Fractional differential equations under stochastic input processes handled by the improved pseudo-force approach

Alba Sofi[1,a*], Giuseppe Muscolino[2,b] and Mario Di Paola[3,c]

[1]Department of Architecture and Territory, University “Mediterranea” of Reggio Calabria, Via dell’Università 25, 89124 Reggio Calabria, Italy

[2]Department of Engineering, University of Messina, Villaggio S. Agata, Messina, Italy

[3]Department of Engineering, University of Palermo, Viale delle Scienze Ed. 8, 90128 Palermo, Italy

[a]alba.sofi@unirc.it, [b]gmuscolino@unime.it, [c]mario.dipaola@unipa.it

*corresponding author

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Abstract. This paper presents a step-by-step procedure for the numerical integration of the fractional differential equation governing the response of a single-degree-of-freedom (SDOF) system with fractional derivative damping. The procedure is developed by extending the *improved pseudo-force method* proposed by the second author for the numerical integration of classical differential equations. To this aim, the *Grünwald–Letnikov* approximation of the fractional derivative is adopted. The proposed numerical procedure is exploited to compute response statistics of a SDOF system subjected to stochastic excitation by applying classical *Monte Carlo Simulation*.

Introduction

Within a deterministic setting, fractional differential equations are usually solved by using the step-by-step *Grünwald–Letnikov* (*GL*) procedure [1,2]. The main issue of this approach is that at each new time step a new term appears in the summation representing the *GL* approximation of the fractional derivative (long tail memory) [2]. Moreover, in order to achieve good accuracy, a very small time step is necessary. This implies an increase of the computational effort which becomes prohibitive in stochastic analysis, especially when *Monte Carlo simulation* (*MCS*) is applied.

In 1996, Muscolino [3] proposed an *improved pseudo-force method* (*IPFM*) for evaluating the solution of linear and non-linear classical differential equations by a step-by-step procedure. This method requires two main steps: i) to consider some (linear or non-linear) terms of the differential equation, depending on the structural response at the current time instant, as pseudo-forces; ii) to accordingly modify the so-called *fundamental* (or *transition*) *matrix* as well as the forcing terms involved in the step-by-step procedure. In this way, it is possible to increase the size of the time step of more than one order of magnitude with respect to classical step-by-step integration schemes, like the *finite difference method* (*FDM*).

In this study, the *IPFM* is revisited and properly adapted to perform the numerical integration of the differential equation governing the response of a single-degree-of-freedom (SDOF) system with fractional derivative damping. Then, the *IPFM* is applied to evaluate response statistics of fractionally damped SDOF systems subjected to stochastic excitation by means of classical *MCS*. The *IPFM* can be extended to multi-DOF systems as well as to non-linear problems.

Grünwald–Letnikov definition of fractional derivative

The *fractional calculus* started with the definition of the *Riemann–Liouville* (*RL*) fractional integral. After that, several definitions for the fractional-order derivative were introduced. Under appropriate conditions, such definitions are equivalent for a wide class of functions [1]. A very useful definition for engineering applications is the *Grünwald–Letnikov* (*GL*) representation of the fractional derivatives, given as:

$$ {}_{a}^{GL}\mathcal{D}_{t}^{\beta}\left\langle f(t)\right\rangle = \lim_{N\to\infty}\left\{\left(\frac{t-a}{n}\right)^{\beta}\frac{1}{\Gamma(-\beta)}\sum_{r=0}^{n-1}\frac{\Gamma(r-\beta)}{\Gamma(r+1)}f\left(t-r\frac{t-a}{n}\right)\right\} \tag{1} $$

where $\Gamma(\bullet)$ is the Euler's gamma function and ${}_{a}^{GL}\mathcal{D}_{t}^{\beta}\left\langle\bullet\right\rangle$ is the *GL* fractional time derivative operator. Notice that Eq. (1) holds if $f(t)$ is continuous and differentiable up to the order $n-1$. Unfortunately, explicit expressions of the *GL* fractional derivative are usually unavailable, so that numerical procedures are needed. To this aim, the time interval $[a=0,\, t_n]$ is subdivided into small intervals of equal length Δt such that $t_0=0,\ t_1=\Delta t, t_2=2\Delta t,\ldots,t_j=j\Delta t,\ldots,\ t_n=n\Delta t$ are the subdivision times. To numerically solve Eq. (1), the *GL* approximation of the *GL* fractional operator based on finite differences [2] can be adopted:

$$ {}_{0}^{GL}\mathcal{D}_{t}^{\beta}\left\langle f(t_n)\right\rangle \cong \frac{1}{(\Delta t)^{\beta}}\sum_{j=1}^{n}\lambda_j(\beta)\, f(t_{n+1-j}) \tag{2} $$

where $\lambda_j(\beta)$ may be easily evaluated in recursive form:

$$ \lambda_1(\beta)=1,\ \lambda_2(\beta)=-\beta,\ldots,\ \lambda_j(\beta)=\left(\frac{j-2-\beta}{j-1}\right)\lambda_{j-1}(\beta),\ldots,\ \ j=3,4,\ldots,n. \tag{3} $$

Note that, the *GL* approximation in Eq. (2) is asymptotically and absolutely stable. Since for $\beta\to 1$ the results reduce to those of the classical Euler methods, the *GL* approximation may be viewed as an extension of the classical explicit and implicit Euler methods [2].

Improved pseudo-force step-by-step integration procedure

In the framework of stochastic dynamics, the most general procedure for evaluating the response of linear and nonlinear structural systems is the *Monte Carlo Simulation* (*MCS*) method. The main advantage of this method is the ability to obtain sufficiently accurate results for any problem for which a deterministic, analytical or numerical, solution is available. Therefore, *MCS* method is currently the only tool available to solve the widest range of stochastic problems involving non-linearities of various kinds, for which there are no analytical solutions, as well as to validate approximate solutions. The main drawback of *MCS* is the heavy computational burden needed to obtain statistically meaningful solutions, which, in some cases, can involve particularly long computation times. This drawback is almost insurmountable for devices having long memory leading to fractional differential equations with stochastic excitations. Indeed, the solution of a problem by *MCS* involves the following three main steps: *i*) *Generation of Samples*

- the number of samples depends on the purposes of the analysis and the precision required in determining response statistics; *ii*) *Deterministic Dynamic Analysis* - for each sample of the forcing process, the corresponding sample of the response process is determined by means of the methods of deterministic dynamics; iii) *Evaluation of Response Statistics* - the various probabilistic characteristics of the response process of interest are estimated "downstream" by means of a statistical analysis.

In this section, a very efficient and accurate method, based on the so-called *improved pseudo-force method* (*IPFM*) [3] to evaluate the response of fractional oscillators is proposed. To illustrate the proposed method, let us consider the following fractional differential equation ruling the motion of a SDOF system:

$$\ddot{x}(t)+c_{\beta}\,{}_{0}^{GL}\mathcal{D}_{t}^{\beta}\left\langle x(t)\right\rangle+\omega_0^2 x(t)=p(t) \tag{4}$$

where ω_0 is the natural circular frequency, c_β is the fractional viscoelastic factor, and $p(t)$ is the forcing function which may be the generic sample of a stochastic process. In order to numerically solve Eq. (4), first the state variable vector $\mathbf{y}(t)=\left[x(t)\;\dot{x}(t)\right]^{\mathrm{T}}$ is introduced. Then, Eq. (4) is rewritten in terms of state variables as follows:

$$\dot{\mathbf{y}}(t)=\mathbf{D}\,\mathbf{y}(t)+\mathbf{v}_{\beta}\,{}_{0}^{GL}\mathcal{D}_{t}^{\beta}\left\langle x(t)\right\rangle+\mathbf{v}_{p}\,p(t) \tag{5}$$

where

$$\mathbf{D}=\left[\begin{array}{c|c}0 & 1\\ \hline -\omega_0^2 & 0\end{array}\right];\quad \mathbf{v}_{\beta}=\left\{\begin{array}{c}0\\ \hline -c_{\beta}\end{array}\right\};\quad \mathbf{v}_{p}=\left\{\begin{array}{c}0\\ \hline 1\end{array}\right\}. \tag{6}$$

The solution of Eq. (5) can be formally written as:

$$\begin{aligned}\mathbf{y}(t)&=\mathbf{\Theta}(t-t_0)\mathbf{y}_0+\int_{t_0}^{t}\mathbf{\Theta}(t-\tau)\left[\mathbf{v}_{\beta}\,{}_{0}^{GL}\mathcal{D}_{\tau}^{\beta}\left\langle x(\tau)\right\rangle+\mathbf{v}_{p}\,p(\tau)\right]\mathrm{d}\tau\\ &=\mathbf{\Theta}(t-t_0)\mathbf{y}_0+\int_{t_0}^{t}\mathbf{\Theta}(t-\tau)\,\mathbf{f}_{\beta}(\tau)\,\mathrm{d}\tau\end{aligned} \tag{7}$$

where the *transition matrix* of the undamped system and the *pseudo-force vector* are introduced:

$$\mathbf{\Theta}(t)=\exp\left(\mathbf{D}t\right)=\begin{bmatrix}\cos(\omega_0 t) & \dfrac{1}{\omega_0}\sin(\omega_0 t)\\ -\omega_0\sin(\omega_0 t) & \cos(\omega_0 t)\end{bmatrix};\quad \mathbf{f}_{\beta}(t)=\mathbf{v}_{\beta}\,{}_{0}^{GL}\mathcal{D}_{t}^{\beta}\left\langle x(t)\right\rangle+\mathbf{v}_{p}\,p(t). \tag{8}$$

It has to be emphasized that in Eq. (8), $p(t)$ is the known external force, while the term ${}_{0}^{GL}\mathcal{D}_{t}^{\beta}\left\langle x(t)\right\rangle$, which depends on the response $x(t)$, is an a priori unknown quantity.

In order to evaluate the integral appearing in Eq. (7), formal solution of the differential Eq. (5) where a fractional derivative appears, a step-by-step procedure is adopted. To this aim, first the time axis is subdivided into small intervals of equal length Δt, with $t_0=0, t_1,\ldots,t_{n-1},t_n,t_{n+1},\ldots$ denoting the subdivision times. Then, the pseudo-force vector, $\mathbf{f}_{\beta}(t)$, in Eq. (8) is assumed piecewise linear in each time interval. Finally, by adopting the *GL* approximation of fractional derivative given by Eq. (2), the following relationships can be written:

$$
\begin{aligned}
&\mathbf{v}_\beta \; {}^{GL}_{\;0}\mathcal{D}_t^\beta \left\langle x(t_n)\right\rangle = \frac{\mathbf{V}_\beta}{(\Delta t)^\beta}\sum_{j=1}^{n}\lambda_j(\beta)\,\mathbf{y}(t_{n+1-j});\\
&\mathbf{v}_\beta \; {}^{GL}_{\;0}\mathcal{D}_t^\beta \left\langle x(t_{n+1})\right\rangle = \frac{\mathbf{V}_\beta}{(\Delta t)^\beta}\sum_{j=1}^{n+1}\lambda_j(\beta)\;\mathbf{y}(t_{n+2-j})\\
&\qquad = \frac{1}{(\Delta t)^\beta}\mathbf{V}_\beta\,\mathbf{y}(t_{n+1}) + \frac{\mathbf{V}_\beta}{(\Delta t)^\beta}\sum_{j=2}^{n+1}\lambda_j(\beta)\;\mathbf{y}(t_{n+2-j})
\end{aligned}
\tag{9}
$$

where $\mathbf{V}_\beta$ is a matrix defined in Eq. (11). By substituting Eq. (9) into Eq. (7), and according to the assumption that the pseudo-force vector $\mathbf{f}_\beta(t)$ is linear in each time step, the following recursive equation, numerical solution of Eq. (5), can be written [3]:

$$
\begin{aligned}
\mathbf{y}(t_{n+1}) = {}& \boldsymbol{\Theta}(\Delta t)\mathbf{y}(t_n) + \boldsymbol{\gamma}_0(\Delta t)\left[\frac{\mathbf{V}_\beta}{(\Delta t)^\beta}\sum_{j=1}^{n}\lambda_j(\beta)\,\mathbf{y}(t_{n+1-j}) + \mathbf{v}_p\; p(t_n)\right]\\
&+\boldsymbol{\gamma}_1(\Delta t)\left[\frac{1}{(\Delta t)^\beta}\mathbf{V}_\beta\,\mathbf{y}(t_{n+1}) + \frac{\mathbf{V}_\beta}{(\Delta t)^\beta}\sum_{j=2}^{n+1}\lambda_j(\beta)\,\mathbf{y}(t_{n+2-j}) + \mathbf{v}_p\; p(t_{n+1})\right]
\end{aligned}
\tag{10}
$$

where $\boldsymbol{\Theta}(\Delta t)$ is the *transition matrix* of the undamped SDOF system and:

$$
\begin{aligned}
&\mathbf{V}_\beta = \begin{bmatrix} 0 & 0\\ -c_\beta & 0\end{bmatrix};\; \mathbf{D}^{-1} = \begin{bmatrix} 0 & -\dfrac{1}{\omega_0^2}\\ 1 & 0\end{bmatrix};\; \mathbf{L}(\Delta t) = \left[\boldsymbol{\Theta}(\Delta t) - \mathbf{I}_2\right]\mathbf{D}^{-1};\\
&\boldsymbol{\gamma}_0(\Delta t) = \left[\boldsymbol{\Theta}(\Delta t) - \frac{1}{\Delta t}\mathbf{L}(\Delta t)\right]\mathbf{D}^{-1};\; \boldsymbol{\gamma}_1(\Delta t) = \left[\frac{1}{\Delta t}\mathbf{L}(\Delta t) - \mathbf{I}_2\right]\mathbf{D}^{-1}.
\end{aligned}
\tag{11}
$$

Equation (10) gives the exact response if the forcing term is piecewise liner. When the latter condition is not strictly verified, Eq. (10) provides much more accurate solutions than all other numerical procedures proposed in literature, with the same integration step.

The main idea of the *IPFM* [3] consists of moving all terms containing the unknown vector $\mathbf{y}(t_{n+1})$ from the right- to the left-hand side, so that, after some algebra, the following step-by-step procedure is obtained:

$$
\begin{aligned}
\mathbf{y}(t_{n+1}) = {}& \tilde{\boldsymbol{\Theta}}(\Delta t)\mathbf{y}(t_n) + \tilde{\boldsymbol{\gamma}}_0(\Delta t)\left[\frac{\mathbf{V}_\beta}{(\Delta t)^\beta}\sum_{j=1}^{n}\lambda_j(\beta)\,\mathbf{y}(t_{n+1-j}) + \mathbf{v}_p\; p(t_n)\right]\\
&+\tilde{\boldsymbol{\gamma}}_1(\Delta t)\left[\frac{\mathbf{V}_\beta}{(\Delta t)^\beta}\sum_{j=2}^{n+1}\lambda_j(\beta)\;\mathbf{y}(t_{n+2-j}) + \mathbf{v}_p\; p(t_{n+1})\right]
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
&\tilde{\boldsymbol{\Theta}}(\Delta t) = \left[\mathbf{I}_2 - \frac{1}{(\Delta t)^\beta}\boldsymbol{\gamma}_1(\Delta t)\mathbf{V}_\beta\right]^{-1}\boldsymbol{\Theta}(\Delta t);\quad \tilde{\boldsymbol{\gamma}}_0(\Delta t) = \left[\mathbf{I}_2 - \frac{1}{(\Delta t)^\beta}\boldsymbol{\gamma}_1(\Delta t)\mathbf{V}_\beta\right]^{-1}\boldsymbol{\gamma}_0(\Delta t);\\
&\tilde{\boldsymbol{\gamma}}_1(\Delta t) = \left[\mathbf{I}_2 - \frac{1}{(\Delta t)^\beta}\boldsymbol{\gamma}_1(\Delta t)\mathbf{V}_\beta\right]^{-1}\boldsymbol{\gamma}_1(\Delta t).
\end{aligned}
\tag{13}
$$

The step-by-step procedure (12) gives the numerical solution $\mathbf{y}(t_{n+1})$ of the set of fractional differential equations (5) at time step t_{n+1}. It has been proved that this procedure is unconditionally stable [3].

The use of the *GL* approximation (2) in the framework of step-by-step integration procedures may be time-consuming. Indeed, due to the memory of the response, the summation in Eq. (3) involves a larger number of terms as time increases. This implies that computational times become prohibitive, especially when *MCS* is applied to perform the stochastic dynamic analysis of structures with fractional damping. To cope with this issue, the decreasing feature of the absolute value of the weights $\lambda_j(\beta)$ (see Eq. (3)) as the number of the time step increases (see e.g. [4,5]) may be exploited to truncate the *GL* approximation. In this way, the calculation at each time step needs not to go back to the beginning of the motion [4], i.e.:

$$\mathbf{v}_\beta \, {}_0^{GL}\mathcal{D}_t^\beta \langle x(t_n) \rangle = \begin{cases} \dfrac{\mathbf{V}_\beta}{(\Delta t)^\beta} \displaystyle\sum_{j=1}^{n} \lambda_j(\beta)\ \mathbf{y}(t_{n+1-j}) & \text{if } n \le n_T \\ \dfrac{\mathbf{V}_\beta}{(\Delta t)^\beta} \displaystyle\sum_{j=1}^{n_T} \lambda_j(\beta)\, \mathbf{y}(t_{n+1-j}) & \text{if } n > n_T \end{cases} \tag{14}$$

where n_T is the number of time instants that must be considered in the analysis to achieve good accuracy. Notice that the truncation can be applied if and only if $\beta < 1$.

Numerical application and discussion

The accuracy of the proposed *IPFM* is assessed by analyzing the response of a SDOF system with fractional damping characterized by the following parameters: $\omega_0 = 1.0$ rad/s and $c_\beta = 1.0$. Different values of the fractional derivative order β are considered. It is assumed that the SDOF system is subjected to a Gaussian white noise $W(t)$ with Power Spectral Density $S_0 = 1/2\pi$. The generic sample of the excitation is generated by means of the well-known spectral representation. The corresponding sample of the response is obtained by applying the proposed *IPFM* with ($n_T = 500$) and without truncation assuming a time step $\Delta t = 0.03$ s. For comparison purposes, the response obtained by applying the finite difference method (*FDM*) with a time step $\Delta t = 0.0001$ s is assumed as reference solution.

Figure 1a displays the time-history of the generic sample of the displacement $x(t)$ for $\beta = 0.2$. The responses provided by the *IPFM* and the *FDM* with a time step $\Delta t = 0.03$ s are contrasted with the reference solution. It is observed that the proposed *IPFM* is much more accurate and efficient than the *FDM* as it yields a solution in excellent agreement with the reference one by using a much larger time step. The enlargement in Fig. 1b shows that the *IPFM* achieves a very good match with the reference solution also when the *GL* approximation of the fractional derivative is truncated to the first $n_T = 500$ terms. This implies a drastic reduction of the computational effort. Numerical investigations, omitted for brevity, have shown that the same degree of accuracy is achieved for other values of β.

Figure 2 shows the time-history of the variance of the displacement provided by *MCS* ($N_s = 5000$ samples) along with the exact steady-state value for $\beta = 0.2$ and $\beta = 0.5$. It can be seen that a very accurate prediction of the displacement variance is provided by the *IPFM* with a time step $\Delta t = 0.03$ s even when a truncation step $n_T = 500$ is considered. Conversely, the *FDM* with $\Delta t = 0.03$ s highly underestimates the variance of the response. A smaller time step is needed to improve the accuracy of the *FDM*.

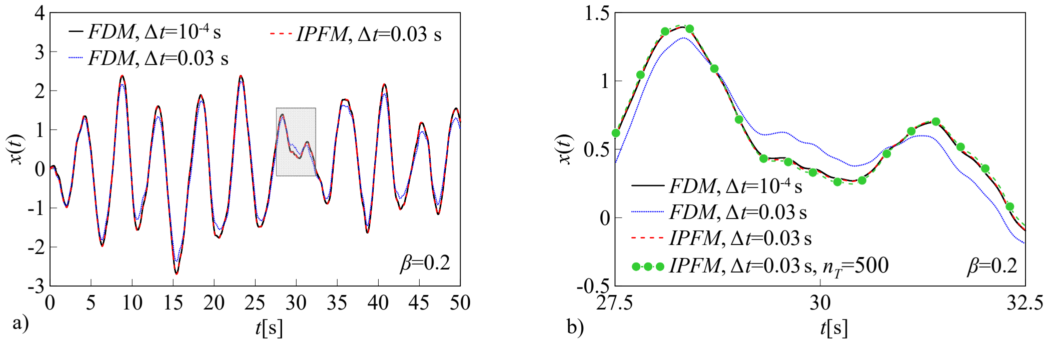

Figure 1. a) Sample of the displacement provided by the *FDM* and the *IPFM*; b) enlargement showing the accuracy of the *IPFM* with truncation ($\beta = 0.2$).

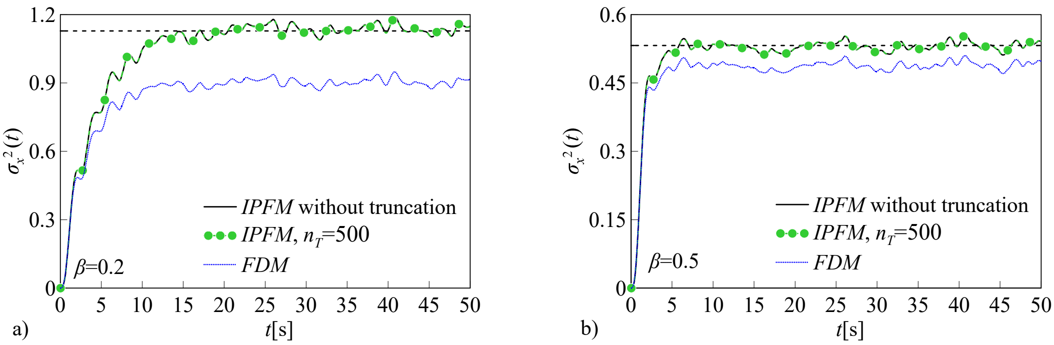

Figure 2. Variance of the displacement provided by *MCS* ($N_s = 5000$ samples) along with the exact steady-state value (black dashed line): a) $\beta = 0.2$ and b) $\beta = 0.5$ ($\Delta t = 0.03$ s).

Numerical results demonstrate that the *IPFM* allows a drastic enhancement of the computational efficiency of *MCS* since it is able to provide accurate estimates of the response using much larger time steps than the classical *FDM*. The computational burden can be further reduced by truncating the *GL* approximation of the fractional derivative.

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