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Research Article

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Existence of two solutions for singular Φ-Laplacian problems

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Abstract: Existence of two solutions to a parametric singular quasi-linear elliptic problem is proved. The equation is driven by the Φ-Laplacian operator, and the reaction term can be nonmonotone. The main tools employed are the local minimum theorem and the Mountain pass theorem, together with the truncation technique. Global $C^{1,\tau}$ regularity of solutions is also investigated, chiefly via *a priori* estimates and perturbation techniques.

Keywords: Φ-Laplacian, Sobolev-Orlicz spaces, singular terms, variational methods

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1 Introduction and main results

In this article, we consider the problem

$$\begin{cases} -\Delta_\Phi u = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{P}_{\lambda,f}$}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, $f : \Omega \times (0,+\infty) \to [0,+\infty)$ is a Carathéodory function, and Δ_Φ is the Φ-Laplacian, namely,

$$\Delta_\Phi u \coloneqq \operatorname{div}(a(|\nabla u|)\nabla u) \tag{1.1}$$

for a suitable C^1 function $a : (0,+\infty) \to (0,+\infty)$. Setting $\varphi(t) = ta(t)$ for all $t > 0$, we denote by Φ the primitive of φ satisfying $\Phi(0) = 0$. With the following hypotheses (compared to also [15, Appendix I]), Φ turns out to be the Young function generated by φ (see [23, Definition 3.2.1]).

Definition 1.1. We say that $u \in W_0^{1,\Phi}(\Omega)$ is a (weak) solution to $(\mathrm{P}_{\lambda,f})$ if $u > 0$ in Ω and, for any $v \in W_0^{1,\Phi}(\Omega)$, one has both $f(\cdot,u)v \in L^1(\Omega)$ and

$$\int_\Omega a(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)\mathrm{d}x = \lambda\int_\Omega f(x,u(x))v(x)\mathrm{d}x.$$

We assume the following hypotheses (the indices i_Ψ, s_Ψ are defined in (2.3)):

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H(a)$_1$:

$$-1 < \inf_{t>0} \frac{ta'(t)}{a(t)} \le \sup_{t>0} \frac{ta'(t)}{a(t)} < +\infty. \tag{1.2}$$

H(a)$_2$: We suppose that

$$\int_1^{+\infty} \Theta_\Phi(t)\mathrm{d}t = +\infty, \quad \text{where } \Theta_\Phi(t) := \frac{\Phi^{-1}(t)}{t^{1+\frac{1}{N}}}. \tag{1.3}$$

Accordingly, the Sobolev-Orlicz conjugate Φ_* is well defined; see Definition 2.3. We also suppose $s_\Phi < i_{\Phi_*}$.
H(f)$_1$: One has

$$\liminf_{s\to 0^+} f(x, s) = +\infty \quad \text{uniformly w.r.t. } x \in \Omega. \tag{1.4}$$

H(f)$_2$: There exist $c_i > 0$, $i = 1, 2$, $\gamma \in (0, 1)$, and a Young function Υ such that $1 < i_\Upsilon \le s_\Upsilon < i_{\Phi_*}$ and

$$f(x, s) \le c_1 \overline{\Upsilon}^{-1}(\Upsilon(s)) + c_2 s^{-\gamma} \tag{1.5}$$

for almost all $x \in \Omega$ and all $s > 0$, where $\overline{\Upsilon}$ is the Young conjugate of Υ (see Definition 2.2).
H(f)$_3$: There exist $R > 0$ and $\mu > s_\Phi$ such that

$$\mu F(x, t) \le tf(x, t), \quad \text{being } F(x, t) := \int_R^t f(x, s)\mathrm{d}s, \tag{1.6}$$

for almost all $x \in \Omega$ and all $t \ge R$.

Remark 1.2. Let us briefly comment the main assumptions we have done.

- Hypothesis H(a)$_1$ is called *ellipticity condition* for operators with the Uhlenbeck structure, namely, in the form (1.1). It implies $1 < i_\Phi \le s_\Phi < +\infty$, which in turn implies $\Phi \in \Delta_2 \cap \nabla_2$ (see (2.2)).
- The first part of H(a)$_2$ is the Sobolev-Orlicz analogue of the Sobolev hypothesis $p < N$. As customary, H(a)$_2$ forces the problem in the worst regularity setting because of the lack of the Morrey-type embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow C^{0,\tau(\cdot)}(\Omega)$ (see [23, Theorem 7.4.4] for a complete statement). Regarding the second part of H(a)$_2$, the condition $s_\Phi < i_{\Phi_*}$ is used only in Section 3.
- The requirement $s_\Upsilon < i_{\Phi_*}$ in H(f)$_2$ is made for the sake of simplicity: indeed, it implies $\Upsilon \ll \Phi_*$ (see (2.9)), which in turn guarantees the compactness of the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Upsilon(\Omega)$. Actually, in Section 3, it suffices to require $s_\Upsilon \le i_{\Phi_*}$. In this respect, see also Remark A.6 in the appendix.
- Condition (1.5) parallels the sub-critical growth condition used in the standard Sobolev setting. We recall that $\overline{\Upsilon}^{-1}(\Upsilon(s))$ can be replaced with $\frac{\Upsilon(s)}{s}$, according to the inequalities:

$$\Psi(s) \le s\overline{\Psi}^{-1}(\Psi(s)) \le 2\Psi(s) \quad \forall s > 0, \tag{1.7}$$

  valid for any Young function Ψ. For a proof of (1.7), vide [31, Proposition 2.1.1].
- Hypothesis H(f)$_3$ is an adaptation of the *Ambrosetti-Rabinowitz unilateral condition* (see, e.g., [29, p. 154]) in the Sobolev-Orlicz setting. Following [12], H(f)$_3$ can be weakened by requiring, instead of $\mu > s_\Phi$,

$$\mu > \limsup_{t\to+\infty} \frac{t\varphi(t)}{\Phi(t)}.$$

The primary aim of the present work is to extend the results of [8,21] to problems driven by non-homogeneous operators as the (p, q)-Laplacian $\Delta_p + \Delta_q$, being $\Delta_r u := |\nabla u|^{r-2}\nabla u$ the classical r-Laplacian, $r \in (1, +\infty)$. A class of operators encompassing the (p, q)-Laplacian is the one described in [15, Appendix I], where ellipticity and Uhlenbeck structure are coupled with a p-growth condition that allows to work in the Sobolev setting $W_0^{1,p}(\Omega)$. This class can be extended further, up to the Φ-Laplacian operator, for which regularity theory and maximum principles are still available (see [25,30]). Existence and regularity results for problems involving the Φ-Laplacian can be found, e.g., in [9,12,14,33]. Dealing with Φ-Laplacian problems requires the usage of Sobolev-Orlicz spaces, since Φ may have the nonstandard growth; this fact is discussed in the appendix, where a class of explicit examples is furnished. An introductory exposition about Orlicz and Sobolev-Orlicz spaces is provided [23, Chapters 3 and 7]; we also address to the monographs [22,31]. The relation between Sobolev-Orlicz spaces and partial differential equations is the subject of [19].

Also singular Φ-Laplacian problems have been studied during the last years. The model case $f(x, u) = a(x)u^{-\gamma}$, with $a \geq 0$ and $\gamma > 0$, was investigated in [32]. A more general problem, including also convection terms (i.e., f depends also on ∇u), was studied in [10]. Due to the lack of variational setting, primarily caused by the strongly singular term (i.e., $\gamma > 1$), both works make use of a generalized Galerkin method to obtain a solution. We are not aware of other existence results pertaining singular Φ-Laplacian problems.

In spite of the aforementioned articles, our approach is variational: first, we construct a sub-solution $\underline{u}$ (Lemma 2.7) and truncate f at the level of $\underline{u}$; then we consider the truncated problem $(\mathrm{P}_{\lambda,\hat{f}})$, which is equivalent to $(\mathrm{P}_{\lambda,f})$ (compared with Lemma 2.9), and find a solution by means of the local minimum theorem reported in Theorem 4.1. To obtain a second solution, we use the Mountain Pass theorem, jointly with the Ambrosetti-Rabinowitz unilateral condition (see $\mathrm{H}(f)_3$), which implies the Palais-Smale condition (vide Lemma 4.6). We highlight that the Ambrosetti-Rabinowitz condition has been used in the context of Φ-Laplacian problems also in [12,33], while [14] uses the Mountain Pass theorem without the Palais-Smale condition. Here, we highlight the fact that we find the first solution without using the $W^{1,\Phi}$ versus C^1 local minimizer technique.

It is worth noticing that the aforementioned works [10,32], which make no use of the Mountain Pass theorem, consider reaction terms with growth not faster than Φ (usually called “linear”); on the contrary, we treat, with the same technique, both linear forcing terms and superlinear ones (but “sub-critical,” i.e., growing slower than Φ_*). This is remarkable in our context since, in a variational setting, linear problems possess coercive energy functionals, allowing to find a solution via the Weierstrass-Tonelli theorem instead of the Mountain Pass one. To the best of our knowledge, this is the first work treating singular Φ-Laplacian problems with this technique, which offers a unified approach to the coercive and the noncoercive cases.

Regularity of solutions is investigated in Section 3. L^∞ estimates are provided in Lemma 3.3 by using a technique introduced by De Giorgi [24, Lemma 2.5.4]. Then $C^{1,\tau}$ regularity is obtained in Theorem 3.5 via the perturbation technique developed by Campanato [6,7], Giaquinta and Giusti [17], combined with a result pertaining solutions to singular semi-linear elliptic problems that traces back to [16] (see also [21]).

In the Appendix, we discuss about the importance of using Sobolev-Orlicz spaces, providing also two examples of problems fulfilling $\mathrm{H}(a)_1$–$\mathrm{H}(a)_2$ and $\mathrm{H}(f)_1$–$\mathrm{H}(f)_3$.

2 Preliminaries

We denote by $d(x)$ the distance of $x \in \Omega$ from $\partial\Omega$, while d_Ω stands for the diameter of Ω. Given any function $u : \Omega \to \mathbb{R}$ and any number $\rho \in \mathbb{R}$, $\{u < \rho\}$ stands for the set $\{x \in \Omega : u(x) < \rho\}$, and the same holds for $\{u \geq \rho\}$, $\{u = \rho\}$, etc.

To avoid unnecessary technicalities, hereafter, we use “for all $x \in \Omega$” instead of “for almost all $x \in \Omega$” when no confusion arises.

Definition 2.1. A function $\Psi : [0, +\infty) \to [0, +\infty)$ is said to be a Young function[1] if it is continuous, strictly increasing, convex, and the following holds true:

$$\lim_{t\to 0^+} \frac{\Psi(t)}{t} = 0, \quad \lim_{t\to +\infty} \frac{\Psi(t)}{t} = +\infty. \tag{2.1}$$

1 Some textbooks, as [1], use the notion of N-function; here, we adopt the nomenclature used in [23]. See [1, Section 8.1] and [23, Remark 3.2.7] for further details.

Definition 2.2. Let Ψ be a Young function. We denote by $\overline{\Psi}$ the Young conjugate of Ψ, defined via Legendre transformation as follows:

$$\overline{\Psi}(t) := \max_{s\geq 0}\{st - \Psi(s)\} \quad \forall t \geq 0.$$

Definition 2.3. Let Ψ be a Young function satisfying (1.3) with Ψ in place of Φ. Suppose also, without loss of generality (compared with [23, Exercise 7.2.2]), that

$$\int_0^1 \Theta_\Psi(s)\mathrm{d}s < +\infty.$$

The Sobolev-Orlicz conjugate of Ψ, indicated as Ψ_*, is defined via its inverse as follows:

$$\Psi_*^{-1}(t) := \int_0^t \Theta_\Psi(s)\mathrm{d}s.$$

Definition 2.4. Let Ψ be a Young function. We write $\Psi \in \Delta_2$ if there exist $k, T > 0$ such that

$$\Psi(2t) \leq k\Psi(t) \quad \forall t \geq T.$$

We write $\Psi \in \nabla_2$ if there exist $\eta > 1$ and $T > 0$ such that

$$\Psi(t) \leq \frac{1}{2\eta}\Psi(\eta t) \quad \forall t \geq T.$$

Equivalent statements are collected in [31, Theorem 2.3.3 and Corollary 2.3.4]; here, we only mention

$$\begin{aligned} \Psi \in \Delta_2 \Leftrightarrow \overline{\Psi} \in \nabla_2 \quad &\Leftrightarrow \quad \limsup_{t\to+\infty} \frac{t\Psi'(t)}{\Psi(t)} < +\infty, \\ \Psi \in \nabla_2 \Leftrightarrow \overline{\Psi} \in \Delta_2 \quad &\Leftrightarrow \quad \liminf_{t\to+\infty} \frac{t\Psi'(t)}{\Psi(t)} > 1. \end{aligned} \tag{2.2}$$

For a comparison with power-law functions, see [31, Corollary 2.3.5].

Let $\Psi \in \Delta_2$. We endow the Orlicz space[2]

$$L^\Psi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega \Psi(|u(x)|)\mathrm{d}x < +\infty \right\}$$

with the Luxembourg norm

$$\|u\|_{L^\Psi(\Omega)} := \inf\left\{ \lambda > 0 : \int_\Omega \Psi\left(\frac{|u(x)|}{\lambda}\right)\mathrm{d}x \leq 1 \right\}.$$

Suppose that

$$1 < i_\Psi := \inf_{t>0} \frac{t\Psi'(t)}{\Psi(t)} \leq \sup_{t>0} \frac{t\Psi'(t)}{\Psi(t)} =: s_\Psi < +\infty, \tag{2.3}$$

which implies $\Psi \in \Delta_2 \cap \nabla_2$ by (2.2). We define the functions $\underline{\zeta}_\Psi, \overline{\zeta}_\Psi : [0, +\infty) \to [0, +\infty)$ as follows:

$$\underline{\zeta}_\Psi(t) := \min\left\{t^{i_\Psi}, t^{s_\Psi}\right\}, \quad \overline{\zeta}_\Psi(t) := \max\left\{t^{i_\Psi}, t^{s_\Psi}\right\}.$$

2 Since $\Psi \in \Delta_2$, we make no distinction between *Orlicz space* and *Orlicz class*; see [23, Theorem 3.7.3].

One has (cf. [14, Lemma 2.1])

$$\underline{\zeta}_\Psi(k)\Psi(t) \le \Psi(kt) \le \overline{\zeta}_\Psi(k)\Psi(t) \quad \forall k, t \ge 0 \tag{2.4}$$

and

$$\underline{\zeta}_\Psi\left(\|w\|_{L^\Psi(\Omega)}\right) \le \int_\Omega \Psi(|w(x)|)\mathrm{d}x \le \overline{\zeta}_\Psi\left(\|w\|_{L^\Psi(\Omega)}\right) \tag{2.5}$$

for all $w \in L^\Psi(\Omega)$. We also recall (see [14, Lemmas 2.4 and 2.5]) that

$$s'_\Psi \le i_{\overline{\Psi}} \le s_{\overline{\Psi}} \le i'_\Psi \tag{2.6}$$

and, provided $s_\Psi < N$,

$$i^*_\Psi \le i_{\Psi_*} \le s_{\Psi_*} \le s^*_\Psi, \tag{2.7}$$

being $r' := \frac{r}{r-1}$ and $r^* := \frac{Nr}{N-r}$, respectively, the Young and the Sobolev conjugates of r.

Definition 2.5. Let Ψ_1, Ψ_2 be two Young functions. We write $\Psi_1 < \Psi_2$ if there exist $c, T > 0$ such that

$$\Psi_1(t) \le \Psi_2(ct) \quad \forall t \ge T. \tag{2.8}$$

We write $\Psi_1 \ll \Psi_2$ if for any $c > 0$ there exists $T = T(c) > 0$ such that (2.8) holds true. Equivalently,

$$\lim_{t\to+\infty} \frac{\Psi_1(t)}{\Psi_2(\eta t)} = 0 \quad \forall \eta > 0.$$

Further characterizations can be found in [31, Theorem 2.2.2]. It is worth recalling the following chain of (nonreversible) implications:

$$s_{\Psi_1} < i_{\Psi_2} \quad \Rightarrow \quad \Psi_1 \ll \Psi_2 \quad \Rightarrow \quad \Psi_1 < \Psi_2. \tag{2.9}$$

We consider the Sobolev-Orlicz space

$$W^{1,\Phi}(\Omega) := \{u \in L^\Phi(\Omega) : |\nabla u| \in L^\Phi(\Omega)\},$$

equipped with the norm $\|u\|_{W^{1,\Phi}(\Omega)} := \|u\|_{L^\Phi(\Omega)} + \|\nabla u\|_{L^\Phi(\Omega)}$, and its subspace $W^{1,\Phi}_0(\Omega)$, which is the closure of $C^\infty_c(\Omega)$ under $\|\cdot\|_{W^{1,\Phi}(\Omega)}$. According to the Poincaré inequality (see, e.g., [10, p. 8]), we are allowed to endow $W^{1,\Phi}_0(\Omega)$ with the norm

$$\|u\|_{W^{1,\Phi}_0(\Omega)} := \|\nabla u\|_{L^\Phi(\Omega)}.$$

Since $\Phi \in \Delta_2 \cap \nabla_2$, the space $W^{1,\Phi}_0(\Omega)$ is separable and reflexive (compared with [1, Theorem 8.31]). Its dual space will be denoted by $W^{-1,\overline{\Phi}}(\Omega)$, while $\langle\cdot,\cdot\rangle$ represent the duality brackets between $W^{-1,\overline{\Phi}}(\Omega)$ and $W^{1,\Phi}_0(\Omega)$. We recall that $W^{1,\Phi}_0(\Omega) \hookrightarrow L^{\Phi_*}(\Omega)$ continuously and $W^{1,\Phi}_0(\Omega) \hookrightarrow L^{\Upsilon}(\Omega)$ compactly for all $\Upsilon \ll \Phi_*$; see [23, Theorems 7.2.3 and 7.4.4].

Although the next result is folklore, we briefly sketch its proof for the sake of completeness.

Lemma 2.6. *Under* H(a)$_1$, *the operator* $-\Delta_\Phi : W^{1,\Phi}_0(\Omega) \to W^{-1,\overline{\Phi}}(\Omega)$ *is defined as*

$$\langle -\Delta_\Phi u, v\rangle := \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v \mathrm{d}x \quad \forall u, v \in W^{1,\Phi}_0(\Omega)$$

is well defined, bounded, continuous, coercive, strictly monotone, and of type (S_+). *Moreover, the functional* $H : W^{1,\Phi}_0(\Omega) \to \mathbb{R}$ *is defined as*

$$H(u) := \int_\Omega \Phi(|\nabla u|)\mathrm{d}x \tag{2.10}$$

is convex, weakly lower semi-continuous, and of class C^1, *with* $H' = -\Delta_\Phi$ *in* $W^{-1,\overline{\Phi}}(\Omega)$.

Proof. According to the Hölder inequality (see [19, p. 62]), we obtain

$$\begin{aligned}|\langle -\Delta_\Phi u, v\rangle| &\le \int_\Omega \varphi(|\nabla u|)|\nabla v|\mathrm{d}x \le s_\Phi \int_\Omega \frac{\Phi(|\nabla u|)}{|\nabla u|}|\nabla v|\mathrm{d}x \\ &\le s_\Phi \left\| \frac{\Phi(|\nabla u|)}{|\nabla u|} \right\|_{L^{\overline{\Phi}}(\Omega)} \|\nabla v\|_{L^\Phi(\Omega)} = s_\Phi \left\| \frac{\Phi(|\nabla u|)}{|\nabla u|} \right\|_{L^{\overline{\Phi}}(\Omega)} \|v\|_{W_0^{1,\Phi}(\Omega)}.\end{aligned} \tag{2.11}$$

By exploiting (1.7), we infer

$$\int_\Omega \overline{\Phi}\left(\frac{\Phi(|\nabla u|)}{|\nabla u|}\right)\mathrm{d}x \le \int_\Omega \Phi(|\nabla u|)\mathrm{d}x < +\infty. \tag{2.12}$$

By (2.11) and (2.12), we deduce that $-\Delta_\Phi$ is well defined. Boundedness and continuity follow from (2.12) and [32, Lemma 7.3].

To prove the coercivity of $-\Delta_\Phi$, we exploit (2.5) to obtain

$$\int_\Omega a(|\nabla u|)|\nabla u|^2\mathrm{d}x = \int_\Omega \varphi(|\nabla u|)|\nabla u|\mathrm{d}x \ge i_\Phi \int_\Omega \Phi(|\nabla u|)\mathrm{d}x \ge i_\Phi \underline{\zeta}_\Phi\big(\|\nabla u\|_{L^\Phi(\Omega)}\big) \tag{2.13}$$

for all $u \in W_0^{1,\Phi}(\Omega)$. Hence,

$$\frac{\langle -\Delta_\Phi u, u\rangle}{\|u\|_{W_0^{1,\Phi}(\Omega)}} \ge i_\Phi \frac{\underline{\zeta}_\Phi(\|u\|_{W_0^{1,\Phi}(\Omega)})}{\|u\|_{W_0^{1,\Phi}(\Omega)}} \to +\infty \quad \text{as } \|u\|_{W_0^{1,\Phi}(\Omega)} \to +\infty.$$

The strict monotonicity and the (S_+) property of $-\Delta_\Phi$ are guaranteed by [9, Propositions A.2 and A.3].

Convexity of H directly follows from convexity of Φ, while Lebesgue's dominated convergence theorem and [32, Lemma 7.3] ensure that H is continuous. As a consequence, H is weakly lower semi-continuous. The fact that H is of class C^1 has been proved in [14, Lemma A.3]. □

First, we consider the problem

$$\begin{cases} -\Delta_\Phi u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathrm{P}_{1,f}$}$$

Lemma 2.7. *Suppose* H(a)$_1$ *and* H(f)$_1$. *Then problem* $(\mathrm{P}_{1,f})$ *admits a subsolution* $\underline{u} \in C_0^{1,\tau}(\overline{\Omega})$, *with* $\tau \in (0,1]$ *opportune, satisfying*

$$k_1 d(x) \le \underline{u}(x) \le k_2 d(x) \quad \forall x \in \Omega \tag{2.14}$$

for suitable $k_1, k_2 > 0$.

Proof. This proof is patterned after the one of [20, Lemma 3.5]. Hypothesis H(f)$_1$ provides $\delta > 0$ such that

$$f(x,s) \ge 1 \quad \text{for all } (x,s) \in \Omega \times (0,\delta). \tag{2.15}$$

For any $n \in \mathbb{N}$, let us consider the following problem:

$$\begin{cases} -\Delta_\Phi u = \dfrac{1}{n} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathrm{P}_{1,\frac{1}{n}}$}$$

By virtue of Lemma 2.6, Minty-Browder's theorem [5, Theorem 5.16] can be applied; thus, $\left(\mathrm{P}_{1,\frac{1}{n}}\right)$ admits a unique solution $u_n \in W_0^{1,\Phi}(\Omega)$. Lieberman's nonlinear regularity theory [25, Theorem 1.7] guarantees that $\{u_n : n \in \mathbb{N}\}$ is bounded in $C_0^{1,\tau}(\overline{\Omega})$ for some $\tau \in (0,1]$. Hence, thanks to the Ascoli-Arzelà theorem and up to subsequences, we obtain $u_n \to u$ in $C_0^1(\overline{\Omega})$ for some $u \in C_0^1(\overline{\Omega})$. Passing to the limit in the weak formulation of $\left(\mathrm{P}_{1,\frac{1}{n}}\right)$ reveals that $u \equiv 0$ in Ω. Hence, it is possible to choose $\hat{n} \in \mathbb{N}$ such that

$$\|u_{\hat{n}}\|_{L^\infty(\Omega)} < \delta. \tag{2.16}$$

Set $\underline{u} := u_{\hat{n}}$. The strong maximum principle [30, Theorem 1.1.1] ensures $\underline{u} > 0$ in Ω. Thus, by (2.15) and (2.16) one has

$$-\Delta_\Phi \underline{u} = \frac{1}{\hat{n}} \le 1 \le f(x, \underline{u})$$

in weak sense. A standard argument involving the Boundary Point lemma [30, Theorem 5.5.1] and the Hölder continuity of $\nabla \underline{u}$ gives (2.14). □

Remark 2.8. Without loss of generality, one can choose $\delta < R$ in (2.15), where $R > 0$ comes from $\mathrm{H(f)}_3$. Hereafter, we make this assumption, which yields $\|\underline{u}\|_{L^\infty(\Omega)} \le \delta < R$, according to (2.16).

Let us consider the auxiliary problem

$$\begin{cases} -\Delta_\Phi u = \hat{f}(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{P}_{1,\hat{f}}$}$$

where $\hat{f} : \Omega \times \mathbb{R} \to [0, +\infty)$ is defined as follows:

$$\hat{f}(x, s) := \begin{cases} f(x, \underline{u}(x)) & \text{if } |s| \le \underline{u}(x), \\ f(x, |s|) & \text{if } |s| > \underline{u}(x), \end{cases}$$

being $\underline{u}$ as in Lemma 2.7. We also define

$$\hat{F}(x, s) := \int_0^s \hat{f}(x, t)\mathrm{d}t.$$

Exploiting (1.5) and (2.14), one has

$$0 \le \hat{f}(x, s) \le c_1 \overline{\Upsilon}^{-1}(\Upsilon(|s|)) + c_1 \overline{\Upsilon}^{-1}(\Upsilon(\underline{u}(x))) + c_2 \underline{u}(x)^{-\gamma} \le c_1 \overline{\Upsilon}^{-1}(\Upsilon(|s|)) + \alpha d(x)^{-\gamma} + \beta, \tag{2.17}$$

being $\alpha, \beta > 0$ such that

$$\alpha := c_2 k_1^{-\gamma}, \quad \beta := c_1 \overline{\Upsilon}^{-1}(\Upsilon(k_2 d_\Omega)).$$

Lemma 2.9. *Let* $\mathrm{H(a)}_1$ *and* $\mathrm{H(f)}_1$ *be satisfied. Then, any* $u \in W_0^{1,\Phi}(\Omega)$ *weak solution to* $(\mathrm{P}_{1,\hat{f}})$ *is a weak solution to* $(\mathrm{P}_{1,f})$ *and vice-versa.*

Proof. To show the equivalence of $(\mathrm{P}_{1,\hat{f}})$ and $(\mathrm{P}_{1,f})$, it suffices to prove that any solution to either $(\mathrm{P}_{1,\hat{f}})$ or $(\mathrm{P}_{1,f})$ is greater than $\underline{u}$; then, the conclusion will follow by the definition of $\hat{f}$.

Let $u \in W_0^{1,\Phi}(\Omega)$ be a weak solution to $(\mathrm{P}_{1,\hat{f}})$. The weak maximum principle, jointly with $\hat{f} \ge 0$, ensures $u \ge 0$. Then Lemma 2.7 and the weak comparison principle (compared with e.g., [30, Theorem 3.4.1]), applied on u and $\underline{u}$, yields $u \ge \underline{u}$: indeed, $-\Delta_\Phi$ is a strictly monotone operator (see Lemma 2.6) and, in weak sense,

$$-\Delta_\Phi \underline{u} \le f(x, \underline{u}) = \hat{f}(x, u) = -\Delta_\Phi u \quad \text{in } \{x \in \Omega : u(x) \le \underline{u}(x)\}.$$

Now let $u \in W_0^{1,\Phi}(\Omega)$ be a weak solution to $(\mathrm{P}_{1,f})$. Recalling that $\|\underline{u}\|_{L^\infty(\Omega)} \le \delta$ by (2.16), from (2.15), we obtain

$$-\Delta_\Phi \underline{u} = \frac{1}{\hat{n}} \le 1 \le f(x, u) = -\Delta_\Phi u \quad \text{in } \{x \in \Omega : u(x) \le \underline{u}(x)\},$$

where δ, $\hat{n}$ come from Lemma 2.7. As mentioned earlier, the weak comparison principle ensures $u \ge \underline{u}$. □

Lemma 2.10. *Suppose* H(a)$_1$–H(a)$_2$ *and* H(f)$_1$–H(f)$_2$. *Then the functional* $K : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ *defined as*

$$K(u) := \int_\Omega \hat{F}(x,u)\mathrm{d}x$$

is well defined, weakly sequentially continuous, and of class C^1, *with*

$$\langle K'(u), v\rangle = \int_\Omega \hat{f}(x,u)v\mathrm{d}x \quad \forall u, v \in W_0^{1,\Phi}(\Omega). \tag{2.18}$$

Moreover, $K' : W_0^{1,\Phi}(\Omega) \to W^{-1,\overline{\Phi}}(\Omega)$ *is a completely continuous operator.*

Proof. By using (1.5) and (1.7), we estimate $\hat{F}$ as follows:

$$\begin{aligned}
|\hat{F}(x,s)| &\le \int_0^{|s|} \hat{f}(x,t)\mathrm{d}t = \int_0^{\underline{u}(x)} f(x,\underline{u}(x))\mathrm{d}t + \int_{\underline{u}(x)}^{|s|} f(x,t)\mathrm{d}t \\
&\le \underline{u}(x)[c_1\overline{\Upsilon}^{-1}(\Upsilon(\underline{u}(x))) + c_2\underline{u}(x)^{-\gamma}] + \int_0^{|s|}[c_1\overline{\Upsilon}^{-1}(\Upsilon(t)) + c_2t^{-\gamma}]\mathrm{d}t \\
&\le c_1\underline{u}(x)\overline{\Upsilon}^{-1}(\Upsilon(\underline{u}(x))) + c_2\underline{u}(x)^{1-\gamma} + c_1|s|\overline{\Upsilon}^{-1}(\Upsilon(|s|)) + \frac{c_2}{1-\gamma}|s|^{1-\gamma} \\
&\le 2c_1\Upsilon(\underline{u}(x)) + c_2\underline{u}(x)^{1-\gamma} + 2c_1\Upsilon(|s|) + \frac{c_2}{1-\gamma}|s|^{1-\gamma}
\end{aligned}$$

for all $(x,s) \in \Omega \times \mathbb{R}$. By exploiting (2.14) and $i_\Upsilon > 1$, we obtain

$$|\hat{F}(x,s)| \le C_1 + C_2\Upsilon(|s|), \tag{2.19}$$

with positive constants

$$\begin{aligned}
C_1 &:= 2c_1\Upsilon(k_2d_\Omega) + c_2(k_2d_\Omega)^{1-\gamma} + \frac{c_2}{1-\gamma}, \\
C_2 &:= 2c_1 + \frac{c_2}{1-\gamma}\frac{1}{\Upsilon(1)}.
\end{aligned} \tag{2.20}$$

Taking into account also that $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Upsilon(\Omega)$ because of $\Upsilon \ll \Phi_*$, we deduce that K is well defined.

Now we compute the Gâteaux derivative of K. We fix $v \in W_0^{1,\Phi}(\Omega)$ and apply Torricelli's theorem to deduce

$$\lim_{t\to 0^+}\frac{K(u+tv)-K(u)}{t} = \lim_{t\to 0^+}\int_\Omega \frac{\hat{F}(x,u+tv)-\hat{F}(x,u)}{t}\mathrm{d}x = \lim_{t\to 0^+}\int_\Omega v\left(\int_0^1 \hat{f}(x,u+stv)\mathrm{d}s\right)\mathrm{d}x. \tag{2.21}$$

According to (2.17), (1.7), the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Upsilon(\Omega)$, and the Hardy inequality [11, Corollary 1][3], besides supposing $t \in (0,1)$, we infer

$$\begin{aligned}
|v\hat{f}(x,u+stv)| &\le c_1|v|\overline{\Upsilon}^{-1}(\Upsilon(|u|+|v|)) + \alpha d^{-\gamma}|v| + \beta|v| \\
&\le c_1(|u|+|v|)\overline{\Upsilon}^{-1}(\Upsilon(|u|+|v|)) + \alpha d^{-\gamma}|v| + \beta|v| \\
&\le 2c_1\Upsilon(|u|+|v|) + \alpha d_\Omega^{1-\gamma}d^{-1}|v| + \beta|v| \in L^1(\Omega).
\end{aligned}$$

3 We use Hardy's inequality in the form

$$\|d^{-1}v\|_{L^\Phi(\Omega)} \le c\|\nabla v\|_{L^\Phi(\Omega)} \quad \forall v \in W_0^{1,\Phi}(\Omega),$$

being $c > 0$ opportune. This inequality is valid since $\Phi \in \nabla_2$.

Hence, we can pass the limit under the integral sign in (2.21) and obtain (2.18). The remaining part of the proof follows exactly as in [8, Lemma 2.3], using the compactness of the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Upsilon}(\Omega)$, as well as (2.17) and [32, Lemma 7.3]. ☐

Remark 2.11. Fix any $u \in W_0^{1,\Phi}(\Omega)$. By (2.19), we deduce that

$$\int_{\Omega\cap\{|u|\leq\rho\}} |\hat{F}(x,u)|\mathrm{d}x \leq (C_1 + C_2\Upsilon(\rho))|\Omega| =: \Pi(\rho) \quad \forall \rho \geq 0. \tag{2.22}$$

3 Regularity of solutions

In this section, we prove $C^{1,\alpha}$ regularity up to the boundary for solutions to $(\mathrm{P}_{1,f})$.

Given a measurable function $u : \Omega \to \mathbb{R}$, for any $k \in \mathbb{R}$, we set

$$\Omega_k := \{x \in \Omega : \ u(x) \geq k\}.$$

Before going on, we discuss a comparison between Moser's iteration method and De Giorgi's technique for L^∞ estimates.

Remark 3.1. A classical way to prove boundedness of solutions to p-Laplace-type equations consists in using Moser's iteration method [27], a technique based on testing the differential equation with suitable powers of the solution, which furnishes a reverse-Hölder inequality to perform a bootstrap argument. In the context of Orlicz spaces, this technique seems to produce, in general, worse results: indeed, the applicability of Moser's method is limited to $s_\Phi \leq i_\Phi^*$, which is in general more restrictive than our hypothesis $s_\Phi \leq i_{\Phi_*}$ (see Remark 1.2, item 3, and (2.7)). Thus, the De Giorgi technique seems to be a more versatile tool in this setting, since it does not make any use of power-law test functions. Anyway, also the following *a priori* estimate proposed seems to be nonoptimal, since it uses powers at the level of (3.10); we expect that the optimal result is represented by the critical growth, i.e., $\Upsilon = \Phi_*$ (paralleling the p-Laplacian case).

De Giorgi's technique basically relies on the following lemma, which is a global version of the local estimate [18, Theorem 7.1] (see also [13, pp. 351–352]).

Lemma 3.2. *Let $p, r > 1$. Suppose that $u \in L^p(\Omega)$ satisfies*

$$\left(\int_{\Omega_k} (u-k)^p \mathrm{d}x\right)^{\frac{1}{r}} \leq c\left[\int_{\Omega_k} (u-k)^p \mathrm{d}x + k^p|\Omega_k|\right] \quad \textit{for all } k \geq K, \tag{3.1}$$

for suitable $c, K > 0$. Then there exists $M > 0$ such that $u \leq M$ in Ω.

Proof. Let us fix $M > 2K$ to be chosen later, and set

$$k_n := M\left(1 - \frac{1}{2^{n+1}}\right) \quad \forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{3.2}$$

By (3.1) and (3.2), we have, for all $n \in \mathbb{N}_0$,

$$\left(\int_{\Omega_{k_{n+1}}} (u-k_{n+1})^p \mathrm{d}x\right)^{\frac{1}{r}} \leq c\left[\int_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x + k_{n+1}^p|\Omega_{k_{n+1}}|\right]. \tag{3.3}$$

Chebichev's inequality entails

$$(k_{n+1}-k_n)^p|\Omega_{k_{n+1}}| \le \int\limits_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x.$$

Thus, recalling (3.2) and $k > 1$,

$$k_{n+1}^p|\Omega_{k_{n+1}}| \le \frac{k_{n+1}^p}{(k_{n+1}-k_n)^p} \int\limits_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x \le 2^{(n+2)p} \int\limits_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x. \tag{3.4}$$

Inserting (3.4) into (3.3) gives

$$\int\limits_{\Omega_{k_{n+1}}} (u-k_{n+1})^p \mathrm{d}x \le \left[c(2^{(n+2)p}+1)\int\limits_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x\right]^r \le Cb^n\left(\int\limits_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x\right)^{1+a} \tag{3.5}$$

for all $n \in \mathbb{N}_0$, where $a := r-1 > 0$, $b := 2^{pr} > 1$, and $C = C(c,p,r) > 0$ is a suitable constant independent of k and n. Now we apply the fast geometric convergence lemma [24, Lemma 2.4.7] to the sequence $y_n := \int_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x$, which ensures $y_n \to 0$ provided

$$y_0 \le C^{-\frac{1}{a}} b^{-\frac{1}{a^2}}. \tag{3.6}$$

We can choose M, independent of n, such that (3.6) holds true: indeed, by (3.2) and the dominated convergence theorem,

$$y_0 = \int\limits_{\Omega_{k_0}} (u-k_0)^p \mathrm{d}x = \int\limits_{\Omega} \left(u - \frac{M}{2}\right)_+^p \mathrm{d}x \overset{M\to+\infty}{\to} 0. \tag{3.7}$$

Keeping M fixed as in (3.6)–(3.7), from $y_n \to 0$, we obtain

$$\int\limits_{\Omega} (u-M)_+^p \mathrm{d}x \le \int\limits_{\Omega} (u-k_n)_+^p \mathrm{d}x = \int\limits_{\Omega_{k_n}} (u-k_n)^p \mathrm{d}x \overset{n\to\infty}{\to} 0,$$

which implies $\int_\Omega (u-M)_+^p \mathrm{d}x = 0$, whence $u \le M$ in Ω. □

Lemma 3.3. *Let* $\mathrm{H(a)}_1$–$\mathrm{H(a)}_2$ *and* $\mathrm{H(f)}_2$ *be satisfied. Then any* $u \in W_0^{1,\Phi}(\Omega)$ *weak solution to* $(\mathrm{P}_{1,f})$ *is essentially bounded in* Ω.

Proof. Pick any $k > 1$. Testing $(\mathrm{P}_{1,f})$ with $(u-k)_+$ and using (1.5) yield

$$\begin{aligned} \int\limits_{\Omega_k} \Phi(|\nabla u|)\mathrm{d}x &\le i_\Phi^{-1} \int\limits_{\Omega_k} \varphi(|\nabla u|)|\nabla u| \mathrm{d}x = i_\Phi^{-1} \int\limits_{\Omega_k} f(x,u)(u-k)\mathrm{d}x \\ &\le i_\Phi^{-1}\left[c_1 \int\limits_{\Omega_k} \overline{\Upsilon}^{-1}(\Upsilon(u))u \mathrm{d}x + c_2 \int\limits_{\Omega_k} u^{1-\gamma} \mathrm{d}x\right]. \end{aligned} \tag{3.8}$$

First, we estimate each term on the right-hand side of (3.8). By convexity of Υ and (1.7), we have, for any k sufficiently large,

$$u^{1-\gamma} \le \Upsilon(u) \le \overline{\Upsilon}^{-1}(\Upsilon(u))u \quad \text{in } \Omega_k. \tag{3.9}$$

Moreover, by (1.7) and (2.4), besides $s_\Upsilon \le i_{\Phi_*}$ and $k > 1$, it turns out that

$$\int_{\Omega_k} \overline{\Upsilon}^{-1}(\Upsilon(u))u\mathrm{d}x \le 2\int_{\Omega_k} \Upsilon(u)\mathrm{d}x \le 2\Upsilon(1)\int_{\Omega_k} u^{s_\Upsilon}\mathrm{d}x \le 2\Upsilon(1)\int_{\Omega_k} u^{i_{\Phi_*}}\mathrm{d}x$$
$$\le 2^{i_{\Phi_*}}\Upsilon(1)\left[\int_{\Omega_k} (u-k)^{i_{\Phi_*}}\mathrm{d}x + k^{i_{\Phi_*}}|\Omega_k|\right]. \tag{3.10}$$

On the other hand, to estimate the left-hand side of (3.8), we observe that the Sobolev embedding theorem and $t^{i_{\Phi_*}} < \Phi_*$ in the sense of (2.8) (see (2.4)), yield

$$W_0^{1,\Phi}(\Omega) \hookrightarrow L^{\Phi_*}(\Omega) \hookrightarrow L^{i_{\Phi_*}}(\Omega),$$

where the latter space is a Lebesgue space. We deduce the embedding inequality

$$c\|w\|_{L^{i_{\Phi_*}}(\Omega)} \le \|\nabla w\|_{L^\Phi(\Omega)} \quad \forall w \in W_0^{1,\Phi}(\Omega)$$

for a suitable $c > 0$. Thus, choosing $w = (u-k)_+$, from (2.5), we obtain

$$\int_{\Omega_k} \Phi(|\nabla u|)\mathrm{d}x \ge \underline{\zeta}_\Phi\Big(\|\nabla u\|_{L^\Phi(\Omega)}\Big) \ge \underline{\zeta}_\Phi\Big(\|\nabla (u-k)_+\|_{L^\Phi(\Omega)}\Big) \ge \underline{\zeta}_\Phi\Big(c\|(u-k)_+\|_{L^{i_{\Phi_*}}(\Omega)}\Big).$$

Reasoning as in (3.7), for any k big enough, we obtain $c\|(u-k)_+\|_{L^{i_{\Phi_*}}(\Omega)} \le 1$, so that

$$\int_{\Omega_k} \Phi(|\nabla u|)\mathrm{d}x \ge c^{s_\Phi}\left(\int_{\Omega_k} (u-k)^{i_{\Phi_*}}\mathrm{d}x\right)^{\frac{s_\Phi}{i_{\Phi_*}}}. \tag{3.11}$$

By inserting (3.9)–(3.11) into (3.8), we deduce

$$\left(\int_{\Omega_k} (u-k)^{i_{\Phi_*}}\mathrm{d}x\right)^{\frac{s_\Phi}{i_{\Phi_*}}} \le C\left[\int_{\Omega_k} (u-k)^{i_{\Phi_*}}\mathrm{d}x + k^{i_{\Phi_*}}|\Omega_k|\right]$$

for a sufficiently large $C > 0$. Hence, applying Lemma 3.1 with $p = i_{\Phi_*} > 1$ and $r = \frac{i_{\Phi_*}}{s_\Phi} > 1$ (compared with $\mathrm{H(a)}_2$) yields the conclusion. □

Remark 3.4. The L^∞ estimate provided in Lemma 3.3 is valid also when $s_\Upsilon = i_{\Phi_*}$ which, in the classical Sobolev setting, represents the *critical case*; hence, M must depend on the solution u. On the other hand, in the *subcritical case* $s_\Upsilon < i_{\Phi_*}$ this estimate can be improved, and it turns out that M depends only on $\|u\|_{W_0^{1,\Phi}(\Omega)}$ instead of u itself.

Theorem 3.5. *Let* $\mathrm{H(a)}_1$–$\mathrm{H(a)}_2$ *and* $\mathrm{H(f)}_1$–$\mathrm{H(f)}_2$ *be satisfied. Then any* $u \in W_0^{1,\Phi}(\Omega)$ *weak solution of* $(\mathrm{P}_{1,f})$ *belongs to* $C_0^{1,\tau}(\overline{\Omega})$ *for some* $\tau \in (0,1]$.

Proof. Lemma 3.3 guarantees that $u \in L^\infty(\Omega)$. In addition, Lemma 2.9 ensures that u solves also $(\mathrm{P}_{1,\hat{f}})$. By using $u \in L^\infty(\Omega)$ and (2.17), we obtain

$$0 \le \hat{f}(x,u(x)) \le Cd(x)^{-\gamma} \quad \forall x \in \Omega,$$

being $C > 0$ sufficiently large. Let us consider the linear problem

$$\begin{cases} -\Delta v = \hat{f}(x,u(x)) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Problem (3.12) admits a unique solution $v \in C_0^{1,\tau}(\overline{\Omega})$, for some $\tau \in (0,1]$, by virtue of Minty-Browder's theorem, Hardy's inequality, and [21, Lemma 3.1]. It turns out that the problem

$$\begin{cases} -\mathrm{div}(a(|\nabla w|)\nabla w - \nabla v(x)) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

admits a unique solution $w \in C_0^{1,\tau}(\overline{\Omega})$, by means of Minty-Browder's theorem and Lieberman's regularity theory, jointly with $\nabla v \in C^{0,\tau}(\overline{\Omega})$. Since u is a solution to (3.13), by uniqueness, we obtain $u = w$, and thus, $u \in C_0^{1,\tau}(\overline{\Omega})$. □

Remark 3.6. We highlight that the $C^{1,\tau}$ global regularity of $\underline{u}$ (in particular its behavior near the boundary; see (2.14)) is crucial in the proof of Theorem 3.5. This is one of the biggest issues in treating singular problems involving operators for which the regularity theory is not fully developed, as the double-phase operator (for an account, see, e.g., [26] and the references therein). Just to give another example, in [15], the continuity of $\underline{u}$ is needed, and no other regularity results are exploited. Incidentally, it is worth mentioning that, although the double-phase operator allows a dependence on x (ruled out by the Φ-Laplacian), it has a very specific structure: it is the sum of a p-Laplacian with a weighted q-Laplacian. On the contrary, the Φ-Laplacian can exhibit a wide range of different structures, not encompassed by the double-phase operator: see, for instance, Example A.5. Accordingly, neither of the aforementioned operators is more general than the other one.

4 Existence and multiplicity results

In this last section, we produce some results about $(\mathrm{P}_{\lambda,f})$. Lemma 2.7 furnishes a subsolution (depending on λ) to $(\mathrm{P}_{\lambda,f})$. Thus, taking into account Lemma 2.9, the solutions to problem

$$\begin{cases} -\Delta_\Phi u = \lambda \hat{f}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{$\mathrm{P}_{\lambda,\hat{f}}$}$$

are exactly the ones of $(\mathrm{P}_{\lambda,f})$. The energy functional associated with $(\mathrm{P}_{\lambda,\hat{f}})$ is

$$J_\lambda := H - \lambda K, \tag{4.1}$$

being H, K as in Lemmas 2.6 and 2.10, respectively. So the solutions to $(\mathrm{P}_{\lambda,\hat{f}})$ are the critical points of J_λ.

Existence of a solution is guaranteed by [8, Theorem 2.1], which we will report later. This result traces back to [2,4].

Theorem 4.1. *Let X be a reflexive Banach space, $H : X \to \mathbb{R}$ and $K : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that H is coercive and sequentially weakly lower semi-continuous, while K is sequentially weakly upper semi-continuous with $\inf_X H = H(0) = K(0)$, and $r > 0$. Then, for every*

$$\lambda \in \left]0, \frac{r}{\sup_{H^{-1}([0,r])} K}\right[, \tag{4.2}$$

the functional $J_\lambda := H - \lambda K$ has a critical point $u_\lambda \in H^{-1}([0,r])$ satisfying $J_\lambda(u_\lambda) \le J_\lambda(v)$ for all $v \in H^{-1}([0,r])$.

A second solution is furnished by the Mountain Pass theorem (vide, e.g., [28, Theorem 5.40]): the applicability of this result relies, in our context, on the Palais-Smale condition.

Definition 4.2. (PS) Let X be a Banach space and $J \in C^1(X)$. We say that J satisfies the Palais-Smale condition if any sequence $\{u_n\} \subseteq X$ such that $\{J(u_n)\}$ is bounded and $\|J'(u_n)\|_{X^*} \to 0$ admits a convergent subsequence.

Theorem 4.3. *Suppose X to be a Banach space, and $J \in C^1(X)$ satisfying* (PS). *Let $u_0, u_1 \in X$, and $\rho > 0$ such that*

$$\max\{J(u_0), J(u_1)\} \leq \inf_{\partial B(u_0,\rho)} J =: \eta_\rho, \quad \|u_1 - u_0\|_X > \rho. \tag{4.3}$$

Set

$$\Gamma := \{\gamma \in C^0([0,1];X) : \gamma(0) = u_0, \gamma(1) = u_1\}, \quad c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)).$$

Then $c \geq \eta_\rho$ and there exists $u \in X$ such that $J(u) = c$ and $J'(u) = 0$. Moreover, if $c = \eta_\rho$, then u can be taken on $\partial B(u_0, \rho)$.

The next theorem concerns the existence of a solution to $(\mathrm{P}_{\lambda,f})$.

Theorem 4.4. *Suppose* $\mathrm{H(a)}_1$–$\mathrm{H(a)}_2$ *and* $\mathrm{H(f)}_1$–$\mathrm{H(f)}_2$. *Then there exists $\lambda^* \in (0, +\infty]$ such that, for all $\lambda \in (0, \lambda^*)$, problem $(\mathrm{P}_{\lambda,f})$ admits a solution $u_\lambda \in C_0^{1,\tau}(\overline{\Omega})$, being $\tau \in (0, 1]$ opportune. Moreover, there exists $r^*_\lambda > 0$ (depending on $\lambda \in (0, \lambda^*)$) such that $\int_\Omega \Phi(|\nabla u_\lambda|)\mathrm{d}x < r^*_\lambda$.*

Proof. We want to apply Theorem 4.1 to J_λ (see (4.1)). As observed earlier, this theorem furnishes a solution $u_\lambda \in W_0^{1,\Phi}(\Omega)$ to $(\mathrm{P}_{\lambda,f})$. Then the regularity of u_λ is a consequence of Theorem 3.5.

To bound from above the ratio

$$\frac{\sup_{H^{-1}([0,r])} K}{r},$$

appearing in (4.2), we exploit (2.19) and estimate K as follows:

$$|K(u)| \leq \int_\Omega |\hat{F}(x, u)|\mathrm{d}x \leq C_1|\Omega| + C_2 \int_\Omega \Upsilon(|u|)\mathrm{d}x. \tag{4.4}$$

Thus, we are led to study the function $\kappa : (0, +\infty) \to (0, +\infty)$ defined as follows:

$$\kappa(r) := \frac{C_1|\Omega|}{r} + \frac{C_2}{r} \sup\left\{ \int_\Omega \Upsilon(|u|)\mathrm{d}x : \int_\Omega \Phi(|\nabla u|)\mathrm{d}x \leq r \right\}.$$

Notice that $\kappa \to +\infty$ as $r \to 0^+$. Now we distinguish four cases, depending on whether $\Upsilon \ll \Phi$, $\Upsilon < \Phi$, $\Upsilon > \Phi$, or $\Upsilon \gg \Phi$. Clearly, some cases overlap.

First case: $\Upsilon \ll \Phi$.

Fix an arbitrary $\varepsilon \in (0, 1]$. There exists $M_\varepsilon > 0$ such that

$$\Upsilon(t) \leq \Phi(\varepsilon t) \leq \varepsilon\Phi(t) \quad \forall t > M_\varepsilon.$$

Hence, by Poincaré's inequality [10, p. 8] and (2.4), we obtain

$$\begin{aligned}
\int_\Omega \Upsilon(|u|)\mathrm{d}x &= \int_{\Omega\cap\{|u|\leq M_\varepsilon\}} \Upsilon(|u|)\mathrm{d}x + \int_{\Omega\cap\{|u|>M_\varepsilon\}} \Upsilon(|u|)\mathrm{d}x \\
&\leq \Upsilon(M_\varepsilon)|\Omega| + \varepsilon \int_\Omega \Phi(|u|)\mathrm{d}x \\
&\leq \Upsilon(M_\varepsilon)|\Omega| + \varepsilon\overline{\zeta}_\Phi(2d_\Omega) \int_\Omega \Phi(|\nabla u|)\mathrm{d}x \\
&\leq \Upsilon(M_\varepsilon)|\Omega| + \varepsilon\overline{\zeta}_\Phi(2d_\Omega) r.
\end{aligned}$$

Thus, κ can be estimated as follows:

$$\kappa(r) \leq \frac{(C_1 + C_2\Upsilon(M_\varepsilon))|\Omega|}{r} + C_2\overline{\zeta}_\Phi(2d_\Omega)\varepsilon. \tag{4.5}$$

Notice that the right-hand side of (4.5) is decreasing in r. Moreover, $\kappa(r) \to 0$ as $r \to +\infty$: indeed, letting $r \to +\infty$ in (4.5) reveals that

$$\limsup_{r\to+\infty}\kappa(r) \leq C_2\overline{\zeta}_\Phi(2d_\Omega)\varepsilon \quad \forall\varepsilon \in (0,1],$$

since ε was arbitrary. We set $\lambda^* = +\infty$. Then, for any $\lambda > 0$, we choose $\varepsilon = \min\{1, (2C_2\overline{\zeta}_\Phi(2d_\Omega)\lambda)^{-1}\}$ and $r_\lambda^* > 2\lambda(C_1 + C_2\Upsilon(M_\varepsilon))|\Omega|$. According to (4.5), these choices guarantee $\kappa(r_\lambda^*) < \lambda^{-1}$, which allows to apply Theorem 4.1 with $r = r_\lambda^*$.

Second case: $\Upsilon < \Phi$.

There exist $M, c > 0$ such that

$$\Upsilon(t) \leq \Phi(ct) \quad \forall t > M.$$

Reasoning as in the first case, we have

$$\begin{aligned}\int_\Omega \Upsilon(|u|)\mathrm{d}x &\leq \Upsilon(M)|\Omega| + \int_\Omega \Phi(c|u|)\mathrm{d}x \leq \Upsilon(M)|\Omega| + \overline{\zeta}_\Phi(2cd_\Omega)\int_\Omega \Phi(|\nabla u|)\mathrm{d}x \\ &\leq \Upsilon(M)|\Omega| + \overline{\zeta}_\Phi(2cd_\Omega)r.\end{aligned}$$

In this case, κ can be estimated as follows:

$$\kappa(r) \leq \frac{(C_1 + C_2\Upsilon(M))|\Omega|}{r} + C_2\overline{\zeta}_\Phi(2cd_\Omega). \tag{4.6}$$

We observe that the right-hand side of (4.6) is decreasing in r and

$$\limsup_{r\to+\infty}\kappa(r) \leq C_2\overline{\zeta}_\Phi(2cd_\Omega).$$

We set $\lambda^* = (C_2\overline{\zeta}_\Phi(2cd_\Omega))^{-1}$ and, for any $\lambda \in (0, \lambda^*)$, we take $r_\lambda^* > \frac{(C_1 + C_2\Upsilon(M))\,|\,\Omega\,|}{\lambda^{-1} - C_2\overline{\zeta}_\Phi(2cd_\Omega)}$. Then one applies Theorem 4.1.

Third case: $\Upsilon > \Phi$.

By loosing information but not generality, we can reduce to the next case, namely, $\Upsilon \gg \Phi$. Indeed, in place of Υ in (2.19), we can consider the intermediate function[4] $\hat{\Upsilon} := \sqrt{\Upsilon\Phi_*}$.

First, we notice that $\Upsilon \ll \Phi_*$ implies

$$\Upsilon(t) \leq \Phi_*(t) \quad \forall t > M,$$

being $M > 0$ opportune. So (2.19) can be rewritten as follows:

$$|\hat{F}(x,s)| \leq C_1 + C_2\Upsilon(M) + C_2\hat{\Upsilon}(|s|) =: \hat{C}_1 + \hat{C}_2\hat{\Upsilon}(|s|).$$

Second, it is readily seen that $\Upsilon \ll \hat{\Upsilon} \ll \Phi_*$ since, for any fixed $\eta > 0$, by (2.4), we have

$$\frac{\hat{\Upsilon}(\eta t)}{\Upsilon(t)} = \sqrt{\frac{\Upsilon(\eta t)}{\Upsilon(t)}}\sqrt{\frac{\Phi_*(\eta t)}{\Upsilon(t)}} \geq \sqrt{\underline{\zeta}_\Upsilon(\eta)}\sqrt{\frac{\Phi_*(\eta t)}{\Upsilon(t)}} \overset{t\to+\infty}{\to} +\infty$$

and

$$\frac{\Phi_*(\eta t)}{\hat{\Upsilon}(t)} = \sqrt{\frac{\Phi_*(\eta t)}{\Phi_*(t)}}\sqrt{\frac{\Phi_*(\eta t)}{\Upsilon(t)}} \geq \sqrt{\underline{\zeta}_{\Phi_*}(\eta)}\sqrt{\frac{\Phi_*(\eta t)}{\Upsilon(t)}} \overset{t\to+\infty}{\to} +\infty.$$

In particular, we obtain $\Phi \ll \hat{\Upsilon}$.

Finally, we notice that $i_{\hat{\Upsilon}}$ and $s_{\hat{\Upsilon}}$ are well defined as in (2.3): indeed,

4 Our definition should not be confused with the one in [31, Definition 6.3.1].

$$\frac{t\hat{\Upsilon}'(t)}{\hat{\Upsilon}(t)} = \frac{1}{2}\frac{t\Upsilon'(t)}{\Upsilon(t)} + \frac{1}{2}\frac{t\Phi_*^*(t)}{\Phi_*(t)} \quad \forall t > 0.$$

We deduce $\frac{i_\Upsilon + i_{\Phi_*}}{2} \le i_{\hat{\Upsilon}} \le s_{\hat{\Upsilon}} \le \frac{s_\Upsilon + s_{\Phi_*}}{2}$.
Fourth case: $\Upsilon \gg \Phi$.

Thanks to (2.5) and the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Upsilon(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega \Upsilon(|u|)\mathrm{d}x &\le \overline{\zeta}_\Upsilon\left(\|u\|_{L^\Upsilon(\Omega)}\right) \le \overline{\zeta}_\Upsilon\left(k\|\nabla u\|_{L^\Phi(\Omega)}\right) \le \overline{\zeta}_\Upsilon(k)\overline{\zeta}_\Upsilon\left(\|\nabla u\|_{L^\Phi(\Omega)}\right)\\ &= \overline{\zeta}_\Upsilon(k)\overline{\zeta}_\Upsilon\left(\underline{\zeta}_\Phi^{-1}\left(\underline{\zeta}_\Phi\left(\|\nabla u\|_{L^\Phi(\Omega)}\right)\right)\right) \le \overline{\zeta}_\Upsilon(k)\overline{\zeta}_\Upsilon\left(\underline{\zeta}_\Phi^{-1}\left(\int_\Omega \Phi(|\nabla u|)\mathrm{d}x\right)\right)\\ &\le \overline{\zeta}_\Upsilon(k)\overline{\zeta}_\Upsilon(\underline{\zeta}_\Phi^{-1}(r)) \le \overline{\zeta}_\Upsilon(k)\left(1 + r^{\frac{s_\Upsilon}{i_\Phi}}\right),\end{aligned}$$

where $k > 0$ is the best constant of the embedding mentioned earlier. So κ is estimated as follows:

$$\kappa(r) \le \frac{C_1|\Omega| + C_2\overline{\zeta}_\Upsilon(k)}{r} + C_2\overline{\zeta}_\Upsilon(k)r^{\frac{s_\Upsilon}{i_\Phi}-1}. \tag{4.7}$$

We observe that $\Upsilon \gg \Phi$ implies $s_\Upsilon > i_\Phi$; otherwise, we have

$$\frac{\Upsilon'(s)}{\Upsilon(s)} \le \frac{\Phi'(s)}{\Phi(s)} \quad \forall s \in (0, +\infty)$$

whence, integrating in $[1, t]$, $t > 1$, and passing to the exponential,

$$\Upsilon(t) \le \frac{\Upsilon(1)}{\Phi(1)}\Phi(t) \quad \forall t \in (1, +\infty),$$

in contrast with $\Upsilon \gg \Phi$. Hence, the right-hand side of (4.7), which can be rewritten as follows:

$$\hat{k}(r) := \frac{A}{r} + Br^\theta, \quad \text{with} \quad A := C_1|\Omega| + C_2\overline{\zeta}_\Upsilon(k),\ B := C_2\overline{\zeta}_\Upsilon(k),\ \theta := \frac{s_\Upsilon}{i_\Phi} - 1 > 0,$$

diverges when $r \to +\infty$. Computing the unique critical point of $\hat{k}$ reveals that

$$\min_{r>0} \hat{k}(r) = \hat{k}\left(\left(\frac{A}{\theta B}\right)^{\frac{1}{\theta+1}}\right) = [A^\theta B(\theta + \theta^{-\theta})]^{\frac{1}{\theta+1}}.$$

In this case, we set $\lambda^* := [A^\theta B(\theta + \theta^{-\theta})]^{-\frac{1}{\theta+1}}$, $r_\lambda^* := \left(\frac{A}{\theta B}\right)^{\frac{1}{\theta+1}}$ and apply Theorem 4.1. □

Remark 4.5. According to (2.5) and Theorem 4.4, we infer

$$\underline{\zeta}_\Phi\left(\|u_\lambda\|_{W_0^{1,\Phi}(\Omega)}\right) \le \int_\Omega \Phi(|\nabla u_\lambda|)\mathrm{d}x < r_\lambda^*.$$

We define the ball

$$B_\lambda := \left\{u \in W_0^{1,\Phi}(\Omega) : \ \|u\|_{W_0^{1,\Phi}(\Omega)} < \underline{\zeta}_\Phi^{-1}(r_\lambda^*)\right\}.$$

Taking into account Theorem 4.4 again, we deduce that u_λ is a minimizer for the restriction of J_λ to $\overline{B}_\lambda$; in particular, u_λ is a local minimizer for J_λ. Incidentally, we stress the fact that this local minimizer has been provided without using any $W^{1,\Phi}$ versus C^1 local minimizer argument.

Lemma 4.6. *Under* H(a)$_1$–H(a)$_2$ *and* H(f)$_1$–H(f)$_3$, *the functional* J_λ *in* (4.1) *satisfies the Palais-Smale condition and is unbounded from below.*

Proof. Let $\{u_n\} \subseteq W_0^{1,\Phi}(\Omega)$ be such that $\{J_\lambda(u_n)\}$ is bounded and $\|J'_\lambda(u_n)\|_{W^{-1,\tilde{\Phi}}(\Omega)} \to 0$ as $n \to \infty$. Hence, for a suitable $c > 0$, up to subsequences, we have

$$\int_\Omega \Phi(|\nabla u_n|)\mathrm{d}x - \lambda \int_\Omega \hat{F}(x, u_n)\mathrm{d}x \le c \tag{4.8}$$

and

$$\left| \int_\Omega a(|\nabla u_n|)\nabla u_n \cdot \nabla v \mathrm{d}x - \lambda \int_\Omega \hat{f}(x, u_n) v \mathrm{d}x \right| \le \|\nabla v\|_{L^\Phi(\Omega)} \tag{4.9}$$

for all $n \in \mathbb{N}$ and $v \in W_0^{1,\Phi}(\Omega)$. We prove that $\{u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$ by showing the boundedness of $\{u_n^-\}$ and $\{u_n^+\}$.

Choosing $v = -u_n^-$ in (4.9) and using (2.13) yield

$$\begin{aligned} i_\Phi \underline{\zeta}_\Phi\left(\|\nabla u_n^-\|_{L^\Phi(\Omega)}\right) &\le \int_\Omega a(|\nabla u_n^-|)|\nabla u_n^-|^2 \mathrm{d}x \\ &\le \int_\Omega a(|\nabla u_n^-|)|\nabla u_n^-|^2 \mathrm{d}x + \lambda \int_\Omega \hat{f}(x, u_n)u_n^- \mathrm{d}x \le \|\nabla u_n^-\|_{L^\Phi(\Omega)}, \end{aligned}$$

whence $\{u_n^-\}$ is bounded in $W_0^{1,\Phi}(\Omega)$.

Exploiting (2.22) and H(f)$_3$, besides Remark 2.8, we have

$$\begin{aligned} \int_\Omega \hat{F}(x, u_n^+)\mathrm{d}x &= \int_{\Omega\cap\{u_n^+\le R\}} \hat{F}(x, u_n^+)\mathrm{d}x + \int_{\Omega\cap\{u_n^+ > R\}} (\hat{F}(x, R) + F(x, u_n^+))\mathrm{d}x \\ &\le 2\Pi(R) + \int_{\Omega\cap\{u_n^+ > R\}} F(x, u_n^+)\mathrm{d}x \\ &\le 2\Pi(R) + \frac{1}{\mu}\int_{\Omega\cap\{u_n^+ > R\}} f(x, u_n^+)u_n^+ \mathrm{d}x \\ &\le 2\Pi(R) + \frac{1}{\mu}\int_\Omega \hat{f}(x, u_n^+)u_n^+ \mathrm{d}x. \end{aligned} \tag{4.10}$$

By (4.8) and (4.10), we deduce

$$\begin{aligned} \int_\Omega \Phi(|\nabla u_n^+|)\mathrm{d}x \le \int_\Omega \Phi(|\nabla u_n|)\mathrm{d}x &\le c + \lambda \int_\Omega \hat{F}(x, u_n)\mathrm{d}x \le c + \lambda \int_\Omega \hat{F}(x, u_n^+)\mathrm{d}x \\ &\le c + 2\lambda\Pi(R) + \frac{\lambda}{\mu}\int_\Omega \hat{f}(x, u_n^+)u_n^+ \mathrm{d}x. \end{aligned} \tag{4.11}$$

On the other hand, choosing $v = u_n^+$ in (4.9) produces

$$\lambda \int_\Omega \hat{f}(x, u_n^+)u_n^+ \mathrm{d}x \le \|\nabla u_n^+\|_{L^\Phi(\Omega)} + \int_\Omega \varphi(|\nabla u_n^+|)|\nabla u_n^+|\mathrm{d}x \le \|\nabla u_n^+\|_{L^\Phi(\Omega)} + s_\Phi \int_\Omega \Phi(|\nabla u_n^+|)\mathrm{d}x. \tag{4.12}$$

By combining (4.11)–(4.12) and rearranging the terms, we obtain

$$\left(1 - \frac{s_\Phi}{\mu}\right)\int_\Omega \Phi(|\nabla u_n^+|)\mathrm{d}x \le c + 2\lambda\Pi(R) + \frac{1}{\mu}\|\nabla u_n^+\|_{L^\Phi(\Omega)}.$$

According to H(f)$_3$ and (2.5), it turns out that $\{u_n^+\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. The (S$_+$) property of H' (see Lemma 2.6) and the compactness of K' (see Lemma 2.10) ensure the Palais-Smale condition for J_λ; see [8, Lemma 3.1] for details.

Now we prove that J_λ is unbounded from below. First, fix any $\overline{R} > R$. Integrating (1.6) in $(\overline{R}, t)$, $t > \overline{R}$, and passing to the exponential yield

$$F(x,t) \geq \frac{F(x,\overline{R})}{\overline{R}^\mu} t^\mu =: c_{\overline{R}} t^\mu \quad \forall (x,t) \in \Omega \times [\overline{R}, +\infty). \tag{4.13}$$

Take any test function $u_0 \in C_c^\infty(\Omega)$ such that $u_0 \geq 0$ in Ω and $u_0 \not\equiv 0$. Then there exists a compact $K \subseteq \Omega$ such that

$$\min_K u_0 > 0 \quad \text{and} \quad |K| > 0. \tag{4.14}$$

For any $M > 0$, set $K_M := \{x \in \Omega : \ Mu_0 > \overline{R}\}$. Observe that $\{K_M\}_{M>0}$ is increasing and $K \subseteq K_M$ for large values of M. By using (4.13) and (2.4), besides recalling Remark 2.8, for M large, we obtain

$$\begin{aligned} J_\lambda(Mu_0) &\leq \int_\Omega \Phi(M|\nabla u_0|)\mathrm{d}x - \lambda \int_{K_M} F(x, Mu_0)\mathrm{d}x \\ &\leq \int_\Omega \Phi(M|\nabla u_0|)\mathrm{d}x - \lambda c_{\overline{R}} M^\mu \int_{K_M} u_0^\mu \mathrm{d}x \\ &\leq M^{s_\Phi} \int_\Omega \Phi(|\nabla u_0|)\mathrm{d}x - \lambda c_{\overline{R}} M^\mu \int_K u_0^\mu \mathrm{d}x. \end{aligned} \tag{4.15}$$

By H(f)$_3$, we have $\mu > s_\Phi$, while (4.14) ensures that $\int_K u_0^\mu \mathrm{d}x > 0$. Hence, $J_\lambda(Mu_0) \to -\infty$ when $M \to +\infty$, as desired. □

Remark 4.7. Incidentally, we notice that (4.13) implies that J_λ is Φ-super-linear, since $\Phi < t^\mu$ in the sense of (2.8).

Theorem 4.8. *Suppose* H(a)$_1$–H(a)$_2$ *and* H(f)$_1$–H(f)$_3$. *Then problem* $(\mathrm{P}_{\lambda,f})$ *admits two distinct solutions in* $C_0^{1,\tau}(\overline{\Omega})$.

Proof. Let λ^*, r^* be given by Theorem 4.4. Fix any $\lambda \in (0, \lambda^*)$. Existence of a solution $u_\lambda \in C_0^{1,\tau}(\Omega)$ to $(\mathrm{P}_{\lambda,f})$ is guaranteed by Theorem 4.4. We want to obtain a second solution $v_\lambda \in C_0^{1,\tau}(\Omega)$ by applying Theorem 4.3 to the functional J_λ defined in (4.1). As in the proof of Theorem 4.4, regularity of v_λ is a consequence of Theorem 3.5.

First, we notice that J_λ is bounded on bounded sets: indeed, by (2.5), (4.4), and the embedding inequality for $W_0^{1,\Phi}(\Omega) \hookrightarrow L^\Upsilon(\Omega)$, we have, for an opportune $C_3 > 0$ independent of u,

$$\begin{aligned} |J_\lambda(u)| &\leq \overline{\zeta}_\Phi\left(\|u\|_{W_0^{1,\Phi}(\Omega)}\right) + C_1|\Omega| + C_2\overline{\zeta}_\Upsilon\left(\|u\|_{L^\Upsilon(\Omega)}\right) \\ &\leq \overline{\zeta}_\Phi\left(\|u\|_{W_0^{1,\Phi}(\Omega)}\right) + C_1|\Omega| + C_3\overline{\zeta}_\Upsilon\left(\|u\|_{W_0^{1,\Phi}(\Omega)}\right). \end{aligned}$$

Taking into account Remark 4.5, we have that u_λ is a local minimizer for J_λ. Since Theorem 4.6 ensures that J_λ is unbounded from below, then u_λ is not a global minimizer. Reasoning as in the first part of the proof of [3, Theorem 2.1] guarantees (4.3) with $u_0 := u_\lambda$. Hence, Theorem 4.3 furnishes $v_\lambda \in W_0^{1,\Phi}(\Omega)$ critical point to J_λ, and thus solution to both $(\mathrm{P}_{\lambda,\hat{f}})$ and $(\mathrm{P}_{\lambda,f})$. Moreover, v_λ fulfills $J_\lambda(v_\lambda) \geq J_\lambda(u_\lambda)$. If $J_\lambda(v_\lambda) > J_\lambda(u_\lambda)$, then $v_\lambda \neq u_\lambda$; else, Theorem 4.3 ensures that v_λ can be taken on ∂B_λ. In any case, we have $v_\lambda \neq u_\lambda$. □

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Appendix

A Examples

In this appendix, we want to show the importance of working in Sobolev-Orlicz spaces instead of classical Sobolev ones. In this sight, we furnish a class of Young functions Φ and reaction terms f whose corresponding problem $(P_{\lambda,f})$ cannot be set in a Sobolev framework, but it fulfills $H(a)_1$–$H(a)_2$ and $H(f)_1$–$H(f)_3$; see Example A.4. Inspiring examples can be found, e.g., in [12], where existence of at least one positive solution is obtained by the Mountain Pass theorem provided $\lambda = 1$ and the reaction term is not affected by singular terms. Following [12, Example 1], at the end of this appendix, we will furnish a more concrete example (Example A.5) satisfying our hypotheses.

First, we construct a class of "pathological" Young functions Ψ (with $1 < i_\Psi < s_\Psi < +\infty$), possessing distinct indices at infinity, that is,

$$\liminf_{t\to+\infty}\frac{t\Psi'(t)}{\Psi(t)} \neq \limsup_{t\to+\infty}\frac{t\Psi'(t)}{\Psi(t)}, \quad \liminf_{t\to+\infty}\frac{t\Psi''(t)}{\Psi'(t)} \neq \limsup_{t\to+\infty}\frac{t\Psi''(t)}{\Psi'(t)}.$$

This could be hopefully useful also in other contexts to construct counterexamples in Orlicz spaces.

Lemma A.1. *Let $1 < q < p < +\infty$. Set $\alpha := \frac{p+q}{2}$ and $\beta := \frac{p-q}{2}$. Then, for any $\varepsilon < \min\left\{4, \frac{q-1}{\beta}\right\}$, the function*

$$\Psi(t) := t^\alpha e^{\eta(t)} \quad \forall t \geq 0, \tag{A1}$$

with

$$\eta(t) = \begin{cases} \frac{\beta\varepsilon}{2e^2}(e-t)^2 - \frac{\beta\varepsilon}{1+\varepsilon^2} & \text{for } 0 \leq t \leq e, \\ \beta\frac{\log t}{1+\varepsilon^2}[\sin(\varepsilon\log(\log t)) - \varepsilon\cos(\varepsilon\log(\log t))] & \text{for } t \geq e, \end{cases} \tag{A2}$$

is a Young function satisfying the following properties:

$$\inf_{t>0}\frac{t\Psi'(t)}{\Psi(t)} = \liminf_{t\to+\infty}\frac{t\Psi'(t)}{\Psi(t)} = q < p = \limsup_{t\to+\infty}\frac{t\Psi'(t)}{\Psi(t)} = \sup_{t>0}\frac{t\Psi'(t)}{\Psi(t)}, \tag{A3}$$

$$q-1-\beta\varepsilon < \inf_{t>0}\frac{t\Psi''(t)}{\Psi'(t)} \leq \liminf_{t\to+\infty}\frac{t\Psi''(t)}{\Psi'(t)} = q-1 < p-1 = \limsup_{t\to+\infty}\frac{t\Psi''(t)}{\Psi'(t)} \leq \sup_{t>0}\frac{t\Psi''(t)}{\Psi'(t)} < p-1+\beta\varepsilon, \tag{A4}$$

$$\liminf_{t\to+\infty}\frac{\Psi(t)}{t^r} = 0 \quad \text{or} \quad \limsup_{t\to+\infty}\frac{\Psi(t)}{t^r} = +\infty \quad \forall r > 1. \tag{A5}$$

If $p < N$, then Ψ satisfies (1.3) with Ψ in place of Φ. If, in addition, $p < q^*$, then $s_\Psi < i_{\Psi_*}$.

Proof. Starting from (A1), let us compute Ψ', Ψ'' in terms of the lower order derivatives:

$$\Psi'(t) = \Psi(t)\left(\frac{\alpha}{t} + \eta'(t)\right) = \frac{\Psi(t)}{t}(\alpha + t\eta'(t)), \tag{A6}$$

$$\begin{aligned}\Psi''(t) &= \Psi'(t)\left(\frac{\Psi'(t)}{\Psi(t)} + \frac{\eta''(t) - \frac{\alpha}{t^2}}{\frac{\alpha}{t} + \eta'(t)}\right) \\ &= \frac{\Psi'(t)}{t}\left(\alpha + t\eta'(t) + \frac{t^2\eta''(t) - \alpha}{\alpha + t\eta'(t)}\right) \\ &= \frac{\Psi'(t)}{t}\left(\alpha - 1 + t\eta'(t) + \frac{t^2\eta''(t) + t\eta'(t)}{\alpha + t\eta'(t)}\right).\end{aligned} \tag{A7}$$

First, we study Ψ in the interval $[0, e]$. We have

$$\eta'(t) = \frac{\beta\varepsilon}{e^2}(t-e) \quad \text{and} \quad \eta''(t) = \frac{\beta\varepsilon}{e^2} \quad \text{for all } t \in (0,e]. \tag{A8}$$

We observe that (A6), (A2), and $\varepsilon < 4$ entail

$$q < \alpha - \frac{\beta\varepsilon}{4} = \alpha + \min_{s\in(0,e]} s\eta'(s) \le \frac{t\Psi'(t)}{\Psi(t)} \le \alpha + \max_{s\in(0,e]} s\eta'(s) = \alpha < p \tag{A9}$$

for all $t \in (0,e]$. Exploiting (A7) and (A9), $\varepsilon < 4$, the monotonicity of $r \mapsto r + \frac{r}{\alpha+r}$, and $\eta' < 0 < \eta''$ in $(0,e]$, we obtain, for all $t \in (0,e]$,

$$\begin{aligned}
\alpha - 1 - \beta\varepsilon &< \alpha - 1 - \frac{\beta\varepsilon}{4}\left(1 + \frac{1}{\alpha - \frac{\beta\varepsilon}{4}}\right) \\
&= \alpha - 1 + \min_{s\in(0,e]}\left(s\eta'(s) + \frac{s\eta'(s)}{\alpha + s\eta'(s)}\right) \\
&\le \frac{t\Psi''(t)}{\Psi'(t)} \le \alpha - 1 + \frac{t^2\eta''(t)}{\alpha + t\eta'(t)} \\
&\le \alpha - 1 + \frac{\beta\varepsilon}{\alpha - \frac{\beta\varepsilon}{4}} < \alpha - 1 + \beta\varepsilon.
\end{aligned} \tag{A10}$$

Now we analyze Ψ in $[e,+\infty)$. We posit $\zeta(t) := \varepsilon(\log(\log t))$ for all $t \ge e$. Integrating by parts twice reveals that

$$\begin{aligned}
\int \sin(\varepsilon \log s)\mathrm{d}s &= s\sin(\varepsilon\log s) - \varepsilon\int\cos(\varepsilon\log s)\mathrm{d}s \\
&= s[\sin(\varepsilon\log s) - \varepsilon\cos(\varepsilon\log s)] - \varepsilon^2\int\sin(\varepsilon\log s)\mathrm{d}s,
\end{aligned}$$

whence, performing the change of variable $s = \log t$ and recalling (A2),

$$\begin{aligned}
\beta\int\frac{\sin(\zeta(t))}{t}\mathrm{d}t = \beta\int\sin(\varepsilon\log s)\mathrm{d}s &= \frac{\beta s}{1+\varepsilon^2}[\sin(\varepsilon\log s) - \varepsilon\cos(\varepsilon\log s)] \\
&= \beta\frac{\log t}{1+\varepsilon^2}[\sin(\varepsilon\log(\log t)) - \varepsilon\cos(\varepsilon\log(\log t))] = \eta(t)
\end{aligned} \tag{A11}$$

for all $t \in [e,+\infty)$. Accordingly, we have

$$\eta'(t) = \beta\frac{\sin(\zeta(t))}{t} \quad \text{and} \quad \eta''(t) = \frac{\beta}{t^2}[t\zeta'(t)\cos(\zeta(t)) - \sin(\zeta(t))] \quad \text{for all } t \ge e. \tag{A12}$$

Hence, we rewrite (A6) and (A7) as follows:

$$\Psi'(t) = \frac{\Psi(t)}{t}(\alpha + \beta\sin(\zeta(t))), \tag{A13}$$

$$\Psi''(t) = \frac{\Psi'(t)}{t}\left(\alpha - 1 + \beta\sin(\zeta(t)) + \frac{\beta t\zeta'(t)\cos(\zeta(t))}{\alpha + \beta\sin(\zeta(t))}\right), \tag{A14}$$

valid for all $t \in [e,+\infty)$.

We observe that $\zeta(t) \to +\infty$ as $t \to +\infty$, so (A9) and (A13) guarantee (A3). Moreover, $t\zeta'(t) \to 0$ as $t \to +\infty$ and

$$0 \le \frac{\beta|\cos(\zeta(t))|}{\alpha + \beta\sin(\zeta(t))} \le \frac{\beta}{\alpha - \beta} \quad \forall t \in [e,+\infty). \tag{A15}$$

Thus, (A14) provides the equalities in (A4). More precisely, we notice that

$$0 < t\zeta'(t) = \frac{\varepsilon}{\log t} \le \varepsilon \quad \forall t \in [e, +\infty). \tag{A16}$$

Exploiting (A14)–(A16), we obtain, for all $t \ge e$,

$$q - 1 - \beta\varepsilon < \alpha - 1 - \beta - \frac{\beta\varepsilon}{\alpha - \beta} \le \frac{t\Psi''(t)}{\Psi'(t)} \le \alpha - 1 + \beta + \frac{\beta\varepsilon}{\alpha - \beta} < p - 1 + \beta\varepsilon. \tag{A17}$$

Because of (A10) and (A17), the inequalities in (A4) hold true.

A direct computation, based on (A8) and (A12), shows that $\eta \in C^2(0, +\infty)$; thus, Ψ enjoys the same property. Moreover, (A3) and (A4) and $\beta\varepsilon < q - 1$ yield $\Psi'(t), \Psi''(t) > 0$ for all $t > 0$. Thus, Ψ is strictly increasing and convex. By using again (A3), together with (2.4), we deduce

$$\Psi(1)\min\{t^p, t^q\} = \Psi(1)\underline{\zeta}_\Psi(t) \le \Psi(t) \le \Psi(1)\overline{\zeta}_\Psi(t) = \Psi(1)\max\{t^p, t^q\} \quad \forall t > 0,$$

which entails (2.1). Hence, Ψ is a Young function.

To prove (A5), let us consider two sequences $h_n, k_n \to +\infty$ such that

$$\sin(\zeta(h_n)) = 1 \quad \text{and} \quad \cos(\zeta(k_n)) = 1 \quad \text{for all } n \in \mathbb{N}.$$

Then (A1) and (A2) give, for any n large enough,

$$\Psi(h_n) = h_n^{\alpha + \frac{\beta}{1+\varepsilon^2}} \quad \text{and} \quad \Psi(k_n) = k_n^{\alpha - \frac{\beta\varepsilon}{1+\varepsilon^2}},$$

ensuring (A5).

Now suppose that $p < N$. Then, setting $\Lambda := t^p$, by (2.4), we infer $\Psi < \Lambda$ in the sense of (2.8). In particular, $\Psi^{-1}(t) \ge c\Lambda^{-1}(t)$ for all $t > 1$, being $c > 0$ small enough. Thus, we obtain

$$\int_1^{+\infty} \Theta_\Psi(t)\mathrm{d}t \ge c\int_1^{+\infty} \Theta_\Lambda(t)\mathrm{d}t = c\int_1^{+\infty} t^{\frac{1}{p^*}-1}\mathrm{d}t = +\infty.$$

The last statement is a direct consequence of (A3) and (2.7). □

Remark A.2. Two motivations suggest to work in Sobolev-Orlicz spaces instead of in the classical Sobolev framework.

The first motivation is structural: if we set the problem in a reflexive Sobolev-Orlicz space $W_0^{1,\Psi}(\Omega)$ (which may be a Sobolev space), the weak formulation of problem $(\mathrm{P}_{\lambda,f})$ requires

$$\int_\Omega a(|\nabla u|)\nabla u \nabla v \mathrm{d}x < +\infty \quad \forall u, v \in W_0^{1,\Psi}(\Omega).$$

This is a *duality* property, which fails whenever $W_0^{1,\Psi}(\Omega) \setminus W_0^{1,\Phi}(\Omega) \neq \emptyset$: indeed, taking $u \in W_0^{1,\Psi}(\Omega) \setminus W_0^{1,\Phi}(\Omega)$ and $v = u$, by (2.3), we obtain

$$\int_\Omega a(|\nabla u|)\nabla u \nabla v \mathrm{d}x = \int_\Omega \varphi(|\nabla u|)|\nabla u|\mathrm{d}x \ge i_\Phi \int_\Omega \Phi(|\nabla u|)\mathrm{d}x = +\infty.$$

Hence, to properly define the concept of “weak solution,” we have to require $W_0^{1,\Psi}(\Omega) \subseteq W_0^{1,\Phi}(\Omega)$, which means $\Phi < \Psi$ (in the sense of (2.8)).

Here comes the second motivation, which is technical: if we suppose $W_0^{1,\Psi}(\Omega) \subsetneq W_0^{1,\Phi}(\Omega)$, then we loose the *coercivity* of $-\Delta_\Phi$. To show this, we pick $u \in W_0^{1,\Phi}(\Omega) \setminus W_0^{1,\Psi}(\Omega)$ and a sequence $\{u_n\} \subseteq W_0^{1,\Psi}(\Omega)$ such that $u_n \to u$ in $W_0^{1,\Phi}(\Omega)$. It turns out that $\|u_n\|_{W_0^{1,\Psi}(\Omega)} \to +\infty$; otherwise, by reflexivity of $W_0^{1,\Psi}(\Omega)$ and up to subsequences, we would have $u_n \rightharpoonup u^*$ in $W_0^{1,\Psi}(\Omega)$ for some $u^* \in W_0^{1,\Psi}(\Omega)$ and, by uniqueness of the weak limit, we would conclude $u = u^* \in W_0^{1,\Psi}(\Omega)$, in contrast with the choice of u. On the other hand, by (2.3),

$$\sup_{n\in\mathbb{N}} \int_\Omega a(|\nabla u_n|)|\nabla u_n|^2 \mathrm{d}x = \sup_{n\in\mathbb{N}} \int_\Omega \varphi(|\nabla u_n|)|\nabla u_n| \mathrm{d}x \leq s_\Phi \sup_{n\in\mathbb{N}} \int_\Omega \Phi(|\nabla u_n|)\mathrm{d}x \leq s_\Phi \sup_{n\in\mathbb{N}} \overline{\zeta}_\Phi\left(\|u_n\|_{W_0^{1,\Phi}(\Omega)}\right) < +\infty,$$

which proves that $-\Delta_\Phi$ is not coercive on $W_0^{1,\Psi}(\Omega)$. Since coercivity of the principal part is an essential ingredient for existence results and, in particular, for our approach, which relies on Theorem 4.1, we adopted the framework $W_0^{1,\Psi}(\Omega) = W_0^{1,\Phi}(\Omega)$.

It remains to prove that $W_0^{1,\Phi}(\Omega)$ is not a Sobolev space in general. To this end, we observe that any Young function given by Lemma A.1, say Φ, furnishes a counterexample. Indeed, suppose by contradiction that $W_0^{1,\Phi}(\Omega) = W_0^{1,r}(\Omega)$ for some $r > 1$. Then we have $\Phi < t^r$ and $t^r < \Phi$ (in the sense of (2.8)), whence

$$\begin{aligned} \Phi < t^r &\Rightarrow \limsup_{t\to+\infty} \frac{\Phi(t)}{t^r} \leq c_1^r < +\infty, \\ t^r < \Phi &\Rightarrow \liminf_{t\to+\infty} \frac{\Phi(t)}{t^r} \geq c_2^{-r} > 0, \end{aligned} \tag{A18}$$

for a suitable $c_1, c_2 > 0$ given by (2.8). Since (A18) contradicts (A5), we deduce that $W_0^{1,\Phi}(\Omega)$ is not a Sobolev space.

Remark A.3. Another important aspect related to the choice of the Sobolev-Orlicz framework is represented by the reaction term: we address the reader to [12, Section 6] for a discussion about this setting and the Ambrosetti-Rabinowitz condition. Here, we limit ourselves to provide an example of nonlinearity $f = f(u)$ fulfilling $H(f)_1$–$H(f)_3$.

Suppose $s_\Phi < i_{\Phi_*}$. By virtue of Lemma A.1, we can construct a Young function Υ satisfying $s_\Phi < i_\Upsilon < s_\Upsilon < i_{\Phi_*}$. Then, fixed $\gamma \in (0, 1)$, we consider

$$f(t) = \frac{\Upsilon(t)}{t} + t^{-\gamma}. \tag{A19}$$

Obviously, f fulfills $H(f)_1$. Observe that (1.7) implies

$$\frac{\Upsilon(t)}{t} \leq \overline{\Upsilon}^{-1}(\Upsilon(t)) \quad \forall t > 0,$$

so that $H(f)_2$ is satisfied with $c_1 = c_2 = 1$. To prove $H(f)_3$, choose any $\mu \in (s_\Phi, i_\Upsilon)$. For all $t > 0$, we have

$$tf(t) = \Upsilon(t) + t^{1-\gamma} \tag{A20}$$

and, given any $R > 0$,

$$F(t) = \int_R^t \left(\frac{\Upsilon(s)}{s} + s^{-\gamma}\right)\mathrm{d}s \leq \int_R^t (i_\Upsilon^{-1}\Upsilon'(s) + s^{-\gamma})\mathrm{d}s \leq \frac{1}{i_\Upsilon}\Upsilon(t) + \frac{t^{1-\gamma}}{1-\gamma}. \tag{A21}$$

Convexity of Υ and $\mu < i_\Upsilon$ guarantee that there exists $R > 0$ such that

$$\frac{\mu}{1-\gamma} t^{1-\gamma} \leq \left(1 - \frac{\mu}{i_\Upsilon}\right)\Upsilon(t) \quad \forall t \geq R. \tag{A22}$$

From (A20)–(A22), we obtain

$$\mu F(t) \leq \frac{\mu}{i_\Upsilon}\Upsilon(t) + \frac{\mu}{1-\gamma} t^{1-\gamma} \leq \Upsilon(t) \leq tf(t) \quad \forall t \geq R,$$

which entails $H(f)_3$.

As announced, we conclude with two examples of problems fulfilling the hypotheses of Theorem 4.8; according to Remark A.2, we stress that it is necessary to set them in the appropriate Sobolev-Orlicz setting. Existence of two solutions for these problems is a consequence of Theorem 4.8.

Example A.4. Take any $r > s > p > q > 1$ such that $p < N$ and $r < q^*$. Let Φ and Υ be given by Lemma A.1 (applied with any sufficiently small $\varepsilon > 0$), such that $i_\Phi = q$, $s_\Phi = p$, $i_\Upsilon = s$, $s_\Upsilon = r$. Let f be defined as in (A19). Then problem $(\mathrm{P}_{\lambda,f})$ admits at least two distinct weak solutions $u, v \in C_0^{1,\tau}(\overline{\Omega})$ for all $\lambda \in (0, \lambda^*)$. Here, $\tau \in (0, 1]$ and $\lambda^* > 0$ are given by Theorems 3.5 and 4.4, respectively.

The hypotheses of Theorem 4.8 are fulfilled: (A4) implies $\mathrm{H(a)}_1$ and the final part of Lemma A.1 gives $\mathrm{H(a)}_2$, while Remark A.3 ensures $\mathrm{H(f)}_1$–$\mathrm{H(f)}_3$.

Example A.5. The same result stated in Example A.4 holds true for the problem

$$\begin{cases} -\mathrm{div}(\log(1 + |\nabla u|)|\nabla u|^{p-2}\nabla u) = \lambda(u^r + u^{-\gamma}) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A23}$$

where $1 < p < N - 1$, $N < p + p^2$, $r \in (p, p^* - 1)$, and $\gamma \in (0, 1)$.

Problem (A23) comes from $(\mathrm{P}_{\lambda,f})$ by choosing, for all $(x, t) \in \Omega \times (0, +\infty)$,

$$a(t) := t^{p-2}\log(1 + t), \quad \Phi(t) := \int_0^t s^{p-1}\log(1 + s)\mathrm{d}s, \quad f(x, t) := t^r + t^{-\gamma}.$$

To verify the assumptions of Theorem 4.8, we explicitly observe that

(1) $H_a(t) := \frac{ta'(t)}{a(t)} = p - 2 + \frac{t}{(t+1)\log(t+1)}$ is a decreasing function in $(0, +\infty)$ with $\lim_{t\to 0^+} H_a(t) = p - 1$ and $\lim_{t\to\infty} H_a(t) = p - 2$. Then we have
$$i_a := \inf_{t>0} H_a(t) = p - 2 < p - 1 = \sup_{t>0} H_a(t) =: s_a. \tag{A24}$$

(2) According to De L'Hôpital's rule, $H_\Phi(t) := \frac{t\Phi'(t)}{\Phi(t)}$ fulfills
$$\lim_{t\to 0^+} H_\Phi(t) = \lim_{t\to 0^+} \frac{ta(t) + t(ta(t))'}{ta(t)} = 2 + \lim_{t\to 0^+} H_a(t) = p + 1 \tag{A25}$$
and
$$\lim_{t\to\infty} H_\Phi(t) = 2 + \lim_{t\to\infty} H_a(t) = p. \tag{A26}$$

(3) One has
$$i_a + 2 \le i_\Phi \le s_\Phi \le s_a + 2. \tag{A27}$$
Indeed, for all $s > 0$, we have $i_a \le \frac{sa'(s)}{a(s)} \le s_a$. Multiplying by $sa(s)$, an integration by parts in $(0, t)$ gives $i_a\Phi(t) \le t^2 a(t) - 2\Phi(t) \le s_a\Phi(t)$ and our claim follows.

From (A24)–(A27), it is readily seen that

$$p = i_a + 2 \le i_\Phi \le p \quad \text{and} \quad p + 1 \le s_\Phi \le s_a + 2 = p + 1,$$

that is,

$$i_\Phi = p < p + 1 = s_\Phi. \tag{A28}$$

Therefore, from (A24), it is clear that $\mathrm{H(a)}_1$ holds if and only if $p > 1$. Bearing in mind (2.7), since we have that $s_\Phi = p + 1 < p^* = i_\Phi^* \le i_{\Phi_*}$, also $\mathrm{H(a)}_2$ is verified. On the other hand, $\mathrm{H(f)}_1$–$\mathrm{H(f)}_3$ follow from Remark A.3 by taking $\Upsilon(t) = t^{r+1}$, being $i_\Upsilon = s_\Upsilon = r + 1$ with $p < r < p^* - 1$.

Remark A.6. Regarding Example A.5, if we drop the condition $N < p + p^2$ and replace $r \in (p, p^* - 1)$ with jointly $p < r$ and $t^r \ll \Phi_*$, we can ensure only that problem (A23) admits at least two distinct weak solutions in $W_0^{1,\Phi}(\Omega)$. In particular, two solutions are obtained in the case $p < r \leq p^* - 1$, since $t^p \ll \Phi$ forces $t^{r+1} < t^{p^*} \ll \Phi_*$ (with an argument similar to the one in the last part of the proof of Lemma A.1). A similar conclusion holds true for Example A.4, replacing $r < q^*$ with $t^r \ll \Phi_*$.