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A Non-Iterative Crosswords-Inspired Approach to the Recovery of 2-D Discrete Signals From Phaseless Fourier Transform Data

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ABSTRACT We propose an innovative approach to the phase retrieval of 2-D discrete complex signals. The solution strategy profitably exploits some fundamental results available for the phase retrieval of one-dimensional discrete signals by following a simple yet effective philosophy resembling the solution of crossword puzzles. The resulting procedure is completely deterministic and allows identifying all the solutions of the problem by using just the spectral amplitude data, knowledge of the support of the source, and a few additional information. As a distinguishing characteristic, it does not exploit neither global-optimization algorithms nor iterative techniques. Numerical examples witnessing the interest of the proposed ideas are given for antenna applications.

INDEX TERMS Phase retrieval, signal recovery, spectral factorization.

I. INTRODUCTION

By DENOTING with f and T, respectively, an arbitrary signal and an operator such that $F(\boldsymbol{u}) = T[f(\boldsymbol{x})] = |F(\mathbf{u})|e^{j\varphi(\boldsymbol{u})}$, $\boldsymbol{x}$ and $\boldsymbol{u}$ being real vector variables spanning either mono-dimensional or multi-dimensional spaces, a general problem underlying very many applications [1]–[28] can be given as follows:

Determine f($\boldsymbol{x}$) starting from the knowledge of $|F(\boldsymbol{u})|$ plus some additional information.

In the closely related case where one is just interested in the complex signal $F(\boldsymbol{u})$, the problem is also known as *Phase Retrieval* (PR) problem [1].

PR solution procedures are essential to achieve the full knowledge of any complex signal whose phase measurements are unavailable or not convenient. As such, they play a key role in a number of canonical problems including inverse electromagnetic scattering [2]–[6] as well as antenna synthesis [7]–[10], diagnostics [11]–[15], and characterization [16], [17]. As a matter of fact, PR applications include astronomy [1], electron microscopy [18], lensless imaging [19], lithography [20], holography [12], optics and crystallography [21], [22], and many other cases.

In the literature, large attention has been devoted to the case where T is a Fourier Transform, which we consider in the following. As well known, solving such a problem requires tackling a number of difficulties, which are briefly recalled below.

In both the one-dimensional and two-dimensional cases (as well as for higher dimensional problems) the solution of the problem is not unique. In fact, the so-called trivial ambiguities, i.e.,

(i) a constant phase on $F(\boldsymbol{u})$,
(ii) a linear phase on $F(\boldsymbol{u})$, which results in a shift on $f(\boldsymbol{x})$,
(iii) a complex conjugation of $F(\boldsymbol{u})$, which results in a reversal plus a conjugation of $f(\boldsymbol{x})$,

as well as any combination of the above, always resulting in an identical $|F(\boldsymbol{u})|^2$ distribution, may come into play.

Things are even worse in the 1-D case. In fact, in such a case, by denoting with u a one-dimensional instance of $\boldsymbol{u}$, $|F(u)|^2$ can be written (but for a constant) as the product of (infinite) factors of the kind $e^{ju} - z_i$, where the roots z_i are organized in couples such that if one of them is relative to $F(u)$, the companion one determines a factor of $F^*(u)$. Then, unless the two elements of a couple have a unitary amplitude, any 'zero flipping' within any couple generates a new solution.[1] Such a difficulty is readily understood in the case of discrete signals having a finite length, where $|F(u)|^2$ can be seen a polynomial in the $z = e^{ju}$ variable [8].

Notably, the last cause for non-uniqueness above is not likely to occur in the 2-D case, as two-dimensional polynomials (for the discrete-signal case) or entire functions of the exponential type (for the continuous source case) are not factorable (but for a zero-measure set of cases). As a consequence, in the 2-D case (which is the one we deal with in the remaining part of the paper) the PR problem has a unique solution but for trivial ambiguities and a zero-measure set of cases.

As one can get rid of ambiguities (i) and (ii) by a-priori fixing, respectively, a phase reference and the support of the source, in the recovery of a 2-D signal from the amplitude of its Fourier transform one has to deal with the (unavoidable) ambiguity amongst the 'true' $F(\boldsymbol{u})$ and its conjugate as well as with the zero-measure set of cases where, due to the fact that $F(\boldsymbol{u})$ can be eventually decomposed into further factors, additional ambiguities could exist. Hence, even in the 2-D case, some additional a-priori information is necessary in order to get a theoretically unique solution.

Even assuming a theoretically unique solution exists in case of ideal data, two further difficulties come into play. In fact, in the 2-D case the (measurement-error-affected) square amplitude distribution may be such that the data do not belong to the range of the quadratic operator relating the unknown signal $f(\boldsymbol{x})$ to the actual $|F(\boldsymbol{u})|^2$ distribution [23]. Consequently, a solution to the problem may not exist at all in cases of actual (i.e., error-affected) data, so that the problem keeps ill-posed. As a possible countermeasure to such an issue, a generalized solution is usually looked for as the global minimum of some cost functional enforcing the best possible fitting amongst the actual and the tentative intensity distributions. Such a functional can be complemented by some penalty function enforcing some expected property of the solution.

Last, but not least, even if the available a-priori information theoretically ensures the uniqueness of the solution, the iterative procedure minimizing the cost function may be trapped into local minima actually different from the ground truth [3], thus resulting into 'false solutions' of the problem.

Notably, additional difficulties may arise in the particular problem one is dealing with. In antenna testing, for example, even assuming the antenna is a uniformly spaced array (so that the spectrum is periodic), depending on the element spacing one may not be able to collect the whole square amplitude distribution of the Fourier Transform of the unknown signal in the basic periodicity cell (see [27] for details).

By taking into account all the above, we propose in the following a new approach to the recovery of 2-D discrete complex signals from the square amplitude of its Fourier Transform. Motivated by the last circumstance above, the approach will be tested with reference to the case of antenna arrays (where the array excitations and the square-amplitude far-field distribution play the role, respectively, of the signal to be recovered and of the known data). By the sake of simplicity, we will start by considering the noiseless case, where effective solution procedures are anyway still lacking, and then we will consider the noisy case at a later stage.

The key idea of the proposed procedure is to recast the 2-D problem at hand as the combination of a number of auxiliary 1-D PR problems. Then, as all the possible solutions of each 'auxiliary' 1-D problem are identified in a fast and effective fashion by using the Spectral Factorization (SF) technique introduced in [8], the final solution of the 2-D PR can be identified by enforcing congruence amongst them. In a nutshell (see below), the procedure resembles the solution of crossword puzzles, where one has to guarantee the congruence amongst across and up-down possible words.

The proposed approach differs from the large variety of existing techniques under many points of view.

First, at least for the noiseless case, it is able to deterministically and uniquely solve the PR problem by exploiting an amount of information coincident with the one theoretically needed to ensure uniqueness of the solution. Note this is a far from trivial capability, as to avoid false solutions one is generally forced to exploit further information.[2]

Obviously, to get the actual solution with the minimal information, one could eventually use global optimization techniques. However, these latter are based on a stochastic search and they have a computational burden growing exponentially with the number of unknowns [30]. Notably, which is a second difference with respect to the literature, the approach we propose and discuss in the following is based instead on a precise deterministic strategy, thus avoiding expensive random searches while still guaranteeing the achievement of the globally optimal solution.

A recently introduced technique for the PR problem is based on the so-called 'Phase Lift' strategy [31]. Roughly

1. By virtue of the Weierstrass Factorization Theorem [29], which may be viewed as an extension of the Fundamental Theorem of Algebra, every entire function can be represented as a (possibly infinite) product involving its roots.

2. For example, in antenna testing one needs either two measurement surfaces, or two different probes, or (in case of reflector diagnostics) two different defocus conditions [24]–[26].

speaking, it amounts to introduce a number of auxiliary unknowns equal to the square of the number of the original problem[3] and on the relaxation of an inherent non-convex constraint into a convex one. As it may be easily understood, the computational complexity of such a technique also grows very rapidly with the number of unknowns (so that it is essentially adopted for 1-D problems [28]). Moreover, which is often overlooked in the literature, Phase Lift-based procedures just find a solution of the problem whereas in some instances (e.g., the 1-D PR problem) the problem may have very many different solutions. Notably, which is a third reason of interest of our approach, in those cases the problem at hand admits multiple different solutions, the technique that we present in the following allows identifying all of them.

In the following, the proposed approach is presented[4] and developed in Section II and assessed in Section III by means of several numerical examples. Conclusions follow.

II. 2-D PHASE RETRIEVAL AS COMBINATORIAL PROBLEM OVER 1-D INSTANCES

By the sake of simplicity, let us consider the case where the unknown 2-D discrete complex signal $f(\boldsymbol{x}) = f(x, y)$ is arranged over a uniformly spaced rectangular grid.[5] Also, let us consider the case where the discrete signal is centered around the origin of its domain, and the number of elements along the x and y coordinates is respectively equal to $N = 2\tilde{N}+1$ and $M = 2\tilde{M}+1$. In such a case,[6] by denoting with u and v the spectral variables in the Fourier transform domain, one will have:

$$F(\mathbf{u}) = F(u, v) = \sum_{n=-\tilde{N}}^{\tilde{N}} \sum_{m=-\tilde{M}}^{\tilde{M}} C_{n,m} e^{jnu} e^{jmv} \tag{1}$$

By simple derivations, it is easily proved that the square amplitude of (1) admits a representation in terms of some auxiliary coefficients $d_{p,q}$ as:

$$|F(u, v)|^2 = \sum_{p=-2\tilde{N}}^{2\tilde{N}} \sum_{q=-2\tilde{M}}^{2\tilde{M}} d_{p,q} e^{jpu} e^{jqv} \tag{2}$$

where, because of the fact that the left-hand member is a real quantity, $\{d_{p,q}\}$ is a Hermitian sequence of complex coefficients.

Notably, the choice of the indices in (1) [as well as the corresponding outcomes in (2)] emphasizes that the square-amplitude distribution has a (spatial) bandwidth that is twice the one pertaining to the corresponding complex signal. Then, if $A^2(u, v)$ denotes the measured *square-amplitude* distribution (which is assumed to be known over all its domain),[7] one can achieve a representation for it (and filter out some noise) by minimizing the following functional:

$$\Phi\{D_{p,q}\} = \left\| A^2(u, v) - \sum_{p=-2\tilde{N}}^{2\tilde{N}} \sum_{q=-2\tilde{M}}^{2\tilde{M}} D_{p,q} e^{jpu} e^{jqv} \right\|^2 \tag{3}$$

where $\{D_{p,q}\}$ are the representation coefficients for the measured power pattern.

Notably, minimization of (3) can be achieved in a fast and effective fashion by expanding the measured $A^2(u, v)$ into a truncated Fourier series, where the spectral variables belong to the range $[-\pi, \pi]$. If some oversampling is used in collecting data, in such a step a filtering of measurement errors on the acquired data is also performed.

A. BASIC IDEA

As anticipated, the proposed procedure resembles the solution of crossword puzzles, where one has to guarantee the congruence amongst solutions across and up-down possible words. In actual facts, advantage is taken from the fact that a very effective procedure exists for the solution of 1-D PR problems for discrete signals, and it is able to give back all the (possibly very many) solutions of the problem. These solutions become then possible ‘words’ to be allocated along the corresponding across, up-down, and eventually oblique directions.[8] Finally, congruence considerations amongst the different possible ‘words’ along the crossing directions will possibly restore uniqueness of the solution (if any) for the 2-D scheme.

A first possible solution procedure, which works nicely for the noiseless case, is given in the Section II-B. Then, in order to counteract the effect of both noise and an increased size of the discrete signals, some alternative solution procedures (implying a minimal additional a-priori information) are given in Section II-C. Finally, in Section II-D a further effective solution procedure is proposed for the important case of interest where the unknown function is known to be *real* or *real and positive* (as for instance in [19] and [22]).

B. FIRST POSSIBLE SOLUTION PROCEDURE

In the noiseless case, and assuming in the following that the $\{D_{p,q}\}$ sequence is known, the proposed approach can be conceptually split into a first part initializing the solution of the ‘crossword’ puzzle plus a second part where the subsequent completion of the scheme is performed. The first part (in the following referred to as *initializing step*) is the most difficult one, and it can be separated in two sub-steps. In particular, in the first sub-step one aims to identify all the

3. One of the authors pursued this strategy (but without any further refinement) in its Master thesis under the guidance of the elder authors of [8].

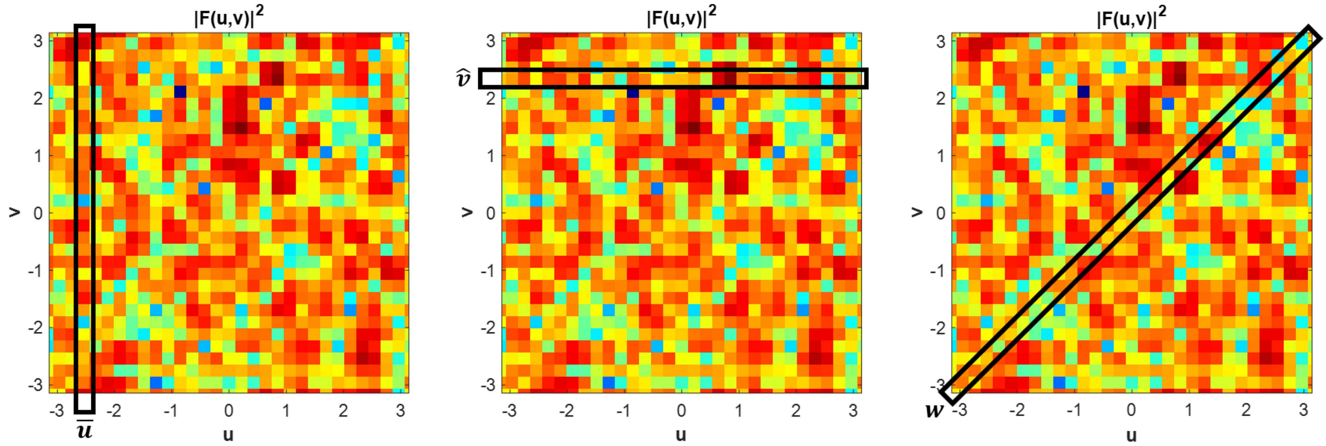
4. The basic idea, resumed in Section II-B, has been presented as a conference contribution in [32], [33]. The present contribution gives new insights on actual capabilities and suggests new ways to overcome limitations both in case of antenna problems as well as for other cases of general interest.

5. In antenna problems, that would be a planar array over a rectangular grid.

6. Similar reasonings hold true in case the number of elements is even.

7. Note that, because of measurement errors, $A(u, v)^2$ will be generally different from the ground-truth $|F(u, v)|^2$.

8. By virtue of bandlimitedness of $F(\boldsymbol{u})$ and $|F(\boldsymbol{u})|^2$, only a finite number of directions has to be actually considered, and because of the periodicity of the involved functions, we just need to consider the portion $[-\pi, \pi]$ of the spectral plane.



(a) (b) (c)

FIGURE 1. Pictorial representation of the crossword-like procedure described in Section II-B. The square amplitude of a random 2-D sequence is considered: (a) vertical, (b) horizontal and (c) oblique line of the spectral plane selected for the solution of the 1-D PR problem via SF [8] for the identification of all possible signal solutions.

(very many) possible complex fields along three intersecting lines (see Fig. 1).

In order to get a better understanding of the overall procedure, let us consider for instance a *vertical* line of the u-v plane (restricted to the $[-\pi, \pi]$ portion) as depicted in Fig. 1(a). By applying the technique [8] one can identify all possible $F(\overline{u}, v)$ behaviors corresponding to the measured square amplitude data along that line. In fact, along the line $u = \overline{u}$, such a 'dictionary' of admissible signals will be determined by applying to the corresponding square amplitude data, i.e.,

$$A^2(\overline{u}, v) = \sum_{q=-2\tilde{M}}^{2\tilde{M}} \overline{D}_q(\overline{u}) e^{jqv} \tag{4.1}$$

the SF technique introduced in [8], where the inherent coefficients are given by [32], [33]:

$$\overline{D}_q(\overline{u}) = \sum_{p=-2\tilde{N}}^{2\tilde{N}} D_{p,q} e^{jp\overline{u}} \tag{4.2}$$

By so doing, a multiplicity of functions defined as:

$$F(\overline{u}, v) = \sum_{m=-\tilde{M}}^{\tilde{M}} \overline{C}_m(\overline{u}) e^{jmv} \tag{5.1}$$

is achieved, where:

$$\overline{C}_m(\overline{u}) = \sum_{n=-\tilde{N}}^{\tilde{N}} C_{n,m} e^{jn\overline{u}} \tag{5.2}$$

Obviously, by following an identical procedure, one can also determine all possible $F(u, \hat{v})$ behaviors corresponding to the measured square amplitude data along a *horizontal* line of the u-v plane [see Fig. 1(b)]. For instance, along the coordinate $v = \hat{v}$, one will achieve:

$$A^2\left(u, \hat{v}\right) = \sum_{p=-2\tilde{N}}^{2\tilde{N}} \hat{D}_p\left(\hat{v}\right) e^{jpu} \tag{6.1}$$

where:

$$\hat{D}_p\left(\hat{v}\right) = \sum_{q=-2\tilde{M}}^{2\tilde{M}} D_{p,q} e^{jq\hat{v}} \tag{6.2}$$

so that the SF technique [8] can again be applied to get a multiplicity of possible behaviors:

$$F\left(u, \hat{v}\right) = \sum_{n=-\tilde{N}}^{\tilde{N}} \hat{C}_n\left(\hat{v}\right) e^{jnu} \tag{7.1}$$

where:

$$\hat{C}_n\left(\hat{v}\right) = \sum_{m=-\tilde{M}}^{\tilde{M}} C_{n,m} e^{jm\hat{v}} \tag{7.2}$$

Then, a similar procedure can also be applied to an oblique line (for instance, the one having $u = v$),[9] so that all possible field behaviors along this oblique line can also be determined. The only additional difficulty is that, along such an

9. By considering the main diagonal u=v [see Fig. 1(c)] and denoting by w the coordinate spanning it, one will achieve:

$$A^2(w) = \sum_{h=-2(\tilde{N}+\tilde{M})}^{2(\tilde{N}+\tilde{M})} \tilde{D}_h e^{jhw} \tag{8}$$

and

$$F(w) = \sum_{h=-(\tilde{N}+\tilde{M})}^{\tilde{N}+\tilde{M}} \tilde{C}_h e^{jhw} \tag{9}$$

where $\tilde{D}_h$ and $\tilde{C}_h$ denote suitable auxiliary sequences. Again, one can deal with an auxiliary 1-D PR problem and, by solving it, he/she will be able to find all the possible F behaviors along the line $u = v$.

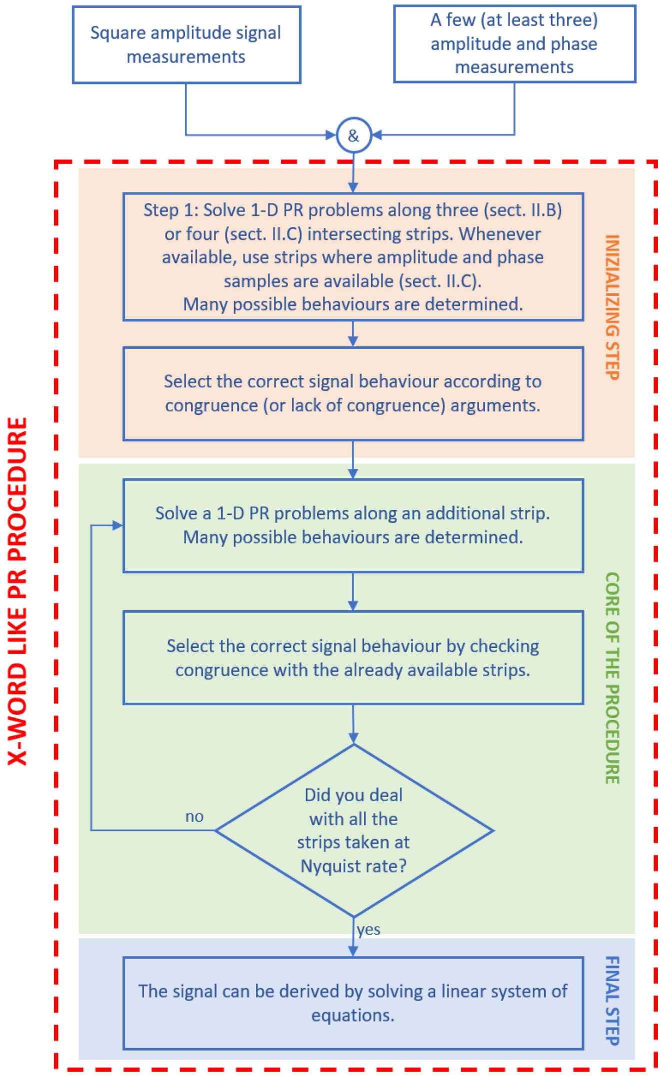


FIGURE 2. Flowchart of the basic idea underlying the proposed crosswords-based PR procedure.

oblique line, the number of terms of summations (and hence the number of zeroes to be considered in the procedure) grows.

Note in case $\overline{u} = 0$, or $\hat{v} = 0$, or for any line passing through the origin, one may also refer to the above 1-D functions (5.2) and (7.2) as to the so-called 'collapsed distributions' [35], which are also related to the Fourier slice theorem [36]. The reason for such a terminology is that expression (5.1) (as well as (7.1)) can be seen as the array factor of a linear array where, for any given m (n), all coefficients $C_{n,m}$ sum up (collapse) into the single excitation $\overline{C}_m(0)$ [$\hat{C}_n(0)$] and similar reasonings apply to all other cuts passing through the origin.

In the second sub-step of the *initializing step*, one can considerably reduce the dictionary of possible words, hopefully getting a single combination amongst the three dictionaries of possible signals, by checking congruence or missed congruence amongst the different possible combinations of signals.

Actually, one can examine any possible combination of horizontal and vertical retrieved signals and, for each couple, looking for the existence of an oblique signal correctly intersecting them. In fact, while a proper choice of the corresponding phase constant will allow any oblique candidate signal to correctly intersect the candidate *vertical* (*horizontal*) one, at the other intersection the phase of the *oblique* signal will generally be different from the one of the *horizontal* (*vertical*) signal (see Fig. 1). As a consequence, one will be able to discard a number of possibilities and hopefully identify the correct triplet of words (or at least to considerably reduce the number of possibilities).

The second and final part of the procedure, i.e., the completion of the scheme, is rather obvious for crosswords solvers. One has to consider additional horizontal, vertical, and oblique lines intersecting the ones already considered and determine the corresponding signals by proceeding to SF along each line, and then pruning the tree of possible solutions by means of congruence arguments. A flowchart summarizing the basic idea of the PR procedure is shown in Fig. 2.

The approach is very different from all existing procedures. In fact, once 1-D factorizations are produced, it is a kind of combinatorial problem. As such, it can be computationally heavy. On the other side, its similarity with crossword puzzles still comes to help by suggesting useful strategies.

For example, in order to reduce the number of combinations to be explored one can start from a line where many zeroes are present. In fact, this reduces the number of ambiguities on the corresponding 1-D PR problems [8]. Hence, a clever choice of the first three lines depicted in Fig. 1 will be of help. In this respect, care must be taken in avoiding that they intersect at a null of $A^2(u, v)$ (wherein phase would make no sense).

As a distinguishing characteristic, if the solution is not unique (which is for instance the case when one deals with u-v factorable patterns [8]) then the approach is able to find in a deterministic fashion all the different solutions of the problem.

C. LIMITATIONS, AND SECOND POSSIBLE SOLUTION PROCEDURE FOR NOISY AND/OR LARGE-SIZED SIGNALS

The above approach sets a deterministic procedure for the noiseless case, which, to the best of our knowledge, was anyway lacking. On the other side, as one can easily understand, limitations arise in case of noisy data and/or large 2-D sequences. In fact, as noisy data imply approximate solutions along the considered lines, some tolerance has to be used at the discrimination points when checking congruence amongst the possible 'words', so that more candidate solutions become eventually admissible. Also, the problem becomes harder and harder with increasing dimensions. In fact, under the actual noisy conditions, the existence of very many candidate words implies that even

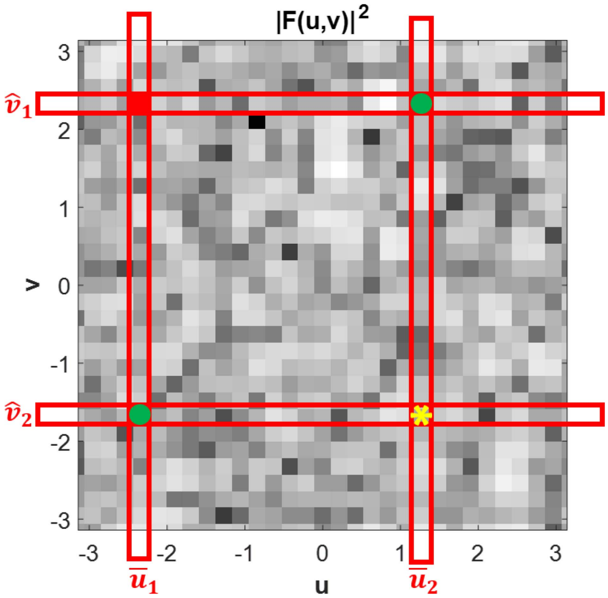


FIGURE 3. Pictorial representation of the crossword-like procedure described in Section II-C as a possible way out for limitations. The square amplitude of a random 2-D sequence is considered. The red square and green circular markers represent the three a-priori known complex data. In particular, the red point $(\overline{u}_1, \hat{v}_1)$ is used to set a reference phase for all the signals along line $v = \hat{v}_1$ and $u = \overline{u}_1$, while the yellow star marker is used as discrimination point to discard the unsuitable solutions and hopefully retrieve the correct signal along the considered lines.

more signals may become 'admissible' at the discrimination points. Accordingly, with increasing dimensions a (possibly exponentially) increasing number of candidate words enter into play, thus negatively affecting the pruning of the tree of possible solutions.

Again, one possible way out comes from the similarity to crossword puzzles. In fact, in those cases, the schemes can be simplified by the availability of some letters in given crossings of rows and columns. In a corresponding fashion, the basic scheme of Section II-B can take advantage from a reduced number of amplitude and phase measurements. In such a way, one can conveniently and effectively initialize the solution of the overall scenario.

One possible strategy, represented in Fig. 3, takes advantage from just three amplitude and phase measurements disposed on three vertices of a rectangle, while the fourth vertex point is used to enforce congruence amongst possible words. By so doing, one can hopefully have an initial pruning of the possible solutions along the two initial rows and columns. Also, by means of congruence arguments one can establish the values of $F(u, \mathrm{v})$ along the four lines determining the rectangle, so that from now on one will have (at least) a couple of amplitude and phase known values of $F(u, \mathrm{v})$ along any row and column.

Obviously, the same statement also holds true in case the fourth vertex can be also considered an a-priori known point, which allows for a significant reduction of the complexity of the initializing step. Also note that PR along diagonals is not anymore strictly required.

As it can be easily understood, the initialization of the procedure is the most difficult part, as once the phase distribution along four or more rows and columns has been correctly retrieved, the pruning of solutions along the remaining rows and columns becomes easier and easier.

D. EXPLOITATION OF 'COLLAPSED' DISTRIBUTIONS, SPECIAL CASES OF ACTUAL INTEREST, AND FINAL REMARKS

As just stated, the triggering/initialization of the procedure is the most critical part of the proposed approach. Interestingly, there are a couple of interesting applications where the initialization can take advantage from the intrinsic nature of the signal to be retrieved, or from some available a-priori information. This is for example the case where the signal $f(\boldsymbol{x})$ is known to be *real* or *real and positive* (which is the case of crystallography, diffraction microscopy, and so on, since such a kind of signal corresponds for instance to the density of microparticles [19] or to nanotubes' atomic dimensions [22]). In such a case, in fact, the property of being real (and positive) of the 2-D signal implies that the corresponding collapsed distribution [see (5.2) and (7.2)] is real (and positive) as well.

Hence, such a property can be conveniently used in solving the 1-D PR problem along all the strips passing through the origin in the transformed domain in order to discard a number of outcomes of the corresponding SF problem, and hence hopefully simplifying the correct initial setting of the procedure. In fact, for example, when $\overline{u} = 0$ or $\hat{v} = 0$, the equations (5.2) and (7.2) simply become respectively:

$$\overline{C}_m(\overline{u} = 0) = \sum_{n=-\tilde{N}}^{\tilde{N}} C_{n,m} \tag{10}$$

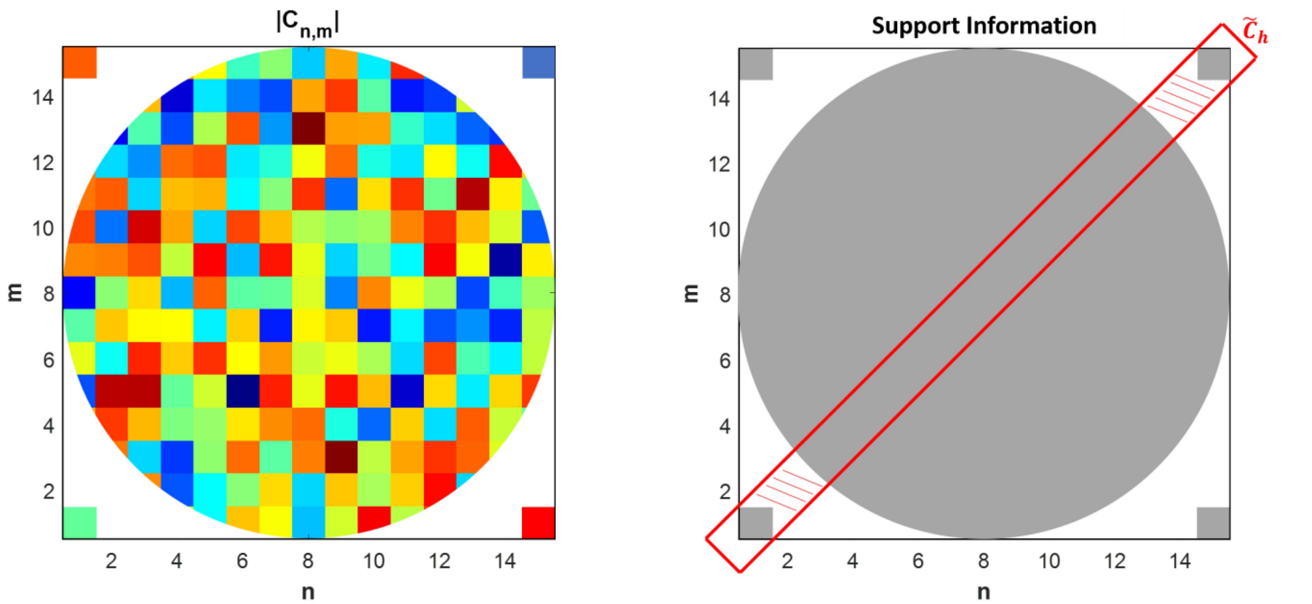
and

$$\hat{C}_n\left(\hat{v} = 0\right) = \sum_{m=-\tilde{M}}^{\tilde{M}} C_{n,m} \tag{11}$$

Consequently, the unknown coefficients at the left hand member of (10) and (11) have to be real and positive, and hence all the solutions that do not meet this requirement can be automatically discarded.

In case of (very peculiar) support information, a similar reasoning is possible. For example, the support information depicted in Fig. 4 implies that the corresponding collapsed distribution along the $u = v$ direction has to be generated by a linear auxiliary array where a number of excitations has to be equal to zero. Obviously, such a circumstance implies that one can hopefully have a univocal phase distribution along the $u = v$ direction, which simplifies the following steps.

As a final comment concerning all the different possible procedures described above, it is worth noting that a suitable alternative to the general solution procedure (i.e., solving the auxiliary 1-D PR problems along *all* the different rows and columns and then exploit the latter in order to discriminate amongst admissible or inadmissible words) is available. In particular, once a sufficient number of rows and



(a) (b)

FIGURE 4. Pictorial representation of the crossword like procedure described in Section II-D for a case where the peculiar signal support is known in advance: (a) amplitude of a random 2-D signal; (b) support information on the collapsed distribution along the main diagonal: null $\tilde{C}_h$ coefficients are expected in correspondence of red-lined regions.

columns have been retrieved, one can conveniently switch to a least square fitting procedure as in [2]. In fact, the already retrieved samples of the spectrum can be used as linear constraints in a more classical PR procedure. Then, provided the number of these constraints is sufficiently large, false solutions can be safely avoided [2], [3].

III. NUMERICAL EXAMPLES

As a proof of concept of the above proposed strategy, we consider in the following some numerical examples. In particular, the 2-D signal to be retrieved corresponds to the excitations of a planar array, while measured data $A^2(u, v)$ correspond to amplitude samples of the corresponding radiated power pattern. Accordingly, we used $u = \beta d_x sin\theta cos\Phi$ and $v = \beta d_y sin\theta sin\Phi$, where $\beta = 2\pi/\lambda$ denotes the wavenumber (λ being the operative wavelength), d_x, d_y are the element spacings along the x and y axis, and θ, Φ respectively denote the elevation and azimuth aperture angles with respect to boresight.

In order to test the proposed procedures in hard phase-retrieval scenarios, all numerical examples (except for the last one) have been performed by adopting, as in [27], *random* array excitations. Moreover, a test case dealing with the PR of a 'structured' electromagnetic field of interest in the applications and having a very large excitations dynamics is also given.

In order to make the whole periodicity range of the power spectrum available for measurements (see [27] for more details), an inter-element spacing of d_x=d_y=0.707λ has been adopted for all the following examples.

By indicating with I= $[I_{11}, I_{12}, \ldots, I_{NM}]$ the vector containing the *NM* array excitations, the Normalized Mean Square Error (NMSE) is computed to quantitatively evaluate the accuracy of the PR procedure:

$$NMSE = \frac{\|\mathbf{I} - \mathbf{I}_{retr}\|_2^2}{\|\mathbf{I}\|_2^2} \tag{12}$$

$\mathbf{I}_{retr}$ being the vector containing the *NM* retrieved array excitations.

The first numerical example deals with a 15x15 elements equally spaced square array with *complex* random excitations. The power pattern has been measured over an equally spaced grid made of 43x43 points in the *u*-*v* plane (which is restricted to the $[-\pi, \pi; -\pi, \pi]$ periodicity range) and has been assumed completely known in four points of the spectral plane in order to properly trigger the recovery procedure and give us the momentum to keep going, see Fig. 5. Consequently, these points allow us to recover the four lines passing through them (two vertical and two horizontal lines), and each point of the retrieved lines becomes a "known" word of our crossword puzzle.

Then, one start working by systematically considering additional horizontal, vertical, and oblique lines of the radiation pattern in order to identify the correct field behavior amongst the very many possible ones. In so doing, the number and locations of the actual lines to be considered is dictated by the bandlimitedness of $F(u,v)$.

A comparison between reference and reconstructed results is given in Fig. 5 in terms of fields [subplots (a) and (b)] and excitations [subplot (c) and (d)], respectively. As it can be seen, the proposed procedure led to a fully satisfactory excitations recovery. The effectiveness of the proposed approach is also testified by the very low NMSE value (i.e., NMSE=1.07x10^{-5}). By exploiting a PC equipped with

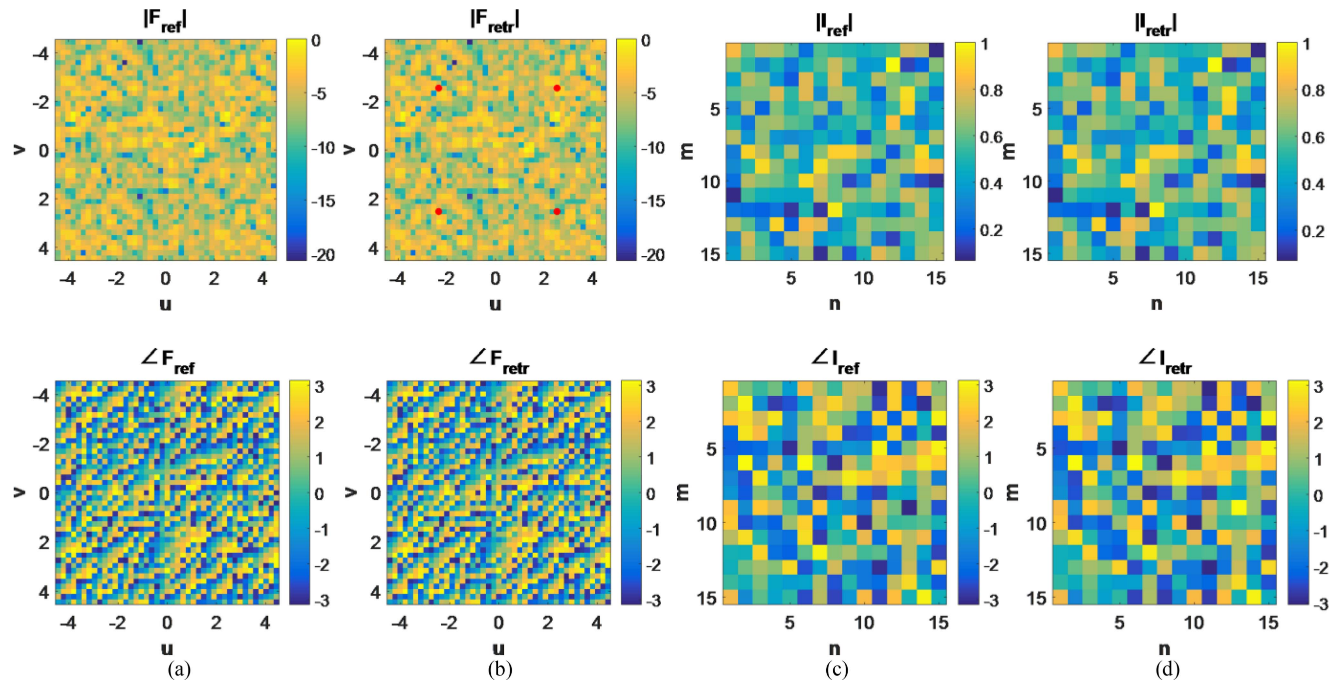


FIGURE 5. Example #1: complex random array excitations. From top to bottom: amplitude and phase of the reference field (a); amplitude and phase of the retrieved field (b); amplitude and phase of the reference excitations (c); amplitude and phase of the retrieved excitations (d). The red circle markers are those point wherein the complex field is supposed a-priori known (in amplitude and phase), and correspond to $(u_1,v_1) = (-2.33, -2.54)$, $(u_2,v_2) = (2.54, -2.54)$, $(u_3,v_3) = (-2.33, 2.54)$, $(u_4,v_4) = (2.54, 2.54)$.

an Intel i7-6700HQ CPU and 16GB RAM, the numerical experiment required roughly[10] 90 minutes.

In the second test case, we used the same signal source of the previous example but enhanced the PR difficulty by corrupting each measured field amplitude by white Gaussian noise with SNR=25dB.[11] Also, the power pattern has been assumed completely known in the same four points of the spectral plane as before in order to properly trigger the recovery procedure.

A comparison between reference and reconstructed results is given in Fig. 6 in terms of fields [subplots (a) and (b)] and excitations [subplot (c) and (d)], respectively. As it can be seen, a satisfactory PR solution has been achieved again, with an NMSE equal to 0.0035. The computational time was approximately equal to the one of the previous noiseless test case.

In the third test case, we considered the case of a *real and positive* signal to test the potentialities of the procedure described in Section II-D. By still referring to antenna problems, we considered a 13x13-elements square array whose excitations have been set as real and positive random variables uniformly distributed in the range [0, 1]. The corresponding power pattern has been measured over an equally spaced grid made of 37x37 points in the *u*-*v* plane at hand, but no complex measurements have been used. In fact, as detailed in Section II-D, dealing with real signals allows to resort to a *collapsed distribution* when $\bar{u} = 0$ and $\hat{v} = 0$ are considered, with the consequent advantage of looking to real excitations sets amongst all gathered solutions through the SF. Once compatible 'words' have been retrieved along the u and v axes, one has the complex spectrum along two sides of a rectangle so that the recovery procedure can continue as in Section II-C, i.e., by determining the complex values of $F(u, \mathrm{v})$ along a rectangle and then performing SF along the remaining rows, columns and diagonals (and enforcing coherence amongst their multiplicity of solutions to discard many of them).

The results for the third example, corresponding to NMSE=$1.27 \mathrm{x} 10^{-4}$, are shown in Fig. 7, and confirm the effectiveness of the proposed PR approach. This numerical experiment required a computational time of 75 minutes.

In the fourth test case, we used the same signal source of the previous example but enhanced the PR difficulty by corrupting each measured field amplitude with SNR=25dB.[12]

A comparison between reference and reconstructed results is given in Fig. 8 in terms of fields [subplots (a) and (b)] and excitations [subplot (c) and (d)], respectively. Also in this case, a satisfactory PR solution has been achieved with a NMSE equal to 0.0018. As in the second test case, no difference in terms of computational time has been experienced with respect to the previous noiseless experiment.

10. The reported computational time includes all the steps of the recovery procedure, i.e., the solution of all the 1-D PR problems and the run time of the algorithm devoted to discard the non-admissible solutions until reaching the final one.

11. Notably, the square amplitude distribution is oversampled with respect to the minimal requirements, which allows a reduction of the overall measurement error.

12. Again, the procedure takes advantage from some oversampling of the square amplitude distribution.

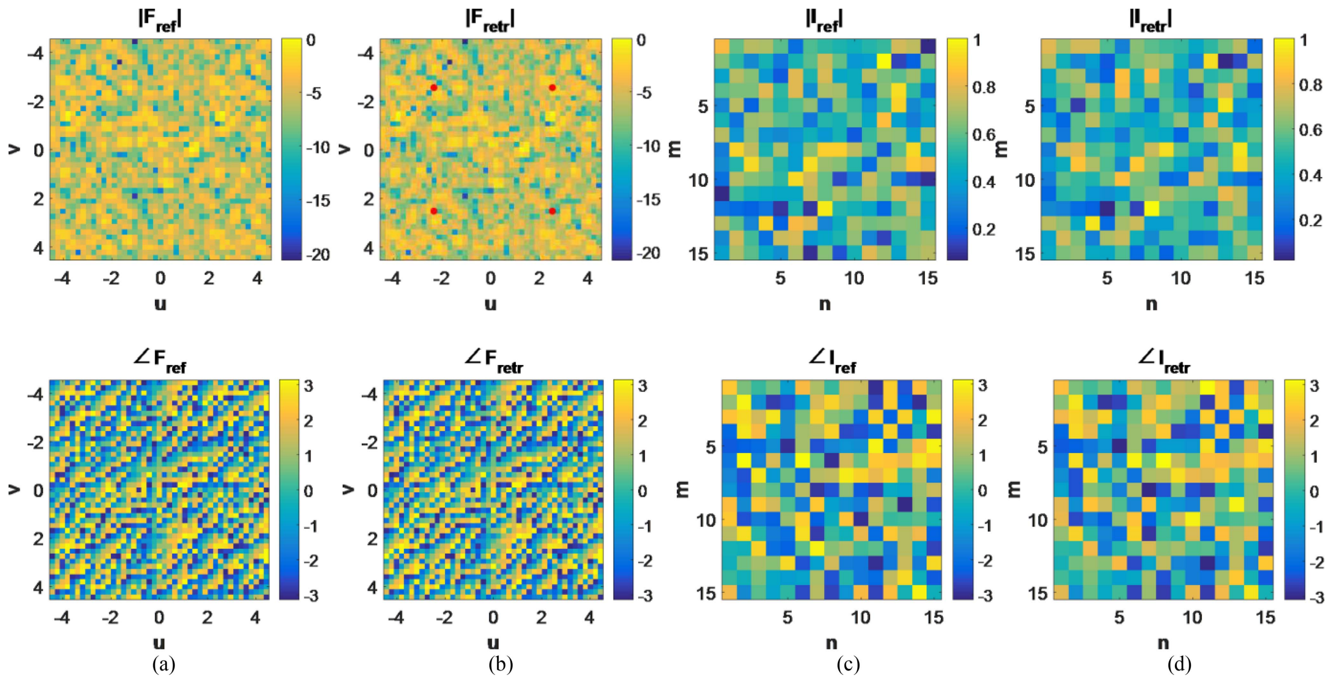


FIGURE 6. Example #2: complex random array excitations and noisy data (SNR=25dB). From top to bottom: amplitude and phase of the reference field (a); amplitude and phase of the retrieved field (b); amplitude and phase of the reference excitations (c); amplitude and phase of the retrieved excitations (d). The red circle markers are those point wherein the complex field is supposed a-priori known (in amplitude and phase), and correspond to $(u_1,v_1) = (-2.33, -2.54)$, $(u_2,v_2) = (2.54, -2.54)$, $(u_3,v_3) = (-2.33, 2.54)$, $(u_4,v_4) = (2.54, 2.54)$.

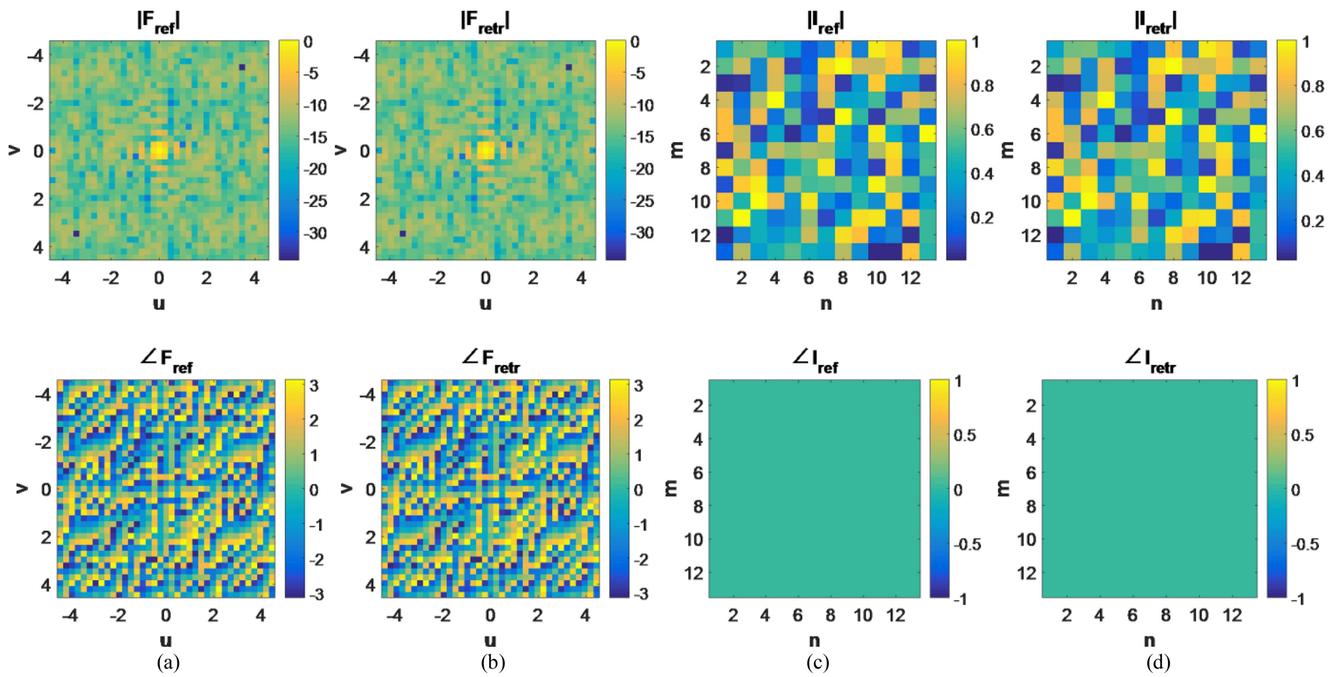


FIGURE 7. Example #3: real and positive random array excitations. From top to bottom: amplitude and phase of the reference field (a); amplitude and phase of the retrieved field (b); amplitude and phase of the reference excitations (c); amplitude and phase of the retrieved excitations (d).

In the last numerical example, we considered a structured pattern where the unknown coefficients have a very large (and possibly the worst) dynamics range ratio (DRR), the latter being defined as the ratio amongst the maximum and minimum amplitudes of the excitations. In particular, the measured power pattern is a flat-top 'shaped' beam having the triangular footprint that appeared in several papers (see for instance [38]) dealing with array antennas optimal synthesis. As it was done in [38], the excitation coefficient of a non-negligible number of radiating elements has been

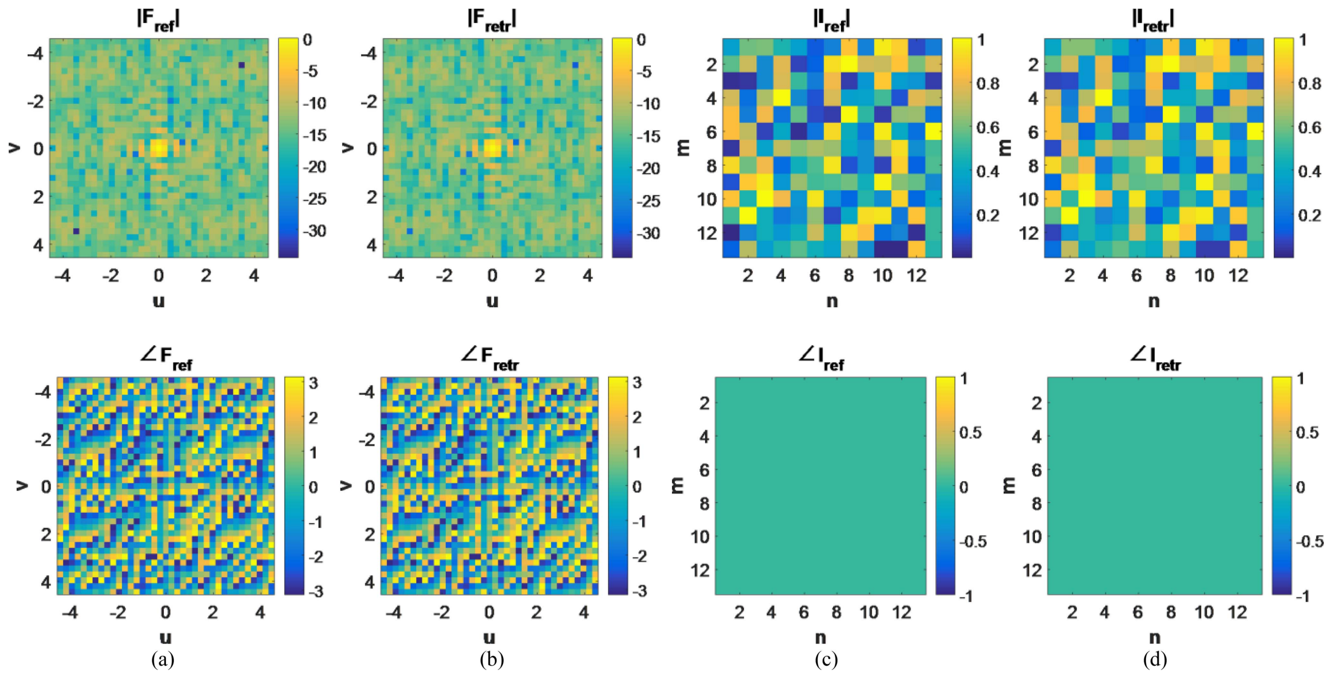


FIGURE 8. Example #4: real and positive random array excitations and noisy data (SNR=25dB). From top to bottom: amplitude and phase of the reference field (a); amplitude and phase of the retrieved field (b); amplitude and phase of the reference excitations (c); amplitude and phase of the retrieved excitations (d).

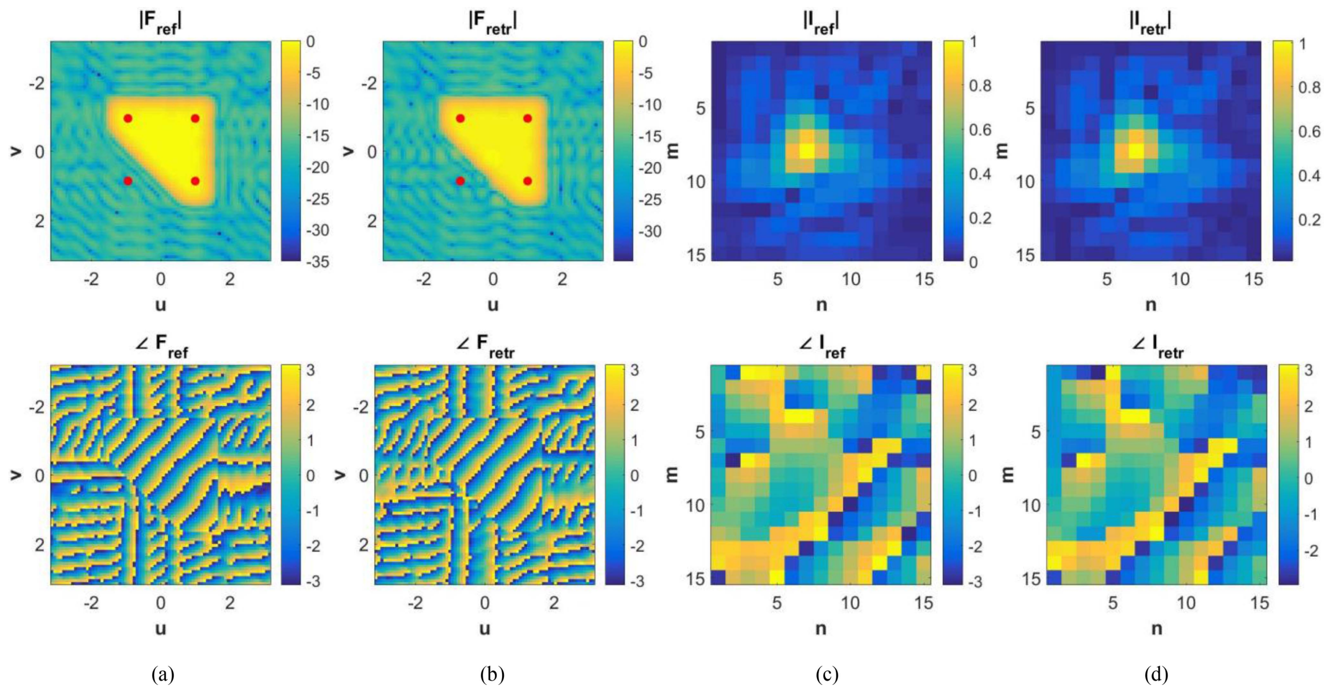


FIGURE 9. Example #5: 'shaped' beam power pattern and optimal complex array excitations [38]. From top to bottom: amplitude and phase of the reference field (a); amplitude and phase of the retrieved field (b); amplitude and phase of the reference excitations (c); amplitude and phase of the retrieved excitations (d). The red circle markers are those point wherein the complex field is supposed a-priori known (in amplitude and phase), and correspond to $(u_1,v_1) = (-0.95, -0.95)$, $(u_2,v_2) = (1, -0.95)$, $(u_3,v_3) = (-0.95, 0.86)$, $(u_4,v_4) = (1, 0.86)$.

set to 0 during the field generation, leading to a virtually 'infinite' DRR value.

The array is composed of 15x15 radiating elements, and the excitations reported in [38] have been used (by assuming this time, differently from [38], d_x=d_y=0.707λ). In the inverse problem, the corresponding power pattern has been considered to be known over an equally spaced grid made of 75x75 points in the *u*-*v* plane, while the radiated field has been assumed completely known in just four points of the spectral plane (see Fig. 9).

A comparison between reference and reconstructed results is given in Fig. 9 in terms of fields [subplots (a) and (b)] and excitations [subplot (c) and (d)], respectively. Once again, the effectiveness of the proposed approach is testified by the very low NMSE value (i.e., NMSE=8.99×10^{-4}), while the DRR of the retrieved excitations turned out being equal to 144.3.

It is worth to underline that, although assessed in the case of array antennas and far field measurements, the given tools can be exploited for the phase recovery of whatever non-superdirective source, and can also be extended to the case of near-field measurements. In such a case, one can in fact exploit the 'reduced radiated field' concept, so that the near field still admits a Fourier series representation in terms of auxiliary variables and a fictitious array [37].

IV. CONCLUSION

A new and original approach to the phase retrieval of two-dimensional complex signals has been presented which allows to set a deterministic and globally effective procedure for the retrieval of 2-D signals from phaseless and noiseless data. The characteristics of the approach have been discussed, emphasizing useful properties as well as the corresponding limitations in the actual case of noisy data. Also, some possible ways out from the limitations, suggesting for the antenna diagnostics problem the possible exploitation of hybrid measurement schemes using a (very) reduced number of amplitude and phase measurements, have been suggested. We have also shown that the basic approach (using just amplitude information) can be of immediate usefulness in those (many) cases where the starting signal is known to be real, or real and positive.

Notably, while (almost) all the existing approaches to antenna characterization problems require the exploitation of two different amplitude distributions [24], or two different probes [25], or two different defocusing conditions [26] in order to avoid the occurrence of false solutions, the proposed one just requires a single measurement surface and few additional information with respect to the one strictly needed for theoretical uniqueness. Also, it is not based in its present version on global optimization.

While having indeed a number of drawbacks, we have shown by means of examples that the approach can be of interest in case of moderate dimension sources, and we are confident that the hybridization of the proposed 'congruence' point of view with other more classical approaches can give rise to further convenient solution procedures.

Another interesting research direction is the application of the above strategy based on 1-D decompositions to antenna synthesis problem and, in particular, to the optimal synthesis of mask constrained power patterns by means of planar arrays. In such a case, in fact, although considerable progress has been made [38], a provably globally optimal synthesis procedure (opposite to the linear case) is still lacking.

REFERENCES

[1] J. R. Fienup and J. C. Dainty, "Phase retrieval and image reconstruction for astronomy," in *Image Recovery: Theory and Application*. Orlando, FL, USA: Academic, 1987, pp. 231–275.

[2] T. Isernia, G. Leone, R. Pierri, and F. Soldovieri, "Role of support information and zero locations in phase retrieval by a quadratic approach," *J. Opt. Soc. America A*, vol. 16, no. 7, pp. 1845–1856, 1999.

[3] T. Isernia, F. Soldovieri, G. Leone, and R. Pierri, "On the local minima in phase reconstruction algorithms," *Radio Sci.*, vol. 31, no. 6, pp. 1887–1899, Nov./Dec. 1996.

[4] F. Soldovieri, R. Solimene, and R. Pierri, "Reconstructing the contour of metallic planar objects from only intensity scattered field data over a single plane," *Opt. Exp.*, vol. 16, no. 13, pp. 9468–9479, 2008.

[5] O. M. Bucci, L. Crocco, M. D'Urso, and T. Isernia, "Inverse scattering from phaseless measurements of the total field on open lines," *J. Opt. Soc. America A*, vol. 23, no. 10, pp. 2566–2577, 2006.

[6] M. D'Urso, K. Belkebir, L. Crocco, T. Isernia, and A. Litman, "Phaseless imaging with experimental data: Facts and challenges," *J. Opt. Soc. America A*, vol. 25, no. 1, pp. 271–281, 2008.

[7] F. Üstüner, E. Aydemir, E. Güleç, M. Ilarslan, M. Çelebi, and E. Demirel, "Antenna radiation pattern measurement using an unmanned aerial vehicle," in *Proc. 31st URSI Gen. Assembly Sci. Symp.*, Beijing, China, Aug. 2014, pp. 1–4.

[8] T. Isernia, O. M. Bucci, and N. Fiorentino, "Shaped beam antenna synthesis problems: Feasibility criteria and new strategies," *J. Electromagn. Waves Appl.*, vol. 12, no. 1, pp. 103–138, 1998.

[9] A. F. Morabito, A. R. Laganà, and T. Isernia, "On the optimal synthesis of ring symmetric shaped patterns by means of uniformly spaced planar arrays," *Progr. Electromagn. Res. B*, vol. 20, pp. 33–48, Mar. 2010.

[10] J. Knapp, A. Paulus, and T. F. Eibert, "Reconstruction of squared field magnitudes and relative phases from magnitude-only near-field measurements," *IEEE Trans. Antennas Propag.*, vol. 67, no. 5, pp. 3397–3409, May 2019.

[11] R. G. Yaccarino and Y. Rahamat-Samii, "Phaseless bi-polar near-field measurements and diagnostics of array antennas," *IEEE Trans. Antennas Propag.*, vol. 47, no. 3, pp. 574–583, Mar. 1999.

[12] D. Morris, "Phase retrieval in the radio holography of reflector antennas and radio telescopes," *IEEE Trans. Antennas Propag.*, vol. AP-33, no. 7, pp. 749–755, Jul. 1985.

[13] Y. Xu, Q. Ye, and G. Meng, "Optimisation of phase factor in phase retrieval for reflector antennas with active surface," *IET Microw. Antennas Propag.*, vol. 12, no. 15, pp. 2285–2291, 2018.

[14] H. Jin, J. Huang, Q. Ye, G. Meng, B. Xiang, and N. Wang, "Surface shape detection with a single far-field intensity by combined amplitude and phase retrieval," *Int. J. Antennas Propag.*, vol. 2019, Aug. 2019, Art. no. 7648604.

[15] R. Palmeri, T. Isernia, and A. F. Morabito, "Diagnosis of planar arrays through phaseless measurements and sparsity promotion," *IEEE Antennas Wireless Propag. Lett.*, vol. 18, no. 6, pp. 1273–1277, Jun. 2019.

[16] T. Brockett and Y. Rahmat-Samii, "Millimeter-wave bi-polar planar phaseless measurements and diagnostics: Characterization of a Ka-band patch array," in *Proc. IEEE Antennas Propag. Soc. Int. Symp.*, 2009, pp. 1–4.

[17] J. Knapp, A. Paulus, J. Kornprobst, U. Siart, and T. F. Eibert, "Multifrequency phase retrieval for antenna measurements," *IEEE Trans. Antennas Propag.*, vol. 69, no. 1, pp. 488–501, Jan. 2021.

[18] A. Grjasnow, A. Wuttig, and R. Riesenberg, "Phase resolving microscopy by multi-plane diffraction detection," *J. Microsc.*, vol. 231, no. 1, pp. 115–123, 2008.

[19] S. Marchesini, "A unified evaluation of iterative projection algorithms for phase retrieval," *Rev. Sci. Instrum.*, vol. 78, no. 1, 2007, Art. no. 011301.

[20] R. M. Von Bunau, H. Fukuda, and T. Terasawa, "Phase retrieval from defocused images and its applications in lithography," *Jpn. J. Appl. Phys.*, vol. 36, no. 12S, p. 7494, 1997.

[21] R. P. Millane, "Phase retrieval in crystallography and optics," *J. Opt. Soc. America A*, vol. 7, no. 3, pp. 394–411, 1990.

[22] J. M. Zuo, I. Vartanyants, M. Gao, R. Zhang, and L. A. Nagahara, "Atomic resolution imaging of a carbon nanotube from diffraction intensities," *Science*, vol. 300, no. 5624, pp. 1419–1421, 2003.

[23] T. Isernia, G. Leone, and R. Pierri, "Phase retrieval of radiated fields," *Inverse Problems*, vol. 11, no. 1, pp. 183–203, 1995.

[24] F. Soldovieri, A. Liseno, G. D'Elia, and R. Pierri, "Global convergence of phase retrieval by quadratic approach," *IEEE Trans. Antennas Propag.*, vol. 53, no. 10, pp. 3135–3141, Oct. 2005.
[25] R. Pierri, G. D'Elia, and F. Soldovieri, "A two probes scanning phaseless near-field far-field transformation technique," *IEEE Trans. Antennas Propag.*, vol. 47, no. 5, pp. 792–802, May 1999.
[26] D. L. Misell, "A method for the solution of the phase problem in electron microscopy," *J. Phys. D, Appl. Phys.*, vol. 6, no. 1, pp. L6–L9, 1973.
[27] A. F. Morabito, R. Palmeri, V. A. Morabito, A. R. Laganà, and T. Isernia, "Single-surface phaseless characterization of antennas via hierarchically-ordered optimizations," *IEEE Trans. Antennas Propag.*, vol. 67, no. 1, pp. 461–474, Jan. 2019
[28] B. Fuchs and L. L. Coq, "Excitation retrieval of microwave linear arrays from phaseless far-field data," *IEEE Trans. Antennas Propag.*, vol. 63, no. 2, pp. 748–754, Feb. 2015.
[29] K. Knopp, "Weierstrass's factor-theorem," in *Theory of Functions: Part II*. New York, NY, USA: Dover, 1996, pp. 1–7.
[30] C.A. Floudas and P. M. Pardalos, *State of the Art in Global Optimization: Computational Methods and Applications IX*. New York, NY, USA: Kluwer Acad. Publ., 1996.
[31] E. J. Candes, Y. C. Eldar, T. Strohmer, and V. Voroninski, "Phase retrieval via matrix completion," *SIAM Rev.*, vol. 57, no. 2, pp. 225–251, 2015.
[32] A. F. Morabito, R. Palmeri, and T. Isernia, "Phase retrieval of scalar fields radiated by finite-dimensional sources through single-surface measurements and a crosswords-like processing," in *Proc. Photon. Electromagn. Res. Symp.*, Rome, Italy, Jun. 2019, pp. 540–546.
[33] A. F. Morabito, P. G. Nicolaci, R. Palmeri, and T. Isernia, "Two-dimensional phase retrieval as a 'crosswords' problem," in *Proc. 13th Eur. Conf. Antennas Propag.*, Krakow, Poland, Mar./Apr. 2019, pp. 1–4.
[34] O. M. Bucci, C. Gennarelli, and C. Savarese, "Representation of electromagnetic fields over arbitrary surfaces by a finite and nonredundant number of samples," *IEEE Trans. Antennas Propag.*, vol. 46, no. 3, pp. 351–359, Mar. 1998.
[35] J. Illade-Quinteiro, J. A. Rodríguez-Gonzalez, and F. Ares-Pena, "Shaped-pattern synthesis by spreading out collapsed distributions," *IEEE Antennas Propag. Mag.*, vol. 52, no. 6, pp. 110–114, Dec. 2010.
[36] E. W. Hansen, *Fourier Transforms: Principles and Applications*. Hoboken, NJ, USA: Wiley, 2014, pp. 281–299.
[37] O. M. Bucci and M. D'Urso, "Power pattern synthesis of given sources exploiting array methods," in *Proc. 2nd Eur. Conf. Antennas Propag.*, Edinburg, U.K., 2007, p. 144.
[38] G. M. Battaglia, A. F. Morabito, G. Sorbello, and T. Isernia, "Mask-constrained power synthesis of large and arbitrary arrays as a few-samples global optimization," *Progr. Electromagn. Res. C*, vol. 98, pp. 69–81, Jan. 2020.