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On the Lifeline Game of the Inertial Players with Integral and Geometric Constraints

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Abstract: In this paper, we consider a pursuit–evasion game of inertial players, where the pursuer's control is subject to integral constraint and the evader's control is subject to geometric constraint. In the pursuit problem, the main tool is the strategy of parallel pursuit. Sufficient conditions are obtained for the solvability of pursuit–evasion problems. Additionally, the main lemma describing the monotonicity of an attainability domain of the evader is proved, and an explicit analytical formula for this domain is given. One of the main results of the paper is the solution of the Isaacs lifeline game for a special case.

Keywords: differential game; integral constraint; geometric constraint; pursuer; evader; strategy; guaranteed capture time; attainability domain; lifeline game

MSC: 91A23; 49N75

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1. Introduction

Game theory studies strategic decision-making in interactive situations. It has found interesting and important applications in various areas, improving strategies and decision-making processes. In games studied in the works [1–3], the tasks of rehabilitation and collaborative manufacturing complementing human and robot capabilities were widely highlighted.

Many problems of conflict-controlled processes can be modeled as differential games. In the middle of the twentieth century, the term "differential games" was first apparent in Isaacs's monograph [4] including exclusive conflict and game problems. The formalization of the theory of differential games has been created by many researchers such as Pontryagin [5] and Krasovskii [6].

In order to implement mathematical models in real-life processes, studies on differential games with various type of restrictions on controls have gained great interest. For example, the works [7,8] are devoted to investigating such a type of game problems. It is essential to mention that games with various types of constraints on controls have not been adequately explored yet. In the theory of differential games, it is not easy to construct players' optimal strategies and to determine the game value. The works [9,10] are especially devoted to establishing the existence of the game value by constructing the players' optimal strategies.

The method of resolving functions for games with integral constraints on control functions was developed by Belousov [11] to obtain a sufficient condition the of completion of the pursuit in a differential game. The solution was further extended to the case of convex integral constraints [12]. The pursuit–evasion and lifeline problems under integral constraints on controls were widely considered by Azamov [13] and Azamov and Samatov [14]. Such differential game problems with integral constraints have been considered for differential-difference equations as well. For instance, the works of Mamadaliev [15–17] are devoted to linear pursuit games under integral constraints on players' controls with delay information. Thereafter, games with linear, linear-geometric, integro-geometric, and mixed constraints on controls were comprehensively solved using the Π-strategy in the works of Samatov et al. [18,19].

Differential games of inertial players are of great significance owing to their many applications to technical and air processes. Ibragimov et al. [20–22] studied a fixed duration differential game of countable number inertial players in Hilbert space with integral constraints. The works [23,24] studied the pursuit–evasion differential games of many pursuers and one evader with geometric constraints for an infinite system of second-order differential equations in Hilbert space. In addition, the papers [25,26] are devoted to investigating nonsmooth and second-order nonlinear aggregative games with multiple players.

A number of concrete differential games were solved by Isaacs, and open problems were formulated in [4]. In particular, the lifeline problem posed in [4] has since been completely solved by Petrosjan [27] when players' controls obey geometric constraints, developing the strategy of parallel approach. Azamov [13] proposed an analytical solution to the lifeline game of multiple pursuers and one evader using the support function of a multi-valued mapping. Azamov [28] investigated the structure of phase space of differential pursuit–evasion games when the evader is under discriminated information, and here, an alternative for differential pursuit–evasion games in $[0, \infty)$ was established by the transfinite iteration method of Pshenichnii's operator.

In addition, Munts and Kumkov [29,30] examined the classic time-optimal differential lifeline games in the formalization of Krasovskii [6], and the authors proposed a numerical method of solving time-optimal differential games with a lifeline. Thereafter, for the cases in which controls of both objects adhering to integral, linear, Grönwall-type or mixed constraints, the pursuit–evasion problem and the kifeline game were solved using the parallel pursuit strategy (Π-strategy) in the works of Samatov et al. [31,32]. Note that the Π-strategy for the pursuer is constructed based on types of constraints on control functions.

In the present paper, we consider a pursuit–evasion differential game of one inertial pursuer and one inertial evader. The control of the pursuer is subject to integral constraints, and the control of the evader is subject to geometric constraints. For the pursuit game, we propose a parallel approach strategy and obtain sufficient conditions of the completion of pursuit. For the evasion game, we obtain a sufficient condition of evasion. Furthermore, we give an explicit formula for the attainability domain of the evader and establish the monotonicity of this attainability domain. One of the important results of our paper is the solution of Isaacs' lifeline game under a condition. All these results are new for the differential game with inertial players and the abovementioned constraints on players' controls.

It is worth mentioning that the results of the present research can be implemented in multiple objective adversarial reach–avoid (RA) games [33,34]. The players of RA games are the evader (attacker) and the defenders (group of pursuers). Additionally, there exists a target zone. The purpose of the evader is to reach the target zone without being caught, while the group of defenders try to postpone or stop the evader from going into the target zone by catching the evader.

This paper is organized as follows. Section 2 is devoted to main definitions and notions. In Section 3, we define the main strategy called the Π-strategy to solve the pursuit game. In Section 4, an evasion game will be considered. Finally, in Section 5, a lifeline game will be solved using the the results of Sections 3 and 4.

2. Statement of Problems

We discuss a differential game of pursuer P and evader E whose control parameters are u and v, respectively. Tthe dynamics are given by the following second-order differential equations:

$$P:\ \ddot{x} = u,\ \ x(0) = x_0,\ \ \dot{x}(0) = x_1, \tag{1}$$

$$E:\ \ddot{y} = v,\ \ y(0) = y_0,\ \ \dot{y}(0) = y_1, \tag{2}$$

where $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$; x_0 and y_0 are players' initial positions, and x_1, y_1 are their initial velocities, respectively. It is assumed that $x_0 \neq y_0$ and $x_1 = y_1$.

Definition 1. *We say that a measurable function $u(\cdot) = (u(t),\ t \geq 0)$ is the admissible control of the pursuer if it satisfies the following integral constraint:*

$$\int_0^t (t-s)|u(s)|^2 ds \leq \rho_0,\ \ t \geq 0, \tag{3}$$

where ρ_0 is a given positive number. We denote the set of all admissible controls $u(\cdot)$ of pursuer by $\mathbb{U}$.

Definition 2. *We say that a measurable function $v(\cdot) = (v(t),\ t \geq 0)$ is the admissible control of the evader if it satisfies the following geometric constraint:*

$$|v(t)| \leq \beta,\quad t \geq 0, \tag{4}$$

where β is a given positive number. We denote the set of all admissible controls $v(\cdot)$ of the evader by $\mathbb{V}$.

Definition 3. *For each pair $(\rho_0, u(\cdot))$, $u(\cdot) \in \mathbb{U}$, we call the quantity*

$$\rho(t) = \rho_0 - \int_0^t (t-s)|u(s)|^2 ds,\ \ \rho(0) = \rho_0 \tag{5}$$

the residual resource of the pursuer at the current time t, $t \geq 0$.

Let $D_\beta = \{d \in \mathbb{R}^n \mid\ |d| \leq \beta\}$.

Definition 4. *We call a mapping $\boldsymbol{u} : D_\beta \to \mathbb{R}^n$ the strategy of the pursuer if the following conditions are satisfied:*

(1) $\boldsymbol{u}(v)$, where $v \in D_\beta$, is a Borel measurable function of v;

(2) For an arbitrary $v(\cdot) \in \mathbb{V}$, the inclusion $(\boldsymbol{u}(v(t)),\ t \geq 0) \in \mathbb{U}$ is satisfied on some time interval $[0, t^*]$.

Let

$$z(t) = x(t) - y(t),\ z_0 = x_0 - y_0,\ \dot{z}(0) = z_1 = x_1 - y_1.$$

From Equations (1) and (2), we then derive the initial value problem:

$$\ddot{z} = u - v,\ \ z(0) = z_0,\ \ \dot{z}(0) = 0.$$

Definition 5. *We call a strategy $\boldsymbol{u}(v)$ the parallel convergence strategy (Π-strategy) if for an arbitrary $v(\cdot) \in \mathbb{V}$, the solution $z(t)$ of the initial value problem*

$$\ddot{z} = \boldsymbol{u}(v(t)) - v(t), \;\; z(0) = z_0, \;\; \dot{z}(0) = 0,$$

has the form

$$z(t) = z_0 \Theta(t, v(\cdot)), \;\; \Theta(0, v(\cdot)) = 1, \;\; t \geq 0,$$

where $\Theta(t, v(\cdot))$ is a scalar function, and this is generally called the convergence function of the pursuer and evader in the pursuit problem.

Definition 6. *We say that the Pursuer wins by using the Π-strategy on a finite time interval $[0, T]$ if, for any $v(\cdot) \in \mathbb{V}$,*

(a) $z(t^*) = 0$ at some instant $t^* \in [0, T]$;

(b) $\boldsymbol{u}(v(t))$, $0 \leq t \leq t^*$, belongs to $\mathbb{U}$ on the interval $[0, t^*]$. In this case, the number T is called a guaranteed capture time.

Definition 7. *We say that the evader wins by using a control $\boldsymbol{v}^*(\cdot) \in \mathbb{V}$ if, for an arbitrary $u(\cdot) \in \mathbb{U}$, the solution $z(t)$ of the initial value problem*

$$\ddot{z} = u(t) - \boldsymbol{v}^*(t), \;\; z(0) = z_0, \;\; \dot{z}(0) = 0,$$

satisfies the condition $z(t) \neq 0$ for all $t \geq 0$.

This paper is dedicated to studying the following game problems where the controls $u(\cdot)$ and $v(\cdot)$ of the players are subject to Constraints (3) and (4), respectively:

Problem 1. *Construct a Π-strategy to ensure the completion of the pursuit (Pursuit Game).*

Problem 2. *Set a special admissible control for the evader and determine the conditions guaranteeing their escape (Evasion Game).*

Problem 3. *Find the sufficient conditions of the completion of pursuit in the lifeline game.*

3. Pursuit Game

We call the function

$$\mathbf{u}(v) = v - \theta(v)\xi_0 \tag{6}$$

the Π-*strategy* of the pursuer, where

$$\theta(v) = \langle v, \xi_0 \rangle + \frac{\eta_0}{2} + \sqrt{\left(\langle v, \xi_0 \rangle + \frac{\eta_0}{2}\right)^2 - |v|^2}, \;\; \xi_0 = \frac{z_0}{|z_0|}, \;\; \eta_0 = \frac{\rho_0}{|z_0|},$$

and $\langle v, \xi_0 \rangle$ is the inner product of the vectors v and ξ_0 in $\mathbb{R}^n$. Here, $\theta(v)$ is usually called the *resolving function*.

Let us present the following important property of the strategy (6) and the resolving function $\theta(v)$.

Proposition 1. *If $\rho_0 \geq 4\beta|z_0|$, then, for all $v \in D_\beta$,*

(a) $\theta(v)$ is well-defined and continuous in D_β,

(b) The following is true:

$$\theta_1 \leq \theta(v) \leq \theta_2, \tag{7}$$

where $\theta_1 = \frac{\eta_0}{2} - \beta + \sqrt{\frac{\eta_0^2}{4} - \eta_0\beta}$, $\theta_2 = \frac{\eta_0}{2} + \beta + \sqrt{\frac{\eta_0^2}{4} + \eta_0\beta}$,

(c) The identity

$$|\boldsymbol{u}(v)|^2 = \eta_0\theta(v), \;\; t \geq 0, \tag{8}$$

holds.

Proof. (a) From the conditions $\rho_0 \geq 4\beta|z_0|$ and $\eta_0 = \frac{\rho_0}{|z_0|}$, we infer $\eta_0\left(\frac{\eta_0}{4} - \beta\right) \geq 0$. Since $|v| \leq \beta$ (see (4)), therefore,

$$\begin{aligned} 0 &\leq \eta_0\left(\frac{\eta_0}{4} - \beta\right) \leq \frac{\eta_0^2}{4} - \eta_0|v| \\ &= \left(-|v| + \frac{\eta_0}{2}\right)^2 - |v|^2 \leq \left(\langle v, \xi_0\rangle + \frac{\eta_0}{2}\right)^2 - |v|^2. \end{aligned}$$

(b) Letting $\tau = \langle v, \xi_0\rangle$ in $\theta(v)$; consider the function

$$f(\tau) = \tau + \frac{\eta_0}{2} + \sqrt{\left(\tau + \frac{\eta_0}{2}\right)^2 - |v|^2}, \quad -\beta \leq \tau \leq \beta.$$

Clearly, $\frac{df(\tau)}{d\tau} > 0$. As a consequence, using (4), it is not hard to obtain that $\min f(\tau) = f(-\beta) = \theta_1$ and $\max f(\tau) = f(\beta) = \theta_2$.

(c) From (6), we obtain

$$\begin{aligned} |\mathbf{u}(v)|^2 &= \langle \mathbf{u}(v), \mathbf{u}(v)\rangle = \Big\langle v - \theta(v)\xi_0, v - \theta(v)\xi_0 \Big\rangle \\ &= |v|^2 + \theta(v)\Big[\theta(v) - 2\langle v, \xi_0\rangle\Big] = \eta_0\theta(v). \end{aligned}$$

and this completes the proof. □

Thanks to Equations (1) and (2), for an arbitrary $v(\cdot) \in \mathbb{V}$ and for the function $\mathbf{u}(v(\cdot)) \in \mathbb{U}$, the pursuer's trajectory is

$$x(t) = x_0 + x_1 t + \int_0^t (t - s)\mathbf{u}(v(s))ds, \tag{9}$$

and the evader's trajectory is

$$y(t) = y_0 + y_1 t + \int_0^t (t - s)v(s)ds. \tag{10}$$

In this case, the goal of Pursuer P is to capture Evader E, i.e., to achieve the equation $x(t) = y(t)$. Evader E strives to avoid an encounter, i.e., to maintain the inequality $x(t) \neq y(t)$ for all $t \geq 0$, and if this cannot be done, to delay the encounter time as long as possible.

If $\rho_0 \geq 4\beta|z_0|$, then the scalar function

$$\Theta(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t (t - s)\theta(v(s))ds, \ \ t \geq 0, \tag{11}$$

is called the *convergence function* of the players in the pursuit game.

Lemma 1. *Let $\rho_0 \geq 4\beta|z_0|$. Then:*

(a) For any $v(\cdot) \in \mathbb{V}$, the function $\Theta(t, v(\cdot))$ is monotonically decreasing in t, $t \geq 0$;

(b) For all $t \in [0, T]$,

$$\Theta^*(t) \leq \Theta(t, v(\cdot)) \leq \Theta^{**}(t), \tag{12}$$

where $\Theta^*(t) = 1 - \frac{\theta_2}{2|z_0|}t^2$ and $\Theta^{**}(t) = 1 - \frac{\theta_1}{2|z_0|}t^2$.

Proof. (a) According to (7) and (11), we have

$$\frac{d\Theta(t,v(\cdot))}{dt} = -\frac{1}{|z_0|}\int_0^t \theta(v(s))ds \le -\frac{\theta_2}{|z_0|}t < 0, \quad t > 0.$$

(b) Clearly [35],

$$\begin{aligned}\Theta(t,v(\cdot)) &\le 1 - \frac{1}{|z_0|}\min_{v(\cdot)\in\mathbb{V}}\int_0^t (t-s)\theta(v(s))ds \\ &\le 1 - \frac{t^2}{2|z_0|}\min_{|v|\le\beta}\theta(v) = \Theta^{**}(t).\end{aligned}$$

Additionally, from (7) we obtain

$$\begin{aligned}\Theta(t,v(\cdot)) &\ge 1 - \frac{1}{|z_0|}\max_{v(\cdot)\in\mathbb{V}}\int_0^t (t-s)\theta(v(s))ds \\ &= 1 - \frac{t^2}{2|z_0|}\max_{|v|\le\beta}\theta(v) = \Theta^{*}(t).\end{aligned}$$

This finishes the proof. □

Theorem 1. *If $\rho_0 \ge 4\beta|z_0|$, then the pursuer wins by using Π-strategy (6) on the time interval $[0,T]$, where $T = \sqrt{2|z_0|/\theta_1}$ and θ_1 is defined by (7).*

Proof. Let $v(\cdot)$ be an arbitrary admissible control of Evader E, and let Pursuer P employ strategy (6). Then, given (9), (10), $z(t) = x(t) - y(t)$, $x_1 = y_1$, we have

$$z(t) = z_0 + \int_0^t (t-s)[\mathbf{u}(v(s)) - v(s)]ds, \;\; z(0) = z_0.$$

Taking into account (6) and (11), we obtain

$$z(t) = z_0\Theta(t,v(\cdot)). \tag{13}$$

Since in view of (12), $\Theta(t,v(\cdot)) \le \Theta^{**}(t)$ and $\Theta^{**}(T) = 0$, we find that there exists time $t^* \in (0,T]$ depending on $v(\cdot)$ such that $\Theta(t^*,v(\cdot)) = 0$. Hence, by virtue of (13), the desired result $z(t^*) = 0$, i.e., $x(t^*) = y(t^*)$, is obtained.

It remains only to verify that the Π-strategy (6) is admissible for each $t \in [0,t^*]$. By (11) $\Theta(t^*,v(\cdot)) = 0$ implies that

$$\int_0^{t^*} (t^*-s)\theta(v(s))ds = |z_0|,$$

and so

$$\begin{aligned}\int_0^t (t-s)|\mathbf{u}(v(s))|^2 ds &\le \int_0^{t^*} (t^*-s)|\mathbf{u}(v(s))|^2 ds \\ &= \eta_0\int_0^{t^*} (t^*-s)\theta(v(s))ds = \eta_0|z_0| = \rho_0.\end{aligned}$$

Thus, Strategy (6) is admissible. The proof of Theorem 1 is complete. □

Observe if $\rho_0 \geq 4\beta|z_0|$ and if the pursuer applies strategy (6); then, for all $t \in [0, t^*]$, for the function (5) by (8), we have $\rho(t) = \Theta(t, v(\cdot))\rho_0$.

Indeed

$$\begin{aligned} \rho(t) &= \rho_0 - \int_0^t (t-s)\eta_0\theta(v(s))ds \\ &= \rho_0\left(1 - \frac{1}{|z_0|}\int_0^t (t-s)\theta(v(s))ds\right) = \rho_0\Theta(t, v(\cdot)). \end{aligned} \quad (14)$$

4. Evasion Game

In this section, an admissible control will be offered for the evader to establish that T is the optimal time of pursuit.

Let the evader employ the following control:

$$\mathbf{v}^*(t) = -\beta\xi_0, \ \ \xi_0 = \frac{z_0}{|z_0|}. \quad (15)$$

In accordance with Equations (1) and (2), for an arbitrary $u(\cdot) \in \mathbb{U}$ and for the control $\mathbf{v}^*(t)$, we obtain the following trajectories of the players:

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s)ds,$$

$$y(t) = y_0 + y_1 t + \int_0^t (t-s)\mathbf{v}^*(s)ds.$$

We prove the following statement.

Theorem 2. *(a) If $\rho_0 \geq 4\beta|z_0|$, then the evader wins by Control (15) in the interval $[0, T)$, where $T = \sqrt{2|z_0|/\theta_1}$.*

(b) If $\rho_0 < 4\beta|z_0|$, then the evader wins by Control (15) in the time interval $[0, \infty)$ and

$$|z(t)| = |y(t) - x(t)| \geq |z_0| - \frac{\rho_0}{4\beta}.$$

Proof. (a) Let $\rho_0 \geq 4\beta|z_0|$. If the evader utilizes Control (15), then for any control of Pursuer $u(\cdot) \in \mathbb{U}$, in view of $x_1 = y_1$, we have

$$z(t) = z_0 + \int_0^t (t-s)u(s)ds - \int_0^t (t-s)\mathbf{v}^*(s)ds.$$

By (15), we have

$$\begin{aligned} |z(t)| &= \left| z_0 + \beta\xi_0 \int_0^t (t-s)ds + \int_0^t (t-s)u(s)ds \right| \\ &\geq |z_0| + \frac{\beta}{2}t^2 - \int_0^t (t-s)|u(s)|ds. \end{aligned} \quad (16)$$

Applying the Cauchy–Schwartz inequality to the last integral in (16), we find

$$\int_0^t (t-s)|u(s)|ds \leq \left(\int_0^t (t-s)ds\right)^{1/2} \left(\int_0^t (t-s)|u(s)|^2 ds\right)^{1/2} \leq \sqrt{\frac{\rho_0}{2}}\, t. \tag{17}$$

As a consequence of (16) and (17), we conclude that

$$|z(t)| \geq \Gamma(t), \quad \Gamma(t) = \frac{\beta}{2}t^2 - \sqrt{\frac{\rho_0}{2}}t + |z_0|. \tag{18}$$

It is not difficult to verify that the smallest positive root of Equation $\Gamma(t) = 0$ is exactly T. Thus, $\Gamma(t) > 0$ for all $t \in [0, T)$, and in consequence, it follows immediately from (18) that $|z(t)| > 0$ on that time interval.

(b) Suppose $\rho_0 < 4\beta|z_0|$. Then, we come to the estimation $|z(t)| \geq \Gamma(t)$ again. It is obvious that

$$\min_{t \geq 0} \Gamma(t) = |z_0| - \rho_0/(4\beta).$$

On account of the condition $\rho_0 < 4\beta|z_0|$, we obtain $\Gamma(t) > 0$ for all $t \geq 0$, and therefore, by (18), we see that $z(t) \neq 0$, i.e., $x(t) \neq y(t)$, $t \geq 0$. The proof is complete. □

5. Lifeline Game

The current section is devoted to investigating the dynamics of the attainability domain of the evader and the lifeline problem of R. Isaacs.

Let a non-empty and closed subset L of the space $\mathbb{R}^n$ be given. Here and subsequently, the area L is called the lifeline.

In the lifeline game L, Pursuer P intends to intercept Evader E to accomplish $x(t_*) = y(t_*)$ at a finite time $t_* > 0$ when Evader E is in $\mathbb{R}^n \setminus L$. Evader E aims to obtain the area L maintaining the condition $x(t) \neq y(t)$, where $t \geq 0$; and if there is no chance of doing this, then Evader E strives to maximize the moment of the encounter with Pursuer P. It should be noted that the area L does not limit the motion of Pursuer P. Furthermore, it is required that the conditions $x_0 \neq y_0$ and $y_0 \notin L$ are satisfied for the initial positions x_0 and y_0.

Definition 8. *Π-strategy (6) is said to be winning on the time interval $[0, T]$ in the lifeline game L, if, for the evader's arbitrary control $v(\cdot) \in \mathbb{V}$, there exists an instant $t_* \in [0, T]$ such that:*
(1) $x(t_*) = y(t_*)$;
(2) $y(t) \notin L$ at each $t \in [0, t_*]$.

Definition 9. *We say that th evader wins in the lifeline game L by a control $v(\cdot) \in \mathbb{V}$ if for every $u(\cdot) \in \mathbb{U}$:*
(1) There exists some moment $\bar{t}$, $\bar{t} > 0$, that $y(\bar{t}) \in L$ and $x(t) \neq y(t)$ while $t \in [0, \bar{t})$; or
(2) $x(t) \neq y(t)$ for all $t \geq 0$.

In the theory of differential games, constructing the attainability domain of the evader in the pursuit game is considered the main step to solve the game with a lifeline, and therefore, we will first study the dynamics of the attainability domain.

If $\rho_0 \geq 4\beta|z_0|$ in Games (1)–(4), then Theorem 1 asserts that by virtue of the Π-strategy (6), the pursuer is able to capture the evader. The players P and E will meet at various points according to the choice of the control $v(\cdot) \in \mathbb{V}$.

Let $M(x, y, \rho)$ be the set consisting of all points μ where the pursuer moving from the position x and consuming the resource ρ should encounter the evader moving from the position y, i.e.,

$$M(x, y, \rho) = \left\{\mu : \ |\mu - x|^2 \geq \frac{\rho}{\beta}|\mu - y|\right\}.$$

When $\rho \neq \beta$, the set $M(x, y, \rho)$ is bounded by the curve

$$\partial M(x, y, \rho) = \left\{ \mu : \ |\mu - x|^2 = \frac{\rho}{\beta} |\mu - y| \right\}. \tag{19}$$

Set (19) in the plane of *Descartes' oval* or *Pascal's snail* [14].

Next, let the pursuer hold the Π-strategy (6) while the evader employs arbitrary control $v(\cdot) \in \mathbb{V}$. Then, t^*, $0 < t^* \leq T$ is the players' meeting time, i.e., $x(t^*) = y(t^*)$. Then, for each triad $\Big(x(t), y(t), \rho(t) \Big)$, $t \in [0, t^*]$, we build the following sets:

$$M\Big(x(t), y(t), \rho(t) \Big) = \left\{ \mu : \ |\mu - x(t)|^2 \geq \frac{\rho(t)}{\beta} |\mu - y(t)| \right\}, \tag{20}$$

$$M(x_0, y_0, \rho_0) = \left\{ \mu : \ |\mu - x_0|^2 \geq \frac{\rho_0}{\beta} |\mu - y_0| \right\}. \tag{21}$$

We can now formulate the following essential statement for Set (20).

Lemma 2. *Let Pursuer apply strategy* (6)*. Then, for any* $v(\cdot) \in V$*,*

$$M\Big(x(t), y(t), \rho(t) \Big) = x(t) + \Theta(t, v(\cdot)) \Big[M(x_0, y_0, \rho_0) - x_0 \Big], \ \ t \in [0, t^*]. \tag{22}$$

Proof. By Set (20), it can be seen that the relationship

$$\mu \in M\Big(x(t), y(t), \rho(t) \Big) - x(t)$$

is identical to

$$|\mu|^2 \geq \frac{\rho(t)}{\beta} |\mu + z(t)|. \tag{23}$$

Clearly, it suffices to analyse (23) for $t \in [0, t^*)$ when $\Theta(t, v(\cdot)) > 0$. As $z(t) = z_0 \Theta(t, v(\cdot))$, where $\Theta(t, v(\cdot))$ is defined by (11), in view of (14), we rewrite (23) in the form

$$\left| \Theta^{-1}\Big(t, v(\cdot) \Big) \mu \right|^2 \geq \frac{\rho_0}{\beta} \left| \Theta^{-1}\Big(t, v(\cdot) \Big) \mu + z_0 \right|.$$

From this, we infer

$$\Theta^{-1}\Big(t, v(\cdot) \Big) \mu \in M(x_0, y_0, \rho_0) - x_0,$$

or, equivalently,

$$\mu \in \Theta(t, v(\cdot)) \Big[M(x_0, y_0, \rho_0) - x_0 \Big].$$

Accordingly, we arrive at the equation

$$\begin{aligned} M\Big(x(t), y(t), \rho(t) \Big) - x(t) &= \left\{ \mu : \ |\mu|^2 \geq \frac{\rho_0}{\beta} |\mu + z_0| \right\} \\ &= \Theta(t, v(\cdot)) \Big[M(x_0, y_0, \rho_0) - x_0 \Big], \end{aligned}$$

which is the desired result. The proof is complete. □

Lemma 3. *The multi-valued mapping* $M\Big(x(t), y(t), \rho(t)\Big) - tx_1$*,* $t \in [0, t^*]$*, is monotonically decreasing with respect to the inclusion, i.e., if* t_1*,* $t_2 \in [0, t^*]$ *and* $t_1 < t_2$*, then* $M\Big(x(t_1), y(t_1), \rho(t_1)\Big) - t_1x_1 \supset M\Big(x(t_2), y(t_2), \rho(t_2)\Big) - t_2x_1$.

Proof. By virtue of (6) and (8), for any $v(\cdot) \in \mathbb{V}$, we have

$$\Big|v(t) - \theta(v(t))\xi_0\Big|^2 = \eta_0\theta(v(t)).$$

Hence, in accordance with (4),

$$\Big|v(t) - \theta(v(t))\xi_0\Big|^2 \geq \frac{\eta_0|v(t)|}{\beta}\theta(v(t)). \tag{24}$$

Multiplying both sides of (24) by $|z_0|^2/\theta^2(v(t))$ and using $\eta_0 = \rho_0/|z_0|$ yields

$$\left|\left(\frac{v(t)|z_0|}{\theta(v(t))} + y_0\right) - x_0\right|^2 \geq \frac{\rho_0}{\beta}\left|\left(\frac{v(t)|z_0|}{\theta(v(t))} + y_0\right) - y_0\right|.$$

From this, letting $\mu = \frac{v(t)|z_0|}{\theta(v(t))} + y_0$, we obtain (21), and so

$$\frac{v(t)|z_0|}{\theta(v(t))} + y_0 \in M(x_0, y_0, \rho_0);$$

hence,

$$v(t)|z_0| + \theta(v(t))y_0 \in \theta(v(t))M(x_0, y_0, \rho_0). \tag{25}$$

For an arbitrary $\psi \in \mathbb{R}^n$, where $|\psi| = 1$, the multi-valued mapping $M\Big(x(t), y(t), \rho(t)\Big)$ has the support function

$$F\Big(M(x(t), y(t), \rho(t)), \psi\Big) = \sup_{\mu \in M(x(t), y(t), \rho(t))} \langle \mu, \psi \rangle.$$

Due to the properties of the Support Function $F\Big(M(x(t), y(t), \rho(t)), \psi\Big)$ (see Property 6 in [36]), the relationship (25) implies that

$$\langle v(t)|z_0|, \psi \rangle - \theta(v(t))F\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big) \leq 0,$$

and so

$$\langle v(t), \psi \rangle - \frac{1}{|z_0|}\theta(v(t))F\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big) \leq 0 \tag{26}$$

for all $\psi \in \mathbb{R}^n$, $|\psi| = 1$.

By integrating both sides of (26) over $[0, t]$ and by the properties of the support function (see Theorem 2 in [36]), we find

$$\left\langle \int_0^t v(s)ds, \psi \right\rangle - \frac{1}{|z_0|}\int_0^t \theta(v(s))dsF\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big) \leq 0. \tag{27}$$

We use (6), (9), (11), and (22) to calculate the derivative of $F(M(x(t), y(t), \rho(t)), \psi)$ with t:

$$\frac{d}{dt}F\Big(M(x(t), y(t), \rho(t)), \psi\Big)$$
$$= \frac{d}{dt}F\left(x_0 + x_1 t + \int_0^t (t-s)\mathbf{u}(v(s))ds + \Theta(t, v(\cdot))\Big[M(x_0, y_0, \rho_0) - x_0\Big], \psi\right)$$

To transform the right-hand side, we use the property of the support function and Equation (11)

$$\langle x_1, \psi\rangle + \left\langle \int_0^t \mathbf{u}(v(s))ds, \psi \right\rangle - \left(\frac{1}{|z_0|}\int_0^t \theta(v(s))ds\right)F\Big(M(x_0, y_0, \rho_0) - x_0, \psi\Big)$$
$$= \langle x_1, \psi\rangle + \left\langle \int_0^t \big(v(s) - \theta(v(s)\big)\xi_0 ds, \psi \right\rangle - \left(\frac{1}{|z_0|}\int_0^t \theta(v(s))ds\right)F\Big(M(x_0, y_0, \rho_0) - x_0, \psi\Big).$$

In view of $z_0 = y_0 - x_0$, this expression takes the form

$$\langle x_1, \psi\rangle + \left\langle \int_0^t v(s)ds, \psi \right\rangle - \left(\frac{z_0}{|z_0|}\int_0^t \theta(v(s))ds, \psi\right)$$
$$- \left(\frac{1}{|z_0|}\int_0^t \theta(v(s))ds\right)F\Big(M(x_0, y_0, \rho_0) - x_0, \psi\Big)$$
$$= \langle x_1, \psi\rangle + \left\langle \int_0^t v(s)ds, \psi \right\rangle - \left(\frac{1}{|z_0|}\int_0^t \theta(v(s))ds\right)F\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big).$$

Thus,

$$\frac{d}{dt}F\Big(M\Big(x(t), y(t), \rho(t)\Big), \psi\Big) = \langle x_1, \psi\rangle + \left\langle \int_0^t v(s)ds, \psi \right\rangle$$
$$- \left(\frac{1}{|z_0|}\int_0^t \theta(v(s))ds\right)F\Big(M(x_0, y_0, \rho_0) - y_0, \psi\Big),$$

and so, for any $\psi \in \mathbb{R}^n$, $|\psi| = 1$, by (27), we obtain

$$\frac{d}{dt}F\Big(M\Big(x(t), y(t), \rho(t)\Big) - tx_1, \psi\Big) \le 0,$$

which completes the proof of Lemma 3. □

Lemma 3 plays a key role in proving the following statements.

Corollary 1. *It can be directly inferred from Lemma 3 that:*

(a) $M\Big(x(t), y(t), \rho(t)\Big) \subset M(x_0, y_0, \rho_0) + tx_1$ *at each* $t \in [0, t^*]$;

(b) $y(t) \in M(x_0, y_0, \rho_0) + tx_1$ *for all* $t \in [0, t^*]$.

We call the set

$$M^*(x_0, y_0, \rho_0, T) = \bigcup_{t=0}^{T} \Big\{ M(x_0, y_0, \rho_0) + tx_1 \Big\}$$

the *attainability domain of the evader* in the pursuit game.

Theorem 3. *Suppose* $\rho_0 \geq 4\beta|z_0|$ *and* $M^*(x_0, y_0, \rho_0, T) \cap L = \varnothing$. *Then* Π*-strategy* (6) *is winning on the time interval* $[0, T]$ *in the the lifeline game* L, *where* $T = \sqrt{2|z_0|/\theta_1}$.

Proof. The proof immediately follows from Theorem 1, Lemma 3, and Corollary 1. □

Theorem 4. *Let* $\rho_0 < 4\beta|z_0|$. *Then, there exists a control* $v(\cdot) \in \mathbb{V}$ *guaranteeing that the evader wins in the lifeline game* L.

Proof. The proof follows directly from Theorem 2. □

6. Conclusions

In the present paper, we have discussed the pursuit–evasion games of one inertial pursuer and one inertial evader with integral and geometrical constraints on the controls, respectively.

In the pursuit game, we have defined the Π-strategy for the pursuer, and we have found a sufficient solvability condition of pursuit. In addition, we have demonstrated that the Π-strategy is optimal, i.e., the evader, using the control $\mathbf{v}^*(t)$, remains uncaught by the guaranteed capture time T. In the Evasion Game, we have obtained the sufficient solvability condition of escape by the control $\mathbf{v}^*(t)$ of the evader.

Moreover, the attainability domain of the evader $M^*(x_0, y_0, \rho_0, T)$ in the Pursuit Game has been constructed. For the case $M^*(x_0, y_0, \rho_0, T) \cap L = \varnothing$ (see Theorem 3), the sufficient solvability condition of the lifeline game for the pursuer has been determined.

In the present paper, the lifeline game is not studied for the case $M^*(x_0, y_0, \rho_0, T) \cap L \neq \varnothing$, and therefore, we suggest the readers study this case. Additionally, to construct the strategy, we allow the pursuer to know the current value of the control v of the evader. For the future work, we recommend the reader to solve a lifeline game when pursuer uses a positional strategy of the form $u(x, y)$.

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