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MULTIPLE WEAK SOLUTIONS FOR A CLASS OF SIXTH ORDER BOUNDARY VALUE PROBLEM: NEW FINDINGS AND APPLICATIONS

Abstract: In this paper, we study the existence of two and infinitely many weak solutions for a class from sixth-order differential equation, in which modelling for describing the behaviour of phase fronts in materials that are undergoing a transition between the liquid and solid. The results are proved by using some critical point theorems.

Key words and phrases: Multiple solutions, Sixth-Order Equations, Variational Methods, Critical Point.

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1. Introduction

In this paper, we study the following problem:

$$\begin{cases} -u^{(vi)}(x) + Au^{(iv)}(x) - Bu''(x) + Cu(x) = \lambda f(x, u(x)),\ x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{1.1}$$

where $A, B, C \in \mathbb{R}$ and parameter $\lambda > 0$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Study sixth-order differential equations was first started by studying the following problem:

$$\frac{\partial u}{\partial x} = \frac{\partial^6 u}{\partial^6 x} + A\,\frac{\partial^4 u}{\partial^4 x} + B\,\frac{\partial^2 u}{\partial^2 x} + f(x, u). \tag{1.2}$$

One of the most important applications problem (1.2) is the model that describes the phase fronts behavior in the materials.

In recent years, BVPs for sixth-order ordinary differential equations have been studied extensively, see [1, 2, 3, 5, 7, 10, 11] and the references therein in [5], Gyulov *et al*. obtained the existence and multiplicity the solutions for the following boundary value problem

$$\begin{cases} -u^{(vi)}(x) + Au^{(iv)}(x) - Bu''(x) + Cu(x) = \lambda f(x, u(x)), 0 < x < L, \\ u(0) = u(L) = u''(0) = u''(L) = u^{(iv)}(0) = u^{(iv)}(L) = 0 \end{cases} \tag{1.3}$$

where $A, B, C \in \mathbb{R}$ and $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [7], Li obtained the existence and multiplicity of positive solutions for the following problem

$$\begin{cases} -u^{(vi)}(x) + A(x)u^{(iv)}(x) + B(x)u''(x) + C(x)u(x) + f(x, u(x)) = 0,\ x \in [0, 1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{1.4}$$

where $A(x), B(x), C(x) \in C([0, 1])$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. Bonanno *et al*. in [1], applied critical point theory and variational methods to prove the existence and multiplicity of solutions for the following problem

$$-u^{(vi)}(x) + Au^{(iv)}(x) - Bu''(x) + Cu(x) = \lambda f(x, u(x)), x \in [a, b] \tag{1.5}$$

where $\lambda > 0$, A, B and C are given real constants, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a function. Recently, Bonanno and Livrea in [2] obtained infinitely many solutions for the nonlinear sixth-order problem (1.1). They used the variational methods and an oscillating behavior on the nonlinear term to demonstrate the existence of these solutions.

In this article, we discuss the existence of two and infinitely many weak solutions for the problem (1.1), under suitable conditions on the nonlinear term. We also present examples to illustrate the results.

2. Preliminaries and Basic Notation

In this section, we first introduce some notations and some necessary definitions. Set

$$X = \{u \in H^3(0,1) \cap H_0^1(0,1) \mid u''(0) = u''(1) = 0\}. \quad (2.1)$$

X is the Sobolev space, consider the inner product

$$< u, v > := \int_0^1 (u'''(x)v'''(x) + u''(x)v''(x) + u'(x)v'(x) + u(x)v(x))dx,$$

which induces the norm

$$||u|| := (|u'''|_2^2 + |u''|_2^2 + |u'|_2^2 + |u|_2^2)^{\frac{1}{2}} \quad (2.2)$$

Proposition 2.1: (see [2]) *If* $k = \frac{1}{\pi^2}$, *for every* $u \in X$, *we have*

$$||u^{(i)}||_2^2 \leq k^{j-i} \, ||u^{(j)}||_2^2 \; i = 0, 1, 2 \;\; j = 1, 2, 3 \text{ with } i < j, \quad (2.3)$$

where $||u||_2 := (\int_0^1 |u(x)|^2 \, dx)^{\frac{1}{2}}$ *is norm in* $L^2(0,1)$.

We introduce the function $N : X \to \mathbb{R}$ as follows,

$$N(u) := ||u'''||_2^2 + A \, ||u''||_2^2 + B \, ||u'||_2^2 + C \, ||u||_2^2, \quad \forall \; u \in X,$$

where A, B and C are real constants and satisfied in the following condition:

$$(H) \max \{ -Ak, -Ak - Bk^2, - Ak - Bk^2 - Ck^3\} < 1.$$

Lemma 2.2: (see [2]) *Put*

$$||u||_X = \sqrt{N(u)}. \; u \in X,$$

and assume that the condition (H) *holds. Then,* $\|u\|_X$ *is a norm equivalent to the norm defined in* (2.2) *and* $(X, \|.\|_X)$ *with following inner product*

$$< u, v >:= \int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))\, dx \ ,$$

is a Hilbert space.

Clearly $(X, \|.\|_X) \to (C^0(0,1). \|.\|_\infty)$ and the embedding is compact.

Lemma 2.3: (*see* [2]) *Assume that* (H) *holds, one has*

$$\|u\|_\infty \leq \frac{k}{2\sqrt{\delta}} \|u\|_X, \ \forall \ u \in X .$$

for every $u \in X$, *and* $\delta > 0$ *is given in* [2].

We say that a function $u \in X$ is called a weak solution of the problem (1.1) if

$$\int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))dx$$

$$- \lambda \int_0^1 f(x, u(x))v(x)dx = 0, \ \forall \ v \in X .$$

Consider $I_\lambda : X \to R$ defined by

$$I_\lambda(u) = \frac{1}{2} \|u\|_X^2 - \lambda \int_0^1 F(x, u(x))dx , \tag{2.4}$$

where

$$F(x,t) = \int_0^t f(x\xi)d\xi \quad \text{for all} \ (x,t) \in [0,1] \times \mathbb{R} .$$

We observe that $I_\lambda \in C^1(X, \mathbb{R})$ for any $v \in X$,

$$I_{\lambda}'(u)v = \int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))dx \tag{2.5}$$

$$-\lambda \int_0^1 f(x, u(x))v(x)dx = 0, \ \forall \ v \in X. \tag{2.6}$$

Thus, the solutions of Problem (1.1) are the critical point of I_{λ}.

Definition 2.4: *Assume X be a real reflexive Banach space. We say J satisfies Palais-Smale condition (denotes by PS condition for short), if any sequence $\{u_k\} \subset X$ for which $\{J(u_k)\}$ is bounded and $J'(u_k) \to 0$ as $k \to 0$ possesses a convergent subsequence.*

The proofs of our results are based the following theorems.

Theorem 2.5: [9, Theorem 4.10] *Let $I_{\lambda} \in C^1(X, \mathbb{R})$, and I_{λ} satisfies the Palais-Smale condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 satisfying $u_1 \notin \Omega$ and*

$$\inf_{\nu \in \partial\Omega} I_{\lambda}(\nu) > \max\{\varphi(u_0), I_{\lambda}(u_1)\},$$

then there exists a critical point u of I_{λ}, i.e., $I_{\lambda}'(u) = 0$ with

$$I_{\lambda}(u) > \max\{I_{\lambda}(u_0), I_{\lambda}(u_1)\}.$$

Theorem 2.6: [15, Theorem 38] *For the functional $I_{\lambda} : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \varnothing$, $\min_{u \in M} I_{\lambda}(u) = \alpha$ has a solution in case the following conditions hold:*

(i_1) *X is a real reflexive Banach space,*

(i_2) *M is bounded and weak sequentially closed,*

(i_3) *I_λ is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $I_\lambda(u) \leq \lim_{n\to\infty} \inf I_\lambda(u_n)$ holds.*

Theorem 2.7: *Consider X be an infinite-dimensional Banach space and $I_\lambda \in C^1(X, \mathbb{R})$ be an even functional which satisfies the (PS)-condition and $I_\lambda(0) = 0$. If $X = V \oplus E$ where V is finite dimensional and I_λ satisfies the conditions*

(j_1) *there are constants $\rho, \alpha > 0$ such that*

$$I_\lambda(u) \geq \alpha, \text{ if } \|u\| = \rho, \ u \in E,$$

(j_2) *for each finite-dimensional subspace $E_n \subseteq X$ there is D_n such that*

$$I_\lambda(u) \geq 0, \text{ if } \|u\| \geq D_n, \ u \in E_n,$$

then I_λ possesses an unbounded sequence of critical points.

We refer the reader to the paper [12, 13] in which Theorem 2.7 was successfully employed to some boundary value problems. To read more on the applications of Theorem 2.5 and 2.6, we refer to the papers [4, 6, 14].

3. Main Results

We utilize the following assumptions throughout this paper.

(f_0) there exist a constants $\nu > 2$ and $T > 0$ such that

$$0 < \nu F(x,t) \leq t f(x,t), \text{ for } |t| > T \text{ and } x \in [0,1].$$

(f_1) $f : V \times \mathbb{R} \to \mathbb{R}$ continues and there exists constant $L > 0$ such that

$$\left|f(x,t)\right| \le c(1+ |t|^{q-1}), \quad \text{for } |t| \le L \text{ and } x \in [0,1]$$

where $q > 2$.

(f_2) $\lim_{t\to 0} \frac{f(x,t)}{t^2} = 0,$ for $x \in [0,1]$ uniformly.

We use the following lemmas to prove our main results.

Lemma 3.1: *Assume that the condition* (f_0) *holds. Then* $I_\lambda(u)$ *satisfies the* (PS)*-condition.*

Proof: Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$. Then, there exists a positive constant c_0 such that $|I_\lambda(u_n)| \le c_0$ and $|I'_\lambda(u_n)| \le c_0$ for all $n \in N$. Therefore, from the definition of I'_λ and (A_1), we have

$$c_0 + c_1 \|u_n\|_X \ge \nu I_\lambda(u_n) - I'_\lambda(u_n)(u_n)$$

$$\ge (\frac{\nu}{2} - 1) \|u_n\|_X^2 + \lambda \int_0^1 (f(x, u_n(x))u_n(x) - \nu F(x, u_n(x)))\, dx$$

$$\ge (\frac{\nu}{2} - 1) \|u_n\|_X^2 . \tag{3.1}$$

therefore for some $c_1 > 0$, since $\nu > 2$ this implies that $\{u_n\}$ is bounded. Since X is Banach space and $\{u_n\}$ is bounded, there exist a subsequence, still denoted by $\{u_n\}$ and a function u in X such that

$$u_n \rightharpoonup u, \text{ in } X, \text{ and } u_n \to u \text{ in } C_1([0,1]). \tag{3.2}$$

By definition $I'_\lambda(u)$, we get

$$< I'_\lambda(u_n), u_n - u > = \int_0^1 \Big(u_n'''(x)(u_n'''(x) - u'''(x)) + Au_n''(x)(u_n''(x) - u''(x))$$

$$+ Bu_n'(x)(u_n'(x) - u(x)) + Cu_n(x)(u_n(x) - u(x))\Big)\, dx$$

$$- \lambda \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\, dx.$$

Therefore, we have

$$< I'_\lambda(u_n) - I'_\lambda(u_n), u_n - u > =$$

$$\int_0^1 \Big(u_n'''(x)(u_n'''(x) - u'''(x)) + Au_n''(x)(u_n''(x) - u''(x))$$

$$+ Bu_n'(x)(u_n'(x) - u(x)) + Cu_n(x)(u_n(x) - u(x))\Big)\, dx$$

$$- \lambda \int_0^1 f(x, u_n(x))(u_n(x) - u(x))\, dx$$

$$-\Big(\int_0^1 \Big(u'''(x)(u_n'''(x) - u'''(x)) + Au''(x)(u_n''(x) - u''(x))$$

$$+ Bu'(x)(u_n'(x) - u'(x)) + Cu(x)(u_n(x) - u(x))\Big)\, dx$$

$$- \lambda \int_0^1 f(x, u(x))(u_n(x) - u(x))\, dx\Big)$$

$$= \int_0^1 \Big((u_n'''(x) - u'''(x))^2) + A(u_n''(x) - u''(x))^2$$

$$+ Bu_n'(x) - u'(x))^2 + Cu_n(x) - u(x))^2 \Big)\, dx$$

$$- \lambda \int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\, dx$$

$$\geq \|u_n - u\|_X^2 - \lambda \int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\, dx.$$

From the continuity of f, we get

$$\int_0^1 \Big((u_n'''(x) - u'''(x))^2 \big) + A(u_n''(x) - u''(x))^2 + B(u_n'(x) - u'(x))^2$$

$$+ C(u_n(x) - u(x))^2 \Big) dx \to 0, \ n \to \infty, \tag{3.3}$$

and

$$\lambda \int_0^1 (f(x, u_n(x)) - f(x, u(x)))(u_n(x) - u(x))\, dx \to 0, n \to \infty, \tag{3.4}$$

from (3.1) , (3.2), we can conclude

$$< I_\lambda'(u_n) - I_\lambda'(u_n), u_n - u > \to 0.$$

Therefore by (3.3) to (3.4), we have

$$\|u_n - u\|_X^2 \to 0.$$

Thus, the sequence u_n converges strongly to u in X. Therefore, I_λ satisfies the (*PS*)-condition. □

Theorem 3.2: *Assume that the assumptions* (f_0), (f_1) *and* (f_2) *hold. Then*:

if $f(x, t) \geq 0$ *for all* $(x, t) \in [0, 1] \times \mathbb{R}$, *the problem* (1.1) *has at least two weak solutions*.

Proof: Clearly, $I_\lambda(0) = 0$. From the Lemma 3.1, we can see I_λ satisfies the (*PS*)-condition. We will show that there exists $R > 0$ such that the functional I_λ has a local minimum $u_0 \in BR = \{u \in X; \|u\|\ X < R\}$. Assume that $\{u_n\} \subseteq \bar{B}_R$ and $u_n \rightharpoonup u$, as $n \to \infty$ by Mazur Theorem [8], there exists sequence $\{v_n\}$ of convex combinations such that

$$v_n = \sum_{j=1}^{n} a_{n_j} u_j, \sum_{j=1}^{n} a_{n_j} = 1, \qquad a_{n_j} \geq 0, j \in N$$

and $v_n \to u$ in X. Clearly, $\bar{B}_R$ is a closed convex set, therefore $\{v_n\} \subseteq \bar{B}_R$ and $u \in \bar{B}_R$. Since, I_λ is weakly sequentially lower semi-continuous on $\bar{B}_R$ and X is a reflexive Banach space, so, from Theorem 2.6 we can know that I_λ has a local minimum $u_0 \in \bar{B}_R$. Assume that $I_\lambda(u_0) = \min_{u\in\bar{B}_R} I_\lambda(u)$, we will show that $I_\lambda(u_0) < \inf_{u\in\partial\bar{B}_R} I_\lambda(u)$. By ($f_1$) and ($f_2$), there exits $\alpha > 0$ such that

$$F(x,t) \le \alpha|t|^2 + c|t|^q \ , \tag{3.5}$$

let $\alpha > 0$ be small enough such that $\alpha < \dfrac{2\delta^2}{\lambda k^2}$, therefore

$$\begin{aligned}
I_\lambda(u) &\ge \frac{1}{2}\|u\|_x^2 - \lambda\alpha\int_0^1 |u(x)|^2\, d\mu - \lambda c\int_0^1 |u(x)|^q\, d\mu \\
&\ge \frac{1}{2}\|u\|_X^2 - \lambda\alpha\|u\|_\infty^2 - \lambda c\int_0^1 |u(x)|^q\, d\mu \\
&\ge \frac{1}{2}\|u\|_X^2 - \lambda\alpha\frac{k^2}{4\delta^2}\|u\|_X^2 - \lambda c\|u\|_X^q \\
&\ge \left(\frac{1}{2} - \lambda\alpha\frac{k^2}{4\delta^2}\right)\|u\|_X^2 - \lambda c\left(\frac{k}{2\delta}\right)^q\|u\|_X^q
\end{aligned}$$

Since, $q > 2$, when $\|u\|_X < 1$ there exist $r > 0$, such that $I_\lambda(u) \ge r > 0$ for every $\|u\|_X = r$, we choosing $R = r$, thus, $I_\lambda(u) > 0 = I_\lambda(0) \ge I_\lambda(u_0)$ for $u \in \partial B_R$. Hence, $u_0 \in B_R$ and $I'_\lambda(u_0) = 0$. Since, u_0 is a minimum point of I_λ on X, there exists $R > 0$ sufficiently large such that $I_\lambda(u_0) \le 0 < \inf_{u\in\partial B_R} I_\lambda(u)$, where $B_R = \{u \in X; \|u\|_X < R\}$. Now, we will show that there exists u_1 with $\|u_1\|_X > R$ such that $I_\lambda(u_1) < \inf_{u\in\partial B_R} I_\lambda(u)$. Letting $k_1 \in X$ and $u_1 = \tau k_1$, $\tau > 0$ and $\|k_1\|_X = 1$. From (f_0) we get there exist constants $a_1,\ a_2 > 0$ such that $F(x,t) \ge a_1\|t\|^\nu - a_2$ for all $x \in [0,1]$. Thus,

$$I_\lambda(u_1) = \frac{1}{2}\,\|\tau k_1\|_X^2\; - \lambda\int_0^1 F(x, \tau k_1)(x))\,d\mu$$

$$\leq \frac{1}{2}\,\tau^2\,\|k_1\|_X^2\, - \lambda\tau^\nu a_1\,\int_0^1 |k_1(x)|^\nu\,d\mu\, + \lambda a_2\,.$$

Since, $\nu > 2$, there exists sufficiently large $\tau > R > 0$ so $I_\lambda(\tau k_1) < 0$. Hence, $\max\{I_\lambda(u_0), I_\lambda(u_1)\} < \inf_{\partial B_R} I_\lambda(u)$. Then, Theorem 2.5 gives the critical point u^*. Therefore, u_0 and u^* are two critical points of I_λ, which are two weak solutions of the Problem (1.1). □

Theorem 3.3: *Assume that the assumption* (f_0) *and the following condition hold*:

(f_4) *there exists* $q > 2$ *such that*

$$\left|f(x, t)\right| \leq c\,|t|^{q-1}, \quad \text{as} \quad |t| \to 0\,.$$

Then Problem (1.1) *has infinitely many pairs of weak solutions.*

Proof: We want to apply Theorem 2.7. By lemma 3.1 the functional I_λ defined in (2.4) satisfies the (*PS*)-condition.

Now, we need to assumptions (j_1) and (j_2) of Theorem 2.7. By condition (f_4) and Lemma 2.3, we have

$$I_\lambda(u) = \frac{1}{2}\,\|u\|_X^2\; - \lambda\int_0^1 F(x, u(x))\,dx$$

$$\geq \frac{1}{2}\,\|u\|_X^2\, - c\int_0^1 |u|^q\,dx$$

$$\geq \frac{1}{2}\,\|u\|_X^2\, - c\,\|u\|_\infty^q$$

$$\geq \frac{1}{2}\,\|u\|_X^2\, - c\frac{k^q}{2^q\,\delta^{\frac{q}{2}}}\,\|u\|_X^q\,.$$

Since, $q > 2$, we have that for $\|u\| = \rho$ sufficiently small $I_\lambda(u) \geq \alpha > 0$. Let E_n be a n-dimensional subspace of *X*, by the equivalence of any two norms on finite-dimensional space, by integrating the condition (f_0) there exist constants $a_1, a_2 > 0$ such that

$$F(t, x) \geq a_1 \, |x|^\nu - a_2$$

for all $t \in [0, 1]$ and $x \in \mathbb{R}$. Now, for any $u \in E_n$, we have

$$I_\lambda(u) \leq \frac{1}{2} \, \|u\|_X^2 - \lambda \int_0^1 F(x, u)(x)) \, dx$$

$$\leq \frac{1}{2} \, \|u\|_X^2 - \lambda \int_0^1 a_1 \, |u(x)|^\nu \, dx + \lambda a_2.$$

Since, $\nu > 2$, there exists sufficiently large $D_n > 0$, such that $I_\lambda(u) \leq 0$ for $\|u\| \geq R_n$. Therefore, all the assumptions of Theorem 2.7 are established. Thus, the functional I_λ possesses an unbounded sequence of critical points on *X*. And it proves the result. □

Now, illustrate our results by the following examples.

Example 3.4: *Consider the following problem*

$$\begin{cases} -u^{(vi)}(x) + 2u^{(iv)}(x) + u''(x) - 3u = \lambda f(x, u(x)), \; x \in [0, 1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{3.6}$$

where $A = 2$, $B = -1$, $C = -3$. *Set* $f(x, t) = t^4$ *for all* $x \in [0, 1]$, *thus, we have* $F(x, t) = \frac{1}{5} t^5$ *for all* $x \in [0, 1]$. *Hence,* $\lim_{\xi \to +\infty} \frac{\xi f(x,\xi)}{F(x,\xi)} = 5 < \infty$, *so, by choosing* $\nu = 5 > 2$ *and* $T = 1$ *the condition* (f_0) *satisfied. Also* $f(x, t) \geq 0$ *for all* $x \in [0, 1]$, *and* $\lim_{t \to 0} \frac{f(x,t)}{t^2} = 0$. *By selecting* $q = 5$ *and* $L = 1$, *we get*

$|f(x,t)| \leq c(1+|t|^4)$ *for* $|t| \leq 1$ *and for same* $c > 0$. *Therefore, all the assumptions in* Theorem 3.2 *are fulfilled. Hence, the* Problem (3.6) *has at least two weak solutions.*

Example 3.5: *Consider the following problem*

$$\begin{cases} -u^{(vi)}(x) + u^{(iv)}(x) - u''(x) + 3u = \lambda f(x, u(x)),\ x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases} \tag{3.7}$$

where $A = 1$, $B = 1$, $C = 3$. *Put*

$$f(x,t) = \begin{cases} 8t^5, & t \leq 1 \\ 8t^7, & t > 1, \end{cases}$$

for all $x \in [0,1]$. *We have*

$$F(x,t) = \begin{cases} \frac{4}{3}t^6, & t \leq 1 \\ t^8 + \frac{1}{3}, & t > 1, \end{cases}$$

for all $x \in [0,1]$. *Hence,* $\lim_{\xi \to +\infty} \frac{\xi f(t,\xi)}{F(t,\xi)} = 8 < \infty$ *and* $\lim_{\xi \to -\infty} \frac{\xi f(t,\xi)}{F(t,\xi)} = 6 < \infty$, *thus by choosing* $\nu = 8 > 2$ *and* $T = 1$ *the condition* (f_0) *satisfied. Also by choosing* $q = 6$ *and* $c = 8$, *we have* $|f(x,t)| \leq 7|t|^5$ *for* $|t| \leq 1$, *therefore, the condition* (f_1) *satisfied. We clearly see that all the assumptions present in* Theorem 3.2 *are established. Thus, the* Problem (3.7) *has infinitely many pairs of weak solution.*

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