Computational Management Science (2024) 21:24
https://doi.org/10.1007/s10287-024-00502-5

ORIGINAL PAPER

Notes on random optimal control equilibrium problem via stochastic inverse variational inequalities

Annamaria Barbagallo[1] · Bruno Antonio Pansera[2] · Massimiliano Ferrara[2,3]

Received: 23 November 2022 / Accepted: 18 January 2024

Abstract
The main objective of the paper is to analyze how policymakers influence the random oligopolistic market equilibrium problem. To this purpose, random optimal control equilibrium conditions are introduced. Since the random optimal regulatory tax is characterized by a stochastic inverse variational inequality, existence and well-posedness results on such an inequality are proved. At last a numerical example is discussed.

Keywords Random optimal control equilibrium problem · Stochastic inverse variational inequalities · Existence results · Well-posedness analysis

1 Introduction

Over the last few decades, random equilibrium problems have focused the interest of many scholars [see for example (Gwinner et al. 2021) and references cited therein]. The definition of random conditions in which the data are affected by a certain degree of uncertainty allows to model and analyze socio-physical phenomena

Bruno Antonio Pansera and Massimiliano Ferrara have been contributed equally to this work.

✉ Annamaria Barbagallo
annamaria.barbagallo@unina.it

Bruno Antonio Pansera
bruno.pansera@unirc.it

Massimiliano Ferrara
massimiliano.ferrara@unirc.it

1 Department of Mathematics and Applications "R. Caccioppoli", University of Naples Federico II, Complesso Universitario Monte Sant'Angelo, via Cintia, 80126 Naples, Italy

2 Department of Law, Economics and Human Sciences & Decisions_lab, University Mediterranea of Reggio Calabria, Via dell'Università 25, 89124 Reggio Calabria, Italy

3 Advanced Soft Computing Lab, Faculty of Engineering and Natural Sciences, Istanbul OKAN University, Istanbul, Turkey

Published online: 11 March 2024

in a more realistic way. The study of random problems has had a remarkable development because the scholars have relevated that the constraints or data of real phenomena are often variable over time in a non-regular and unpredictable manner. Examples in which that happens are unpredictable events or sudden accidents. For these reasons, we propose an oligopolistic market equilibrium model capable of managing random constraints.

In the early 1950s, Nash (1950, 1951) presented the so-called non-cooperative game. In Dafermos and Nagurney (1987), Nagurney (1998a, 1998b), Nagurney et al. (1994), the problem was addressed in the static case by using a finite dimensional variational approach. Barbagallo et al. (2009), considered the time dependence in the model obtaining its evolutionary variational equivalent formulation. Moreover the market has been described making use of the Lagrange multipliers in Barbagallo and Maugeri (2011). In Barbagallo and Mauro (2012a, 2012b), the dynamic model has been improved by considering production excesses and both production and demand excesses, respectively. In order to find approximate equilibrium solutions, a numerical method is analyzed in Barbagallo (2012). Then, in Barbagallo and Mauro (2014), the authors left the point of view of the producer whose aim is to maximize his profit, analyzing the policymaker one whose aim is to control exports of goods by imposing taxes or incentives. The optimal control model is characterized by an inverse variational inequality. In addition Barbagallo et al. (2013, 2016), considered the model in which the set of constraints depends on the expected equilibrium distribution. Such a problem is called elastic or with set adaptive constraint. In particular, in this model, the capacity constraint set is defined by means of a multifunction and the equilibrium conditions are expressed by an evolutionary quasi-variational inequality. Finally, in Barbagallo et al. (2021, 2023), the uncertainty in the oligopolistic market equilibrium problem is taken into account thanks to the introduction of random constraints. The random firms' point of view is governed by a random Nash principle which is expressed by a stochastic variational inequality in a Hilbert space setting. Thanks to the variational formulation the existence and the uniqueness of the equilibrium solution has been investigated [see also Dorta-González et al. (2004); Muu et al. (2008); Xian et al. (2004); Zhou et al. (2005)]. Furthermore, in Barbagallo et al. (2021), the random optimal control problem is introduced and characterized by a stochastic inverse variational inequality.

Now we are interested to study the existence and well-posedness of the random optimal control equilibirum problem. In particular, we present the producer's point of view of the random oligopolistic market equilibrium problem in presence of production excesses. We underline that the production excesses occur when, in a period of economic crisis, the firms cannot sell all the amounts of the commodity producted to the demand markets. The equilibrium condition of such a problem is expressed by a stochastic variational inequality. After that, the policymaker's point of view is discussed. In particular, control policies are considered by imposing higher taxes or subsidies in order to reduce or encourage the commodity exportations in a stochastic framework. Therefore, this model is a policymaker optimization problem and, in this setting, we establish the equivalence between the random taxes system that controls the commodity exportations and a stochastic inverse variational inequality. Thanks to this characterization, we show under which assumptions the random optimal control regulatory tax exists. Since the equivalence between the stochastic inverse variational inequality and a standard stochastic variational inequality is established, it is worth introducing a numerical scheme for solving the last one.

Numerical methods for variational inequalities in the deterministic setting have been extensively studied [see, for instance, Facchinei and Pang (2003)]. If the expected value of the operator is completly available, then stochastic variational inequalities can be solved by these methods. On the contrary, when such a value is not available, the sampling of the random variable and the use of values of the operator given a sample (the procedure is called a "stochastic oracle" call) are requested. In this situation, there are two methodologies for solving stochastic variational inequalities: sample average approximation and stochastic approximation. In this paper, we consider the stochastic approximation approach. The stochastic approximation methodology is a projection-type method where the exact mean of the expected value of the operator is replaced along the iterations by a random sample of the operator. This method causes a stochastic error in the trajectory of the method. We stress that the generated sequence is a stochastic process which updates iteratively according to the chosen projection algorithm and the sampling information used in each iteration. Therefore, asymptotic convergence of the stochastic approximation method guarantees a solution with total probability. The first analysis of the stochastic approximation approach has been recently carried out in Jiang and Xu (2008). The stochastic approximation methodology has been first proposed by Robbins and Monro (1951) for stochastic equations. After this fundamental work, such a methodology has been used by several scholars for solving stochastic variational inequalities [see, for instance, Juditsky et al. (2011), Koshal et al. (2013), Wang and Bertsekas (2016)]. In this paper, we propose a new stochastic approximation methodology which generates a sequence updating iteratively according to the projected reflected gradient algorithm and the sampling fixed in every iteration. Such a projection algorithm has been studied by Malitsky in the deterministic framework [see Malitsky (2015)]. In addition, starting form the convergence analysis in Malitsky (2015), we deduce that our numerical method has a solution with total probability.

Another purpose of the paper is to generalize the notion of well-posedness for a variational inequality, introduced in Lucchetti and Patrone (1981), to the class of stochastic inverse variational inequalities which express the random elastic oligopolistic market equilibrium conditions. The well-posedness has a crucial role in the study of optimization problems. The first scholar who introduced the concept of well-posedness was Tykhonov in Tykhonov (1966) for a global minimization problem, well-known as well-posedness. The well-posedness of a global minimization problem requires the existence and uniqueness of minimizer, and the convergence of every minimizing sequence to the unique minimizer. The concept of well-posedness can also be used in a constrained minimization problem in an abstract way. We highlight that the well-posedness of a constrained minimization problem requires that every minimizing sequence should be cointained in the constraint set. This sequence is called a generalized minimizing sequence for constrained minimization problems. Since a minimization problem is equivalent to a variational inequality under convexity and differentiability assumptions, it worth studing the well-posedness also for variational inequalities. Here, the stochastic formulation of generalized minimizing sequence is given. In addition, the concept of well-posedness to a stochastic inverse variational inequality is presented and some metric characterizations are established. Under suitable conditions, the equivalence between the well-posedness of a stochastic inverse variational inequality and the existence and uniqueness of its solution is proved. Finally, the well-posedness of a stochastic inverse variational inequality is characterized with the well-posedness of a suitable classical stochastic variational inequality.

The paper is organized as follows. In Sect. 2, the random oligopolistic market equilibrium model is presented. In particular, the equivalence between the random Nash equilibrium condition and a stochastic variational inequality is shown. Furthermore, the random optimal control problem is studied by using a stochastic inverse variational inequality. In Sect. 3 an existence result of the random optimal regulatory tax is proved. Section 4 is devoted to the well-posedness of the stochastic inverse variational inequality. Finally, in Sect. 5 a numerical example is provided.

2 The random optimal control equilibrium model

This section aims to explore the random optimal control equilibrium problem. Before going into the specific problem, let us first present in detail the firms' point of view of the oligopolistic market equilibrium problem in which the data are affected by a certain degree of uncertainty and production excesses occur.

Let $(\Omega, A, \mathbb{P})$ be a probability space. Let $L^2(\Omega, \mathbb{R}^k, \mathbb{P})$ be the Hilbert space of random vectors $v : \Omega \to \mathbb{R}^k$ such that the expectation

$$\mathbb{E}\|v\|^2 = \int_\Omega \|v(\omega)\|^2 d\mathbb{P}$$

is finite. We introduce the bilinear form on $(L^2(\Omega, \mathbb{R}^k, \mathbb{P}))^* \times L^2(\Omega, \mathbb{R}^k, \mathbb{P})$ as

$$\ll \phi, w \gg_{\mathbb{E}} = \int_\Omega \langle \phi(\omega), w(\omega) \rangle d\mathbb{P},$$

where $\phi \in (L^2(\Omega, \mathbb{R}^k, \mathbb{P}))^* = L^2(\Omega, \mathbb{R}^k, \mathbb{P})$, $w \in L^2(\Omega, \mathbb{R}^k, \mathbb{P})$ and

$$\langle \phi(\omega), w(\omega) \rangle = \sum_{l=1}^{k} \phi_l(\omega) w_l(\omega).$$

We consider m firms P_i, $i = 1, \dots, m$, which produce a homogeneous commodity and n demand markets Q_j, $j = 1, \dots, n$, which are generally spatially separated. Assume that the homogeneous commodity, produced by the m firms and consumed by the n markets, depends on random variables. We denote:

- the random variable expressing the nonnegative commodity output produced by the firm P_i by $p_i = p_i(\omega)$, $\omega \in \Omega$, $i = 1, \dots, m$;

- the random variable expressing the nonnegative demand for the commodity of the demand market Q_j by $q_j = q_j(\omega), \omega \in \Omega, j = 1, \dots, n$;
- the random variable expressing the nonnegative commodity shipment between the supply producer P_i and the demand market Q_j by $x_{ij} = x_{ij}(\omega), \omega \in \Omega$, $i = 1, \dots, m, j = 1, \dots, n$;
- the random variable expressing the nonnegative strategy of the firm P_i by $x_i(\omega) = (x_{i1}(\omega), \dots, x_{in}(\omega)), \omega \in \Omega, i = 1, \dots, m$;
- the random variable expressing the nonnegative production excess for the commodity of the firm P_i by $\varepsilon_i = \varepsilon_i(\omega), \omega \in \Omega, i = 1, \dots, m$.

For technical reasons, we assume that the random commodities belong to the Hilbert space $L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$.

We suppose that the following feasibility conditions hold:

$$p_i(\omega) = \sum_{j=1}^{n} x_{ij}(\omega) + \varepsilon_i(\omega), \quad i = 1, \dots, m, \ \mathbb{P} - \text{a.s.}, \tag{1}$$

Thus the random quantity produced by each firm P_i has to be equal to the random commodity shipments from that firm to all the demand markets plus the random production excesses.

Since the random production excesses are nonnegative, we can rewrite (1), as

$$\sum_{j=1}^{n} x_{ij}(\omega) \le p_i(\omega), \quad i = 1, \dots, m, \ \mathbb{P} - \text{a.s.} \tag{2}$$

Moreover, we assume that there exist two nonnegative random variables $\underline{x}, \overline{x} \in L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ such that

$$0 \le \underline{x}_{ij}(\omega) \le x_{ij}(\omega) \le \overline{x}_{ij}(\omega), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \mathbb{P} - \text{a.s.} \tag{3}$$

Hence, the set of feasible distributions $x \in L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ is

$$\mathbb{K}^* = \Big\{ x \in L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P}) : \quad 0 \le \underline{x}_{ij}(\omega) \le x_{ij}(\omega) \le \overline{x}_{ij}(\omega), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \mathbb{P} - \text{a.s.}, \quad \sum_{j=1}^{n} x_{ij}(\omega) \le p_i(\omega), \quad i = 1, \dots, m, \ \mathbb{P} - \text{a.s.} \Big\}, \tag{4}$$

which is a convex closed bounded subset of $L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$.

We introduce:

- the random variable f_i denoting the production cost for each firm P_i, such that $f_i = f_i(\omega, x(\omega)), \omega \in \Omega, i = 1, \dots, m$;
- the random variable d_j denoting the demand price of the commodity for each demand market Q_j, such that $d_j = d_j(\omega, x(\omega)), \omega \in \Omega, j = 1, \dots, n$;

- the random variable g_i expressing the storage cost of the commodity produced by the firm P_i, such that $g_i = g_i(\omega, x(\omega))$, $\omega \in \Omega, i = 1, \dots, m$;
- the random variable c_{ij} expressing the transaction cost, which includes the transportation cost associated with trading the commodity between the firm P_i and the demand market Q_j, such that $c_{ij} = c_{ij}(\omega, x(\omega))$, $\omega \in \Omega, i = 1, \dots, m$, $j = 1, \dots, n$;
- the random variable η_{ij} expressing the supply or resource tax imposed on the supply market P_i for the transaction with the demand market Q_j, such that $\eta_{ij} = \eta_{ij}(\omega)$, $\omega \in \Omega, i = 1, \dots, m$, $j = 1, \dots, n$;
- the random variable λ_{ij} expressing the incentive pay imposed on the supply market P_i for the transaction with the demand market Q_j, such that $\lambda_{ij} = \lambda_{ij}(\omega)$, $\omega \in \Omega, i = 1, \dots, m$, $j = 1, \dots, n$;
- the random variable h_{ij} expressing the difference between the supply tax and the incentive pay imposed on the supply market P_i for the transaction with the demand market Q_j, namely $h_{ij} = h_{ij}(\omega) = \eta_{ij}(\omega) - \lambda_{ij}(\omega)$, $\omega \in \Omega$, $i = 1, \dots, m$, $j = 1, \dots, n$.

We underline that $\eta, \lambda \in L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ and, hence, also $h \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$.

The profit v_i of the firm P_i is, then, given by

$$\begin{aligned} v_i(\omega, x(\omega)) = & \sum_{j=1}^{n} d_j(\omega, x(\omega)) x_{ij}(\omega) - f_i(\omega, x(\omega)) - \sum_{j=1}^{n} c_{ij}(\omega, x(\omega)) x_{ij}(\omega) \\ & - g_i(\omega, x(\omega)) - \sum_{j=1}^{n} h_{ij}(\omega) x_{ij}(\omega), \quad i = 1, \dots, m, \ \mathbb{P} - \text{a.s.}, \end{aligned}$$

namely it is equal to the price which the demand markets are disposed to pay minus the production cost, the transportation cost, the storage cost and the taxes. These kind of costs all together determine the total cost must be considered.

Assuming that the profit function v_i is continuously differentiable for each $i = 1, \dots, m$, let us indicate with $\nabla_D v = \left(\frac{\partial v_i}{\partial x_{ij}} \right)_{\substack{i = 1, \dots, m \\ j = 1, \dots, n}}$. Let us assume that:

1. $\nabla_D v$ is a Carathéodory mapping and there exists $b \in L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P})$ such that

$$\left\| \nabla_D v(\omega, x(\omega)) \right\| \leq b(\omega) \| x(\omega) \|, \quad \forall x \in L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P}), \ \mathbb{P} - \text{ a.s. };$$

2. v_i is pseudoconcave[1] with respect to the variables $x_i, i = 1, \dots, m$.

The firms supply the commodity in a noncooperative fashion, namely each one tries to maximize its own profit function considered the optimal distribution pattern for the other firms. Hence, we state the random equilibrium condition by means of the random Nash principle.

[1] A function v_i, continuously differentiable, is called *pseudoconcave* with respect to $x_i, i = 1, \dots, m$, [see Mangasarian (1965)] if and only if

$$\left\langle \frac{\partial v_i}{\partial x_i}(x_1, \dots, x_i, \dots, x_m), x_i - y_i \right\rangle \geq 0 \ \Rightarrow \ v_i(x_1, \dots, x_i, \dots, x_m) \geq v_i(x_1, \dots, y_i, \dots, x_m).$$

Definition 1 The feasible mapping $x^* \in \mathbb{K}$ is a random oligopolistic market equilibrium distribution if and only if for each $i = 1, \dots, m$ and $\mathbb{P}$-a.s. it results

$$v_i(\omega, x^*(\omega)) \geq v_i(\omega, x_i(\omega), x^*_{-i}(\omega)), \tag{5}$$

where $x^*_{-i}(\omega) = (x^*_1(\omega), \dots, x^*_{i-1}(\omega), x^*_{i+1}(\omega), \dots, x^*_m(\omega))$, $i = 1, \dots, m$, $\mathbb{P}$-a.s., and $x_i(\omega) = (x_{i1}(\omega), \dots, x_{in}(\omega))$, $i = 1, \dots, m$, $\mathbb{P}$-a.s.

It is possible to prove that under Assumptions 1 and 2, Definition 1 is equivalently expressed by the following stochastic variational inequality [see Barbagallo et al. (2021)]

$$\text{Find } x^* \in \mathbb{K}: \quad \ll -\nabla_D v(x^*), x - x^* \gg_{\mathbb{E}} \geq 0, \quad \forall x \in \mathbb{K}, \tag{6}$$

namely

$$\text{Find } x^* \in \mathbb{K}: \quad \int_\Omega \sum_{i=1}^m \sum_{j=1}^n \frac{\partial v_i(\omega, x^*(\omega))}{\partial x_{ij}} \left(x_{ij}(\omega) - x^*(\omega) \right) d\mathbb{P} \geq 0, \quad \forall x \in \mathbb{K}.$$

In the sequel we present the random optimal control problem. Let us start to remark that here the random variable h has a different meaning. Precisely, while h is a fixed parameter in the producers' perspective, it is a variable in the policymaker's one.

Let $x(h) = x(\omega, h(\omega))$ be the random function of regulatory taxes, with $h \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$, $\mathbb{P}$-a.s. We assume that

(a) $x : \Omega \times L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is a Carathéodory function and there exists a function $\gamma \in L^2(\Omega, \mathbb{P})$ such that

$$\|x(\omega, h(\omega))\| \leq \gamma(\omega) + \|h(\omega)\|, \quad \forall h \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}), \ \mathbb{P} - \text{a.s.} \tag{7}$$

The set of feasible states is given by

$$W = \Big\{ w \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) : \quad 0 \leq \underline{x}_{ij}(\omega) \leq w_{ij}(\omega) \leq \overline{x}_{ij}(\omega), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \mathbb{P} - \text{a.s.} \Big\}. \tag{8}$$

Definition 2 A random regulatory tax $h^* \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is a random optimal regulatory tax if $x(h^*) \in W$ and, for $i = 1, \dots, m$, $j = 1, \dots, n$, $\mathbb{P}$-a.s., the following implications hold:

$$\begin{aligned} x_{ij}(\omega, h^*(\omega)) = \underline{x}_{ij}(\omega) \ &\Rightarrow \ h^*_{ij}(\omega) \leq 0, \\ \underline{x}_{ij}(\omega) < x_{ij}(\omega, h^*(\omega)) < \overline{x}_{ij}(\omega) \ &\Rightarrow \ h^*_{ij}(\omega) = 0, \\ x_{ij}(\omega, h^*(\omega)) = \overline{x}_{ij}(\omega) \ &\Rightarrow \ h^*_{ij}(\omega) \geq 0. \end{aligned}$$

Definition 2 means that the random optimal regulatory tax h^* is such that the corresponding state $x(h^*)$ has to satisfy capacity constraints, namely $x(h^*) \in W$. Furthermore if $x_{ij}(\omega, h^*(\omega)) = \underline{x}_{ij}(\omega)$, then the random exportations must be encouraged, namely the random taxes must be less than or equal to the random incentive pays. If $x_{ij}(\omega, h^*(\omega)) = \overline{x}_{ij}(\omega)$, then the random exportations must be reduced, hence the random taxes must be greater than or equal to the random incentive pays. At last if $\underline{x}_{ij}(\omega) < x_{ij}(t, h^*(\omega)) < \overline{x}_{ij}(\omega)$ is satisfied, so the random taxes must be equal to the random incentive pays.

Definition 2 is characterized by the following stochastic inverse variational inequality [see Barbagallo et al. (2021)]

$$\begin{aligned} &\text{Find } h^* \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) : \quad x(h^*) \in W, \\ &\int_\Omega \sum_{i=1}^m \sum_{j=1}^n \big(w_{ij}(\omega) - x_{ij}(\omega, h^*(\omega))\big) h^*_{ij}(\omega) d\mathbb{P} \leq 0, \quad \forall w \in W. \end{aligned} \tag{9}$$

Let us set $Z = L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) \times W$ and indicate with

$$z(\omega) = \begin{pmatrix} h(\omega) \\ w(\omega) \end{pmatrix} \in Z.$$

Therefore, we consider the mapping $F : \Omega \times Z \to L^2(\Omega, \mathbb{R}^{2mn}, \mathbb{P})$, defined as

$$F(\omega, z(\omega)) = \begin{pmatrix} w(\omega) - x(\omega, h(\omega)) \\ -h(\omega) \end{pmatrix}, \quad \forall z \in Z, \ \mathbb{P} - \text{a.s.}$$

we note that Z is a closed convex and unbounded subset of $L^2(\Omega, \mathbb{R}^{2mn}, \mathbb{P})$. The following result holds (see Barbagallo et al. (2021)).

Theorem 1 *The stochastic inverse variational inequality* (9) *is equivalent to the following stochastic variational inequality*

$$\text{Find } z^* \in Z : \quad \int_\Omega \sum_{r=1}^{2m} \sum_{s=1}^n F_{rs}(\omega, z^*(\omega))\big(z_{rs}(\omega) - z^*_{rs}(\omega)\big) d\mathbb{P} \geq 0, \quad \forall z \in Z. \tag{10}$$

3 Existence results

In this section we show an existence result for the stochastic inverse variational inequality (9). For this purpose, we suppose that

(b) $x : \Omega \times L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is anti-monotone with respect to h, namely

$$\langle h_1(\omega) - h_2(\omega), x(\omega, h_1(\omega)) - x(\omega, h_2(\omega))\rangle \leq 0,$$
$$\forall h_1, h_2 \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}),\ \mathbb{P} - \text{a.s.};$$

(c) There exists a constant $M > 0$ such that for any $h \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$, with $\|h\|_{L^2(\Omega,\mathbb{R}^{mn},\mathbb{P})} > M$, it results

$$\int_\Omega \langle h(\omega), w_0^{proj}(\omega) - x(\omega, h(\omega))\rangle d\mathbb{P} \geq 0, \tag{11}$$

where w_0^{proj} is the projection of $w_0(\omega) = x(\omega, 0)$ into the feasible set W, namely

$$w_0^{proj}(\omega) = \begin{cases} \underline{x}(\omega), & \text{if } w_0(\omega) \leq \underline{x}(\omega),\ \mathbb{P} - \text{a.s.}, \\ w_0(\omega), & \text{if } \underline{x}(\omega) < w_0(\omega) < \overline{x}(\omega),\ \mathbb{P} - \text{a.s.}, \\ \overline{x}(\omega), & \text{if } w_0(\omega) \geq \overline{x}(\omega),\ \mathbb{P} - \text{a.s.} \end{cases}$$

Assumption (11) implies that if we assume the random price large enough ($w_0(\omega)$ positive enough for $w_0(\omega) \leq \overline{x}$ and negative enough for $w_0(\omega) \leq \underline{x}$), then the resulted load curve $x(\omega, h(\omega))$ will be strictly controlled in the interior of the feasible set W. More precisely, since $x(h)$ is nonincreasing with respect to h, which is a result of the market law, if we take $h_{ij} < 0$ for those i with $w_{ij}^0 \geq \overline{x}_{ij}$, certainly $x_{ij}(h) \geq \overline{x}_{ij}$ will not be changed; if we increase h_{ij} from 0 to a value large enough, then $x_{ij}(h) \leq \overline{x}_{ij}$ will happen. As such, we assume reasonably that there exists $M > 0$ such that if $\text{abs}\,(h_{ij}) > M$ then it holds that $h_{ij}(\overline{x}_{ij} - x_{ij}(h)) \geq 0$, for every $i = 1, \dots, m$, $j = 1, \dots, n$, with $w_{ij}^0 \geq \overline{x}_{ij}$. Similarly, for those i and j with $w_{ij}^0 \leq \underline{x}_{ij}$, if we take $h_{ij} > 0$, then $x_{ij}(h) \leq \underline{x}_{ij}$ will not vary; if h_{ij} is decreased from zero to a sufficiently negative value, then $x_{ij}(h) \geq \underline{x}_{ij}$ will happen. In this case, we may assume that there exists $M > 0$ such that if $\text{abs}\,(h_{ij}) > M$ then it holds that $h_{ij}(\underline{x}_{ij} - x_{ij}(h)) \geq 0$, for every $i = 1, \dots, m$, $j = 1, \dots, n$, with $w_{ij}^0 \leq \underline{x}_{ij}$. For those i and j such that $\underline{x}_{ij} < w_{ij}^0 < \overline{x}_{ij}$, considering the monotonicity of $x(h)$ with respect to h, it is obvious that $h_{ij}(w_{ij}^0 - x_{ij}(h)) \geq 0$.

Making use of Theorem 3 in Barbagallo and Guarino Lo Bianco (2023), we can establish the following existence result.

Theorem 2 *Let us suppose that conditions* (*a*), (*b*) *and* (*c*) *hold. Then the stochastic inverse variational inequality* (9) *admits a solution.*

3.1 Stochastic approximation method

Since Theorem 1 establishes the equivalence between the stochastic inverse variational inequality (9) and the stochastic variational inequality (10), it is also worth investigating on (10).

Stochastic variational inequalities have been studied not only for the theoretical point of view but also for what concerns numerical aspects. The first stochastic approximation methods was proposed by Jiang and Xu (2008). The algorithm iteratively updates z_n according to the formula:

$$z_{n+1} = P_Z[z_n - \alpha_n F(\omega_n, z_n)],$$

where P_Z is the Euclidean projection operator onto K, $\{z_n\}$ is an approximation of x and $\{\alpha_n\}$ is a sequence of positive stepsizes. If A is strongly monotone or strictly monotone, L-Lipschitz continuous and the stepsizes satisfy $\sum_n \alpha_n = \infty$, $\sum_n \alpha_n^2 < \infty$ (with $0 < \alpha_n < 2\nu/L^2$ in the case where A is ν-strongly monotone) and an unbiased oracle with uniform variance (namely there exists $\sigma > 0$ such that $\mathbb{E}\left[\|F(\omega, z) - \mathbb{E}[F(\omega, z)]\|^2\right] \leq \sigma^2$, for all $z \in Z$), then the method determines a sequence almost surely convergent. Later many scholars analyzed different stochastic approximation methods based on the projection operator, see for instance (Juditsky et al. 2011; Koshal et al. 2013; Wang and Bertsekas 2016).

Now, we introduce a new projection method for monotone and Lipschitz-continuous mapping with constant $L > 0$. More precisely, it is a projected reflected gradient algorithm with a constant stepsize which requires only one projection onto the feasible set and only one value of the mapping per iteration. The method for the deterministic case was studied in Malitsky (2015). Let us denote the residual function by

$$R(z, y) = \|y - P_Z(z - \alpha F(\omega, y))\| + \|z - y\|.$$

Now we formally state our iterative scheme.

3.2 Algorithm

1. Choose $z_0 = y_0 \in Z$, $\omega_0 \in \Omega$ and $\alpha \in \left(0, \frac{\sqrt{2}-1}{L}\right)$.
2. Given z_n, y_n and ω_n, compute
$$z_{n+1} = P_Z(z_n - \alpha F(\omega_n, y_n)).$$
3. If $R(z_n, y_n) = 0$ then stop: $z_n = y_n = z_{n+1}$ is a solution. Otherwise compute
$$y_{n+1} = 2z_{n+1} - z_n,$$
set $n = n + 1$ and return to step 2.

Assuming that the mapping A is monotone and Lipschitz continuous with constant $L > 0$, and the unbiased oracle has uniform variance, the algorithm generates a sequence $\{z_n\}$ weakly convergent to a solution to the stochastic variational inequality.

4 Well-posedness conditions

The purpose of this section is to investigate on the well-posedness of the random optimal control problem.

We say that a sequence $\{h_n\} \subset L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is an approximating sequence for (9) if and only if there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, as $n \to +\infty$, such that

$$\int_\Omega \langle h_n(\omega), w(\omega) - x(\omega, h_n(\omega))\rangle \, d\mathbb{P} \leq \varepsilon_n, \quad \forall w \in W, \ \forall n \in \mathbb{N}.$$

Definition 3 The stochastic inverse variational inequality (9) is well-posed if and only if (9) has a unique solution and every approximating sequence converges to the unique solution.

Proceeding with the same arguments of Theorem 4 in Barbagallo and Guarino Lo Bianco (2023), the next well-posedness result can be proved.

Theorem 3 *Let $x : \Omega \times L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ be an hemicontinuous along line segments and anti-monotone mapping. Then, (9) is well-posed if and only if it has a unique solution.*

The well-posedness for the stochastic variational inequality (10) can be introduced following analogous statements as in Definition 3. More precisely, a sequence $\{z_n\} \subset L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is called an approximating sequence for (10), if and only if there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, as $n \to +\infty$, such that

$$\int_\Omega \langle F(\omega, z_n(\omega)), z_n(\omega) - z(\omega)\rangle \, d\mathbb{P} \leq \varepsilon_n, \quad \forall z \in Z, \ \forall n \in \mathbb{N}.$$

In addition, we say that (10) is well-posed if and only if (10) has a unique solution and every approximating sequence converges to the unique solution.

We can establish the bridge between the well-posedness of (10) and the one of (10), following the same technique used to prove Theorem 5 in Barbagallo and Guarino Lo Bianco (2023).

Theorem 4 *Let $x : \Omega \times L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ be a continuous mapping. Then, (9) is well-posed if and only if (10) is well-posed.*

Let $\alpha > 0$. A sequence $\{h_n\} \subset L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is said to be an α-approximating sequence for the stochastic inverse variational inequality (9) if and only if there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, as $n \to +\infty$, such that

$$\int_\Omega \langle h_n(\omega), w(\omega) - x(\omega, h_n(\omega))\rangle d\mathbb{P} \leq \frac{\alpha}{2} \|w - x(h_n)\|^2 + \varepsilon_n, \quad \forall w \in W, \ \forall n \in \mathbb{N}.$$

Definition 4 We say that the stochastic inverse variational inequality (9) is α-well-posed in the generalized sense if and only if it has a nonempty solution set

S and every α-approximating sequence has some subsequence which converges to an element of S. If the solution set S has only one element, we say that (9) is α -well-posed.

In the following, 0-well-posedness in the generalized sense is simply said as well-posedness in the generalized sense, similarly for the 0-well-posedness. To prove our main results, we first establish the following lemma.

Lemma 5 *Let W be a nonempty convex subset of $L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$. Let $\alpha \geq 0$ and $h^* \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$. Then h^* is a solution to (9) if and only if*

$$\int_\Omega \sum_{i=1}^m \sum_{j=1}^n h^*(\omega)(w_{ij}(\omega) - x_{ij}(\omega, h^*(\omega)))\, d\mathbb{P} \leq \frac{\alpha}{2} \|w - x(\omega, h^*(\omega))\|^2, \quad \forall w \in W.$$

Proof The necessary condition is trivial. Let us fix $w \in W$. For the convexity of W, we have that $x(\omega, h^*(\omega)) + t(w - x(\omega, h^*(\omega))) \in W$. Thus, it results

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^m \sum_{j=1}^m (w_{ij}(\omega) - x_{ij}(\omega, h^*(\omega)))h^*_{ij}(\omega)\, d\mathbb{P} \\
&= \int_\Omega \sum_{i=1}^m \sum_{j=1}^m (w_{ij}(\omega) - t(w_{ij}(\omega) - x_{ij}(\omega, h^*(\omega))))h^*_{ij}(\omega)\, d\mathbb{P} \\
&\leq \frac{\alpha}{2} \|t(w_{ij}(\omega) - x_{ij}(\omega, h^*(\omega)))\|^2 \\
&= \frac{\alpha t^2}{2} \|w_{ij}(\omega) - x_{ij}(\omega, h^*(\omega))\|^2.
\end{aligned}$$

Then the claim follows by passing to the limit as $t \to 0$. □

Likewise, a sequence $\{z_n\} \subset L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ is called α-approximating sequence for the stochastic variational inequality (10) if and only if there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, as $n \to +\infty$, such that

$$\int_\Omega \langle F(\omega, z_n(\omega)), z_n(\omega) - z(\omega) \rangle\, d\mathbb{P} \leq \frac{\alpha}{2} \|z_n - z\| + \varepsilon_n, \quad \forall z \in Z,\ \forall n \in \mathbb{N}.$$

Definition 5 The stochastic variational inequality (10) is said well-posed in the generalized sense if and only if (10) has a nonempty solution set S and every approximating sequence has a subsequence which converges to an element of S.

By using the same arguments proposed in the proof of Theorem 5 in Barbagallo and Guarino Lo Bianco (2023), we can obtain the following result.

Theorem 6 *Let W be a closed subset of $L^2([0, T], \mathbb{R}^{mn}, \mathbb{P})$ and let $x : \Omega \times L^2([0, T], \mathbb{R}^{mn}, \mathbb{P}) \to L^2([0, T], \mathbb{R}^{mn}, \mathbb{P})$ be a continuous mapping. Then,*

(9) *is well-posed in the generalized sense if and only if* (10) *is well-posed in the generalized sense.*

Let W be a nonempty closed convex subset of $L^2([0,T],\mathbb{R}^{mn},\mathbb{P})$. The α-approximating solution set $T_\alpha(\varepsilon)$ of (9), for every $\varepsilon > 0$, is defined as

$$T_\alpha(\varepsilon) = \Big\{ h \in L^2(\Omega,\mathbb{R}^{mn},\mathbb{P}) : \ x(h) \in W, \\ \int_\Omega \langle h(\omega), w(\omega) - x(\omega, h(\omega))\rangle \, d\mathbb{P} \le \frac{\alpha}{2}\|w - x(h)\|^2 + \varepsilon, \quad \forall w \in W \Big\}$$

To state the results of well-posedness it is necessary to introduce a measure of noncompactness, that is, a measure that associates zero to each compact set and a positive number to each other set according to “how far" they are from the compactness. There are at least two possible measures of noncompactness. In this paper we consider the Kuratovski one [see Kuratowski (1968)].

Definition 6 Let W be a nonempty subset of $L^2(\Omega,\mathbb{R}^{mn},\mathbb{P})$. The (Kuratovski) noncompactness measure μ of the set W is

$$\mu(W) = \inf \left\{ \varepsilon > 0 : \ W \subset \cup_{i=1}^n W_i, \ \operatorname{diam} W_i < \varepsilon, \ i = 1, \dots, n \right\},$$

where every $\{W_i\}_{i=1,\dots,n}$, is a finite covering of the set W.

We recall the definition of the Hausdorff distance.

Definition 7 Let W_1 and W_2 be two nonempty subsets of $L^2(\Omega,\mathbb{R}^{mn},\mathbb{P})$. We define the surplus of W_1 over W_2 as

$$e(W_1, W_2) = \sup \left\{ d(A, W_2) : \ A \in W_1 \right\}.$$

The Hausdorff distance between W_1 and W_2 is given by

$$\mathbb{H}(W_1, W_2) = \max \left\{ e(W_1, W_2), e(W_2, W_1) \right\}.$$

In the following we show a metric characterization of the α-well-posedness of (9) in terms of the diameter of the set $T_\alpha(\varepsilon)$. A similar result for inverse tensor variational inequalities is obtained in Anceschi et al. (2023). Since we are studing stochastic inverse variational inequalities, and so we have to consider the inner product in the probability space $L^2(\Omega,\mathbb{R}^{mn},\mathbb{P})$, we prove the metric characterization in detail. Therefore, by using the same arguments of the proof of Theorem 4.1 in Anceschi et al. (2023), we can obtain the following result.

Theorem 7 *Let W be a nonempty closed convex subset of $L^2(\Omega,\mathbb{R}^{mn},\mathbb{P})$. Let $x : \Omega \times L^2(\Omega,\mathbb{R}^{mn},\mathbb{P}) \to L^2(\Omega,\mathbb{R}^{mn},\mathbb{P})$ be a continuous mapping. Then* (9) *is α-well-posed if and only if*

$$T_\alpha(\varepsilon) \neq \emptyset, \ \forall \varepsilon > 0, \quad \text{and} \quad \text{diam } T_\alpha(\varepsilon) \to 0, \ \text{as } \varepsilon \to 0. \tag{12}$$

Proof We start assuming that (9) is α-well-posed. Hence, (9) has a unique solution h^* and, in particular, $h^* \in T_\alpha(\varepsilon)$ for every $\varepsilon > 0$. By contradiction, if diam $T_\alpha(\varepsilon) \nrightarrow 0$, as $\varepsilon \to 0$, there exist $l > 0$, a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, $\varepsilon_n \to 0$, as $n \to +\infty$, and $u_n, v_n \in T_\alpha(\varepsilon_n)$, for every $n \in \mathbb{N}$, such that

$$\|v_n - u_n\| > l, \quad \forall n \in \mathbb{N}. \tag{13}$$

Since $u_n, v_n \in T_\alpha(\varepsilon_n)$, for every $n \in \mathbb{N}$, both $\{u_n\}$ and $\{v_n\}$ are α-approximating sequences for (9). By the assumption of α-well-posedness, both two sequences converge to the unique solution h^* to (9), which is in contradiction with (13).

Vice versa, we assume that (12) holds. Let $\{h_n\} \subset L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ be an α-approximating sequence for (9). Then there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, as $n \to +\infty$, such that $x(h_n) \in W$, for every $n \in \mathbb{N}$, and

$$\int_\Omega \langle h_n(\omega), w(\omega) - x(\omega, h_n(\omega)) \rangle \, d\mathbb{P} \leq \frac{\alpha}{2} \|w - x(h_n)\|^2 + \varepsilon_n, \quad \forall w \in W, \ \forall n \in \mathbb{N}. \tag{14}$$

Consequently $h_n \in T_\alpha(\varepsilon_n)$, for every $n \in \mathbb{N}$, and by (12), $\{h_n\}$ is a Cauchy sequence which converges to an element $\overline{h} \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$. Being x continuous and W a closed set, we deduce $x(\overline{h}) \in W$ and

$$\int_\Omega \langle \overline{h}(\omega), w(\omega) - x(\omega, \overline{h}(\omega)) \rangle d\mathbb{P} \leq \frac{\alpha}{2} \|w - x(\overline{h})\|^2, \quad \forall w \in W.$$

By using Lemma 5, we have that $\overline{h}$ is a solution to (9). In order to conclude the proof, we must show that (9) has a unique solution. Indeed, assuming by contradiction that the problem has two distinct solutions h_1 and h_2. As a consequence, it follows that $h_1, h_2 \in T_\alpha(\varepsilon)$, for every $\varepsilon > 0$, and, then $0 < \|h_1 - h_2\| \leq \text{diam } T_\alpha(\varepsilon)$, in contradiction with (12). □

Now we establish the metric characterization of the α-well-posedness in the generalized sense of (9) in terms of the measure of noncompactness of the set $T_\alpha(\varepsilon)$. An anologous result for inverse tensor variational inequalities is proved in Anceschi et al. (2023). However, for the reader's convenience, we report the proof details to highlight the differences in the stochastic setting. Thus, by using the same arguments of the proof of Theorem 4.2 in Anceschi et al. (2023), we can obtain the following result.

Theorem 8 *Let W be a nonempty closed convex subset of $L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$. Let $x : \Omega \times L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) \to L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ be a continuous mapping. Then* (9) *is α-well-posed in the generalized sense if and only if*

$$T_\alpha(\varepsilon) \neq \emptyset, \ \forall \varepsilon > 0, \quad \text{and} \quad \mu\big(T_\alpha(\varepsilon)\big) \to 0, \ \text{as } \varepsilon \to 0. \tag{15}$$

Proof At first, we suppose that (9) is α-well-posed in the generalized sense. Then its solution set S is nonempty and compact. Indeed, if $\{h_n\}$ is a sequence in S, then it is an α-approximating sequence for (9). By virtue of the α-well-posedness in the generalized sense, we deduce that $\{h_n\}$ has a subsequence which converges to an element of S. Thus, S is compact. In addition, it results that $\emptyset \neq S \subset T_\alpha(\varepsilon)$, for every $\varepsilon > 0$. Consequently, it follows

$$\mathbb{H}(T_\alpha(\varepsilon), S) = \max\left\{e(T_\alpha(\varepsilon), S), e(S, T_\alpha(\varepsilon))\right\} = e(T_\alpha(\varepsilon), S), \quad \forall \varepsilon > 0.$$

Hence, by using the compactness of S, we have

$$\mu(T_\alpha(\varepsilon)) \leq 2\mathbb{H}(T_\alpha(\varepsilon), S) + \mu(S) = 2e(T_\alpha(\varepsilon), S).$$

We argue by contradiction, assuming that $e(T_\alpha(\varepsilon), S) \nrightarrow 0$, as $\varepsilon \to 0$. Therefore, there exist $l > 0$, a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, $\varepsilon_n \to 0$, as $n \to +\infty$, and $h_n \in T_\alpha(\varepsilon_n)$, for every $n \in \mathbb{N}$, such that

$$h_n \notin S + B(0, l), \quad \forall n \in \mathbb{N}, \tag{16}$$

where $B(0,\ l)$ is the closed ball centered at zero with radius l in the space $L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$. Since $h_n \in T_\alpha(\varepsilon)$, for every $n \in \mathbb{N}$, $\{h_n\}$ is an α-approximating sequence for (9). Therefore, there exists a subsequence $\{h_{n_k}\}$ which converges to an element of S. Hence, it is a contradiction with (16).

Vice versa, we assume that (15) holds. Being x continuous and W a closed set, it follows that $T_\alpha(\varepsilon)$ is closed and nonempty, for every $\varepsilon > 0$. We consider

$$\begin{aligned} S' &= \cap_{\varepsilon>0} T_\alpha(\varepsilon) \\ &= \Big\{ h \in L^2(\Omega, \mathbb{R}^{mn}_+, \mathbb{P}) : \ x(h) \in W, \\ &\qquad\qquad \int_\Omega \langle h(\omega), w(\omega) - x(\omega, h(\omega))\rangle d\mathbb{P} \leq \frac{\alpha}{2}\|w - x(h)\|^2 \Big\}. \end{aligned}$$

Taking into account of Lemma 5, we deduce $S' = S$.

By assumption (15) and making use of Theorem of p. 412 of Kuratowski (1968), we can conclude that S is nonempty compact and $e(T_\alpha(\varepsilon), S) = \mathbb{H}(T_\alpha(\varepsilon), S) \to 0$, as $\varepsilon \to 0$.

Let $\{g_n\} \subset L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P})$ be an α-approximating sequence for (9). Then, there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n > 0$, for every $n \in \mathbb{N}$, and $\varepsilon_n \to 0$, as $n \to +\infty$, such that $x(g_n) \in W$, for every $n \in \mathbb{N}$, and

$$\int_\Omega \langle g_n(\omega), w(\omega) - f(\omega, g_n(\omega))\rangle d\mathbb{P} \leq \frac{\alpha}{2}\, \|w - x(g_n)\|^2 + \varepsilon_n, \quad \forall w \in W,\ \forall n \in \mathbb{N}.$$

As a consequence, $g_n \in T_\alpha(\varepsilon_n)$, for every $n \in \mathbb{N}$, and, hence, we have $d(g_n, S) \leq e(T_\alpha(\varepsilon_n), S) \to 0$. Since S is compact, there exists $\overline{h}_n \in S$ such that $\|g_n - \overline{h}_n\| = d(g_n, S) \to 0$, as $n \to +\infty$. By the same assumption, we also deduce that the sequence $\{\overline{h}_n\}$ has a subsequence which converges to $\overline{h} \in S$. Therefore, the corresponding subsequence $\{g_{n_k}\}$ converges to $\overline{h}$. Thus, the claim is completely achieved. □

5 Numerical example

In this section, we provide a numerical example concerning a random oligopolistic market equilibrium problem in order to verify the applicability of the proposed model to real situations. For this purpose, we consider a simple economic network done of two firms and two demand markets, as Fig. 1 shows. Let us assume that the following capacity constraints:

$$0 \leq x_{ij}(\omega) \leq \overline{x}_{ij}, \quad \forall i = 1, 2, \ \forall j = 1, 2, \ \mathbb{P} - a.s.,$$

hold, where the random variables $\overline{x}_{ij}$, $i = 1, 2$, $j = 1, 2$, are uniformly distributed with probability density functions:

$$f_{\overline{x}_{11}}(t) = \begin{cases} \dfrac{1}{50}, & \text{if } 50 \leq t \leq 100, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{\overline{x}_{12}}(t) = \begin{cases} \dfrac{1}{60}, & 90 \leq t \leq 150, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{\overline{x}_{21}}(t) = \begin{cases} \dfrac{1}{30}, & 10 \leq t \leq 40, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{\overline{x}_{22}}(t) = \begin{cases} \dfrac{1}{30}, & 70 \leq t \leq 100, \\ 0, & \text{elsewhere.} \end{cases}$$

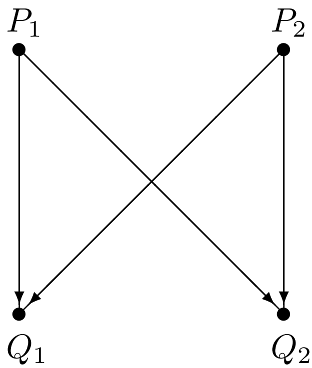

Fig. 1 Numerical random oligopolistic market network

Furthermore, let p_i be the random variables expressing the commodity production of the firm P_i, $i = 1, 2$, so that the commodity shipments have to satisfy:

$$x_{11}(\omega) + x_{12}(\omega) \leq p_1(\omega), \mathbb{P} - \text{a.s.},$$
$$x_{21}(\omega) + x_{22}(\omega) \leq p_2(\omega), \mathbb{P} - \text{a.s.}$$

where the random variables p_i, $i = 1, 2$, are uniformly distributed with probability density functions:

$$f_{p_1}(t) = \begin{cases} \frac{1}{200}, & \text{if } 300 \leq t \leq 500, \\ 0, & \text{otherwise}, \end{cases}$$
$$f_{p_2}(t) = \begin{cases} \frac{1}{200}, & \text{if } 500 \leq t \leq 700, \\ 0, & \text{otherwise}. \end{cases}$$

As a consequence, the set of feasible distributions:

$$\mathbb{K} = \Big\{ x \in L^2(\Omega, \mathbb{R}^4, \mathbb{P}) : \quad 0 \leq x_{ij}(\omega) \leq \bar{x}_{ij}, \quad \forall i = 1, 2, \ \forall j = 1, 2, \ \mathbb{P} - \text{a.s.},$$
$$x_{11}(\omega) + x_{12}(\omega) \leq p_1(\omega), \quad \mathbb{P} - \text{a.s.},$$
$$x_{21}(\omega) + x_{22}(\omega) \leq p_2(\omega), \quad \mathbb{P} - \text{a.s.} \Big\}.$$

Now let us consider the profit function v_i for the firms P_i, $i = 1, 2$, as

$$\begin{aligned} v_1(\omega, x(\omega)) = & \frac{3}{2} x_{11}^2(\omega) + x_{12}^2(\omega) - x_{11}(\omega) x_{22}(\omega) - 4 x_{12}(\omega) x_{21}(\omega) \\ & - h_{11}(\omega) x_{11}(\omega) - h_{12}(\omega) x_{12}(\omega), , \quad \mathbb{P} - \text{a.s.}, \\ v_2(\omega, x(\omega)) = & x_{21}^2(\omega) + 2 x_{22}^2(\omega) - h_{21}(\omega) x_{21}(\omega) - h_{22}(\omega) x_{22}(\omega), \quad \mathbb{P} - \text{a.s.} \end{aligned}$$

Let us compute the operator $\nabla_D v$:

$$\nabla_D v(\omega, x(\omega)) = \begin{pmatrix} 3x_{11}(\omega) - x_{22}(\omega) - h_{11}(\omega) & 2x_{12}(\omega) - 4x_{21}(\omega) - h_{12}(\omega) \\ 2x_{21}(\omega) - h_{21}(\omega) & 4x_{22}(\omega) - h_{22}(\omega) \end{pmatrix}, \quad \mathbb{P} - \text{a.s.}$$

The equilibrium distribution is a solution to the following governing stochastic variational inequality:

$$\ll -\nabla_D v(x^*), x - x^* \gg_{\mathbb{E}} \geq 0, \quad \forall x \in \mathbb{K}. \tag{17}$$

We compute the equilibrium solution by using the direct method [see Barbagallo et al. (2021)]. Therefore, solving the following system

$$\begin{cases} -3x^*_{11}(\omega) + x^*_{22}(\omega) + h_{11}(\omega) = 0 \\ -2x^*_{12}(\omega) + 4x^*_{21}(\omega) + h_{12}(\omega) = 0 \\ -2x^*_{21}(\omega) + h_{21}(\omega) = 0 \\ -4x^*_{22}(\omega) + h_{22}(\omega) = 0 \end{cases} \tag{18}$$

we obtain that the following solution

$$x^*(\omega) = \begin{pmatrix} \frac{1}{3}h_{11}(\omega) + \frac{1}{12}h_{22}(\omega) & \frac{1}{2}h_{12}(\omega) + h_{21}(\omega) \\ \frac{1}{2}h_{21}(\omega) & \frac{1}{4}h_{22}(\omega) \end{pmatrix}, \quad \mathbb{P} - \text{a.s.}$$

Let us start to analyze the random optimal control problem in order to investigate how the policymarker influence the strategy choise of the firms. To this aim, we consider the set of feasible states

$$W = \left\{ w \in L^2(\Omega, \mathbb{R}^4, \mathbb{P}) \,:\, 0 \le w_{ij}(\omega) \le \bar{x}_{ij}(\omega), \quad \forall i = 1, 2, \; \forall j = 1, 2, \; \mathbb{P} - \text{a.s.} \right\}.$$

and the following stochastic inverse variational inquality

$$\begin{aligned} &\ll w - x(h^*), h - h^* \gg_{\mathbb{E}} - \ll h^*, w - w^* \gg_{\mathbb{E}} \ge 0, \\ &\forall (h, w) \in L^2(\Omega, \mathbb{R}^{mn}, \mathbb{P}) \times W. \end{aligned} \tag{19}$$

For $w(\omega) = w^*(\omega)$, $\mathbb{P}$-a.s., we solve the following system

$$\begin{cases} 12w^*_{11}(\omega) - 4h^*_{11}(\omega) - h^*_{22}(\omega) = 0 \\ 2w^*_{12}(\omega) - h^*_{12}(\omega) - 2h^*_{21}(\omega) = 0 \\ 2w^*_{21}(\omega) - h^*_{21}(\omega) = 0 \\ 4w^*_{22}(\omega) - h^*_{22}(\omega) = 0 \end{cases}$$

We obtain the following solution

$$h^*(\omega) = \begin{pmatrix} 3w^*_{11}(\omega) - w^*_{22}(\omega) & 2w^*_{12}(\omega) - 4w^*_{21}(\omega) \\ 2w^*_{21}(\omega) & 4w^*_{22}(\omega) \end{pmatrix}, \quad \mathbb{P} - \text{a.s.}$$

To compute the solution to (19), we consider, for example, that the random variables w^*_{1j}, $j = 1, 2$, are uniformly distribuited in the interval [50, 100], while w^*_{2j}, $j = 1, 2$, in the interval [70, 100]. Therefore, the random variables h^*_{21} and h^*_{22} are uniformly distribuited with probability density functions:

$$f_{h^*_{21}}(x) = \begin{cases} \frac{1}{60}, & \text{if } 20 \le x \le 80, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{h^*_{22}}(x) = \begin{cases} \frac{1}{120}, & \text{if } 280 \le x \le 400, \\ 0, & \text{elsewhere,} \end{cases}$$

whereas the others are distribuited in trapezoidal manner with probability density functions:

$$f_{h^*_{11}}(x) = \begin{cases} \dfrac{x-50}{4500}, & \text{if } 50 \le x < 80, \\ \dfrac{1}{150}, & \text{if } 80 \le x \le 200, \\ \dfrac{230-x}{4500}, & \text{if } 200 < x \le 230, \\ 0, & \text{elsewhere,} \end{cases}$$

$$f_{h^*_{12}}(x) = \begin{cases} \dfrac{x-20}{14400}, & \text{if } 20 \le x \le 140, \\ \dfrac{260-x}{14400}, & \text{if } 140 < x \le 260, \\ 0, & \text{elsewhere.} \end{cases}$$

In the same way, we can analyze other states.

Acknowledgements The first author is a member of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM. The research of the first author has been partially supported by the Research Project PRIN-PNRR P2022XSF5H "Stochastic models in biomathematics and applications".

Author contributions All authors contributed equally to this work.

Funding Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare no competing interests.

References

Anceschi F, Barbagallo A, Guarino Lo Bianco S (2023) Inverse tensor variational inequalities and applications. J Optim Theory Appl 196:570–589

Barbagallo A (2012) Advanced results on variational inequality formulation in oligopolistic market equilirbrium problem. Filomat 5:935–947

Barbagallo A, Cojocaru MG (2009) Dynamic equilibrium formulation of oligopolistic market problem. Math Comput Model 49:966–976

Barbagallo A, Di Meglio G, Ferrara M (2023) Random oligopolistic market equilibrium model with excesses: variational formulation and inverse problem. Comput Optim Appl 84(1):27–52

Barbagallo A, Ferrara M, Mauro P (2021) Stochastic variational approach for random Cournot-Nash principle. In: Jadamba B, Khan AA, Migórski S, Sama M (eds) Deterministic and stochastic optimal control and inverse problems. CRC Press, Taylor & Francis Group, Boca Raton, pp 241–269

Barbagallo A, Guarino Lo Bianco S (2023) A random time-dependent noncooperative equilibrium problem. Comput Optim Appl 84:27–52

Barbagallo A, Maugeri A (2011) Duality theory for the dynamic oligopolistic market equilibrium problem. Optim 60:29–52

Barbagallo A, Mauro P (2012) Evolutionary variational formulation for oligopolistic market equilibrium problems with production excesses. J Optim Theory Appl 155:1–27

Barbagallo A, Mauro P (2012) Time-dependent variational inequality for an oligopolistic market equilibrium problem with production and demand excesses. Abstr Appl Anal 2012:651975

Barbagallo A, Mauro P (2014) Inverse variational inequality approach and applications. Numer Funct Anal Optim 35:851–867

Barbagallo A, Mauro P (2013) A quasi variational approach for the dynamic oligopolistic market equilibrium problem. Abstr Appl Anal 2013:952915

Barbagallo A, Mauro P (2016) A general quasi-variational problem of Cournot-Nash type and its inverse formulation. J Optim Theory Appl 170:476–492

Dafermos S, Nagurney A (1987) Oligopolistic and competitive behavior of spatially separated markets. Reg Sci Urban Econ 17:245–254

Dorta-González P, Santos-Penate D, Suárez-Vega R (2004) Cournot oligopolistic competition in spatially separated markets: the Stackelberg equilibrium. Ann Reg Sci 38:499–511

Facchinei F, Pang J-S (2003) Finite-dimensional variational inequalities and complementarity problems. Springer, New York

Gwinner J, Jadamba B, Khan AA, Raciti F (2021) Uncertainty quantification in variational inequalities: theory, numerics, and applications. Chapman and Hall/CRC Press, Boca Raton

Jiang H, Xu H (2008) Stochastic approximation approaches to the stochastic variational inequality problem. IEEE Trans Autom Control 53:1462–1475

Juditsky A, Nemirovski A, Tauvel C (2011) Solving variational inequalities with stochastic mirror-prox algorithm. Stoch Syst 1:17–58

Koshal J, Nedić A, Shanbhag UV (2013) Regularized iterative stochastic approximation methods for stochastic variational inequality problems. IEEE Trans Autom Control 58:594–608

Kuratowski K (1968) Topology. Academic Press, New York

Lucchetti R, Patrone F (1981) A characterization of Tykhonov well-posedness for minimum problems, with applications to variational inequalities. Numer Funct Anal Optim 3:461–476

Malitsky Yu (2015) Projected reflected gradient methods for monotone variational inequalities. SIAM J Optim 25:502–520

Mangasarian O (1965) Pseudoconvex functions. J Soc Ind Appl Math Ser A Control 3:281–290

Maugeri A, Raciti F (2009) On existence theorems for monotone and nonmonotone variational inequalities. J Convex Anal 16:899–911

Muu LD, Nguyen VH, Quy NV (2008) On Nash-Cournot oligopolistic market equilibrium models with concave cost functions. J Glob Optim 41:351–364

Nagurney A (1998) Algorithms for oligopolistic market equilibrium problems. Reg Sci Urban Econ 18:425–445

Nagurney A (1998) Network economics: a variational inequality approach. Kluwer Academic Publishers, Boston

Nagurney A, Dupuis P, Zhang D (1994) A dynamical systems approach for network oligopolies and variational inequalities. Ann Reg Sci 28:263–283

Nash JF (1950) Equilibrium points in *n*-person games. Proc Natl Acad Sci USA 36:48–49
Nash JF (1951) Non-cooperative games. Ann Math 54:286–295
Robbins H, Monro S (1951) A stochastic approximation method. Ann Math Stat 22:400–407
Scrimali L (2012) An inverse variational inequality approach to the evolutionary spatial price equilibrium problem. Optim Eng 13:375–387
Tykhonov AN (1966) On the stability of the functional optimization problem. USSR J Comput Math Math Phys 6:631–634
Xian W, Yuzeng L, Shaohua Z (2004) Oligopolistic equilibrium analysis for electricity markets: a nonlinear complementarity approach. IEEE Trans Power Syst 19(3):1348–1355
Wang M, Bertsekas D (2016) Stochastic first-order methods with random constraint projection. SIAM J Optim 26:681–717
Zhou J, Lam WHK, Heydecker BG (2005) The generalized Nash equilibrium model for oligopolistic transit market with elastic demand. Transp Res Part B Methodol 39:519–544