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A novel interval matrix stiffness method for the analysis of steel frames with uncertain semi-rigid connections

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ABSTRACT

The static analysis of steel frames with uncertain semi-rigid connections is addressed. The effects of connection flexibility are incorporated into the frame model by means of rotational springs at the end nodes of beams. Assuming an idealized linear-elastic behaviour, the initial stiffness of rotational springs is expressed in terms of the so-called *fixity factor* which is considered uncertain to take into account the large scatter of connection stiffness and capacity, evidenced by experimental measures. Specifically, the *fixity factors* are modelled as interval variables whose bounds are consistent with the limits assigned by Eurocode 3 for different types of steel semi-rigid connections. Within the framework of the so-called *Improved Interval Analysis* via *Extra Unitary Interval*, a novel interval matrix stiffness method is developed by incorporating the interval *fixity factors* into the classical deterministic formulation. The lower bound and upper bound of the interval structural response are estimated by applying a sensitivity-based procedure. A single-storey frame and a five-storey frame with uncertain semi-rigid beam-to-column connections are analyzed to validate the proposed approach as well as to investigate the influence of the uncertain connection stiffness on structural behaviour.

1. Introduction

It is now widely recognized that connection design plays a crucial role in the prediction of the behaviour of frame structures. To simplify the analysis and design, connections are commonly idealized as perfectly rigid or ideally hinged. However, numerous experimental investigations [1] have shown that the actual behaviour ranges between these extreme cases so that semi-rigid or partially restrained connections are more realistic idealizations (see e.g., [2]). Current building codes for steel structures such as Eurocode 3 (EC3) [3] take into account the flexibility of connections.

The inherently nonlinear flexural behaviour of semi-rigid connections is described by the moment-rotation curve, which can be determined using analytical, empirical, experimental, mechanical, numerical, and information-based models [4]. The effects of connection flexibility are commonly incorporated into structural analysis by means of rotational springs at the end nodes of beams. In the context of a simplified linear-elastic analysis, such springs are characterized by their initial rotational stiffness, which can be conveniently expressed in terms of the so-called *fixity factors* (see e.g., [5,6]), whose values range between 0 and 1, corresponding to the ideally pinned and rigid connection, respectively. Corrective matrices whose entries depend on dimensionless parameters e.g., the *fixity factors*, are often used to modify the classical stiffness matrix of the beam element with fixed ends (see e.g., [7,8]). Flexible connections affect the distribution of internal forces in the members of a frame structure [9]. The influence of semi-rigid connections on the design, analysis, and reliability of frame structures has been extensively studied in the literature (see e.g., [9–14]).

Experimental tests often show a large scatter of stiffness and capacity measures, even for the same type of connection. Indeed, the behaviour of connections is affected by several sources of uncertainty such as the weld quality, inaccuracies, or errors in the manufacturing process, etc. Therefore, a non-deterministic characterization of flexible joints is needed to obtain accurate predictions of the structural response.

Several studies in the literature have investigated the influence of uncertain properties of semi-rigid connections on the overall behaviour of frames using the classical probabilistic model (see e.g., [15–21]). As observed in Ref. [22], the common scenario is such that the structural design has to be completed before the types of connections are specified, and sometimes even before the steel fabricator has been appointed. This

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implies that the traditional probabilistic model may not be the most suitable choice to describe the scatter of the stiffness of semi-rigid connections. Indeed, in practical situations, available data may not be sufficient to provide an accurate probabilistic characterization of the uncertain parameters. Conversely, non-probabilistic approaches (see e. g., [23–26]), such as the interval model [27] or fuzzy sets theory [28], allow the designer to describe the uncertain properties of partially restrained connections relying on poor or vague information.

Over the last decade, the analysis of frame structures with uncertain semi-rigid connections based on non-probabilistic uncertainty models has attracted increasing research interest (see e.g., [22,29-32]). Tangaramvong et al. [29] developed a mathematical programming approach for obtaining the extreme responses of frames with semi-rigid connections characterized by an interval moment-rotation relationship. De Luca di Roseto et al. [22] presented a new performance-based fuzzy design procedure for steel moment-resisting frames where the non-deterministic behaviour of the connections has been taken into account by modelling their *fixity factors* as fuzzy variables with a triangular membership function. Pham et al. [30] developed a fuzzy finite element approach for the static analysis of functionally graded material (FGM) frame structures involving fuzzy semi-rigid connections and system parameters. The fuzzy static displacement responses were determined by a methodology based on the α-level optimization approach and first-order Taylor's approximation. This approach was extended to the fuzzy free vibration analysis of FGM frame structures by formulating a novel FGM structural member with semi-rigid connections based on the Timoshenko beam theory [31]. More recently, Pham and Truong [32] developed a novel method for the fuzzy analysis of nonlinear inelastic semi-rigid steel frames which is able to capture the variation of the load-carrying capacity due to fuzziness in the structural properties and applied loads.

Despite the above-mentioned progress in the application of fuzzy sets theory, much research effort is still needed to assess the effectiveness of non-probabilistic approaches in capturing the non-deterministic behaviour of semi-rigid frame structures. To enhance the application of non-deterministic approaches in the design practice, special attention should be devoted to the development of efficient uncertainty propagation strategies which rely on suitable extensions of classical structural analysis methods. In this context, the present study addresses the static analysis of steel frames with semi-rigid connections modelled as rotational springs, characterized by a linear-elastic behaviour with uncertain stiffness. Under the reasonable assumption that available data are limited, the interval model of uncertainty is adopted. Specifically, the uncertain stiffness of the generic rotational spring, expressed in terms of the associated *fixity factor*, is modelled as an interval variable within the framework of the *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*) [33]. The uncertain stiffness of the partially restrained connections is incorporated into the classical matrix stiffness method by interval extension [27]. The interval stiffness matrix and nodal force vector of the beam-type element are defined as explicit functions of the interval *fixity factors*. The lower bound (LB) and upper bound (UB) of the response are evaluated by applying a sensitivity-based procedure (see e. g., [34–38]). As a first step, this procedure involves the study of the sign of response sensitivities to predict the monotonic increasing or decreasing behaviour within a small range around the nominal values of the uncertain parameters. Based on the outcomes of this preliminary study, one can identify the combinations of the endpoints of the interval parameters, which yield accurate estimates of response bounds provided that the behaviour is monotonic over the assigned range of parameter values. Then, response bounds can be efficiently evaluated by performing only two deterministic analyses, one for each of the two sets of endpoints of the interval parameters determined by sensitivity analysis.

Single-bay one-storey and five-storey frames with uncertain semi-rigid beam-to-column connections are selected as case studies. The accuracy of the presented procedure is assessed by comparison with a time-consuming combinatorial procedure, known as *vertex method* [39], which yields the exact bounds of the response for monotonic problems.

The rest of the paper is organized as follows: in Section 2, the model of semi-rigid connections with interval stiffness is defined; Section 3 is devoted to the formulation of a novel interval matrix stiffness method for the static analysis of frames with partially restrained connections; in Section 4, a sensitivity-based procedure for the evaluation of the bounds of the interval response is developed; in Section 5, numerical results are presented and discussed; and some conclusions are drawn in Section 6.

2. Uncertain semi-rigid connections

2.1. Semi-rigid connections

Frame structures are conventionally analyzed assuming an ideal model of the connections, i.e. ideally pinned or perfectly rigid connections. However, it is widely recognized that the actual behaviour of the connections is semi-rigid. Neglecting the effects of axial and shear deformations compared to flexural ones, the rotational behaviour of connections can be described by their moment-rotation relationship $M - \varphi_r$, which relates the bending moment transmitted by the connection, M, to the relative rotation, φ_r, experienced by the connection due to its flexibility. Typical moment-rotation curves for a variety of commonly used semi-rigid connections are shown in Fig. 1.

Semi-rigid or partially restrained connections are usually incorporated into the design of frame structures by considering rotational springs ideally placed between beams and columns or at the base of columns (see Fig. 2).

Assuming a linearized model i.e., considering the first branch of the actual nonlinear moment-rotation curve (see Fig. 1), the j − th semi-rigid connection ($j = 1, 2$) of the h − th beam element (see Fig. 2) of a frame structure is characterized by a constant stiffness (the *initial stiffness*) $k_j^{(h)}$ such that the relationship between the bending moment at the j − th beam end $M_j^{(h)}$ and the relative rotation $\varphi_{rj}^{(h)}$ is given by:

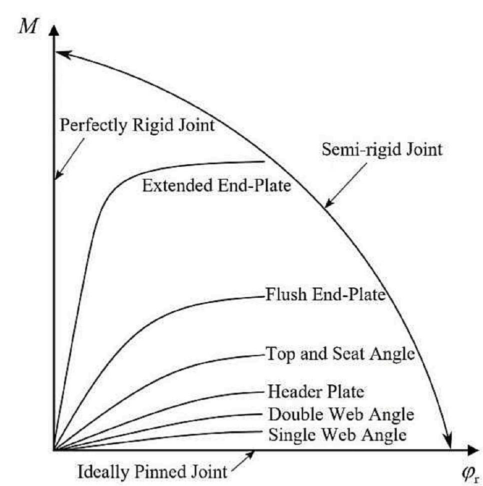

Fig. 1. Typical moment-rotation curves for various types of connections (adapted from [22]).

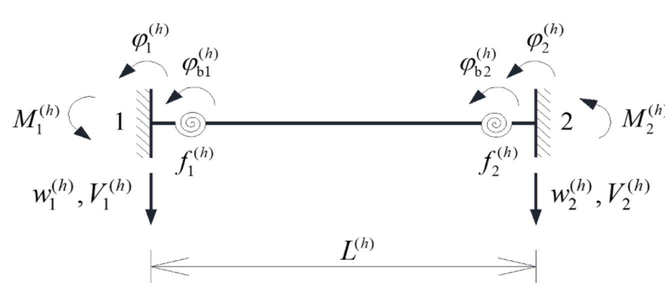

Fig. 2. Sketch of the frame element with semi-rigid connections.

$$M_j^{(h)} = -k_j^{(h)}\varphi_{rj}^{(h)}. \tag{1}$$

Due to the flexibility of the connection, the rotation of the nodal restraints can be expressed as $\varphi_j^{(h)} = \varphi_{bj}^{(h)} + \varphi_{rj}^{(h)}$ where $\varphi_{bj}^{(h)}$ is the rotation at the beam ends.

The constant stiffness of the rotational spring $k_j^{(h)}$ can be expressed as follows (see e.g., [5,6]):

$$k_j^{(h)}\left(f_j^{(h)}\right) = \frac{4E^{(h)}I^{(h)}}{L^{(h)}}\frac{3f_j^{(h)}}{4\left(1-f_j^{(h)}\right)} \tag{2}$$

where $I^{(h)}$, $L^{(h)}$, and $E^{(h)}$ are the moment of inertia, length, and Young's modulus of the member; $f_j^{(h)}$ is the dimensionless *fixity factor* which varies in the range [0, 1]. The ideally pinned and perfectly rigid connections are retrieved as limit cases when $f_j^{(h)} \to 0$ and $f_j^{(h)} \to 1$, respectively. For steel frame structures, the limit values of the *fixity factors* established by EC3 [3] for pinned and rigid joints are 0.14 and 0.89, respectively.

Based on Eq. (2), the rotational deformability of the semi-rigid connection can be defined as:

$$\rho_j^{(h)} = \frac{1}{k_j^{(h)}} = \frac{L^{(h)}}{4E^{(h)}I^{(h)}}\lambda_j^{(h)} \tag{3}$$

where

$$\lambda_j^{(h)} = \frac{4\left(1-f_j^{(h)}\right)}{3f_j^{(h)}} \tag{4}$$

is a dimensionless quantity such that $\lambda_j^{(h)} \to 0$ and $\lambda_j^{(h)} \to \infty$ in the limit cases of perfectly rigid ($f_j^{(h)} \to 1$) and ideally pinned ($f_j^{(h)} \to 0$) connections. For the limit values of the *fixity factors* established by EC3, $\lambda_j^{(h)} \in [0.16, 8.19]$.

2.2. Interval fixity factor

Connection rotational stiffness is affected by a high degree of uncertainty so that a non-deterministic model is needed to obtain accurate estimates of structural response. In the present study, uncertainty affecting the behaviour of semi-rigid connections is taken into account by modelling the *fixity factors* as interval variables [27] with given lower bound (LB) and upper bound (UB). Specifically, the *fixity factor* for the j-th connection of the h-th element of a frame is expressed as follows:

$$f_j^{(h)I} = f_{0j}^{(h)}\left(1+\alpha_j^{(h)I}\right) \tag{5}$$

where the apex denotes interval variables; $\alpha_j^{(h)I} = \left[\underline{\alpha}_j^{(h)}, \bar{\alpha}_j^{(h)}\right] \in \mathbb{IR}$ is a symmetric interval variable ($\underline{\alpha}_j^{(h)} = -\bar{\alpha}_j^{(h)}$) representing the dimensionless fluctuation of the *fixity factor* around the nominal value, $f_{0j}^{(h)}$; the symbols $\underline{\alpha}_j^{(h)}$ and $\bar{\alpha}_j^{(h)}$ denote the LB and UB of the interval variable; $\mathbb{IR}$ is the set of all real interval numbers. Following the *Improved Interval Analysis* via *Extra Unitary Interval* (*IIA* via *EUI*) [33], the symmetric interval variable $\alpha_j^{(h)I}$ can be expressed as:

$$\alpha_j^{(h)I} = \Delta\alpha_j^{(h)}\hat{e}_j^{(h)I} \tag{6}$$

where

$$\Delta\alpha_j^{(h)} = \frac{\Delta f_j^{(h)}}{f_{0j}^{(h)}} = \frac{\bar{\alpha}_j^{(h)} - \underline{\alpha}_j^{(h)}}{2} \tag{7}$$

is the normalized deviation amplitude which satisfies the condition $\Delta\alpha_j^{(h)} < 1$ in order to ensure positive values of the *fixity factor*; $\hat{e}_j^{(h)I} = [-1, 1]$ is the so-called *EUI* [33] associated with the fluctuation of the *fixity factor* of the $j-$th connection of the $h-$th element. In order to guarantee that the values of the *fixity factors* are always within the interval [0, 1], the normalized deviation amplitude should also satisfy the condition $\Delta\alpha_j^{(h)} < (1-f_{0j}^{(h)})/f_{0j}^{(h)}$.

Based on Eq. (5), the rotational stiffness and deformability of the j-th connection in Eqs. (2) and (3) have an interval nature and are functions of the interval variable $\alpha_j^{(h)I}$:

$$k_j^{(h)I} \equiv k_j^{(h)}\left(\alpha_j^{(h)I}\right) = \frac{4E^{(h)}I^{(h)}}{L^{(h)}}\frac{1}{\lambda_j\left(\alpha_j^{(h)I}\right)};$$
$$\rho_j^{(h)I} \equiv \rho_j^{(h)}\left(\alpha_j^{(h)I}\right) = \frac{1}{k_j^{(h)I}} = \frac{L^{(h)}}{4E^{(h)}I^{(h)}}\lambda_j\left(\alpha_j^{(h)I}\right) \tag{8a,b}$$

where

$$\lambda_j^{(h)I} \equiv \lambda_j^{(h)}\left(\alpha_j^{(h)I}\right) = \frac{4\left(1-f_j^{(h)I}\right)}{3f_j^{(h)I}} = \frac{4\left[1-f_{0j}^{(h)}\left(1+\alpha_j^{(h)I}\right)\right]}{3f_{0j}^{(h)}\left(1+\alpha_j^{(h)I}\right)} \tag{9}$$

is the rotational spring deformability normalized with respect to $L^{(h)}/(4E^{(h)}I^{(h)})$.

Fig. 3 shows the LB, UB, and nominal value of the interval dimensionless deformability $\lambda_j^{(h)I}$ (Eq. (9)), for $\Delta\alpha_j^{(h)} = 0.1$, versus the nominal value of the *fixity factor* within the range prescribed by EC3 [3] i.e., $f_{0j}^{(h)} \in [f_{0j,\min}^{(h)}, f_{0j,\max}^{(h)}] = [0.14, 0.89]$. The extreme values of the nominal dimensionless deformability, $\lambda_{0j,\max}^{(h)} = 8.19$ and $\lambda_{0j,\min}^{(h)} = 0.16$, corresponding to $f_{0j,\min}^{(h)}$ and $f_{0j,\max}^{(h)}$, respectively, are also displayed. It can be observed that the nominal value of the deformability tends to zero and infinity, respectively, when $f_{0j}^{(h)} \to 1$ and $f_{0j}^{(h)} \to 0$, i.e. when the reference connection is perfectly rigid and ideally pinned. Due to the uncertain initial rotational stiffness, the dimensionless flexibility of the actual connection ranges between a LB and an UB, $\underline{\lambda}_j^{(h)}$ and $\bar{\lambda}_j^{(h)}$, which correspond to the UB and LB of the *fixity factor* i.e., $\bar{f}_j^{(h)} = f_{0j}^{(h)}(1+\Delta\alpha_j^{(h)})$ and $\underline{f}_j^{(h)} = f_{0j}^{(h)}(1-\Delta\alpha_j^{(h)})$.

3. Novel interval matrix stiffness method

3.1. Beam element with uncertain semi-rigid connections

Let us consider the $h-$th beam element of a frame structure with

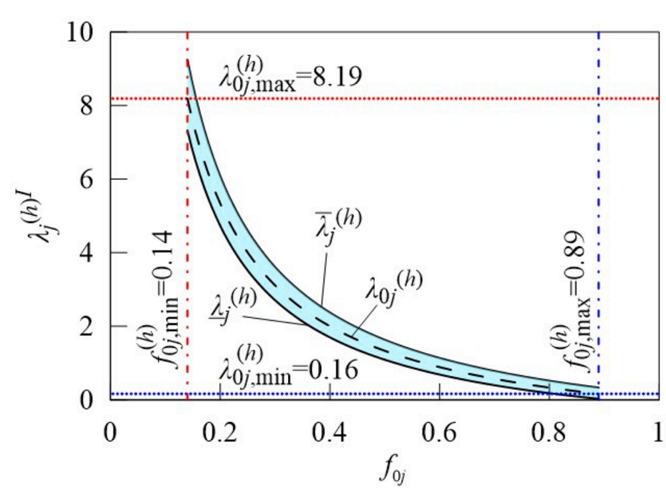

Fig. 3. Interval dimensionless deformability of the $j-$th semi-rigid connection ($\Delta\alpha_j^{(h)} = 0.1$) versus the nominal value of the *fixity factor*.

semi-rigid restraints at the end nodes 1 and 2 described by rotational springs (see Fig. 2) with uncertain stiffness depending on the interval *fixity factors* $f_j^{(h)I}$ $(j = 1, 2)$, defined in Eq. (5). Due to the uncertainty affecting semi-rigid connections, all response quantities have an interval nature.

By interval extension of Eq. (1), the interval bending moment at the beam ends depends on the interval relative rotation due to the flexibility of the rotational springs through the following relationship:

$$M_j^{(h)I} = -k_j^{(h)I}\varphi_{rj}^{(h)I} = -\frac{\varphi_{rj}^{(h)I}}{\rho_j^{(h)I}}. \tag{10}$$

In the previous equation, $\varphi_{rj}^{(h)I} = \varphi_j^{(h)I} - \varphi_{bj}^{(h)I}$ is the relative rotation, with $\varphi_j^{(h)I}$ and $\varphi_{bj}^{(h)I}$ $(j = 1, 2)$ denoting the interval rotations at the nodal restraints and the beam ends (see Fig. 2); $k_j^{(h)I}$ and $\rho_j^{(h)I}$ are the interval rotational stiffness and deformability defined by Eqs. (8a,b). Based on Eq. (10), the moment capacity of the connection ranges between an UB and a LB depending on the value of the uncertain initial stiffness.

Let $\mathbf{u}^{(h)I} = \left[w_1^{(h)I} \quad \varphi_1^{(h)I} \quad w_2^{(h)I} \quad \varphi_2^{(h)I} \right]^{\mathrm{T}}$ be the interval vector which collects the nodal displacements and rotations describing the flexural behaviour of the $h-$ th beam element with partially restrained nodes and $\mathbf{q}^{(h)I} = \left[V_1^{(h)I} \quad M_1^{(h)I} \quad V_2^{(h)I} \quad M_2^{(h)I} \right]^{\mathrm{T}}$ the interval vector listing the transversal forces and bending moment reactions at nodes (see Fig. 2). Furthermore, the interval fluctuations of the *fixity factors* $f_j^{(h)I} \equiv f_j^{(h)}(\alpha_j^{(h)I})$, $(j = 1, 2)$, around the nominal values are collected into the interval vector $\boldsymbol{\alpha}^{(h)I} \equiv \left[\alpha_1^{(h)I} \quad \alpha_2^{(h)I} \right]^{\mathrm{T}}$.

The element stiffness matrix in the local reference system can be defined by interval extension of the deterministic stiffness matrix available in the relevant literature (see e.g., [8]):

$$\begin{aligned}\mathbf{k}^{(h)I} &\equiv \mathbf{k}^{(h)}\left(\boldsymbol{\alpha}^{(h)I}\right)\\ &= \frac{E^{(h)}I^{(h)}}{L^{(h)3}}\begin{bmatrix} 12g_{11}^{(h)I} & -6L^{(h)}g_{12}^{(h)I} & -12g_{13}^{(h)I} & -6L^{(h)}g_{14}^{(h)I} \\ -6L^{(h)}g_{21}^{(h)I} & 4L^{(h)2}g_{22}^{(h)I} & 6L^{(h)}g_{23}^{(h)I} & 2L^{(h)2}g_{24}^{(h)I} \\ -12g_{31}^{(h)I} & 6L^{(h)}g_{32}^{(h)I} & 12g_{33}^{(h)I} & 6L^{(h)}g_{34}^{(h)I} \\ -6L^{(h)}g_{41}^{(h)I} & 2L^{(h)2}g_{42}^{(h)I} & 6L^{(h)}g_{43}^{(h)I} & 4L^{(h)2}g_{44}^{(h)I} \end{bmatrix}\end{aligned} \tag{11}$$

where the entries of the classical stiffness matrix of the beam element with perfectly fixed end nodes are multiplied by the following dimensionless interval functions:

$$\begin{aligned}
g_{11}^{(h)I} &= g_{33}^{(h)I} = g_{13}^{(h)I} = g_{31}^{(h)I} = \frac{1}{\Psi^I}\left\{1 + \frac{1}{4}\left(\lambda_1^{(h)I} + \lambda_2^{(h)I}\right)\right\};\\
g_{12}^{(h)I} &= g_{21}^{(h)I} = g_{32}^{(h)I} = g_{23}^{(h)I} = \frac{1}{\Psi^I}\left(1 + \frac{1}{2}\lambda_2^{(h)I}\right);\\
g_{41}^{(h)I} &= g_{14}^{(h)I} = g_{34}^{(h)I} = g_{43}^{(h)I} = \frac{1}{\Psi^I}\left(1 + \frac{1}{2}\lambda_1^{(h)I}\right);\\
g_{22}^{(h)I} &= \frac{1}{\Psi^I}\left(1 + \frac{3}{4}\lambda_2^{(h)I}\right);\\
g_{44}^{(h)I} &= \frac{1}{\Psi^I}\left(1 + \frac{3}{4}\lambda_1^{(h)I}\right);\\
g_{24}^{(h)I} &= g_{42}^{(h)I} = \frac{1}{\Psi^I}
\end{aligned} \tag{12a-f}$$

with

$$\Psi^I \equiv \Psi\left(\boldsymbol{\alpha}^{(h)I}\right) = 1 + \left(\lambda_1^{(h)I} + \lambda_2^{(h)I}\right) + \frac{3}{4}\lambda_1^{(h)I}\lambda_2^{(h)I}. \tag{13}$$

It can be readily verified that the stiffness matrix of the beam with perfectly rigid and ideally hinged nodes can be retrieved from Eq. (11) as the limit case when $\lambda_j^{(h)} \to 0$ and $\lambda_j^{(h)} \to \infty$ $(j = 1, 2)$.

In the context of the matrix stiffness method, the nodal actions due to external loads acting along the beam axis need to be evaluated. For instance, in the case of a beam subjected to a uniformly distributed transversal load of intensity q, the interval vector collecting nodal actions takes the following form (see e.g., [8]):

$$\mathbf{f}^{(h)}\left(\boldsymbol{\alpha}^{(h)I}\right) \equiv \mathbf{f}^{(h)I} = -\begin{bmatrix} V_1^{q(h)}\left(\boldsymbol{\alpha}^{(h)I}\right) \\ M_1^{q(h)}\left(\boldsymbol{\alpha}^{(h)I}\right) \\ V_2^{q(h)}\left(\boldsymbol{\alpha}^{(h)I}\right) \\ M_2^{q(h)}\left(\boldsymbol{\alpha}^{(h)I}\right) \end{bmatrix} = -\frac{qL^{(h)}}{12}\begin{bmatrix} -6v_1^{(h)I} \\ L^{(h)}m_1^{(h)I} \\ -6v_2^{(h)I} \\ -L^{(h)}m_2^{(h)I} \end{bmatrix} \tag{14}$$

where $V_j^{q(h)}(\boldsymbol{\alpha}^{(h)I})$ and $M_j^{q(h)}(\boldsymbol{\alpha}^{(h)I})$, $(j = 1, 2)$, are the transversal and moment reactions at the beam end nodes. In Eq. (14), the entries pertaining to the beam with perfectly fixed end nodes are multiplied by the following dimensionless interval functions:

$$\begin{aligned}
v_1^{(h)I} &= \frac{1}{\Psi^I}\left(\eta^I + \frac{1}{2}\lambda_2^{(h)I}\right);\\
m_1^{(h)I} &= \frac{1}{\Psi^I}\left(1 + \frac{3}{2}\lambda_2^{(h)I}\right);\\
v_2^{(h)I} &= \frac{1}{\Psi^I}\left(\eta^I + \frac{1}{2}\lambda_1^{(h)I}\right);\\
m_2^{(h)I} &= \frac{1}{\Psi^I}\left(1 + \frac{3}{2}\lambda_1^{(h)I}\right)
\end{aligned} \tag{15a-d}$$

with:

$$\eta\left(\boldsymbol{\alpha}^{(h)I}\right) = 1 + \frac{3}{4}\left[\lambda_1^{(h)I} + \lambda_2^{(h)I}\right] + \frac{3}{4}\lambda_1^{(h)I}\lambda_2^{(h)I}. \tag{16}$$

It is worth remarking that the functions in Eqs. (12), (13), (15) and (16) depend only on the interval fluctuations collected into the interval vector $\boldsymbol{\alpha}^{(h)I} \equiv \left[\alpha_1^{(h)I} \quad \alpha_2^{(h)I} \right]^{\mathrm{T}}$ and on the nominal value of the interval *fixity factors* $f_{0j}^{(h)}$, $(j = 1, 2)$.

3.2. *Interval equilibrium equations*

By performing the standard coordinate transformation from the local to the global reference system and assembly procedure, the following set of linear interval algebraic equations governing the equilibrium of the frame structure is obtained:

$$\mathbf{K}\left(\boldsymbol{\alpha}^I\right)\mathbf{U}\left(\boldsymbol{\alpha}^I\right) = \mathbf{F}\left(\boldsymbol{\alpha}^I\right) \tag{17}$$

where $\mathbf{U}(\boldsymbol{\alpha}^I)$ is the interval n-vector of global nodal displacements; $\mathbf{K}(\boldsymbol{\alpha}^I)$ is the $(n \times n)$ interval global stiffness matrix, formally expressed as follows:

$$\mathbf{K}\left(\boldsymbol{\alpha}^I\right) = \sum_{h=1}^{N_e^{(ID)}} \mathbf{L}^{(h)\mathrm{T}}\mathbf{k}^{(h)}\mathbf{L}^{(h)} + \sum_{h=1}^{N_e^{(PR)}} \mathbf{L}^{(h)\mathrm{T}}\mathbf{k}^{(h)}\left(\boldsymbol{\alpha}^{(h)I}\right)\mathbf{L}^{(h)}, \tag{18}$$

and $\mathbf{F}(\boldsymbol{\alpha}^I)$ is the interval n-vector collecting the nodal global external loads, given by:

$$\mathbf{F}\left(\boldsymbol{\alpha}^I\right) = \sum_{h=1}^{N_e^{(ID)}} \mathbf{L}^{(h)\mathrm{T}}\mathbf{f}^{(h)} + \sum_{h=1}^{N_e^{(PR)}} \mathbf{L}^{(h)\mathrm{T}}\mathbf{f}^{(h)}\left(\boldsymbol{\alpha}^{(h)I}\right). \tag{19}$$

For notation conciseness, it is assumed that the matrix $\mathbf{L}^{(h)}$ accounts for both connectivity and coordinate transformation of the element properties from the local to the global reference system. In the previous equations, $N_e^{(ID)}$ and $N_e^{(PR)}$ indicate the number of elements with *ideal* (*ID*) and *partially restrained* (*PR*) connections; $\boldsymbol{\alpha}^I$ is the vector collecting the interval fluctuations of the *fixity factors* of all the semi-rigid connections around the nominal values:

$$\boldsymbol{\alpha}^I = \left[\left(\boldsymbol{\alpha}^{(1)I}\right)^{\mathrm{T}} \quad \left(\boldsymbol{\alpha}^{(2)I}\right)^{\mathrm{T}} \quad \cdots \quad \left(\boldsymbol{\alpha}^{(N_e^{(PR)})I}\right)^{\mathrm{T}} \right]^{\mathrm{T}} \tag{20}$$

where $\boldsymbol{\alpha}^{(h)I} \equiv \left[\alpha_1^{(h)I} \quad \alpha_2^{(h)I} \right]^{\mathrm{T}}$, $(h = 1, 2, \ldots, N_e^{(PR)})$.

It is worth noting that in Eqs. (18) and (19) the contributions associated with elements having ideal and partially restrained connections are separated in order to emphasize that the uncertain rotational stiffness of the joints only affects the stiffness matrix and the nodal force vector of the beam elements with flexible restraints.

4. Sensitivity-based procedure

4.1. Bounds of interval generalized displacements

All possible solutions of the interval global equilibrium (see Eq. (17)), obtained as the uncertain parameters vary within their intervals, are contained in a solution set, Σ, formally defined as:

$$\Sigma = \left\{ \mathbf{U} \in \mathbb{R}^n | \mathbf{K}(\boldsymbol{\alpha})\mathbf{U}(\boldsymbol{\alpha}) = \mathbf{F}(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \boldsymbol{\alpha}^I \right\} \tag{21}$$

where n denotes the order of the global displacement vector $\mathbf{U}(\boldsymbol{\alpha})$. The exact evaluation of the solution set is very difficult since it is typically described by a complicated region in the output space (see e.g., [25]). To address this issue, in the framework of interval structural analysis, the interval displacement vector $\mathbf{U}^I$, containing the solution set Σ, which has the narrowest interval components, is commonly determined. In the literature, several strategies have been proposed to evaluate the LB and UB of the interval displacement vector $\mathbf{U}^I$ (see e.g., [25,26,34-36,40,41]).

In the context of linear interval structural analysis, the monotonic behaviour of the response with respect to the uncertain parameters is often exploited to estimate its bounds. When the response is a monotonic function of the interval parameters, the exact bounds can be evaluated by applying a time-consuming combinatorial procedure, known as *vertex method* [39]. The key idea of this procedure is to seek the bounds of the response among the 2^{N_u} solutions pertaining to all possible combinations of the endpoints of the N_u uncertain parameters. The computational burden of the *vertex method* becomes prohibitive as the number of uncertain parameters increases, so that it is commonly used to derive benchmark solutions for validation purposes.

In the present study, a sensitivity-based procedure (see e.g., [34–38]) is applied for the evaluation of the bounds of the response of frame structures with semi-rigid connections, characterized by interval initial rotational stiffness. Without loss of generality, elements with partially restrained connections at both ends will be considered so that the governing equations involve $N_u = 2N_e^{(PR)}$ uncertain parameters.

The sensitivity of the interval global displacement vector with respect to the $i-$ th uncertain parameter $\alpha_i = \alpha_p^{(h)} \in \alpha_i^I = \left[\underline{\alpha}_i, \bar{\alpha}_i\right]$, $(i = 2h - 2 + p;\ p = 1,\ 2;\ h = 1, 2, \ldots, N_e^{(PR)})$, can be obtained by direct differentiation of the equilibrium equations in Eq. (17) i.e.:

$$\mathbf{s}_{\mathbf{U},i} = \left.\frac{\partial \mathbf{U}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \mathbf{K}_0^{-1}(\mathbf{F}_i - \mathbf{K}_i\mathbf{U}_0) \tag{22}$$

where

$$\mathbf{K}_0 = \mathbf{K}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\mathbf{0}};\ \mathbf{F}_0 = \mathbf{F}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\mathbf{0}};\ \mathbf{U}_0 = \mathbf{K}_0^{-1}\mathbf{F}_0 \tag{23a-c}$$

are the nominal global stiffness matrix, nodal force vector and displacement vector.

Furthermore, the $(n \times n)$ matrix $\mathbf{K}_i$ in Eq. (22) is defined as:

$$\mathbf{K}_i = \left.\frac{\partial \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left.\frac{\partial \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_p^{(h)}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \mathbf{L}^{(h)\mathrm{T}}\mathbf{s}_{\mathbf{k},p}^{(h)}\mathbf{L}^{(h)} \tag{24}$$

where $\mathbf{s}_{\mathbf{k},p}^{(h)}$ is the sensitivity of the stiffness matrix of the $h-$ th element to the parameter $\alpha_i = \alpha_p^{(h)}$ which can be evaluated by direct differentiation of Eq. (11):

$$\begin{aligned}\mathbf{s}_{\mathbf{k},p}^{(h)} &= \left.\frac{\partial \mathbf{k}^{(h)}\left(\boldsymbol{\alpha}^{(h)}\right)}{\partial \alpha_p^{(h)}}\right|_{\boldsymbol{\alpha}^{(h)}=\mathbf{0}} \\ &= \frac{E^{(h)}I^{(h)}}{L^{(h)3}}\begin{bmatrix} 12g_{11,p}^{(h)} & -6L^{(h)}g_{12,p}^{(h)} & -12g_{13,p}^{(h)} & -6L^{(h)}g_{14,p}^{(h)} \\ -6L^{(h)}g_{21,p}^{(h)} & 4L^{(h)2}g_{22,p}^{(h)} & 6L^{(h)}g_{23,p}^{(h)} & 2L^{(h)2}g_{24,p}^{(h)} \\ -12g_{31,p}^{(h)} & 6L^{(h)}g_{32,p}^{(h)} & 12g_{33,p}^{(h)} & 6L^{(h)}g_{34,p}^{(h)} \\ -6L^{(h)}g_{41,p}^{(h)} & 2L^{(h)2}g_{42,p}^{(h)} & 6L^{(h)}g_{43,p}^{(h)} & 4L^{(h)2}g_{44,p}^{(h)} \end{bmatrix}\end{aligned} \tag{25}$$

with

$$g_{rs,p}^{(h)} = \left.\frac{\partial g_{rs}^{(h)}\left(\boldsymbol{\alpha}^{(h)}\right)}{\partial \alpha_p^{(h)}}\right|_{\boldsymbol{\alpha}^{(h)}=\mathbf{0}} \tag{26}$$

where $g_{rs}^{(h)}(\boldsymbol{\alpha}^{(h)})$, $(r,s = 1,2,3,4;\ h = 1,2,\cdots,N_e^{(PR)})$, are the dimensionless interval functions defined in Eqs. (12a-f). It is worth remarking that such functions depend on the nominal value of the *fixity factors* and the interval fluctuations. It follows that, for elements with semi-rigid connections of the same reference type i.e. characterized by the same nominal value of the *fixity factor*, the partial derivatives in Eq. (26) need to be evaluated only once. This provides substantial computational savings.

The n-vector $\mathbf{F}_i$ in Eq. (22) collects the sensitivities of the global nodal forces to the $i-$ th uncertain parameter $\alpha_i = \alpha_p^{(h)} \in \alpha_i^I = \left[\underline{\alpha}_i, \bar{\alpha}_i\right]$, $(i = 2h - 2 + p;\ p = 1,\ 2;\ h = 1, 2, \ldots, N_e^{(PR)})$, i.e.:

$$\mathbf{F}_i = \left.\frac{\partial \mathbf{F}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left.\frac{\partial \mathbf{F}(\boldsymbol{\alpha})}{\partial \alpha_p^{(h)}}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \mathbf{L}^{(h)\mathrm{T}}\mathbf{s}_{\mathbf{f},p}^{(h)} \tag{27}$$

where $\mathbf{s}_{\mathbf{f},p}^{(h)}$ is the sensitivity vector of the nodal forces of the $h-$ th element to the parameter $\alpha_i = \alpha_p^{(h)}$. In the case of a beam element subjected to a uniformly distributed transversal load of intensity q, the vector $\mathbf{s}_{\mathbf{f},p}^{(h)}$ can be obtained by direct differentiation of the vector $\mathbf{f}^{(h)}(\boldsymbol{\alpha}^{(h)})$ defined in Eq. (14):

$$\mathbf{s}_{\mathbf{f},p}^{(h)} = \left.\frac{\partial \mathbf{f}^{(h)}\left(\boldsymbol{\alpha}^{(h)}\right)}{\partial \alpha_p^{(h)}}\right|_{\boldsymbol{\alpha}^{(h)}=\mathbf{0}} = -\begin{bmatrix} V_{1,p}^{q(h)} \\ M_{1,p}^{q(h)} \\ V_{2,p}^{q(h)} \\ M_{2,p}^{q(h)} \end{bmatrix} \tag{28}$$

with

$$V_{r,p}^{q(h)} = \left.\frac{\partial V_r^{q(h)}\left(\boldsymbol{\alpha}^{(h)}\right)}{\partial \alpha_p^{(h)}}\right|_{\boldsymbol{\alpha}^{(h)}=\mathbf{0}};\qquad M_{r,p}^{q(h)} = \left.\frac{\partial M_r^{q(h)}\left(\boldsymbol{\alpha}^{(h)}\right)}{\partial \alpha_p^{(h)}}\right|_{\boldsymbol{\alpha}^{(h)}=\mathbf{0}} \tag{29a,b}$$

where $r = 1,\ 2;\ p = 1,\ 2;\ V_r^{q(h)}(\boldsymbol{\alpha}^{(h)})$ and $M_r^{q(h)}(\boldsymbol{\alpha}^{(h)})$, $(r = 1,\ 2)$, are the transversal forces and bending moments at beam end nodes defined in Eq.(14).

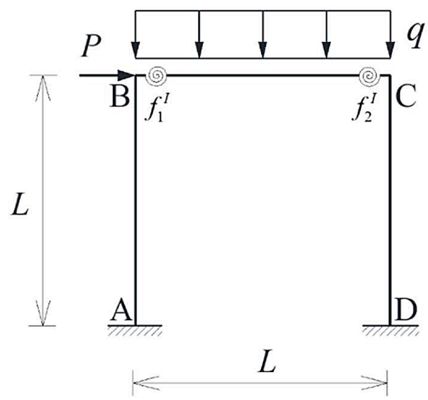

Fig. 4. One-storey frame with semi-rigid beam-to-column connections.

By examining the sign of the $i-$ th sensitivity of the $j-$ th displacement component, $s_{U_j,i}$, the monotonic increasing or decreasing behaviour of U_j^I for small variations of the $i-$ th uncertain parameter around the nominal value can be predicted. Hence, the combinations of the endpoints of the uncertain parameters, $\alpha_{(j)i}^{(\mathrm{LB})}$ and $\alpha_{(j)i}^{(\mathrm{UB})}$, which yield accurate estimates of the LB and UB of U_j^I can be determined as follows:

$$\begin{aligned}
&\text{if} \quad s_{U_j,i} > 0, \quad \text{then} \quad \alpha_{(j)i}^{(\mathrm{UB})} = \overline{\alpha}_i, \quad \alpha_{(j),i}^{(\mathrm{LB})} = \underline{\alpha}_i; \\
&\text{if} \quad s_{U_j,i} < 0, \quad \text{then} \quad \alpha_{(j)i}^{(\mathrm{UB})} = \underline{\alpha}_i, \quad \alpha_{(j)i}^{(\mathrm{LB})} = \overline{\alpha}_i, \\
&\qquad (j = 1, 2, \cdots, n; i = 1, 2, \cdots, N_u).
\end{aligned} \tag{30a,b}$$

The parameters $\alpha_{(j)i}^{(\mathrm{LB})}$ and $\alpha_{(j)i}^{(\mathrm{UB})}$ can be collected into the following vectors:

$$\begin{aligned}
\boldsymbol{\alpha}_{(j)}^{(\mathrm{LB})} &= \left[\alpha_{(j)1}^{(\mathrm{LB})} \quad \alpha_{(j)2}^{(\mathrm{LB})} \quad \cdots \quad \alpha_{(j)N_u}^{(\mathrm{LB})} \right]^{\mathrm{T}}; \\
\boldsymbol{\alpha}_{(j)}^{(\mathrm{UB})} &= \left[\alpha_{(j)1}^{(\mathrm{UB})} \quad \alpha_{(j)2}^{(\mathrm{UB})} \quad \cdots \quad \alpha_{(j)N_u}^{(\mathrm{UB})} \right]^{\mathrm{T}}, \ (j = 1, 2, \cdots, n).
\end{aligned} \tag{31a,b}$$

The LB and UB of the $j-$ th interval displacement component U_j^I can be obtained by solving the equilibrium (see Eq. (17)) for the two sets of parameters, $\boldsymbol{\alpha}_{(j)}^{(\mathrm{LB})}$ and $\boldsymbol{\alpha}_{(j)}^{(\mathrm{UB})}$, identified by sensitivity analysis, i.e.:

$$\begin{aligned}
\underline{U}_j &= \left\{ \mathbf{U}\left(\boldsymbol{\alpha}_{(j)}^{(\mathrm{LB})}\right) \right\}_j = \left\{ \mathbf{K}^{-1}\left(\boldsymbol{\alpha}_{(j)}^{(\mathrm{LB})}\right) \mathbf{F}\left(\boldsymbol{\alpha}_{(j)}^{(\mathrm{LB})}\right) \right\}_j; \\
\overline{U}_j &= \left\{ \mathbf{U}\left(\boldsymbol{\alpha}_{(j)}^{(\mathrm{UB})}\right) \right\}_j = \left\{ \mathbf{K}^{-1}\left(\boldsymbol{\alpha}_{(j)}^{(\mathrm{UB})}\right) \mathbf{F}\left(\boldsymbol{\alpha}_{(j)}^{(\mathrm{UB})}\right) \right\}_j
\end{aligned} \tag{32a,b}$$

where $\{ \cdot \}_j$ means the $j-$ th component of the vector between curly brackets.

4.2. Bounds of interval nodal forces

The sensitivity-based procedure outlined in the previous sub-section can also be applied to evaluate the bounds of the transversal forces and bending moments at the end nodes of the $h-$ th beam element, collected into the vector $\mathbf{q}^{(h)I} \equiv \mathbf{q}^{(h)}(\boldsymbol{\alpha}^I) = \left[V_1^{(h)I} \quad M_1^{(h)I} \quad V_2^{(h)I} \quad M_2^{(h)I} \right]^{\mathrm{T}}$. For a beam element with semi-rigid connections, such a vector reads:

$$\mathbf{q}^{(h)}(\boldsymbol{\alpha}^I) = \mathbf{k}^{(h)}(\boldsymbol{\alpha}^{(h)I}) \mathbf{L}^{(h)} \mathbf{U}(\boldsymbol{\alpha}^I) - \mathbf{f}^{(h)}(\boldsymbol{\alpha}^{(h)I}) \tag{33}$$

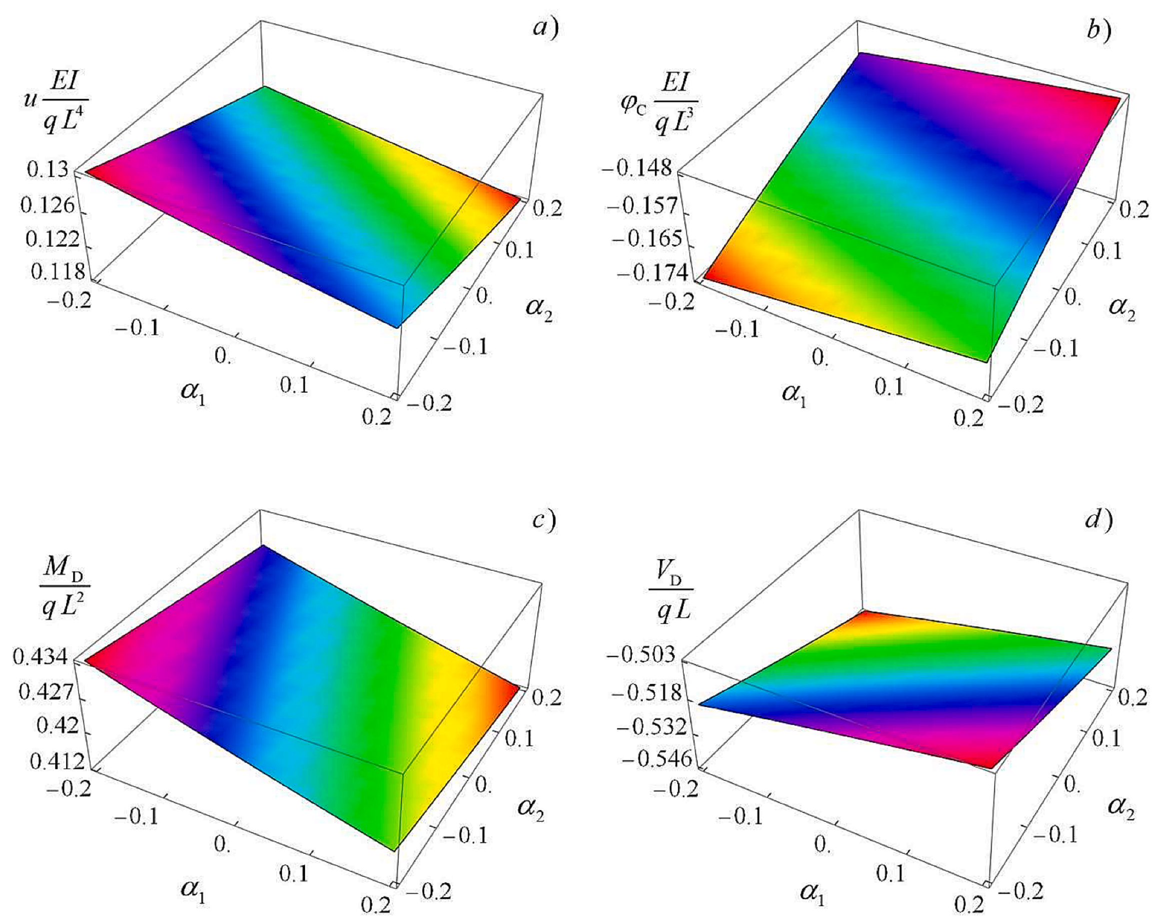

Fig. 5. Normalized responses of the one-storey frame versus the dimensionless fluctuations, α_1 and α_2, of the interval *fixity factors: a*) horizontal displacement; *b*) rotation of node C; *c*) bending moment at node D; *d*) transversal force at node D ($f_0 = 0.16$).

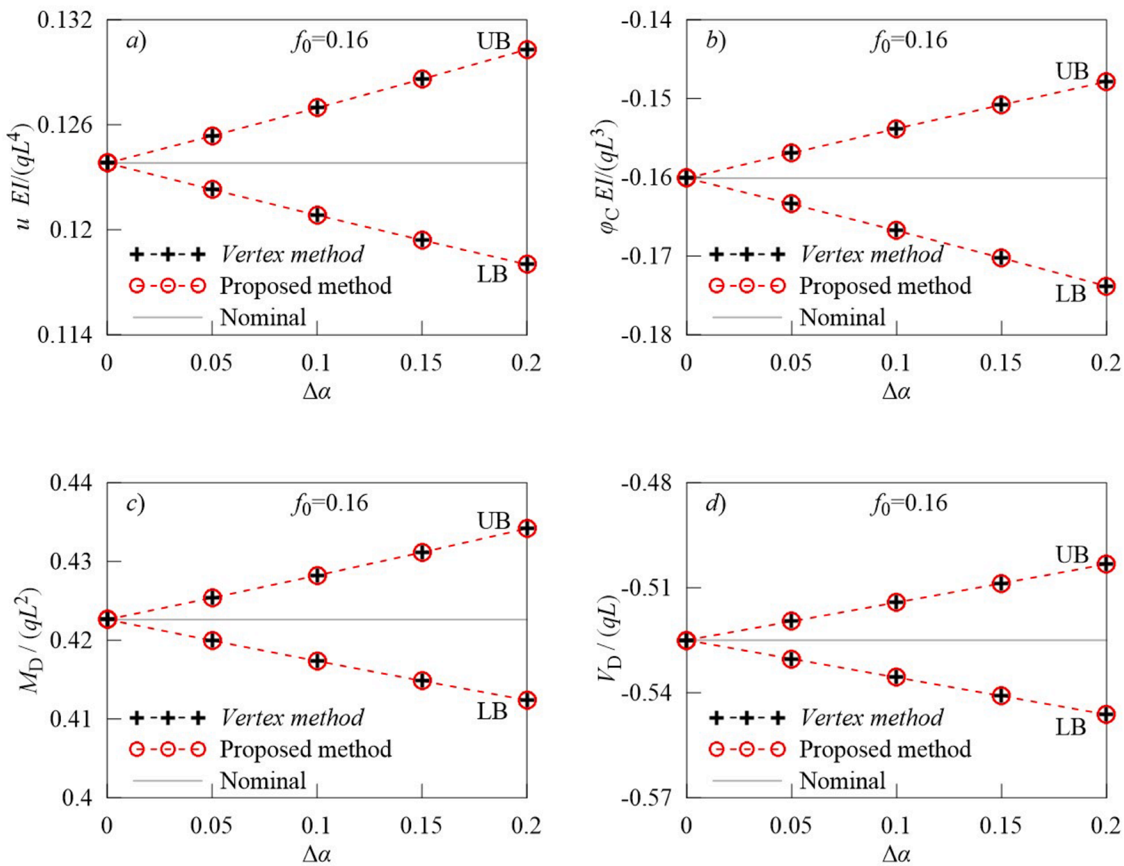

Fig. 6. Bounds of the normalized responses of the one-storey frame evaluated by the proposed sensitivity-based procedure and the *vertex method* along with the nominal solutions versus the normalized deviation amplitude of the uncertain parameters: *a*) horizontal displacement; *b*) rotation at node C; *c*) bending moment at node D; *d*) shear force at node D ($f_0 = 0.16$).

where $\mathbf{k}^{(h)}(\boldsymbol{\alpha}^{(h)I})$ is the interval element stiffness matrix (Eq. (11)) and $\mathbf{f}^{(h)}(\boldsymbol{\alpha}^{(h)I})$ is the vector collecting nodal actions due to external loads (see e.g., Eq. (14)).

The sensitivity of $\mathbf{q}^{(h)}(\boldsymbol{\alpha}^I)$ to the $i-$ th uncertain parameter $\alpha_i \in \alpha_i^I = \left[\underline{\alpha}_i, \bar{\alpha}_i\right]$ can be derived by direct differentiation of the previous equation, i.e.:

$$\mathbf{s}_{\mathbf{q},i}^{(h)} = \left.\frac{\partial \mathbf{q}^{(h)}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \mathbf{s}_{\mathbf{k},i}^{(h)}\mathbf{L}^{(h)}\mathbf{U}_0 + \mathbf{k}_0^{(h)}\mathbf{L}^{(h)}\mathbf{s}_{\mathbf{U},i} - \mathbf{s}_{\mathbf{f},i}^{(h)} \tag{34}$$

where $\mathbf{s}_{\mathbf{k},i}^{(h)}$, $\mathbf{s}_{\mathbf{U},i}$ and $\mathbf{s}_{\mathbf{f},i}^{(h)}$ are the $i-$ th sensitivities of the stiffness matrix of the h $-$ th element (see Eq. (25)), of the global nodal displacement vector (see Eq. (22)) and of the nodal actions due to external loads (see Eq. (28)); $\mathbf{k}_0^{(h)}$ and $\mathbf{U}_0$ are the nominal element stiffness matrix and nominal global displacement vector.

In the case of beam elements with ideal connections, the vector collecting transversal forces and bending moments at beam end nodes takes the following form:

$$\mathbf{q}^{(h)}\left(\boldsymbol{\alpha}^I\right) = \mathbf{k}^{(h)}\mathbf{L}^{(h)}\mathbf{U}\left(\boldsymbol{\alpha}^I\right) - \mathbf{f}^{(h)} \tag{35}$$

where the classical element stiffness matrix and force vector are involved, and all the quantities on the right-hand side are deterministic except the global displacement vector. It follows that the sensitivity to the $i-$ th uncertain parameter $\alpha_i \in \alpha_i^I = \left[\underline{\alpha}_i, \bar{\alpha}_i\right]$ reads:

$$\mathbf{s}_{\mathbf{q},i}^{(h)} = \left.\frac{\partial \mathbf{q}^{(h)}(\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\mathbf{0}} = \mathbf{k}^{(h)}\mathbf{L}^{(h)}\mathbf{s}_{\mathbf{U},i}. \tag{36}$$

The combinations of the endpoints of the uncertain parameters which yield accurate estimates of the LB and UB of the $j-$ th component of the force vector q_j^I can be determined by examining the sign of sensitivities, as outlined in Eq. (30) for the nodal displacements.

Compared with previous developments relying on the monotonicity assumption (see e.g., [34,35]), this study presents the formulation of a novel interval matrix stiffness method based on the *IIA* via *EUI* [33] and highlights the potential of the sensitivity-based approach when response sensitivities can be evaluated by direct differentiation. In this regard, both the implementation and computational efficiency of the method benefit from the explicit dependence of the interval element stiffness matrix and force vector on the interval *fixity factors*. Furthermore, the formulation includes the evaluation of the bounds of nodal forces which is quite a challenging task in the context of interval structural analysis due to overestimation issues.

It is worth remarking that the applicability of the presented procedure is not restricted to steel frame structures with uncertain semi-rigid connections. Different materials as well as other sources of uncertainty e.g., material and geometrical properties of the members or applied loads, can be considered. Within the more general framework of the interval finite element formulation (see e.g., [25,26,36,40,41]), the proposed approach can be applied to arbitrary linear discretized structures, provided that the response is a monotonic function of the interval parameters.

5. Numerical applications

5.1. One-storey frame with semi-rigid beam-to-column connections

The single-bay one-storey steel frame with semi-rigid beam-to-

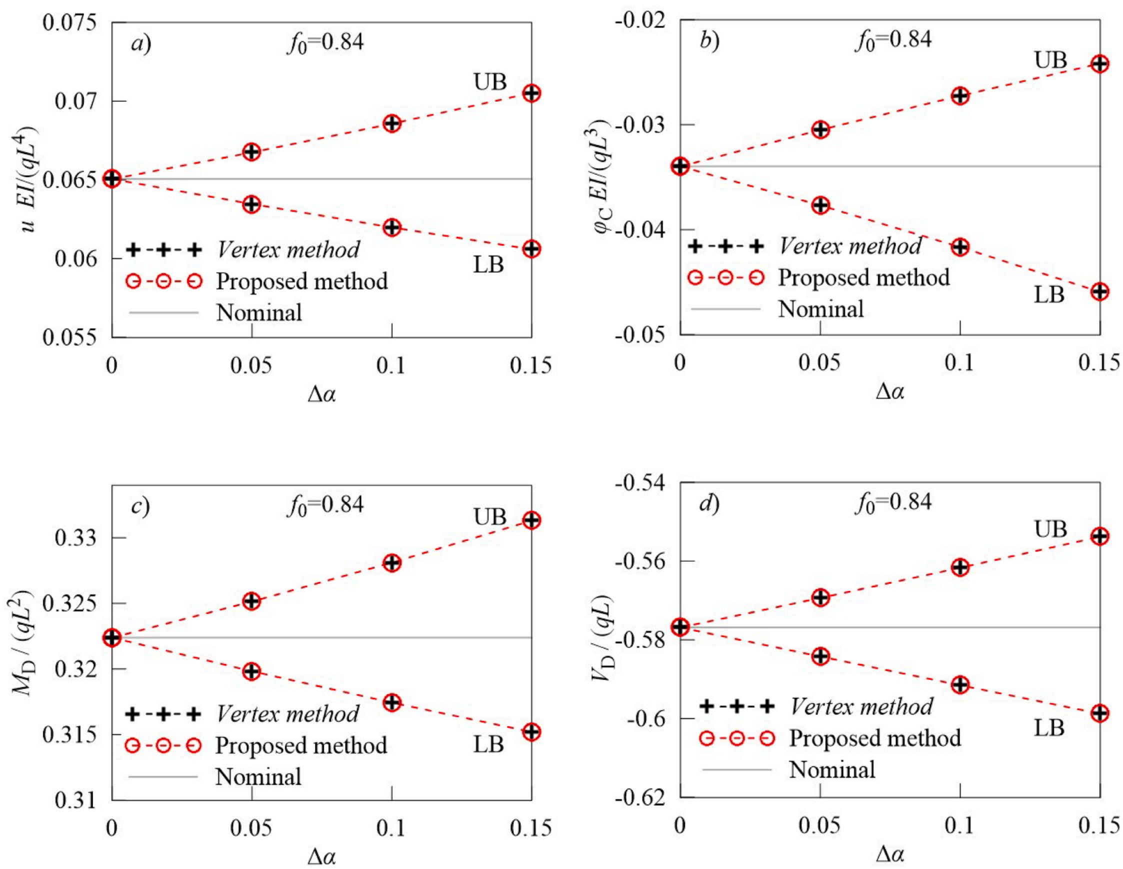

Fig. 7. Bounds of the normalized responses of the one-storey frame evaluated by the proposed sensitivity-based procedure and the *vertex method* along with the nominal solutions versus the normalized deviation amplitude of the uncertain parameters: *a*) horizontal displacement; *b*) rotation at node C; *c*) bending moment at node D; *d*) shear force at node D ($f_0 = 0.84$).

column connections depicted in Fig. 4 is selected as the first case study. The moment of inertia, I, the length of the members, L, and Young's modulus of the material, E, are assumed the same for the beam and columns. The beam is subjected to a uniformly distributed transversal load of intensity q; a horizontal force of magnitude $P = qL$ is applied to node B.

The partially restrained beam-to-column connections are characterized by interval *fixity factors* (see Eq.(5)) $f_j^I = f_0(1 + \alpha_j^I) = f_0(1 + \Delta\alpha\widehat{e}_j^I)$, $(j = 1, 2)$, with the same nominal value f_0 and normalized deviation amplitude $\Delta\alpha$.

The normalized horizontal displacement, $uEI/(qL^4)$, rotation of node C, $\varphi_C EI/(qL^3)$, bending moment, $M_D/(qL^2)$, and shear force, $V_D/(qL)$, at node D, are selected as response quantities of interest. As a first step, the monotonic behaviour of such quantities with respect to the uncertain parameters is assessed.

Fig. 5 displays the normalized response quantities of interest versus the dimensionless fluctuations $\alpha_j \in \alpha_j^I = [-\Delta\alpha, \Delta\alpha]$, $(j = 1, 2)$, of the two uncertain parameters for $\Delta\alpha = 0.2$. The nominal value of the interval *fixity factors* is set equal to $f_0 = 0.16$ which could correspond to a top-and-seat angle connection [2]. It can be observed that both the normalized horizontal displacement, $uEI/(qL^4)$, and bending moment at node D, $M_D/(qL^2)$, are monotonic decreasing functions of α_1 and α_2. Furthermore, the normalized rotation of node C, $\varphi_C EI/(qL^3)$, is a monotonic increasing function of both the uncertain parameters, while the normalized shear force at node D, $V_D/(qL)$, is a monotonic increasing function of α_1 and a decreasing function of α_2. Since the four functions exhibit a monotonic behaviour, their bounds are achieved for suitable combinations of the endpoints of the interval parameters which can also be predicted by applying the sensitivity-based approach described in Section 4. It is worth mentioning that response samples provided by the presented stiffness matrix method for assigned values of the uncertain *fixity factors* of the semi-rigid connections are in agreement with those obtained by applying the finite element software SAP2000.

The accuracy of the proposed method is demonstrated by performing appropriate comparisons with the exact bounds of the response provided by the *vertex method*. The latter requires 2^2 deterministic analyses, since $N_u = 2$ uncertain parameters are involved. Two values of the nominal *fixity factors* i.e., $f_0 = 0.16$ and $f_0 = 0.84$, are considered to represent the behaviour of nearly-pinned and nearly-rigid connections [22], respectively. Indeed, $f_0 = 0.84$ could correspond to the *fixity factor* of either a T-stub or an extended end-plate connection [2]. Figs. 6 and 7 display the bounds of the normalized response quantities of interest versus the normalized deviation amplitude of the uncertain parameters, $\Delta\alpha$, evaluated by applying the proposed sensitivity-based procedure and the *vertex method* for $f_0 = 0.16$ and $f_0 = 0.84$, respectively. The nominal solution is also plotted. It can be observed that the proposed bounds are the same as the exact ones obtained by the *vertex method*. As expected, the region of the response becomes wider as the normalized deviation amplitude $\Delta\alpha$ increases. It is worth remarking that the sensitivity-based procedure is able to predict the exact bounds of interval internal forces which are more affected by the *dependency phenomenon* than displacements.

Once the accuracy of the sensitivity-based procedure has been assessed, the influence of uncertainty on different types of semi-rigid connections is investigated by varying both the nominal value and the deviation amplitude of the interval *fixity factors*. The 3D plots in Fig. 8 display the proposed LB and UB of the normalized response quantities of interest versus the nominal value $0.14 \le f_0 \le 0.89$ [3] and the normalized deviation amplitude $0 \le \Delta\alpha \le 0.12$ of the *fixity factors* of the

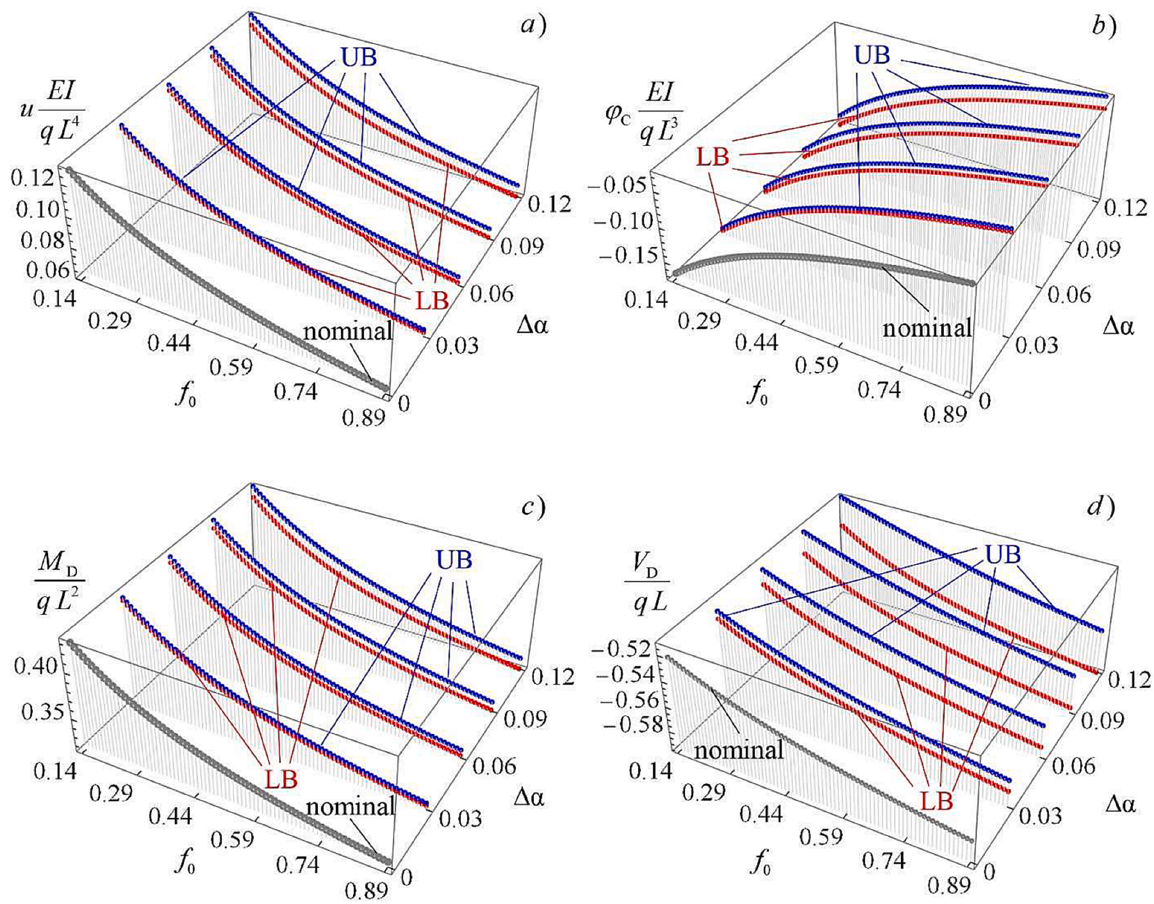

Fig. 8. Bounds of the normalized responses of the one-storey frame versus the nominal value and normalized deviation amplitude of the *fixity factors:* *a*) horizontal displacement; *b*) rotation of node C; *c*) bending moment at node D; *d*) transversal force at node D.

partially restrained beam-to-column connections along with the nominal solutions. Besides the increase of the width of the response region with $\Delta\alpha$, the 3D plots capture a significant change in the response as the behaviour of the nominal semi-rigid connections varies within the limits established by EC3 [3]. The nominal solutions pertaining to $\Delta\alpha = 0$ fall between the LB and UB for any value of $\Delta\alpha$, as shown in Figs. 6 and 7.

5.2. Five-storey frame with semi-rigid beam-to-column connections

The second numerical example concerns the single-bay five-storey frame with beam-to-column semi-rigid connections depicted in Fig. 9. The moment of inertia, I, the length of the members, L, and Young's modulus of the material, E, are assumed the same for all the beams and columns. All the beams are subjected to a uniformly distributed transversal load of intensity q; a horizontal force of magnitude $P = qL$ is applied at each floor (see Fig. 9).

The ten partially restrained beam-to-column connections are characterized by interval *fixity factors* (see Eq.(5)) $f_j^{(h)I} = f_0(1 + \Delta\alpha\widehat{e}_j^{(h)I})$, ($h = 1, 2, \ldots, 5$; $j = 1, 2$), with the same nominal value f_0 and normalized deviation amplitude $\Delta\alpha$.

The response quantities of interest are: the normalized horizontal displacement and drift ratio at the top floor, $u_5EI/(qL^4)$ and $\delta_5EI/(qL^3)$; the normalized maximum bending moment and shear force (at the base of the frame, right column), $M_{\max}/(qL^2)$ and $V_{\max}/(qL)$. Preliminary investigations, omitted for conciseness, have demonstrated that the selected response quantities are monotonic functions of the uncertain parameters.

For validation purposes, Fig. 10 displays the proposed bounds of the four response quantities contrasted to the exact ones provided by the

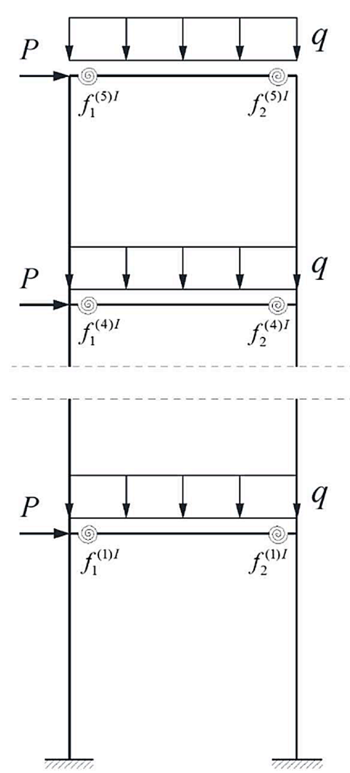

Fig. 9. Five-storey frame with partially restrained beam-to-column connections.

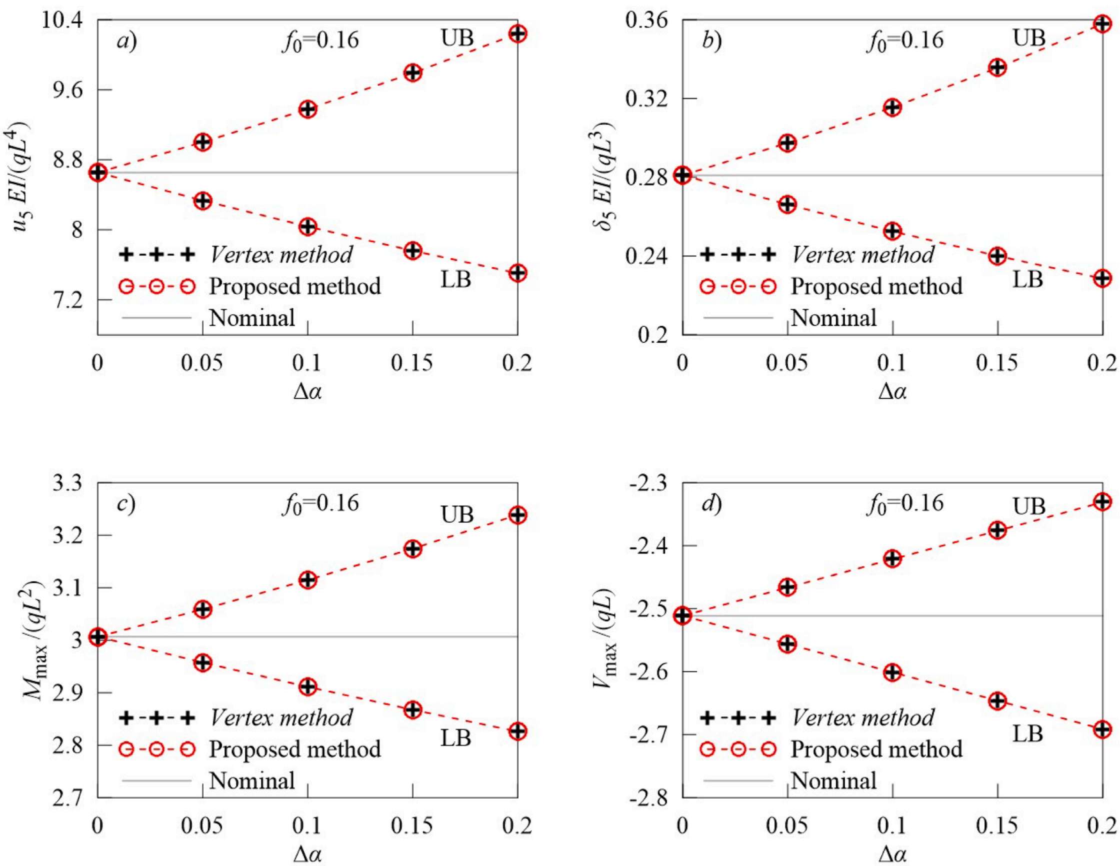

Fig. 10. Bounds of the normalized responses of the five-storey frame evaluated by the proposed sensitivity-based procedure and the *vertex method* along with the nominal solutions versus the normalized deviation amplitude of the uncertain parameters: *a*) horizontal displacement of the top floor; *b*) interstory drift ratio of the top floor; *c*) maximum bending moment; *d*) maximum transversal force ($f_0 = 0.16$).

vertex method, versus the normalized deviation amplitude of the uncertain parameters $\Delta\alpha$, for $f_0 = 0.16$. An excellent agreement can be observed even for relatively large degrees of uncertainty. It is observed that the sensitivity-based procedure is able to accurately predict the bounds of interval internal forces which are more affected by the *dependency phenomenon* than displacements. Furthermore, it is worth emphasizing that the evaluation of response bounds by the proposed approach involves only 2 deterministic analyses whatever the number of the uncertain parameters is. Conversely, for the selected case study, the *vertex method* requires 2^{10}=1024 deterministic analyses to be

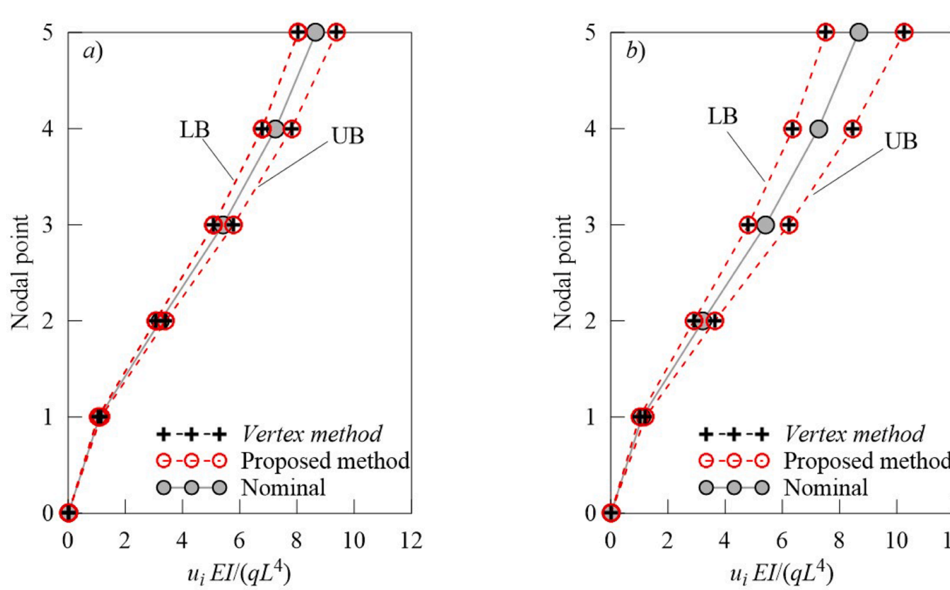

Fig. 11. Proposed bounds and nominal values of the normalized horizontal displacements at nodal points of the five-storey frame ($f_0 = 0.16$): *a*) $\Delta\alpha = 0.1$; *b*) $\Delta\alpha = 0.2$.

performed, as many as all possible combinations of the endpoints of the ten interval *fixity factors*. Furthermore, it can be observed that the region enclosed by the LB and UB of the response becomes wider as larger deviation amplitudes of the *fixity factors* are considered. Similar conclusions can be drawn from Figs. 11 and 12 which display the comparison between the proposed and exact LB and UB of the interval response in terms of the overall normalized horizontal displacements at each floor and interstory drift ratios at nodal points of the frame, for $\Delta\alpha = 0.1$ and $\Delta\alpha = 0.2$.

The influence of uncertainty on different types of semi-rigid connections can be inferred from the 3D plots in Fig. 13, which display the proposed LB and UB of the normalized response quantities of interest

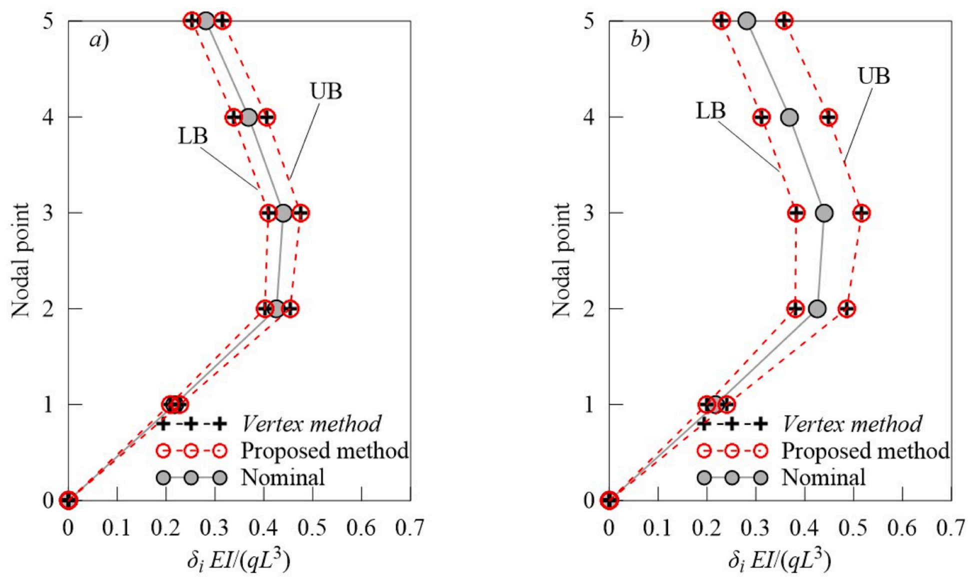

Fig. 12. Proposed bounds and nominal values of the interstory drift ratios of the five-storey frame ($f_0 = 0.16$): *a*) $\Delta\alpha = 0.1$; *b*) $\Delta\alpha = 0.2$.

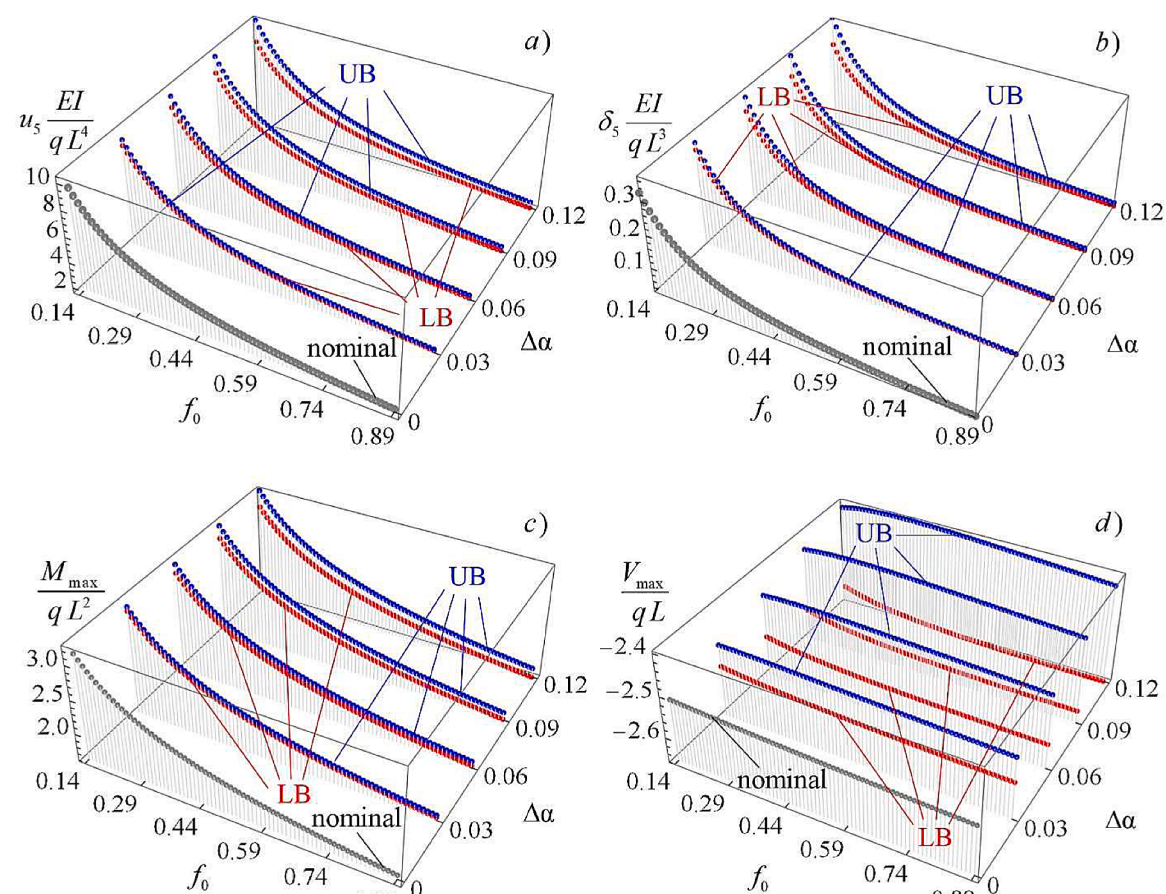

Fig. 13. Bounds of the normalized responses of the five-storey frame versus the nominal value and normalized deviation amplitude of the *fixity factors: a*) horizontal displacement of the top floor; *b*) interstory drift ratio of the top floor; *c*) maximum bending moment; *d*) maximum transversal force.

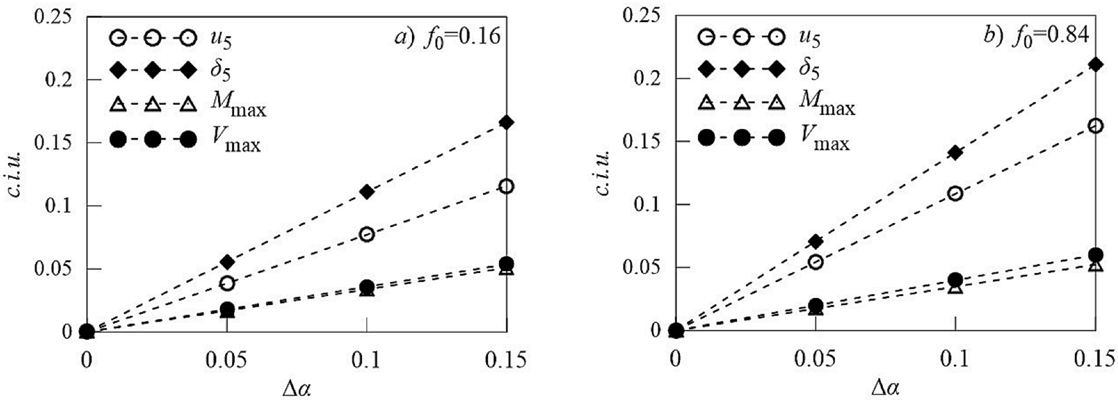

Fig. 14. *Coefficient of interval uncertainty* of the selected response quantities of the five-storey frame versus the normalized deviation amplitude of the interval *fixity factors* obtained by the proposed method: *a)* $f_0 = 0.16$; *b)* $f_0 = 0.84$.

versus the nominal value $0.14 \leq f_0 \leq 0.89$ [3] and the normalized deviation amplitude $0 \leq \Delta\alpha \leq 0.12$ of the *fixity factors* along with the nominal solutions. As for the previous case study, the 3D plots show both the increase in the width of the response region with $\Delta\alpha$ and the remarkable change in the response as the nominal value of the *fixity factor* varies within the limits established by EC3 [3] for pinned and rigid joints.

A measure of the dispersion of the interval response around the midpoint value is provided by the *coefficient of interval uncertainty* (*c.i.u.*), defined as [36]:

$$c.i.u.\left[R^I\right] = \frac{\Delta R}{\left|\mathrm{mid}\left\{R^I\right\}\right|} = \frac{\bar{R} - \underline{R}}{\left|\bar{R} + \underline{R}\right|} \tag{37}$$

where $R^I = \left[\underline{R}, \bar{R}\right]$ is the generic interval response quantity.

Fig. 14 shows the *c.i.u.* (Eq. (37)) of the selected response quantities versus the normalized deviation amplitude of the uncertain parameters $\Delta\alpha$, for two different types of semi-rigid connections characterized by nominal *fixity factors* $f_0 = 0.16$ and $f_0 = 0.84$. In both cases, it is observed that the *c.i.u.* increases linearly with $\Delta\alpha$ for all the response quantities. Furthermore, the drift ratio and horizontal displacement at the top floor exhibit the largest dispersion around the nominal value. In particular, for both these response quantities, uncertainty in the *fixity factors* is amplified through the propagation process when a reference connection type with $f_0 = 0.84$ i.e. with a nominal behaviour close to the one of rigid joints, is considered.

6. Conclusions

A novel interval matrix stiffness method for the static analysis of steel frames with uncertain semi-rigid connections has been presented. The non-deterministic behaviour of the connections, evidenced by the large scatter of experimental measures, has been described by modelling the *fixity factors* as interval variables with assigned lower bound and upper bound. The presented model allows the analyst to select a reference type of connection by assigning the nominal value of the *fixity factor*. Then, interval fluctuations around the selected nominal *fixity factor* enable one to describe the uncertain behaviour of the actual connection as ranging between two extremes corresponding to the upper bound and the lower bound of the moment capacity. The proposed interval matrix stiffness method has been derived by incorporating the interval stiffness of the semi-rigid connections into the classical deterministic formulation. Then, a sensitivity-based procedure for estimating the bounds of the response has been presented. Sensitivities of the response to the uncertain parameters have been efficiently derived by direct differentiation taking advantage of the explicit dependence of the stiffness matrix and nodal force vector of the elements with semi-rigid connections on the uncertain *fixity factors*. This remarkable feature facilitates computer implementation and enhances computational efficiency. The knowledge of response sensitivities can also be exploited to identify the semi-rigid connections whose changes in stiffness significantly affect the selected response quantity.

Numerical results have demonstrated the accuracy of the presented procedure and the significant effect of the uncertain initial stiffness of the semi-rigid connections on structural response. By varying the nominal value of the *fixity factor*, it has been observed that uncertainty has a different influence on the response for various types of reference connections. A detailed design of joints as well as accurate manufacturing and construction processes are recommended to reduce uncertainties in the behaviour of joints.

The proposed interval matrix stiffness method represents an effective tool to predict the response of frame structures under possible variations in the initial stiffness of partially restrained connections and aid designers in selecting the most suitable type of joint. The adoption of the interval model appears particularly appropriate to deal with the limited information on steel frame connections typically available to designers. Before joints are detailed and realized, by applying the proposed approach in conjunction with expert judgment, structural engineers can select the reference connection type and assume a possible range of the initial stiffness to account for the various sources of uncertainty. The corresponding range of structural response provides designers with more confidence than a crisp value in the context of decision-making.

It is worth remarking that the applicability of the presented method is not restricted to steel frames with uncertain semi-rigid connections as different materials can be considered in the analysis and uncertainties affecting material and geometrical properties of members or applied loads can also be included. Furthermore, the proposed procedure can be applied to arbitrary linear discretized structures within the general framework of interval finite element analysis.

Ongoing research is focusing on the extension of the presented formulation to include the inherently nonlinear behaviour of semi-rigid connections. To this aim, a suitable nonlinear moment-rotation curve has to be incorporated into the formulation of the interval matrix stiffness method, and efficient propagation strategies are needed to predict the bounds of the nonlinear response.

CRediT authorship contribution statement

Federica Genovese: Writing – original draft, Visualization, Validation, Software. **Alba Sofi:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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