

Integrating (3.5) from  $\varepsilon$  to  $s < \frac{b_1 \bar{\delta}}{p_A - 1}$  and passing to the limit as  $\varepsilon \rightarrow 0^+$  we obtain condition (1.1.5) of Theorem 5.3.1 of [PS]. Thus  $u > 0$  in  $\Omega$ .  $\square$

Now we accomplish with the proof of Theorems 3.2 and 3.3.

**Proof of Theorem 3.2.** From the proof of Theorem 3.3 of [BaTo1] we know that there exists a nontrivial solution  $\bar{u} \geq 0$ , to the problem

$$-\Delta_A(u) = \rho_1(x) + g_1(|u|).$$

Theorem 3 of [Ci2] guarantees that  $\bar{u}$  is bounded. Finally, from Proposition 2.5, we have that  $\bar{u} \in C_0^{1,\beta}(\bar{\Omega})$ . The inequalities in (3.1) show that  $\bar{u}$  is a supersolution to problem (1.1) and  $\underline{u} = 0$  is a subsolution to problem (1.1). The assumptions on  $\rho_2$  guarantee that  $\underline{u} = 0$  is not a solution. If we put  $\sigma(x) = \max\{\rho_1(x) + g_1(\bar{u}(x)), \rho_2(x) + g_2(\bar{u}(x))\}$  for a.e.  $x \in \Omega$ , then (3.1) leads to

$$|f(x, s, \xi)| \leq \sigma(x) + a(s)A'(|\xi|)|\xi| \quad \text{for a.e. } x \in \Omega, \text{ all } s \in [0, \bar{u}(x)], \text{ all } \xi \in \mathbb{R}^n.$$

Let  $I = [0, \sup_{\Omega} \bar{u}]$  and  $a = \|a\|_{L^\infty(I)}$ . Then  $a < +\infty$  and  $a(s) \leq a$  for a.e.  $s \in I$ . Let  $I_0 \subset I$  be a set of null measure, such that  $a(s) > a$  for all  $s \in I_0$ . For  $s \in I_0$  it holds

$$(3.6) \quad \begin{aligned} |f(x, s, \xi)| &= \lim_{t \rightarrow s} |f(x, t, \xi)| = \liminf_{t \rightarrow s} |f(x, t, \xi)| \leq \sigma(x) + \liminf_{t \rightarrow s} a(t)A'(|\xi|)|\xi| \\ &\leq \sigma(x) + aA'(|\xi|)|\xi| \quad \text{for a.e. } x \in \Omega, \xi \in \mathbb{R}^n. \end{aligned}$$

Thus

$$(3.7) \quad |f(x, s, \xi)| \leq \sigma(x) + aA'(|\xi|)|\xi| \quad \text{for a.e. } x \in \Omega, \quad s \in [0, \bar{u}(x)], \quad \xi \in \mathbb{R}^n.$$

So, from Theorem 3.5, problem (1.1) admits at least a nontrivial solution  $u \in C_0^{1,\beta}(\bar{\Omega})$  such that  $0 \leq u \leq \bar{u}$ .

Now we prove that, under the additional condition on  $g_2$ ,  $u > 0$  in  $\Omega$ . We set  $b(s) = \max\{g_2(s), \frac{A(k_3 s)}{s}\}$ , for  $s \geq 0$  and observe that the left inequality in (3.1) guarantees that we can apply Corollary 3.7.  $\square$

**Proof of Theorem 3.3.** It is enough to put  $f_1(x, s, \xi) = -f(x, -s, -\xi)$  and to use Theorem 3.2 for  $f_1$ .  $\square$

## 4 Examples

This section is devoted to some examples with different Young functions and various nonlinearities. In the first two examples we consider the problem

$$(4.1) \quad \begin{cases} -\operatorname{div} ( (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u ) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $1 < q < p < \infty$ . Here  $A(t) = \frac{t^p}{p} + \frac{t^q}{q}$  for all  $t \geq 0$ , and (2.7) holds with  $\delta = q - 1$ ,  $g_0 = p - 1$ .

**Example 4.1** Let  $a : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be two continuous functions and let  $h : \Omega \rightarrow \mathbb{R}$  be an essentially bounded function. Assume that there exist  $s_1, s_2 \in \mathbb{R}$ , such that  $g(s_1) = g(s_2) = 0$  for some  $s_1 < 0 < s_2$ ,  $g(s) \neq 0$  for all  $s \in ]s_1, s_2[$ , and  $|\{x \in \Omega : h > 0\}| > 0$ ,  $|\{x \in \Omega : h < 0\}| > 0$ .

Set

$$f(x, s, \xi) = g(s)h(x) + a(s)A'(|\xi|)(|\xi|) \quad \text{for } (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

Then  $u_1 = s_1$  and  $u_2 = s_2$  are a subsolution and a supersolution to (4.1), respectively. Also  $u \equiv 0$  is not a solution nor a sub or a supersolution and  $f$  satisfies condition (H) with  $\sigma(x) = |h(x)| \max_{[s_1, s_2]} |g(s)|$  and  $a = \max_{[s_1, s_2]} |a(s)|$ . By Theorem 3.5, problem (4.1) has a nontrivial solution  $u \in C_0^{1,\beta}(\bar{\Omega})$  with  $s_1 \leq u \leq s_2$  a.e. in  $\Omega$ .

**Example 4.2** Let  $a : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be two continuous functions and let  $h : \Omega \rightarrow \mathbb{R}$  be an essentially bounded function. Assume that  $h(x) \geq 0$  (or  $h(x) \leq 0$ ) in  $\Omega$ ,  $h(x) > 0$  on a set of positive measure,  $g(s_1) = 0$  for some  $s_1 > 0$ ,  $g(s) \neq 0$  for all  $s \in [0, s_1[$ , and  $g(0)h(x) \geq 0$  in  $\Omega$ .

Set

$$f(x, s, \xi) = g(s)h(x) + a(s)A'(|\xi|)(|\xi|) \quad \text{for } (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.$$

Then  $u_1 \equiv 0$  and  $u_2 \equiv s_1$  are a subsolution and a supersolution to (4.1), respectively. Also  $u \equiv 0$  is not a solution and  $f$  satisfies condition (H) with  $\sigma(x) = |h(x)| \max_{[0, s_1]} |g(s)|$  and  $a = \max_{[0, s_1]} |a(s)|$ . By Theorem 3.5, problem (4.1) has a nontrivial solution  $u \in [0, s_1]$ ,  $u \in C_0^{1,\beta}(\bar{\Omega})$ . From Theorem 5.3.1 in [PS],  $u > 0$  in  $\Omega$  (note that  $g(s)h(x) \geq 0$  in  $\Omega \times [0, s_1]$ ).

**Example 4.3** Consider the problem

$$(4.2) \quad \begin{cases} -\operatorname{div} (\lg(1 + |\nabla u|^q) |\nabla u|^{p-2} \nabla u) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with  $q > 0$ ,  $p > 1$ .

Let  $r \geq q$ ,  $\bar{\delta} > 0$  and define  $b : [0, +\infty[ \rightarrow [0, +\infty[$  as

$$b(s) = \begin{cases} s^{p+q-1} & \text{if } s \in [0, \bar{\delta}], \\ s^{p+r-1} & \text{if } s > \bar{\delta}. \end{cases}$$

Assume that  $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a Carathéodory function satisfying

(f<sub>0</sub>) there exists  $\sigma > 0$  such that  $f(x, \sigma, 0) \leq 0$  a.e. in  $\Omega$ ;

(f<sub>1</sub>)  $f(x, 0, 0) \geq 0$  a.e. in  $\Omega$ , with strict inequality on a set of positive measure;

(f<sub>2</sub>) there exist  $k > 0$  such that

$$f(x, s, \xi) \geq -k(|\xi|^{p-1} \lg(1 + |\xi|^q) + b(s)) \text{ for } x \in \Omega, \text{ all } s \geq 0 \text{ and all } \xi \in \mathbb{R}^n, \text{ with } |\xi| \leq 1.$$

(f<sub>3</sub>) there exists  $c > 0$  such that  $|f(x, s, \xi)| \leq c(1 + |\xi|^p \lg(1 + |\xi|^q))$  for all  $x \in \Omega$ ,  $s \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^n$ .

For the function  $A'$  in problem (4.2) condition (2.7) holds with  $\delta = p - 1$ ,  $g_0 = p - 1 + q$  and  $A(t) \approx t^{p+q}$  for  $t$  small. We can apply Theorem 3.5 and Corollary 3.7 to obtain the existence of a positive solution  $u \in C_0^{1,\beta}(\bar{\Omega})$ , and  $u \leq \sigma$  in  $\Omega$ .

The example above extends in two directions Theorem 6 of [FMMT]: it allows an higher growth for  $f$  and permit also the choice  $r = q$  in the lower bound for  $f$ .

**Example 4.4** Consider the problem

$$(4.3) \quad \begin{cases} -\operatorname{div} \left( \frac{|\nabla u|^{p-2} \nabla u}{\lg^q(1 + |\nabla u|)} \right) = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with  $p > 1$ ,  $p - q - 1 > 0$ . Let  $\rho \in L^\infty(\Omega)$  and  $g_1, g_2 : [0, +\infty[ \rightarrow [0, +\infty[$  be two unbounded, nondecreasing functions, such that  $g_1(0) = g_2(0) = 0$ . Also, let  $a_1, a_2 : \mathbb{R} \rightarrow [0, +\infty[$  be two locally essentially bounded functions and let  $c_1, c_2 > 0$ .

Assume that  $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a Carathéodory function satisfying

$$-c_1 + g_1(|s|) - a_1(s) \frac{|\xi|^p}{\lg^q(1 + |\xi|)} \leq f(x, s, \xi) \leq -c_2 + g_2(|s|)\rho(x) + a_2(s) \frac{|\xi|^p}{\lg^q(1 + |\xi|)}$$

for  $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ .

We show that problem (4.3) has a nontrivial solution  $u \leq 0$  in  $\Omega$ .

For the function  $A'$  in problem (4.3) condition (2.7) holds with  $\delta = p - 1 - q$ ,  $g_0 = p - 1$ . If  $k := \inf\{s > 0 : g_1(s) \geq c_1\}$ , then  $\underline{u} \equiv -k$  is a subsolution to (4.3), and  $\bar{u} \equiv 0$  is a supersolution but not a solution to (4.3). Let  $a = \max\{\|a_1\|_{L^\infty([-k, 0])}, \|a_2\|_{L^\infty([-k, 0])}\}$ ,  $\sigma(x) = \max\{c_1, -c_2 + g_2(k)\rho(x)\}$ .

Then

$$|f(x, s, \xi)| \leq \sigma(x) + aA'(|\xi|)|\xi| \text{ for } x \in \Omega, s \in [-k, 0], \xi \in \mathbb{R}^n.$$

By Theorem 3.5, problem (4.3) has a nontrivial solution  $u \in [-k, 0]$ .

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