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Lagrange Multipliers and Nonlinear Variational Inequalities with Gradient Constraints

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The paper deals with nonlinear monotone variational inequalities with gradient constraints. In particular, using a new strong duality principle, the equivalence between the problem under consideration and a suitable double obstacle problem is proved. Moreover, the existence of L^2 Lagrange multipliers is achieved.

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1. Introduction

Aim of the paper is to study a nonlinear variational problem with gradient constraints and homogeneous Dirichlet boundary condition.

Variational problems with gradient constraints have been intensively studied a few decades ago and have seen many progresses also recently.

Several related questions have been investigated, as regularity of the solution, existence of Lagrange multipliers, connection with other variational problems, numerical aspects and so on.

Variational problems with gradient constraints arise in elastoplasticity of materials, optimal control problems, mathematical finance, sandpiles, superconductors (see [19,38] and reference therein). An interesting property of variational problems with gradient constraints is the connection with double obstacle problems, even if this equivalence is not true in the general case (see [18], and counterexamples in [26,37] in the case of nonconstant gradient constraint $|Du| \le g(x)$).

Indeed, an important example among them is the well-kwon elastic-plastic torsion problem, which is the problem of finding a function u belonging to the convex set

$$\mathbb{K} = \left\{ v \in H_0^{1,2}(\Omega) : |Dv| \le 1 \text{ a.e. in } \Omega \right\}$$

such that

$$\int_\Omega \nabla u \, \nabla(v-u)\, dx \ge \int_\Omega f(v-u)\, dx, \quad \forall v \in \mathbb{K}. \tag{1.1}$$

Existence and uniqueness of the solution to (1.1) easily follows (see [39]). In 1972 Brezis proved in [2] that there exists a unique Lagrange multiplier $\lambda \in L^\infty(\Omega)$, when f is a positive constant. Moroever, in [3] the authors prove the equivalence between the elastic-plastic torsion problem (1.1) and an obstacle problem, where the obstacle is the distance function.

In the paper we study a nonlinear monotone variational problem, subject to a strictly convex gradient constraint. In particular, we give the following contributions: first, we provide an equivalence result (Theorem 3.1) between the problem with gradient constraints and a suitable double obstacle problem, using a viscosity solution idea for Hamilton-Jacobi equation as in [9,30]. This connection allows us to apply a new duality principle, holding under a constraint qualification condition (Assumption S), that is verified at the solution of the double obstacle problem. As a second result, we obtain the existence of L^2-Lagrange multipliers.

Let us introduce the problem in more detail.

Let $a(p) : \mathbb{R}^n \to \mathbb{R}^n$ be an operator of class C^1, strongly monotone and let $G : \mathbb{R}^n \to \mathbb{R}$ be a C^2 nonnegative uniformly strictly convex function. Setting

$$K = \left\{ v \in H_0^{1,2}(\Omega) : G(Dv) \le M \text{ a.e. on } \Omega \right\},$$

from the monotonicity of the operator easily follows the existence and the uniqueness of the solution u to the problem:

$$\mathit{Find}\ u \in K : \int_\Omega \sum_{i=1}^n a_i(Du) \left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \ge \int_\Omega f(v-u)dx, \quad \forall v \in K. \tag{1.2}$$

In the paper, in a first step, we prove, under strong monotonicity assumption (3.1), strict convexity (3.2) and assuming f to be a positive constant, the equivalence between problem (1.2) and a suitable double obstacle problem:

$$\mathit{Find}\ u \in K_{w^-,w^+} : \int_\Omega \sum_{i=1}^n a_i(Du) \left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \ge \int_\Omega f(v-u)dx, \quad \forall v \in K_{w^-,w^+}, \tag{1.3}$$

where

$$K_{w^-,w^+} = \left\{ v \in H_0^{1,2}(\Omega) : w^- \le v(x) \le w^+(x) \text{ a.e. on } \Omega \right\},$$

and

$$w^+ = \sup_{v \in K} v(x), \quad w^- = \inf_{v \in K} v(x). \tag{1.4}$$

(see Theorem 3.1).

It should be pointed out that the functions w^+, w^- are well defined. As observed in [30] (pp. 409, 413), in general w^+ and w^- only are uniformly Lipschitz continuous, but the strict convexity assumption on G (3.2) guarantees that w^+ and w^- are bounded.

The properties of the obstacles follow from the theory of viscosity solutions to Hamilton-Jacobi equations (see [32], p. 133, [31]).

Then, in a second step, an existence theorem of Lagrange multipliers is proved together with the following relationship between the coincidence set I and the plastic region P

$$I = \{x \in \Omega : u(x) = w^+(x) \text{ or } u(x) = w^-(x)\} = P = \{x \in \Omega : G(Du) = M\}. \tag{1.5}$$

Let us remark that we denote by P the plastic region in virtue of physical properties of the elastic-plastic torsion problem.

The paper is organized as follows. In Section 2 we discuss some related works. In Section 3 we illustrate the problems in details and state the main results of the paper. In Section 4 we remind some notations and some results of duality theory, that we need in the sequel. In Section 5 we introduce a suitable double obstacle problem and we investigate the connection between this problem and the problem with gradient constraints. Moreover, in Section 6 we prove the existence of Lagrange multipliers. Finally, in Section 7 we draw some conclusions and discuss our ongoing research.

2. Related work

The gradient constrained problem is a classical problem intensively studied a few decades ago.

As already recalled in Section 1, the problem with constant gradient constraint $|Du| \le 1$, the well-known elastic-plastic torsion problem, has been deeply studied, together with its equivalence with the obstacle problem and with a Lagrange multiplier problem (see [2,3,5,6,20–25]).

The regularity of the solution, the equivalence with other variational problems and with a Lagrange multiplier problem have been investigated in the case of nonconstant gradient constraints too.

In [18] the author studied general linear elliptic equations with a non-constant gradient constraint, $|Du| \le g(x)$, with $g(x) \in C^2(\Omega)$, and proved an existence and uniqueness result in the space $W^{2,p}(\Omega) \cap W^{1,\infty}(\Omega)$, with $1 < p < \infty$. He also obtained $W^{2,\infty}$ regularity under some additional restrictions. Those restrictions were removed by M. Wiegner in [41]. Some extended results were obtained by H. Ishii and S. Koike in [28], who considered the existence and uniqueness of the solution of the variational inequalities of the forms, which are considered by Evans. They proved that, if f and g are allowed to vanish simultaneously, uniqueness of solutions may fail. Other regularity results are contained in [12,30].

In [37] L. Santos obtained, under an extra condition on the constraint g, the equivalence of an evolutive variational inequality with non-constant gradient constraints associated with the Laplacian with a double obstacle variational problem (see also [38] for existence of Lagrange multipliers for a non-constant gradient constraint problem associated with the Laplacian).

Choe and Shim in [10] proved $C^{1,\alpha}$ regularity for the solution to a quasilinear variational inequality subject to a C^2 strictly convex gradient constraint (see [9] for the gradient constraint

$G(Du) \le M$). In [11] the authors generalized the regularity results in [10], by allowing more general constraints.

Moreover, in [40] the relationship between a gradient constrained problem associated with an operator of type $F(Du) + G(x, u)$ and a suitable obstacle problem is investigated. Under some suitable regularity assumptions on G and the convexity condition on F, the authors prove the equivalence of the problems.

Finally, in the recent paper [36] the author proves the optimal $W^{2,\infty}$- regularity for variational problems with non-striclty convex gradient constraints.

3. Main results

In this section we describe the assumptions and the main results of the paper.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set with lipschitz boundary $\partial\Omega$ and a be an operator of class C^1. In addition, the operator is assumed to be strongly monotone, namely there exists $\lambda > 0$, such that

$$(a(P) - a(Q), P - Q) \ge \lambda \|P - Q\|^2 \quad \forall P, Q \in \mathbb{R}^n, P \ne Q, \tag{3.1}$$

Let $G : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative C^2 strictly convex function such that

$$\sum_{i,j=1}^{n} \frac{\partial G}{\partial A_i \partial A_j} \xi_i \xi_j \ge c\|\xi\|^2 \tag{3.2}$$

$\forall A, \xi \in \mathbb{R}^n$ and for some positive constant c.

Moreover, we assume $G(0) = a(0) = 0$.

The first result we are able to prove is the equivalence between the gradient constrained problem and a suitable double obstacle problem.

Theorem 3.1. *Under the above assumptions on Ω, a and G, if $f \equiv const. > 0$, the solution u of (1.2) coincides with the solution of (1.3).*

In particular, the following relationship between the coincidence set I and the plastic region P.

Theorem 3.2. *Under the same assumptions as in Theorem 3.1, let $u \in K \cap W^{2,p}(\Omega)$ be the solution to problem (1.2). Setting*

$$I = \{x \in \Omega : u(x) = w^+(x) \text{ or } u(x) = w^-(x)\}$$
$$\Lambda = \{x \in \Omega : w^-(x) < u(x) < w^+(x)\}$$

it results

$$P = \{x \in \Omega : G(Du) = M\} = I,$$
$$E = \{x \in \Omega : G(Du) < M\} = \Lambda.$$

Finally, the following result on the existence of Lagrange multipliers holds.

Theorem 3.3. *Under the same assumptions as in Theorem 3.1, let $u \in K \cap W^{2,p}(\Omega)$ be the solution to problem (1.2). Then, there exists $\bar{\nu} \in L^2(\Omega)$ such that*

$$\begin{cases} \bar{\nu} \geq 0 \ a.e. \ in \ \Omega \\ \bar{\nu}(G(Du) - M) = 0 \ a.e. \ in \ \Omega \\ \displaystyle\sum_{i=1}^{n} \frac{\partial a_i(Du)}{\partial x_i} + f + \bar{\nu} = 0 \ a.e. \ in \ \Omega. \end{cases} \tag{3.3}$$

4. Preliminary results

In the sequel we use the strong duality theory, in order to prove our results.

We note that, in our settings, we cannot apply the strong duality property in the classical sense (see [29]), since the ordering cone, which defines the sign constraints, has an empty interior.

So, we now recall a recent strong duality principle, obtained using new separation theorems based on the notion of quasi-relative interior. To this aim, we start presenting the problem formulation and we remind some preliminary concepts.

Let S be a nonempty subset of a real linear space X; $(Y, \|\cdot\|)$ be a partially ordered real normed space with ordering cone C, with $C^* = \{\lambda \in Y^* : \ \langle \lambda, y \rangle \geq 0 \ \forall y \in C\}$ the dual cone of C, Y^* topological dual of Y.

Consider the primal problem

$$\min_{v \in \mathbb{K}} \psi(v), \tag{4.1}$$

where $\mathbb{K} = \{v \in S : \ g(v) \in -C\}$ and the dual problem

$$\max_{\mu \in C^*} \inf_{v \in S} [\psi(v) + \mu(g(v))], \tag{4.2}$$

where $\psi : S \to \mathbb{R}$ is a given objective functional, $g : S \to Y$ is a given constraint mapping and μ is the Lagrange multiplier associated with the sign constraints.

As it is well known (see [29]), the weak duality always holds:

$$\max_{\mu \in C^*} \inf_{v \in S} [\psi(v) + \mu(g(v))] \leq \min_{v \in \mathbb{K}} \psi(v). \tag{4.3}$$

Moreover, if problem (4.1) is solvable and in (4.3) the equality holds, we say that the strong duality between primal problem (4.1) and dual problem (4.2) holds.

So, strong duality is not always fulfilled and it is necessary to assume some conditions, called "constraints qualification conditions", to guarantee it.

The classical theory requires that the interior of the ordering cone is empty.

However, the ordering cone of almost all the known problems, stated in infinite dimensional spaces, has the interior (and generalized interior concepts) empty.

In [13] the authors introduced a condition called Assumption S (from Separation), which ensures strong duality and has revealed itself very effective in applications.

In the infinite dimensional settings Assumption S results to be a necessary and sufficient condition for the strong duality (see [1,14–16,27]).

We recall that, given a point $x \in X$ and a subset M of X, the set

$$T_M(x) := \left\{ h \in X : h = \lim_n \lambda_n (x_n - x), \lambda_n > 0, \ x_n \in M \ \forall n \in N, \ \lim_n x_n = x \right\}$$

is called the tangent cone to M at x.

Definition 4.1 (Assumption S)**.** *Assumption S is fulfilled at a point* $x_0 \in \mathbb{K}$ *if it results*

$$T_{\widetilde{M}}(0,\theta_Y)\cap\Big(]-\infty,0[\times\{\theta_Y\}\Big)=\emptyset, \tag{4.4}$$

where

$$\widetilde{M}=\{(\psi(x)-\psi(x_0)+\alpha, g(x)+y:\ x\in S\setminus\mathbb{K},\ \alpha\geq 0,\ y\in C\}.$$

Now, we may present the following strong duality principle (see Theorem 1.1 in [15]):

Theorem 4.1. *Assume that the functions $\psi:S\to\mathbb{R}$, $g:S\to Y$ are convex. If problem (4.1) is solvable and Assumption S is fulfilled at the extremal solution $u\in\mathbb{K}$, then also problem (4.2) is solvable.*
Moreover, if $\bar{\mu}\in C^*$ verifies (4.2), we have

$$\langle\bar{\mu}, g(u)\rangle=0 \tag{4.5}$$

and the optimal values of the two problems coincide; namely

$$\psi(u)=\min_{v\in\mathbb{K}}\psi(v)=\max_{\mu\in C^*}\inf_{v\in S}[\psi(v)+\mu(g(v))]=\psi(u)+\langle\bar{\mu},g(u)\rangle$$

Let us recall that the following one is the so-called Lagrange functional

$$L(v,\mu)=\psi(v)+\langle\mu,g(v)\rangle\quad\forall v\in S,\ \forall\mu\in C^*. \tag{4.6}$$

By means of Theorem 4.1, it is possible to show the usual relationship between a saddle point of the Lagrange functional and the solution of the constraint optimization problem (4.1) (see Theorem 5 in [13]).

Theorem 4.2. *Let us assume that the assumptions of Theorem 4.1 are satisfied. Then $u\in\mathbb{K}$ is a minimal solution to problem (4.1) if and only if there exist $\bar{\mu}\in C^*$ such that $(u,\bar{\mu})$ is a saddle point of the Lagrange functional (4.6), namely*

$$L(u,\mu)\leq L(u,\bar{\mu})\leq L(v,\bar{\mu}),\quad\forall v\in S,\ \mu\in C^*.$$

and

$$\langle\bar{\mu},g(u)\rangle=0.$$

Let us remark that, recently, other constraint qualification assumptions, which ensure strong duality, have been also introduced in [33,34].

5. Equivalence of two variational problems

The first contribution of the paper is an equivalence result between the variational problems (1.2) and (1.3).

To this aim, first we prove the following estimate for the solution u to the double obstacle problem (1.3).

Theorem 5.1. *Under the same assumptions as in Theorem 3.1, let u be the solution to the double obstacle (1.3), then*

$$|u|_1 = \sup\left\{ \frac{|u(x)-u(y)|}{|x-y|} : x, y, \in \overline{\Omega},\ x \neq y \right\} \leq L = \max\{|w^+|_1, |w^-|_1\}. \tag{5.1}$$

Proof. We generalize the result, proved in [35] in the case $a(Du) = \Delta u$ and with gradient constraint $G(Du) = |Du|^2$.

Let $\tilde{u}$ be the extension by zero of $u \in H_0^{1,2}(\Omega)$ to the whole space $\mathbb{R}^n$.

If u is the solution to (1.3), we set

$$u_h(x) = \tilde{u}(x+h) - \tilde{u}(x) - L|h| \quad \forall x, h \in R^n \tag{5.2}$$

and

$$u_h^+(x) = \max\{\tilde{u}(x+h) - \tilde{u}(x) - L|h|, 0\}. \tag{5.3}$$

We define

$$v_1(x) = \max\{\tilde{u}(x), \tilde{u}(x+h) - L\,|h|\} = \tilde{u}(x) + u_h^+(x)$$

and

$$v_2(x) = \min\{\tilde{u}(x), \tilde{u}(x-h) + L\,|h|\} = \tilde{u}(x) - u_h^+(x-h).$$

It is easily proved that

$$\begin{aligned} &v_1(x) \leq \max\{\tilde{w}^+(x), \tilde{w}^+(x+h) - L|h|\} = \tilde{w}^+(x), \\ &v_2(x) \geq \min\{\tilde{w}^-(x), \tilde{w}^-(x-h) + L\,|h|\} = \tilde{w}^-(x), \\ &v_1(x) \leq v_2(x), \end{aligned}$$

namely $v_{1/\Omega}$ and $v_{2/\Omega}$ belong to K_{w^-,w^+}.

Then, we may use $v_{1/\Omega}$ and $v_{2/\Omega}$ as admissible functions in (1.3), and taking into account that $u_h^+(x) = u_h^+(x-h) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$, we get

$$\int_{\mathbb{R}^n} \sum_{i=1}^n a_i(D\tilde{u}(x)) \frac{\partial u_h^+(x)}{\partial x_i} dx \geq \int_{\mathbb{R}^n} f u_h^+(x) dx, \tag{5.4}$$

$$-\int_{\mathbb{R}^n} \sum_{i=1}^n a_i(D\tilde{u}(x)) \frac{\partial u_h^+(x-h)}{\partial x_i} dx \geq -\int_{\mathbb{R}^n} f u_h^+(x-h) dx. \tag{5.5}$$

Making the change of variables $x \to x+h$ in (5.5), and, keeping in mind that f is constant, we have

$$\int_{\mathbb{R}^n} \sum_{i=1}^n a_i(D\tilde{u}(x+h)) \frac{\partial u_h^+(x)}{\partial x_i} dx \leq \int_{\mathbb{R}^n} f u_h^+(x) dx. \tag{5.6}$$

Adding (5.4) and (5.6), we get

$$\int_{\mathbb{R}^n} \sum_{i=1}^n \left(a_i(D\tilde{u}(x+h)) - a_i(D\tilde{u}(x))\right) \frac{\partial u_h^+(x)}{\partial x_i} dx \leq 0. \tag{5.7}$$

If we set $E_h^+ = \{x \in \mathbb{R}^n \ :\ u_h^+(x) = \tilde{u}(x+h) - \tilde{u}(x) - L|h| \geq 0\}$, it follows

$$\int_{E_h^+} \sum_{i=1}^{n} (a_i(D\tilde{u}(x+h)) - a_i(D\tilde{u}(x))) \frac{\partial u_h^+(x)}{\partial x_i} dx$$
$$+ \int_{\mathbb{R}^n \setminus E_h^+} \sum_{i=1}^{n} (a_i(D\tilde{u}(x+h)) - a_i(D\tilde{u}(x))) \frac{\partial u_h^+(x)}{\partial x_i} dx \leq 0.$$

Bearing in mind (5.2), (5.3), it means

$$\int_{E_h^+} \sum_{i=1}^{n} (a_i(D\tilde{u}(x+h)) - a_i(D\tilde{u}(x))) \left(\frac{\partial \tilde{u}(x+h)}{\partial x_i} - \frac{\partial \tilde{u}(x)}{\partial x_i} \right) dx \leq 0. \tag{5.8}$$

In virtue of strong monotonicity assumption (3.1), inequality (5.8) implies $u_h^+ = 0$ in E_h^+ and then

$$\tilde{u}(x+h) - \tilde{u}(x) \leq L|h| \quad \forall x, h \in \mathbb{R}^n,$$

namely, our thesis.

□

Now, we are in position to prove Theorem 3.1. First, we prove that, if u is the solution to problem (1.3), then $u \in K$. The result is obtained by means of the strong duality theory and a maximum principle.

Let us stress that, during the proof, we obtain a regularity result for the solution u to the problem (1.3), namely $u \in W^{2,p}(\Omega)$, $p > 1$.

Indeed, consider the following elastic-plastic torsion problem

$$\text{Find } w \in \tilde{K} : \int_\Omega \sum_{i=1}^{n} a_i(Dw) \left(\frac{\partial v}{\partial x_i} - \frac{\partial w}{\partial x_i} \right) dx \geq \int_\Omega f(v-w) dx, \quad \forall v \in \tilde{K}, \tag{5.9}$$

where

$$\tilde{K} = K_{w^-,w^+} \cap \{v \in H_0^1 : \; |Dv| \leq L \; a.e. \; in \; \Omega\}.$$

$\tilde{K}$ is a convex, bounded, closed set and, in virtue of Theorem 5.1, $u \in \tilde{K}$.

From the classical theory (see [4]), the unique solution to the variational inequality (5.9) belongs to $W^{2,p}(\Omega)$.

Since u verifies (5.9), by uniqueness, then $u \in W^{2,p}(\Omega)$ (for a different approach in the case of homogeneous variational inequality see [9]).

Now, we can rewrite problem (1.3) as a minization problem. Indeed, u is a solution to

$$\min_{v \in K_{w^-,w^+}} \psi(v), \tag{5.10}$$

where

$$\psi(v) = \int_\Omega \left\{ \sum_{i=1}^{n} a_i(Du) \left(\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) - f(v-u) \right\} dx.$$

Problem (5.10) verifies the Assumption S at the minimal point $u \in K_{w^-,w^+}$. Indeed, setting

$$X = S = L^2(\Omega),\; Y = L^2(\Omega) \times L^2(\Omega),\; C = C^* = \left\{ (\alpha, \beta) \in L^2(\Omega) \times L^2(\Omega) : \alpha(x), \beta(x) \geq 0 \text{ a.e. in } \Omega \right\},$$
$$g(v) = (g_1(v), g_2(v)) = (w^- - v, v - w^+),$$

it results

$$\tilde{M} = \{(\psi(v) + \alpha, w^- - v + y_1, v - w^+ + y_2), v \in L^2 \setminus K_{w^-,w^+}, \alpha \geq 0, y = (y_1, y_2) \in C\}.$$

In order to achieve the Assumption S, we must show that, if we have $\left(l, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)}\right) \in T_{\tilde{M}}(0, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)})$, namely,

$$\left(l, \theta_{L^2(\Omega)}, \theta_{L^2(\Omega)}\right) = \lim_n [\lambda_n(\psi(v_n) + \alpha_n, w^- - v_n + y_{1n}, v_n - w^+ + y_{2n})],$$

with $\lambda_n > 0$, $\lim_n(\psi(v_n) + \alpha_n) = 0$, $\alpha_n \geq 0$, $v_n \in L^2(\Omega) \setminus K_{w^-,w^+}$, $\lim_n \lambda_n(w^- - v_n + y_{1n}) = \theta_{L^2(\Omega)}$, $\lim_n \lambda_n(v_n - w^+ + y_{2n}) = \theta_{L^2(\Omega)}$, $y_n = (y_{1n}, y_{2n}) \in C$, then l must be nonnegative.

If we argue as in Lemma 1 and Lemma 2 in [17], we get $l \geq 0$, that is the double obstacle problem fulfills the Assumption S and then, by Theorem 4.1, the strong duality holds.

Then, if we define, $\forall v \in L^2(\Omega)$, $\forall(\lambda, \mu) \in C$, the Lagrange functional

$$L(v, \lambda, \mu) = \tag{5.11}$$

$$= \int_\Omega (-\sum_{i=1}^n \frac{\partial a_i(Du)}{\partial x_i} - f)(v - u)\, dx + \int_\Omega \lambda(w^-(x) - v(x))\, dx + \int_\Omega \mu(v(x) - w^+(x))\, dx,$$

Theorem 4.2 ensures that there exists $(\bar{\lambda}, \bar{\mu}) \in C$ such that

$$\int_\Omega \bar{\lambda}(w^-(x) - u(x))\, dx = 0, \quad \int_\Omega \bar{\mu}(u(x) - w^+(x))\, dx = 0,$$

namely

$$\bar{\lambda}(w^-(x) - u(x)) = 0, \quad \bar{\mu}(u(x) - w^+(x)) = 0, \quad a.e.\ in\ \Omega. \tag{5.12}$$

Moreover, $(u, \bar{\lambda}, \bar{\mu})$ is a saddle point of the Lagrange functional (5.11), that is

$$L(u, \lambda, \mu) \leq L(u, \bar{\lambda}, \bar{\mu}) \leq L(v, \bar{\lambda}, \bar{\mu}) \quad \forall v \in L^2(\Omega),\ \forall(\lambda, \mu) \in C. \tag{5.13}$$

Considering the right-hand side of (5.13), we get

$$\int_\Omega (-\sum_{i=1}^n \frac{\partial a_i(Du)}{\partial x_i} - f)(v - u)\, dx + \int_\Omega \bar{\lambda}(w^- - v)\, dx + \int_\Omega \bar{\mu}(v - w^+)\, dx \geq 0 \quad \forall v \in L^2(\Omega). \tag{5.14}$$

Thanks to (5.12), it follows from (5.14)

$$\int_\Omega (-\sum_{i=1}^n \frac{\partial a_i(Du)}{\partial x_i} - f - \bar{\lambda} + \bar{\mu})(v - u)\, dx \geq 0 \quad \forall v \in L^2(\Omega),$$

and then

$$-\sum_{i=1}^n \frac{\partial a_i(Du)}{\partial x_i} - f - \bar{\lambda} + \bar{\mu} = 0 \quad \text{a.e. in } \Omega. \tag{5.15}$$

Consider now the coincidence set, $I = I^+ \cup I^-$, with $I^+ = \{x \in \Omega :\ u(x) = w^+(x)\}$ and $I^- = \{x \in \Omega :\ u(x) = w^-(x)\}$. Moreover, we set $\Lambda = \{x \in \Omega :\ w^-(x) < u(x) < w^+(x)\}$. We have to prove that $G(Du) \leq M$ *a.e. in* Ω.

First, we recall that w^+, w^- are differentiable at x if $u(x)=w^+(x)$ or $u(x)=w^-(x)$ and $G(Dw^+)=G(Dw^-)=M$, a.a. $x\in\Omega$, (see [8], [32] page 133), then we may conclude that

$$G(Du)=M \ a.a.\ x\in I \tag{5.16}$$

On the other hand, for a.a. $x\in\Lambda$ from (5.12) and (5.15) it follows that $\bar\lambda=\bar\mu=0$ and, then,

$$-\sum_{i=1}^{n}\frac{\partial a_i(Du)}{\partial x_i}=f \quad \text{a.e. in } \Lambda. \tag{5.17}$$

We follow the method used in [4], Lemma III.10, since $f=const.$ and taking into account the regularity of u, that we have proved during the proof.

Differentiate (5.17) with respect to x_k, multiply by $\frac{\partial G(Du)}{\partial p_k}$ and sum over k, then

$$\sum_{i,j,k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_j\partial x_i}\frac{\partial G(Du)}{\partial p_k}+\sum_{i,j,k=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial^3 u}{\partial x_j\partial x_i\partial x_k}\frac{\partial G(Du)}{\partial p_k}=0,$$

namely,

$$\sum_{i,j,k=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial^3 u}{\partial x_j\partial x_i\partial x_k}\frac{\partial u}{\partial x_k}=-\sum_{i,j,k=1}^{n}\frac{\partial}{\partial x_k}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_j\partial x_i}\frac{\partial G(Du)}{\partial p_k}. \tag{5.18}$$

It results

$$\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial G(Du)}{\partial x_j}\right]=\sum_{i,j,k=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial G(Du)}{\partial p_k}\frac{\partial^2 u}{\partial x_k\partial x_j}$$
$$+\sum_{i,j,k=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial^2 u}{\partial x_k\partial x_j}\frac{\partial}{\partial x_i}\left(\frac{\partial G(Du)}{\partial p_k}\right)+\sum_{i,j,k=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial G(Du)}{\partial p_k}\frac{\partial^3 u}{\partial x_j\partial x_i\partial x_k}.$$

Then, bearing in mind (5.18), we get

$$\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial G(Du)}{\partial x_j}\right]=\sum_{i,j,k=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial^2 u}{\partial x_k\partial x_j}\frac{\partial}{\partial x_i}\left(\frac{\partial G(Du)}{\partial p_k}\right) \tag{5.19}$$
$$+\sum_{i,j,k=1}^{n}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_k\partial x_j}-\frac{\partial}{\partial x_k}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_j\partial x_i}\right]\frac{\partial G(Du)}{\partial p_k}.$$

Since $a(Du)$ is a monotone operator and by virtue the strict convexity of $G(Du)$ (see also [9] Lemma 4 and [10] Lemma 6), we can conclude that

$$\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n}\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial G(Du)}{\partial x_j}\right]$$
$$\geq\sum_{i,j,k=1}^{n}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_k\partial x_j}-\frac{\partial}{\partial x_k}\left(\frac{\partial a_i(Du)}{\partial p_j}\right)\frac{\partial^2 u}{\partial x_j\partial x_i}\right]\frac{\partial G(Du)}{\partial p_k}$$
$$=\sum_{i,j,k=1}^{n}\left[\sum_{l=1}^{n}\frac{\partial^2 a_i(Du)}{\partial p_j\partial p_l}\frac{\partial^2 u}{\partial x_l\partial x_i}\frac{\partial^2 u}{\partial x_k\partial x_j}-\sum_{l=1}^{n}\frac{\partial^2 a_i(Du)}{\partial p_j\partial p_l}\frac{\partial^2 u}{\partial x_k\partial x_l}\frac{\partial^2 u}{\partial x_j\partial x_i}\right]\frac{\partial G(Du)}{\partial p_k}=0.$$

Now, we may apply the maximum principle to operator

$$-\mathcal{B}(\varphi)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial a_i(Du)}{\partial p_j}\frac{\partial \varphi}{\partial x_j}\right),$$

acting on $G(Du)$ on Λ, since the coefficients of the operator $\mathcal{B}$ are bounded. Then, taking into account that Λ is an open set, by the maximum principle, the value M cannot be attained at interior points and one concludes that

$$G(Du)<M \quad a.a\ x\in\Lambda, \tag{5.20}$$

namely,

$$\Lambda\subset E=\{x\in\Omega:\ G(Du)<M\}. \tag{5.21}$$

So, from (5.16), (5.20) we get that, if u is the solution to (1.3), then

$$u\in K.$$

Since, $K\subset K_{w^-,w^+}$, by uniqueness we may conclude that u is the solution to problem (1.2), that is, the thesis of Theorem 3.1.

We can now obtain Theorem 3.2.

Indeed, as we already noted, $G(Dw^+)=G(Dw^-)=M$, a.a. $x\in\Omega$, then (5.16) holds and the coincidence set I is a subset of the plastic region $P=\{x\in\Omega:G(Du)=M\}$, namely

$$E=\Omega\setminus P\subset\Omega\setminus I=\Lambda.$$

Taking into account (5.21), Theorem 3.2 is achieved.

6. Existence of Lagrange multipliers

This section is devoted to the existence of Lagrange multipliers associated to problem (1.2).

So, let $u\in K$ be the solution to problem (1.2).

We have already proved in Theorem 3.1 that u is also a solution to the double obstacle problem (1.3). Moreover, $u\in W^{2,p}(\Omega)$, $p>1$.

We may argue as in Theorem 3.1. Indeed, we may rewrite problem ((1.3) as a minimization problem and we may prove that Assumption S is fulfilled at the solution u.

Then, from Theorem 4.1 it follows that there exists $(\bar\lambda,\bar\mu)\in C$, such that

$$\begin{cases}\bar\lambda(w^-(x)-u(x))=0, \quad \bar\mu(u(x)-w^+(x))=0, \quad a.e.\ in\ \Omega\\[2ex] \displaystyle\sum_{i=1}^{n}\frac{\partial a_i(Du)}{\partial x_i}+f-\bar\lambda+\bar\mu=0\ a.e.\ in\ \Omega.\end{cases} \tag{6.1}$$

Bearing in mind Theorem 3.2, choosing $\bar\nu=\bar\mu-\bar\lambda$, we get

$$\begin{cases}\bar\nu(G(Du)-M)=0, \quad a.e.\ in\ \Omega\\[2ex] \displaystyle\sum_{i=1}^{n}\frac{\partial a_i(Du)}{\partial x_i}+f+\bar\nu=0\ a.e.\ in\ \Omega\end{cases} \tag{6.2}$$

and Theorem 3.3 is achieved.

7. Conclusion

In this paper we have focused on a nonlinear monotone variational problem with strictly convex gradient constraints and homogeneous boundary condition.

We have introduced a suitable double-obstacle problem, obtained the regularity of the solution and proved the equivalence between the two problems.

Moreover, we have achieved the equivalence between the gradient constrained problem and a Lagrange multiplier problem.

Main tool in proving the results is a strong duality theory, that holds in infinite-dimensional settings.

Further work is the investigation of existence of Lagrange multipliers (even as Radon measures as in [7,24,26]) of the gradient constrained variational problem weakening the assumptions on the operator a and on the constraints mapping G or considering a L^2-free term f.

Competing Interests. The authors declare that they have no competing interests.

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References

1. Bot RI, Csetnek ER, Moldovan A. 2008. Revisiting some duality theorems via the quasirelative interior in convex optimization. *J. Optim. Theory Appl.* **139** (1).
2. Brezis H. 1972. Multiplicateur de Lagrange en torsion elasto-plastique. *Arch. Ration. Mech. Anal.* **49**.
3. Brezis H, Sibony M. 1971. Equivalence de Deux Inéquation Variationnelles et Application. *Arch. Rational Mech. Anal.* **41**.
4. Brezis H, Stampacchia G. 1968. Sur la régularité de la solution d'inéquations elliptiques. *Bull. Soc. Math. France* **96**.
5. Caffarelli LA, Friedman A. 1980. Reinforcement problems in elasto-plasti- city. *Roc. Mount. J. Math.* **10**.
6. Caffarelli LA, Riviére N.M. 1979. The Lipschitz character of the stress tensor, when twisting an elastic plastic bar. *Arch. Rational Mech. Anal.* **69** (1).
7. Chiadó-Piat V, Percivale D. 1994. Generalized Lagrange multipliers in elastoplastic torsion. *J. Differential Equations* **114**.
8. Cannarsa P, Soner HM. 1987. On the singularities of the viscosity solutions to Hamilton-Jacobi-Bellman Equations, *Indiana Univ. Math. J.* **36**.
9. Choe HJ. 1991. Regularity for Solutions of Nonlinear Variational Inequalities with Gradient Constraints. *IMA Preprint Series* **801**.
10. Choe HJ, Shim YS. 1995 Degenerate variational inequalities with gradient constraints, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **22** (1).
11. Choe HJ, Souksomvang P. 2016. Elliptic gradient constraint problem *Commun. Partial Differ. Equ.* **41** (12).
12. Cimatti G. 1976. The plane stress problem of Ghizetti in elastoplasticity, *Appl. Math. Optim.* **3** (1).
13. Daniele P, Giuffrè S, Idone G, Maugeri A. 2007. Infinite Dimensional Duality and Applications. *Math. Ann.* **339**.
14. Daniele P, Giuffrè S. 2007. General infinite dimensional duality and applications to evolutionary network equilibrium problems. *Optim. Lett.* **1**.
15. Daniele P, Giuffrè S, Maugeri A. 2009. Remarks on General Infinite Dimensional Duality with Cone and Equality Constraints, *Commun. Appl. Anal.* **13**.
16. Daniele P, Giuffrè S, Maugeri A, Raciti F. 2014. Duality theory and applications to unilateral problems, *J. Optim. Theory Appl.*, **162** (3).
17. Donato MB. 2011. The infinite dimensional Lagrange multiplier rule for convex optimization problems. *J. Funct. Anal.* **261** (8).
18. Evans L. 1979. A second order elliptic equation with gradient constraint. *Comm.Part. Diff. Eq.* **4**.
19. Figalli A, Shahgholian H. 2015. An overview of unconstrained free boundary problems. *Philos. Trans. Roy. Soc. A* **373**.

20. Gerhardt C. 1975. On the Existence and Uniqueness of a Warpening Function in the Elastic-plastic Torsion of a Cylindrical Bar with a Multiply Connected Cross- section, Proceedings of the "Joint Symposium IUTAM/IMU (A. Dold, B. Eckm- nann, P. Germain and B. Nayroles, Eds.), Marseille.
21. Giuffrè S, Maugeri A. 2012. New results on infinite dimensional duality in elastic-plastic torsion. *Filomat* **26** (5).
22. Giuffrè S, Maugeri A. 2013. Lagrange multipliers in elastic-plastic torsion, *AIP Conf. Proc.* **1558**.
23. Giuffrè S, Maugeri A. 2014. A measure-type Lagrange multiplier for the elastic-plastic torsion. *Nonlinear Anal.* **102**.
24. Giuffrè S, Maugeri A, Puglisi D. 2015. Lagrange multipliers in elastic-plastic torsion problem for nonlinear monotone operators. *J. Differ. Equ.* **259** (3).
25. Giuffrè S, Pratelli A, Puglisi D. 2018. Radial solutions and free boundary of the elastic-plastic torsion problem, *J. Convex Anal.* **25** (2).
26. Giuffrè S. 2020. Lagrange multipliers and non-constant gradient constrained problem. *J. Differential Equations* **269** (1).
27. Giuffrè S., Marcianò A. 2021. Duality Minimax and Applications. *Minimax Theory Appl.* **6** 2.
28. Ishii H, Koike S. 1983. Boundary regularity and uniqueness for an elliptic equation with gradient constraint. *Commun. Partial Differ. Equ.* **8** (4).
29. Jahn J. 1996 *Introduction to the theory of nonlinear optimization*, Berlin: Springer-Verlag.
30. Jensen R. 1983. Regularity for elastoplastic type variational inequalities, *Indiana Univ. Math. J.* **32**.
31. Kruzkov S.N. 1975. Generalized solutions of Hamilton Jacobi equations of Eikonal type I. *Math USSR Sb.* **27**.
32. Lions PL. 1982 *Generalized Solutions for Hamilton-Jacobi Equations*, Boston-London-Melbourne: Pitman Advanced Publishing Program.
33. Maugeri A, Puglisi D. 2014. A new necessary and sufficient condition for the strong duality and the infinite dimensional Lagrange multiplier rule, *J. Math. Anal. Appl.* **415**.
34. Maugeri A, Puglisi D. 2015. A new necessary and sufficient condition for non-convex strong duality via subdifferential, *Numer. Funct. Anal. Optim.* **35**.
35. Rodriguez JF. 1987 *Obstacle Problems in Mathematical Physics*, Amsterdam: North-Holland.
36. Safdari M. 2021. Global optimal regularity for variational problems with nonsmooth non-strictly convex gradient constraints, *J. Differ. Equ.* **279**.
37. Santos L. 2002. Variational problems with non-constant gradient constraints. *Port. Math.* **59** (2).
38. Santos L. 2017. Lagrange multipliers and transport densities. *Math. Pures Appl.* **108**.
39. Stampacchia G. 1964. Formes bilinraires coercitives sur les ensembles convexes. *C.R. Acad. Sci. Paris* **258**.
40. Treu G, Vornicescu M. 2000. On the equivalence of two variational problems. *Calc. Var.* **11**.
41. Wiegner M. 1981. The $C^{1,1}$-character of solutions of second order elliptic equations with gradient constraint. *Comm.Part. Diff. Eq.* **6**.