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Modeling financial leasing by optimal stopping approach

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Abstract
Leasing valuation is a topic that has aroused considerable interest in business circles. This paper examines leasing from the point of view of the lessor who can decide to leave the contract due to default. We analyze in introducing a model in which the lessor decides whether or not to terminate the contract at a given point in time, comparing it with the cost of capital of alternative investments. The proposed model is stochastic, and it is strongly based on correlated random walks, making it more adaptable to real-world circumstances. Furthermore, we propose a recombinant binomial tree based on correlated random walks, performing numerical simulations starting from CIR and Vasicek models. We will point out that as the rate of cost of capital of an alternative investment increases, the optimal boundary curve decreases, so the lessor leaves, while as the past interest rates increases, the curve rises and the lessor will have a concrete interest in maintaining the contract.

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1 Introduction

Capital investment is not the result of today's culture and approach; rather, it has been and continues to be universally acknowledged as the primary tool for the advancement of economic growth. Because of this, the transformation of traditional contracts was necessary in order to facilitate the development of new structures that are more effective and are in line with the most recent tactics of a market economy that is continually expanding. Due to these factors, a new strategy of investing financial resources, known as a leasing operation, has developed all over the world. Leasing was shown to be a brilliant solution, which was able to adjust properly to the requirements of a constantly shifting economy. Leasing provides a sustainable and dependable alternative to the traditional credit or lease agreements, making it an attractive choice. In point of fact, the examination of leasing evolution trends is connected to the development of industrial technology, shifts in fiscal and monetary policies, as well as the general transformation of the global economy. All of these elements have had a distinct impact on the level of each state, which is evidence that the leasing market reflects some special characteristics that vary throughout continents, regions, and countries. Leasing activities, on the other hand, have been a constant of the technical-scientific revolution. This is true despite the fact that the global economy may be expanding or contracting, and this holds true both during times of economic boom and during periods of economic recession. Leasing is treated economically in the majority of international legal systems, and it is recognized as a contemporary form of commercial relations. Leasing contracts are extensively used in durable goods markets. A third of the capital equipment in U.S. corporations is leased. The topic of leasing contracts has been the subject of a significant amount of research and writing in the field of finance. A great part of this literature operates on the assumption that the structure of the leasing contract is already known, and it derives the equilibrium lease value and the rental rate for a diverse range of leasing contracts (see, e.g., McConnell et al. (1983); Grenadier (1995) and Grenadier (2005)). The most comprehensive examination of the factors that determine the corporate leasing policy may be found in Smith and Wakeman (1985) work. In the case of bankruptcy, it is far simpler for a lessor to reclaim an asset than it is for the holder of a secured debt to do so. So, it is possible that businesses who have trouble securing finance may be expected to lease. Sharpe and Nguyen (1995) show evidence that businesses that are expected to incur high financial contracting costs lease a greater proportion of their capital equipment. Smith and Wakeman (1985) also explore the reasoning behind a number of conditions that are commonly included in lease agreements. Their explanation of the many choices to purchase at the conclusion of the lease is the one that is most pertinent to our investigation. They contend that the purpose of this clause is to provide the lessee with an incentive to maintain proper care of the asset. Leasing has been widely recognized as one of the most lucrative methods for financing productive ventures. This provides an additional layer of protection to individuals who lack sufficient financial resources.

The rise that leasing has had in the capital markets of the world economy is evidence of the benefits that leasing operations bring to the parties involved. These benefits are evidenced both by the practical results that leasing operations have had over the course of time and by the rise that leasing has had in the capital markets. The dynamics of social development, as well as the specific expansion of a market economy that is in a continuous process of restructuring, in the midst of changes in economic, social, and institutional structures, require a reform of national legislation that is in line with both the realities of the present and the objective of achieving a European and international economic and monetary unit. The modeling of the existing legislative framework in accordance with the requirements of society will have as its primary interest, not only to provide appropriate and immediate solutions to economic and social realities, but also to eliminate legal traditions that have destabilized this market segment in recent years. This will have as a secondary interest, providing adequate and immediate solutions to economic and social realities, see, e.g., Tomescu (2022). The vast majority of the literature, already available, focuses on asset leasing arrangements. Just to cite a few examples: Franks and Hodges (1978) derived a simple formula for valuing leases and studied the impact on their analysis of different tax rates and the different amounts of debt that can be shifted by the lease; Myers et al. (1976) examined the company's rent in case of debt problems; Realdon (2006) has provided closed-end solutions for pricing secured bank loans and finance leases subject to default risk; Dmitrieva et al. (2019) studied the effect of the lease. Grenadier (2005) offered a complete picture in one of his first publications for estimating the equilibrium credit spread on leases subject to risk of default. His approach is adaptable to a broad range of real-world leasing structures, including usage-indexed leases, security deposits, required upfront payments, embedded lease options, and lease credit insurance agreements, to name a few. Grenadier (1995) builds a single pricing framework for a wide variety of lease agreements by applying a real options strategy to endogenously determine the full term structure of lease rates. This allows you to evaluate lease agreements more accurately. The structure of the model is very similar to that of more conventional representations of the term structure of interest rates. The author also demonstrates that the model is adaptable enough to calculate equilibrium lease rates for leases of virtually any structure and virtually any term, including term leases, leases with options to renew or cancel, lease insurance contracts, variable rate leases and leases with payments that depend on the use of assets. Foo Sing and Liang Tang (2004) model the lessee's default options and estimate the economic value of the options using an American binomial discrete-time option pricing model. The results show a positive relationship between the option premium and the original fee and a negative relationship between the transfer costs and the risk-free rate. Trigeorgis (1996) discusses the numerical valuation of lease agreements with a variety of embedded operating options, using a contingent loss analysis (CCA) (contingent claim analysis) of operating lease options and a CCA-based numerical analysis method. Returning to the definition of finance leasing understood as a transaction in which the lessor purchases an asset from a supplier and leases it to the lessee for most of its useful life. Leasing is characterized by an original financing system, followed by an agreement by which the lessor transmits to the lessee, in exchange for payments, the right to use an asset for an agreed upon period of time; this is subject to the condition that at the end of this period, the financier/lessor respects the right of the user/lessee to opt for the extension of the contract see Baber (2019). During the lease term, the lessee enjoys possession and profits and is responsible for maintenance. At the end of the lease period, the lessee continues to take over the asset or it is sold, see, e.g., Simon (2007). Therefore, in this study we will generally treat leasing from the point of view of the lessor who cancelled the contract because the lessee has not paid the expected fees; in this case, a lease agreement is defined as defaulting. So we are going to analyze a model in which the lessor decides whether or not to rescind the contract in a given time making a comparison with the capital cost of an alternative investments. For many years, leasing evaluation is a subject that has drawn significant interest in economic environments. Franks and Hodges (1978) have approached the topic from a variety of approaches.

The aim of this paper is to introduce in the related literature concerning the leasing issues new advances strictly connected with some real evidences regarding practitioner's problem solving. The theoretical framework is the basis to start in working in this direction for building a decision making platform. The paper is organized as follows. In section 2, we analyze a leasing contract in which, as of a certain date, the lessee no longer pays the rent. We study the optimal moment to terminate the lease when the default interest is higher than the interest rate of outstanding expenses in the event that the return on alternative investments is stochastic. In section 3 we formulate the problem as a finite-term optimal stopping problem, and motivate it that it is treated directly with a numerical method. Finally, in section 4, we demonstrate the efficiency of our model by performing numerical simulations to calculate the optimal stopping value.

1.1 Motivation of the study

Our study has found particular attention from the international legislator, see for more e.g. Safarova (2021). International regulatory interventions, in addition to being an effective testimony of the persistent topicality of some issues related to the financial leasing operation, see Nechaev et al. (2022), it is believed necessary for verifying the impact and effects of the products themselves in relation to the figure of the finance lease and the qualifying features of the same. Thus, the methodology of the proposed study is a mathematical approach, which allows us to study the experience of the legal regulation of leasing from the point of view of international legal traditions and modern realities in order to identify general and international trends. Starting from Cananà et al. (2022) the authors present a model by which to find the conditions determining the optimal time to rescind the contract at issue, when the arrears interest rate is higher than the interest rate of the outstanding fees established in the contract and the return of alternative investments is purely deterministic. By this paper we are going to analyze a similar context of analysis into a stochastic environment more adaptable to some real circumstances useful not only for an academician but also for financial corporate practitioners, in particular. Motivated by the valuation of American option prices, in this paper we study the leasing contract as an optimal stopping problem (Dayanik and Karatzas 2003) in which the stochastic dynamics of the opportunity cost of alternative investment for the residual time follows an affine term structure models.

It is well-known that the Vasicek model and Cox, Ingersoll and Ross (CIR) model are both affine models. At this end, we study numerical schemes for optimal stopping in the framework of affine term structure model. We propose a numerical scheme is based on correlated random walks In conclusions, the regulatory interventions are deemed necessary to verify the impact and effects of the same products in relation to the figure of the financial leasing and the qualifying features of the same, as well as a survey on the possible emergence of new or revitalized problems resulting precisely from these more recent events of the economic crisis. The economic crisis has undoubtedly created difficulties across the board for all sectors and legislative interventions have not always been sufficient to find effective solutions without harming the interests of the two counterparts. The economic decision whether to lease or alternative investment remains a challenging task for many companies willing to review their decisions. Hence, the need and the idea to provide, after an introduction to the leasing problem, a mathematical analysis of the problem supported by numerical simulations which allow the two counterparties to decide whether or not to continue with the contract compatibly with compliance with the international regulatory rules. We therefore want to support a widely detailed Law items with numerical evidence that supports any decisions of the lessee and lessor. The results obtained, in fact, show that as the cost of capital rate of an alternative investment increases, the optimal contour curve decreases, therefore the landlord cancels the lease, while as the arrears interest rate increases, the curve rises and the lessor will have an interest accrued in maintaining the lease.

2 Searching for the optimal time to leave the contract: some mathematical aspects

In this paper, we consider a leasing contract in which, from a certain date forward, the lessee no longer pays the fees. As a consequence, the lessor decides whether or not to rescind the contract at date z; the lessor, in fact, compares it with the capital cost of capital of an alternative investment. Mananà et al. (2022) study the optimal time to terminate the lease contract when the arrears interest is higher than outstanding fees interest rate in the case that the return alternative investments is deterministic. We analyze the same problem but we suppose that the capital cost of an alternative investment is stochastic.

As already mentioned (see Mananà et al. 2022), in the case the lessee is insolvent the credit at time z for the lessor is the sum of the amount of the expired fees capitalized in simple interest accumulation at the arrears interest rate m and the outstanding fees $\gamma(s)$ with the redemption price R discounted with a lower force of interest δ. Hence, the lessor credit in case of insolvency is

$$C(z) := \int_0^z \gamma(s)(1 + m(z - s))\mathrm{d}s + \int_z^T \gamma(s)e^{-\delta(s-z)}\mathrm{d}s + Re^{-\delta(T-z)}.$$

To describing the stochastic dynamics of the opportunity cost of alternative investment for the residual time $T - z$, we assume that the lessor can invests at the time $z \in [0, T]$ the amount $C(z)$ in a bond market. We assume the term structure of interest rates of the bond market is driven by one-dimensional stochastic process for the instantaneous spot rate r given by:

$$\mathrm{d}r(z) = \hat{\mu}(r(z))\mathrm{d}z + \varphi(r(z))\,\mathrm{d}\tilde{W}_z \tag{1}$$

where $\tilde{W}$ is a standard Wiener process on the filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_z)_{z\in[0,T]}, \mathbb{P}\big)$ where the filtration $(\mathcal{F}_z)_{z\in[0,T]}$ satisfies the usual conditions and $\hat{\mu}$ and φ are real valued functions such that stochastic differential equation (SDE) (1) have a unique strong solution. Moreover, we assume that the interest rate model possesses a time-homogeneous affine term structure, i.e., the continuously compounded spot rate, $R(z, T)$, is an affine function of $r(z)$ (see Brigo and Mercurio (2006)):

$$R(z, T) = \hat{A}(z, T) + \hat{B}(z, T)r(t) \tag{2}$$

where $\hat{A}(z, T)$ and $\hat{B}(z, T)$ are deterministic functions of time. Note that the discount factor, i.e., the value at time z of one unit of currency at time T, is

$$P(z, T) = e^{-R(z,T)(T-z)}.$$

As usual, we assume that the bond market is free of arbitrage. Therefore, there exists a so called market price of risk process which defines a Girsanov change of measure Q such that

$$P(z, T) = E^{Q}\left(e^{-\int_z^T r(u)\mathrm{d}u}|\mathcal{F}_z\right).$$

In other words, Q is a martingale measure for the bond market (see Bjork (1997)). We do not specify the market price of risk but we assume directly that the dynamics of the instantaneous spot rate is

$$\mathrm{d}r(z) = \mu(r(z))\mathrm{d}z + \varphi(r(z))\,\mathrm{d}W_z, \qquad r(0) = r_0, \tag{3}$$

where W is a Wiener process with respect to a fixed martingale measure Q and r_0 is a positive constant. For a time-homogeneous affine-term structure, we have the following result (see proposition 3.5 in Bjork (1997)).

Theorem 1 *Suppose that process r is the solution of the SDE (3). The corresponding interest rate model has a time-homogeneous affine-term structure if and only if there exist some constants c_i, $i = 0, \ldots, 3$ such that $\mu(r) = c_0 + c_1 r$ and $\varphi^2(r) = c_2 + c_3 r$. Moreover, if*

$$P(z, T) = A(z, T)e^{-B(z,T)r(z)}$$

the function A and B are the solutions of the following differential equations

$$\begin{cases} \frac{\partial}{\partial z} B(z,T) = \frac{1}{2} c_3 B^2(z,T) - c_1 B(z,T) - 1 \\ B(T,T) = 0 \\ \frac{\partial}{\partial z} (\log A(z,T)) = -\frac{1}{2} c_0 B^2(z,T) + c_2 B(z,T) \\ A(T,T) = 1 \end{cases} \tag{4}$$

We build our problem assuming that r is given by Vasicek's model or Cox, Ingersoll and Ross's (CIR's) model. We note that both models have the same drift factor $\mu(r) = \alpha(\theta - r)$, thus ensuring that the mean reversion of the spot rate toward the long run value θ with an adjustment speed given by the parameter $\alpha > 0$. For both models $c_0 = -\alpha$ and $c_1 = \alpha\theta$.

The diffusion term is $\varphi(r) = \sigma$ for Vasicek's model and $\varphi(r) = \sigma\sqrt{r}$ for CIR's model, where σ is a positive constant. Therefore, we have

- Vasicek $c_2 = 0$ and $c_3 = \sigma^2$;
- CIR $c_2 = \sigma^2$ and $c_3 = 0$.

The first differential equation of system (4) is a Riccati equation which in general we have to solve numerically. Instead, for Vasicek's and CIR's models it is easy to find the analytical solution. For the Vasicek model, we have

$$A(z,T) = exp\left\{\left(\theta - \frac{\sigma^2}{2\alpha^2}\right)[B(z,T) - T + z] - \frac{\sigma^2}{4\alpha} B(z,T)^2\right\}$$

$$B(z,T) = \frac{1}{\alpha}\left[1 - e^{\alpha(T-z)}\right]$$

and for the CIR model, we have

$$A(z,T) = \left[\frac{2hrmexp\left\{(\alpha+h)\frac{(T-z)}{2}\right\}}{2h + (\alpha+h)(rmexp(T-z)h - 1)}\right]^{\frac{2\alpha\theta}{\sigma^2}}$$

$$B(z,T) = \frac{2(rmexp(T-z)h - 1)}{2h + (\alpha+h)(rmexp(T-z)h - 1)}$$

where $h = \sqrt{\alpha^2 + 2\sigma^2}$.

By convention, we can define the early exercise boundary as the optimal solution for the following problem of first passage time through a boundary. The lessor's problem can be stated as a finite-maturity optimal stopping problem and be expressed as

$$\sup_{\tau\in\mathcal{S}} \mathrm{E}u\left(\frac{C(\tau)}{P(\tau,T)}\right) = \sup_{\tau\in\mathcal{S}} \mathrm{E}u\left(\frac{C(\tau)}{A(\tau,T)} e^{B(\tau,T)r(\tau)}\right)$$

where $\mathcal{S}$ is the set of stopping time with respect to the filtration $(\mathcal{F}_z)_{z\in[0,T]}$. Its aim is to maximize the final wealth measured by some utility function. The utility function

u is defined as a function that is strictly increases, strictly concave and continuously differentiable. Based on the wording of the model, it's easy to prove that the optimal boundary curve does not depend on $u(x)$.

Even for relatively simple problems, there are no explicit solutions for an optimal stopping problem driven by a one-dimensional diffusion process. We perform some numerical simulations to calculate the optimal stopping value depending on the initial spot rate r_0.

We also calculate the optimal boundary curve separating the region enticing to uphold the contract from the region enticing that doesn't. We use the numerical methods developed by Bayraktar et al. (2018) to construct recombining tree approximations that apply the Skorokhod embedding technique for small perturbations of the correlated random walks.

3 Model mechanism design and its tools

In this paper, we propose a recombining binomial tree who is based on correlated random walks. The optimal stopping value is given by:

$$V = \sup_{\tau \in \mathcal{S}} Ef(\tau, r(\tau)) \tag{5}$$

where $\tau \in \mathcal{S}$ is the set of stopping times on $\mathcal{S}$ with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. In this case the function $f : [0, T] \to \mathbb{R}$ is:

$$f(\tau, r) = \frac{C(\tau)}{A(\tau, T)} e^{B(\tau,T)r}$$

As a first step, we have detailed the construction of a binomial tree based on the discretization of the process $(r(t))_{t\in[0,T]}$. Especially the algorithm foresees the approximation of the stochastic process $(r(t))_{t\in[0,T]}$ by a correlated random walks that build a recombining binomial tree that is space lattice and on uniform time.

After this, we approximate the optimal stopping problem with a discrete time problem by using standard dynamical programming (5). We suppose that the stochastic process $(r(t))_{t\in[0,T]}$ is one-dimensional diffusion process and it is weak solution of following autonomous stochastic differential equation:

$$\begin{cases} rmdr(t) = \mu(r(t))rmdt + \sigma(r(t))rmdW_t \\ r(0) = \overline{r}_0 \in \mathbb{R} \end{cases}$$

where $(W_t)_{t\in[0,T]}$ is a standard one dimensional Brownian motion with respect to a filtration $(\mathcal{F}_z)_{z\in[0,T]}$ which satisfies the usual conditions.

The functions μ e σ are bounded and Lipschitz continuous on $]D, U[$, with $D < U$, $D \geq -\infty$ and $U \leq +\infty$. Moreover we assume that there is $\epsilon > 0$ such that $\sigma(x) > \epsilon$, for all $x \in]D, U[$. In the case of D or U which is finite number, the process $(r(t))_{t\in[0,T]}$ must be truncated. Therefore, by assuming that $\overline{r}_0 \in]D, U[$, we build a process $X = (X_z)_{z\in[0,T]}$ which has absorbing barriers D and U. The absorbed stochastic process is:

$$X_z = \mathbb{I}_{z<\inf\{t:\, r_t\notin]D,U[\}}\; r(t) + \mathbb{I}_{z\geq\inf\{t:\, r(t)\notin]D,U[\}}\; r\left(\inf\{t:\; r(t)\notin]D,U[\}\right) \qquad (6)$$

Let $n \in \mathbb{N}$ and let $h = \frac{T}{n}$ be the time discretization. Let $\xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ be a sequence of random variables, each of which assumes value -1 or 1. Let us build a random walk:

$$\hat{X}_k = \overline{r}_0 + \sqrt{h}\;\sum_{i=1}^{k}\xi_i, \quad k = 0, 1, \ldots, n$$

with following absorbing barriers:

$$D_n = \overline{r}_0 + \sqrt{h}\min_{k\in\mathbb{Z}\cap[-n-1,+\infty[}\left\{\overline{r}_0 + \sqrt{h}\;k > D + \sqrt[3]{h}\right\}$$
$$U_n = \overline{r}_0 + \sqrt{h}\max_{k\in\mathbb{Z}\cap]-\infty,n+1]}\left\{\overline{r}_0 + \sqrt{h}\;k < U - \sqrt[3]{h}\right\}$$

Let $\mathcal{G}_k = \sigma\{\xi_1, \ldots, \xi_k\}$, $k = 1, \ldots, n$, the filtration generated by the sequence of random variables $\xi_1, \ldots \xi_k$. The goal is to determine a probability $\mathbb{P}_n$ on $\mathcal{G}_n$ and a predictable process $(\alpha_k)_{0\leq k\leq n}$ with respect the filtration $(\mathcal{G}_k)_{1\leq k\leq n}$ so that the process $(\hat{X}_k)_{0\leq k\leq n}$ is a Markov chain which weakly approximating the truncated process (6). To this end we construct a perturbation of the process $\left(\hat{X}_k\right)_{0\leq k\leq n}$:

$$Y = \left(\hat{X}_k + \sqrt{h}\;\alpha_k\xi_k\right)_{0\leq k\leq n}$$

By imposing the following conditions on the first two moments of Y:

$$E_n(Y_k - Y_{k-1}|\mathcal{G}_{k-1}) = h\mu(\hat{X}_{k-1})$$

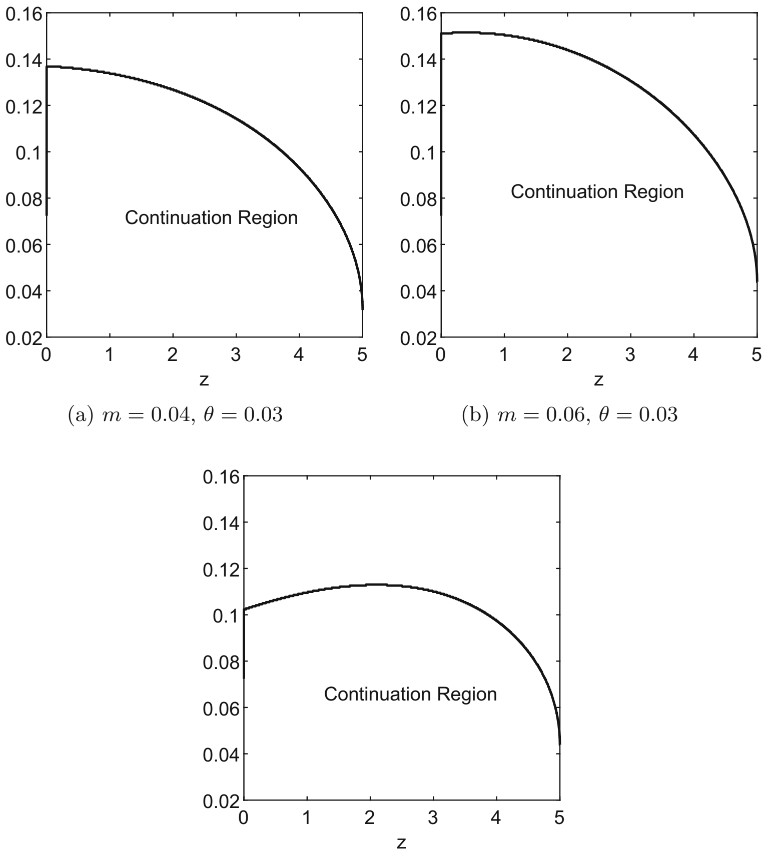
and

$$E_n((Y_k - Y_{k-1})^2|\mathcal{G}_{k-1}) = h\sigma^2(\hat{X}_{k-1})$$

from which follows, (for details see, for example, the article Bayraktar et al. (2018)):

$$\alpha_k = \frac{\sigma^2(\hat{X}_{k-1}) - 1}{2}, \quad k = 0, \ldots, n$$
$$P_n\left(\xi_k = \pm 1|\mathcal{G}_{k-1}\right) = \frac{1}{2}\left(1 \pm \frac{a_{k-1}\;\xi_{k-1} + \sqrt{h}\;\mu(\hat{X}_{k-1})}{1+\alpha_k}\right), \quad k = 1, \ldots, n$$

where $\xi_0 := 0$, $\hat{X}_{-1} := \overline{r}_0 - \sqrt{h}$ and n is large enough.



(a) $m = 0.04$, $\theta = 0.03$ (b) $m = 0.06$, $\theta = 0.03$

(c) $m = 0.06$, $\theta = 0.06$

Fig. 1 Optimal stopping boundary curve under Vasicek model's

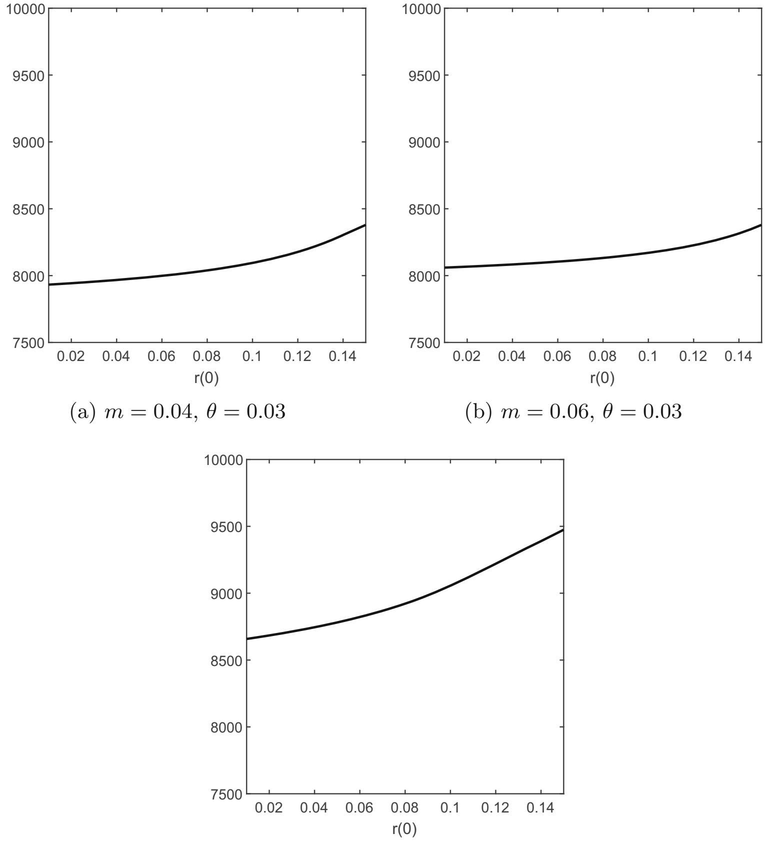
Let $\mathcal{S}_n$ be the set of all stopping times with respect to filtration $(\mathcal{G}_k)_{1 \le k \le n}$. Consider the discrete-time optimal stopping problem:

$$V_n = \sup_{\tau \in \mathcal{S}_n} E_n f(\tau h, \hat{X}_\tau).$$

This problem can be solved by backward induction using the well-known Bellman Dynamic Programming Principle (for details see Peskir and Shiryaev (2006)). Therefore, it is possible to determine an optimal stopping time τ_n such that

$$V_n = E_n(f(\tau_n h, \hat{X}_{\tau_n})).$$

By using Skorokhod embedding technique, we can prove that V_n approximates V.



(a) $m = 0.04,\ \theta = 0.03$

(b) $m = 0.06,\ \theta = 0.03$

(c) $m = 0.06,\ \theta = 0.06$

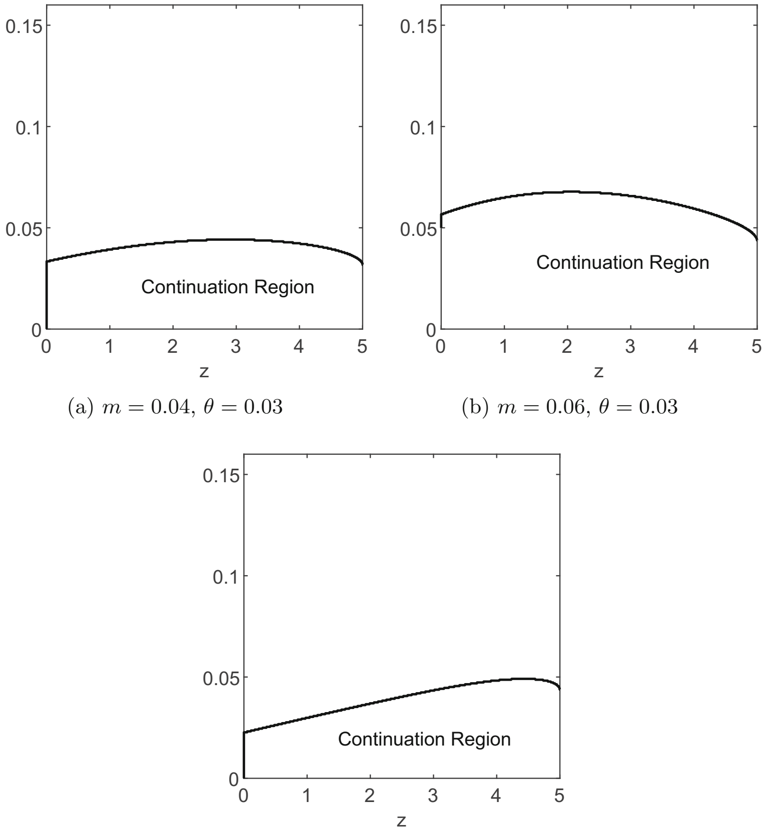
Fig. 2 The value function under Vasicek model's

4 Numerical simulations

In this paragraph, we present the optimal stopping simulations as well as the optimal boundary curve simulations, referring both to CIR's and Vasicek's models. Therefore, the recombinant binomial tree and the backward recursion procedure described above can be easily implemented. In particular, in the simulations performed we used C++. Let us remember that r and m are the rates relating to the contract. Numerical simulations are performed using the following sets of parameters:

$$\begin{array}{llll} m = \{0.04, 0.06\} & \delta = 0.02 & \gamma = 1000 & R = 2000 \\ \alpha = 1.1 & \sigma = 0.1 & \theta = \{0.04, 0.06\} & \end{array}$$

The decision to terminate the contract early at time z depends only on spot rate $r(z)$. Namely, if $r(z)$ is above the optimal boundary curve it is better to terminate the contract otherwise the opposite is better. We indicate the region where it is best to



(a) $m = 0.04$, $\theta = 0.03$

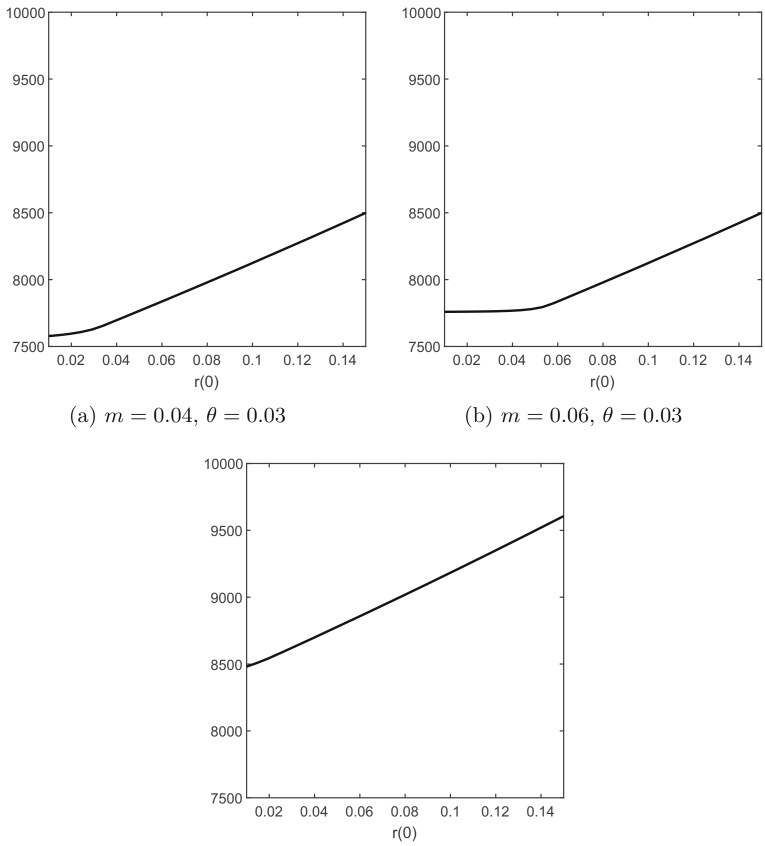
(b) $m = 0.06$, $\theta = 0.03$

(c) $m = 0.06$, $\theta = 0.06$

Fig. 3 Optimal stopping boundary curve under CIR model's

close the contract as early exercise region and the region where the opposite happens as continuation region (see Figs. 1 and 3). Therefore, if the initial spot rate $r(0)$ is in the early exercise region, it is convenient to terminate the contract immediately. Otherwise, if the initial spot rate belongs to the continuation region, two scenarios are possible. In the first case, if the random path of r hits the optimal boundary curve before the time T, then the lessor terminates the contract before the expiration date. In the second case, if the random path never hits it, then he will terminate the contract at date T. Roughly speaking, if the optimal boundary curve rises, i.e., the continuation region widens, the lessor will have a vested interest in closing the outstanding contract when the spot rate $r(z)$ reaches higher values.

In our simulations, we show the behavior of the continuation region by varying the long term interest rate θ and the arrears interest rate m. Looking at Figs. 1 and 3, we notice that m and θ have a different role on the amplitude of the continuation region. By increasing the arrears interest rate m, the continuation region widens (see Fig. 1, panel (a) and (b), and Fig. 3, panel (a) and (b)). The opposite happens for the long-term



(a) $m = 0.04,\ \theta = 0.03$ (b) $m = 0.06,\ \theta = 0.03$

(c) $m = 0.06,\ \theta = 0.06$

Fig. 4 The value function under CIR model's

interest rate θ (see Fig. 1, panel (b) and (c), and Fig. 3, panel (b) and (c)). In both CIR and Vasicek models, the continuation region is convex and, under equal conditions, it is larger for the Vasicek model.

Figures 2 and 4 show the value of the lessor's credit, at the time the lessee ceases to pay the fees, including the option to terminate the contract early and invest proceeds in the cost opportunity of the alternative investment.

5 Conclusions

A new method of investing financial resources, known as a leasing operation, has arisen as a response to the recognition that financial capital investment is necessary for economic expansion. Leasing is an appealing option since it gives a sustainable and stable alternative to conventional credit or lease agreements. This makes leasing an attractive choice. Leasing has been a consistent feature throughout the technical-scientific revolution, and the majority of international law systems analyze it from an economic perspective. It is distinguished by an initial financing method, which is then followed by an agreement in which the lender transfers the borrower the right to use an asset for a length of time that has been previously agreed upon. Contracts for leasing are commonplace in the markets for durable goods. The subject of the value of leasing has been one that has garnered a significant amount of interest in the business world for a significant number of years. The vast majority of models that have been proposed in the relevant research have taken several different approaches to the problem at hand. As a result, over the course of our research, we have focused on leasing from the perspective of the lessor. More specifically, we have investigated a model in which the lessor determines whether or not to terminate the contract at the end of a specified period by contrasting the current cost of capital with the cost of capital associated with an alternative investment. When the default interest rate is higher than the interest rate of the outstanding payments established in the contract and the return on alternative investments is purely stochastic, we present a model that can be used to find the conditions that determine the optimal moment to terminate the contract in question. We do this when the default interest rate is higher than the interest rate of the outstanding payments established in the contract. A model like this is well-suited for application in an analysis context within a stochastic environment since it is more adaptive to some real-world scenarios and can thus be of service not just to academics but also to financial firms that are actively engaged in business. A recombinant binomial tree that is based on correlated random walks that are correlated to numerical simulations that are based on CIR and Vasicek models is something that we proposed in our study. The analysis demonstrated that when the rate r increases, the ideal boundary curve drops. when a result, the lessor will terminate the lease if the rate r continues to rise; but, if the rate m continues to rise, the curve will rise, and the lessor will have a vested interest in keeping the lease. In conclusion, we demonstrate that, all other factors being held constant, the lower the market interest rates are, the greater the values of, and the cheaper it is to close the contract.

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