mathematics

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Evasion Differential Game of Multiple Pursuers and One Evader for an Infinite System of Binary Differential Equations

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Abstract: We study a differential evasion game of multiple pursuers and an evader governed by several infinite systems of two-block differential equations in the Hilbert space l_2. Geometric constraints are imposed on the players' control functions. If the state of a controlled system falls into the origin of the space l_2 at some finite time, then pursuit is said to be completed in a differential game. The aim of the pursuers is to transfer the state of at least one of the systems into the origin of the space l_2, while the purpose of the evader is to prevent it. A sufficient evasion condition is obtained from any of the players' initial states and an evasion strategy is constructed for the evader.

Keywords: infinite system of differential equations; differential game; strategy; control; geometric constraint

MSC: 91A23; 49N75

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Citation: Ibragimov, G.; Kazimirova, R.; Pansera, B.A. Evasion Differential Game of Multiple Pursuers and One Evader for an Infinite System of Binary Differential Equations. *Mathematics* **2022**, *10*, 4448. https://doi.org/10.3390/math10234448

Academic Editor: David Barilla

Received: 25 October 2022
Accepted: 24 November 2022
Published: 25 November 2022

1. Introduction

Differential games have been the subject of research since the 1960s (see, for example, [1–12]). In recent years, there has been a growing interest in differential games (see, for example, [13–20]. However, most of the studies in the field of differential games have focused only on games of finite-dimensional Euclidean spaces.

Many real-world problems can be modeled as control problems for partial differential equations. A large and growing body of literature has studied the control problems described by partial differential equations. In [21], a time-optimal control problem for the parabolic equation has been studied for the first time. For more detailed information, we refer readers to [22].

Two earlier works, [23,24], studied the differential game problems for partial differential equations. The decomposition method is one of the most practical ways to study partial differential equations. Using the decomposition method allows us to obtain a control or differential game problem governed by a system of ordinary differential equations (see, e.g., [25–34]).

For example, the work [31] considers the following controlled system

$$z_t - Az = -u + v, \;\; z(x,0) = z_0(x), \;\; z|_{S_T} = 0, \tag{1}$$

where $z = z(x,t)$, $z_0(x) \in L_2(\Omega)$, $x = (x_1, x_2, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $n \geq 1$, Ω is bounded, T is a given positive number, $t \in [0,T]$, the operator A has the form $Az = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x) z_{x_j})$, $a_{ij}(x) = a_{ji}(x)$, $a_{ij}(x)$ are measurable bounded functions that satisfy the condition

$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq K\sum_{i=1}^{n}\xi_i^2$ for some $K > 0$ and for all $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $x \in \Omega$, $u(x,t)$, $v(x,t) \in L_2(Q_T)$ are the control functions of pursuer and evader, respectively, and $Q_T = \{(x,t)|\ x \in \Omega, 0 < t < T\}$, $S_T = \{(x,t)|\ x \in \partial\Omega, 0 < t < T\}$.

Using the decompositions

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)\psi(t),\ \ z_0(x) = \sum_{i=1}^{\infty} z_{i0}(t)\psi(t),\ \ u(x,t) = \sum_{i=1}^{\infty} u_i(t)\psi(t),\ \ v(x,t) = \sum_{i=1}^{\infty} v_i(t)\psi(t)$$

Equation (1) is reduced to the following infinite system of ordinary differential equations

$$\dot{z}_i = -\lambda_i z_i - u_i + v_i,\ \ z_i(0) = z_{i0},\ \ i = 1,2,\ldots$$

Although the decomposition method provides countably many differential equations in the Hilbert space l_2, the simplicity of the equations attracts the attention of the authors. The works [30,31,33] suggest considering differential game problems for the infinite system of differential equations independently of partial differential equations. The work [35] proposes a differential game for such an infinite system of differential equations. Several studies on control problems and differential game problems have been carried out on infinite systems of differential equations (see, for example, [26,36–38]).

Papers [39,40] are dedicated to the evasion differential game in the finite-dimensional Euclidean space $\mathbb{R}^n$. It has been shown [39] that one evader can avoid many slow pursuers. This result has been extended to the case where the pursuers' control set is a subset of the interior of the evader control set [40].

For the multi-pursuer simple motion differential game,

$$\begin{array}{llll} \dot{x}_i = u_i, & x_i(0) = x_{i0}, & |u_i| \leq 1, & \\ \dot{y} = v, & y(0) = y_0, & |v| \leq 1, & x_{i0} \neq y_0, \end{array} \tag{2}$$

where $i \in I = \{1, 2, \ldots, m\}$, $m \geq 1$, $x_i, y \in \mathbb{R}^n$, $i \in I$, and it is known [41] that if y_0 belongs to the interior of the convex hull of the points x_{i0}, $i \in I$, then one can construct the strategies of pursuers $x_1, x_2, \ldots, x_m$ such that, for the arbitrary control of the evader, the pursuers can capture the evader y, that is, $x_i(t_1) = y(t_1)$ at some index $i \in \{1, 2, \ldots, m\}$ and $t_1 > 0$; if y_0 outside of the interior of the convex hull of the initial states x_{i0}, $i \in I$, then evasion is possible in game (2), i.e., one can construct a control for the evader such that $x_i(t) \neq y(t)$, $t > 0$, for any controls of the pursuers and for all $i = 1, 2, \ldots, m$.

If we consider differential game (2) in the Hilbert space l_2 with countably many pursuers x_i, $i = 1, 2, \ldots$, and one evader $y \in l_2$, then we can specify some initial states of players for which evasion is possible. However, no previous studies have investigated pursuit differential game (2) in l_2. Despite the simplicity of the players' movements, so far no initial position $(y_0, x_{10}, x_{20}, \ldots)$ is specified from which the pursuit in game (2) can be completed with countably many pursuers in l_2.

For an infinite system of binary differential equations in the Hilbert space l_2, the pursuit differential game of one pursuer and one evader was studied in [42], when the pursuer's control set contains the evader's control set.

For the same infinite system of binary differential equations in the Hilbert space l_2, an evasion differential game of multiple pursuers and one evader is studied in the present paper. We prove that if the evader's control set contains or coincides with the control set of any pursuer, then evasion is possible for any number of pursuers and from any initial position of the game. We also build an evasion strategy for the evader. As mentioned above, in the case of the finite-dimensional space, for some initial states of the players, the pursuit can be completed [41]. The main result of the present paper shows that the infinite dimensionality of the space l_2 is itself an advantage for the evader and evasion is possible from any finite number of pursuers.

2. Statement of Problem

We recall the vector space of all sequences of real numbers

$$l_2 = \left\{ \xi = (\xi_1, \xi_2, \ldots) \mid \sum_{n=1}^{\infty} \xi_n^2 < \infty \right\}$$

is a Hilbert space where the inner product of the vectors $\mu, \nu \in l_2$ and the norm of μ are given by

$$\langle \mu, \nu \rangle = \sum_{n=1}^{\infty} \mu_n \nu_n, \ \ ||\mu|| = \sqrt{\langle \mu, \mu \rangle}.$$

We consider a differential game governed by the following infinite system of binary differential equations

$$\begin{aligned} \dot{x}_{ij} &= -\alpha_j x_{ij} - \beta_j y_{ij} + u_{ij}^1 - v_j^1, \quad x_{ij}(0) = x_{ij}^0, \\ \dot{y}_{ij} &= \ \ \beta_j x_{ij} - \alpha_j y_{ij} + u_{ij}^2 - v_j^2, \quad y_{ij}(0) = y_{ij}^0, \end{aligned} \tag{3}$$

where $x_{ij}, y_{ij} \in \mathbb{R}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots$, are the state variables, α_j, β_j are real numbers and $\alpha_j \geq 0$, u_{ij}^1, u_{ij}^2, $j = 1, 2, \ldots$, are the control parameters of the i-th pursuer, $i = 1, 2, \ldots, m$, and v_j^1, v_j^2, $j = 1, 2, \ldots$, are the evader's control parameters. We assume that

$$x_i^0 = (x_{i1}^0, x_{i2}^0, \ldots) \in l_2, \ \ y_i^0 = (y_{i1}^0, y_{i2}^0, \ldots) \in l_2, \ \ i = 1, 2, \ldots, m.$$

Let, for $j = 1, 2, \ldots$, $i = 1, 2, \ldots, m$,

$$u_{ij} = (u_{ij}^1, u_{ij}^2), \ \ v_j = (v_j^1, v_j^2), \ \ z_{ij}^0 = (x_{ij}^0, y_{ij}^0),$$

$$z_i^0 = (z_{i1}^0, z_{i2}^0, \ldots), \ \ \|z_i^0\| = \left(\sum_{j=1}^{\infty} \left(\left(x_{ij}^0 \right)^2 + \left(y_{ij}^0 \right)^2 \right) \right)^{1/2},$$

$$u_i(t) = (u_{i1}, u_{i2}, \ldots), \ \ ||u_i(t)|| = \left(\sum_{j=1}^{\infty} \left(\left(u_{ij}^1(t) \right)^2 + \left(u_{ij}^2(t) \right)^2 \right) \right)^{1/2},$$

$$v = (v_1, v_2, \ldots), \ \ ||v|| = \left(\sum_{j=1}^{\infty} \left(\left(v_j^1(t) \right)^2 + \left(v_j^2(t) \right)^2 \right) \right)^{1/2},$$

We assume that $z_i^0 \neq 0$ for all $i = 1, 2, \ldots, m$.

Definition 1. *We call the function $u_i(t) = (u_{i1}(t), u_{i2}(t), \ldots)$, $i \in \{1, 2, \ldots, m\}$, with measurable coordinates $u_{ij}(t)$, $j = 1, 2, \ldots$, such that $||u_i(t)|| \leq \rho_i$, $0 \leq t \leq T$, the i-th pursuer's admissible control, where the positive number ρ_i is a given, and T is a sufficiently large positive number.*

Definition 2. *We call the function $v(t) = (v_1(t), v_2(t), \ldots)$, $||v(t)|| \leq \sigma$, with measurable coordinates $v_j(t)$, $0 \leq t \leq T$, $j = 1, 2, \ldots$, the evader's admissible control, where the positive number σ is a given.*

We consider the case where $\sigma \geq \rho_i$ for all $i = 1, \ldots, m$. The solutions $z_i(t) = (z_{i1}(t), z_{i2}(t), \ldots)$, $0 \leq t \leq T$, of infinite system of differential equations (3) are considered in the space of functions $h(t) = (h_1(t), h_2(t), \ldots) \in l_2$ whose coordinates $h_i(t)$ are absolutely continuous and defined on the interval $0 \leq t \leq T$.

Definition 3. *We say that evasion is possible in game (3) if there exists a control $v_0(t)$, $0 \leq t \leq T$, of evader such that for any controls $u_i(t)$, $0 \leq t \leq T$, of the pursuers, the solutions of initial value problems (3), $z_i(t) = (z_{i1}(t), z_{i2}(t), \ldots)$, $i = 1, \ldots, m$, are nonzero for all t, $0 \leq t \leq T$, in other words, on the interval $0 \leq t \leq T$, we have $z_i(t) \neq 0$ for all $i = 1, 2, \ldots, m$.*

It should be noted that the evader uses the control $v_0(t)$ on the time interval $[0, T]$ and the pursuers apply arbitrary controls $u_i(t)$, $i = 1, 2, \dots, m$, on that interval.

Problem 1. *The problem is to find a control $v_0(t)$, $0 \le t \le T$, of evader such that evasion is possible in game* (3).

3. The Main Result

In this section, we prove the following statement.

Theorem 1. *For any initial states z_i^0, $i = 1, 2, \dots, m$, evasion is possible in the game described by the infinite system of equations* (3).

Proof. Clearly, system (3) can be rewritten as follows

$$\dot{z}_{ij} = A_j z_{ij} + u_{ij} - v_j, \quad z_{ij}(0) = z_{ij}^0, \;\; i = 1, 2, \dots, m, \;\; j = 1, 2, \dots, \tag{4}$$

where

$$A_j = \begin{bmatrix} -\alpha_j & -\beta_j \\ \beta_j & -\alpha_j \end{bmatrix}, \;\; j = 1, 2, \dots.$$

It is not difficult to verify that

$$e^{A_j t} = e^{-\alpha_j t} \begin{bmatrix} \cos \beta_j t & -\sin \beta_j t \\ \sin \beta_j t & \cos \beta_j t \end{bmatrix}, \;\; j = 1, 2, \dots.$$

For the solution of (3) or the same of (4), we have

$$z_{ij}(t) = e^{A_j t} \psi_{ij}(t), \;\; \psi_{ij}(t) = z_{ij}^0 + \int_0^t e^{-A_j s} (u_{ij}(s) - v_j(s)) ds. \tag{5}$$

Note that the matrices $e^{A_j t}$ are not singular, and so, the equation $z_{ij}(t) = 0$ is equivalent to the equation $\psi_{ij}(t) = 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots$.

Therefore, to prove the theorem, it is sufficient to construct an admissible control $v_0(t)$, $0 \le t \le T$, of the evader such that $\psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), \dots) \ne 0$ for all $i = 1, 2, \dots, m$ and $0 \le t \le T$.

Indeed, because $z_1^0 = (z_{11}^0, z_{12}^0, \dots) \ne 0$, therefore at least one of the components $z_{1j}^0 \in \mathbb{R}^2$, $j = 1, 2, \dots$, of z_1^0 is not equal to 0. In other words, there exists a positive integer n_1 such that $z_{1n_1}^0 \ne 0$. Similarly, the inequality $z_2^0 = (z_{21}^0, z_{22}^0, \dots) \ne 0$ implies that $z_{2n_2}^0 \ne 0$ for some positive integer n_2. Finally, $z_m^0 = (z_{m1}^0, z_{m2}^0, \dots) \ne 0$ implies that $z_{mn_m}^0 \ne 0$ for some positive integer n_m.

Let $n = \max\limits_{i=1,\dots,m} n_i$. Then, clearly, $Z_i^0 = (z_{i1}^0, z_{i2}^0, \dots, z_{in}^0) \ne 0$ for all $i = 1, 2, \dots, m$. We can assume, by increasing n if necessary, that $2n \ge m$. Thus, we have

$$Z_i^0 = (z_{i1}^0, z_{i2}^0, \dots, z_{in}^0) \ne 0, \;\; Z_i^0 \in \mathbb{R}^{2n}, \;\; i = 1, 2, \dots, m. \tag{6}$$

Let

$$\Psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), \dots, \psi_{in}(t)), \;\; i = 1, \dots, m,$$

where

$$\psi_{ij}(t) = z_{ij}^0 + \int_0^t e^{-A_j s} (u_{ij}(s) - v_j(s)) ds, \;\; i = 1, 2, \dots, m, \;\; j = 1, 2, \dots, n.$$

The proof of the theorem is completed by showing that $\Psi_i(t) \ne 0$, $i = 1, \dots, m$, because these relations imply that

$$\psi_i(t) = (\psi_{i1}(t), \psi_{i2}(t), \dots) \neq 0, \ \ i = 1,2,\dots,m, \ \ 0 \leq t \leq T,$$

hence that $z_i(t) \neq 0$, $i = 1,2,..,m$. In other words, we have reduced the differential game in the Hilbert space l_2 to a differential game in the finite-dimensional Euclidean space $\mathbb{R}^{2n}$.

We recall the number m of points $Z_i^0 \in \mathbb{R}^{2n}$ does not exceed the dimension $2n$ of the space $\mathbb{R}^{2n}$, i.e., $m \leq 2n$. From this, we conclude that there is a unit vector

$$e = (e_1, e_2, \dots, e_n) \in \mathbb{R}^{2n}, \ \ |e| = 1, \ \ e_j \in \mathbb{R}^2,$$

such that the inner product $\langle Z_i^0, e\rangle \geq 0$ for all $i = 1,\dots,m$. As the vector e, we can take an orthonormal vector to the hyperplane in $\mathbb{R}^{2n}$ passing through the points Z_i^0, $i = 1,2,\dots,m$.

Next, we construct a control for the evader as follows

$$v_j(t) = -\frac{e^{-A_j^* t} e_j}{\sqrt{\sum\limits_{k=1}^{n} |e^{-A_k^* t} e_k|^2}}\sigma, \ \ j = 1,\dots,n, \ \ v_j(t) = 0, \ \ j = n+1, n+2, \dots, \ \ t \geq 0, \tag{7}$$

where A^* denotes the transpose of the matrix A, and show that the control (7) of the evader guarantees the evasion on the time interval $[0, T]$. Note that the denominator in (7) does not equal zero, i.e.,

$$\sum_{k=1}^{n} |e^{-A_k^* t} e_k|^2 \neq 0, \ \ t \geq 0,$$

because otherwise we would have

$$e^{-A_k^* t} e_k = 0, \ \ k = 1,2,\dots,n,$$

which implies that $e_k = 0$, $k = 1,2,\dots,n$, that is, $e = 0$, which is impossible because $e = (e_1, e_2, \dots, e_n)$ is a unit vector.

To prove the theorem, we assume the contrary. There exist admissible controls of pursuers such that when the evader applies the control (7), $\Psi_p(\tau) = 0$ for some $p \in \{1,\dots,m\}$ and $\tau > 0$. We conclude from the inequalities $\langle Z_i^0, e\rangle \geq 0$, $i = 1,2,\dots,m$, that

$$\sum_{j=1}^{n} \langle z_{pj}^0, e_j\rangle \geq 0, \ \ i = 1,2,\dots,m, \tag{8}$$

hence that

$$\begin{aligned}
\langle \Psi_p(\tau), e\rangle &= \sum_{j=1}^{n} \psi_{pj}(\tau) e_j & (9)\\
&= \sum_{j=1}^{n} \langle z_{pj}^0, e_j\rangle + \sum_{j=1}^{n} \int_0^\tau \langle e^{-A_j s}(u_{pj}(s) - v_j(s)), e_j\rangle ds & (10)\\
&\geq \int_0^\tau \sum_{j=1}^{n} \langle e^{-A_j s} u_{pj}(s), e_j\rangle ds - \int_0^\tau \sum_{j=1}^{n} \langle e^{-A_j s} v_j(s), e_j\rangle ds & (11)
\end{aligned}$$

We use the Cauchy–Schwartz inequality to estimate the first integral in (9)

$$\begin{aligned}
\int_0^\tau \sum_{j=1}^{n} \langle e^{-A_j s} u_{pj}(s), e_j\rangle ds &= \int_0^\tau \sum_{j=1}^{n} \langle u_{pj}(s), e^{-A_j^* s} e_j\rangle ds & (12)\\
&\geq -\rho_p \int_0^\tau \sqrt{\sum_{j=1}^{n} |e^{-A_j^* s} e_j|^2} ds, & (13)
\end{aligned}$$

where the equality sign holds at

$$u_{pl}(t) = -\frac{e^{-A_l^* t} e_l}{\sqrt{\sum\limits_{j=1}^{n} |e^{-A_j^* t} e_j|^2}} \rho_p, \ \ l = 1, 2, \dots, n, \ \ 0 \le t \le \tau. \tag{14}$$

Using (7) and (12) leads to the inequality

$$\langle \Psi_p(\tau), e \rangle = \int\limits_0^\tau \sum_{j=1}^{n} \langle u_{pj}(s), e^{-A_j^* s} e_j \rangle ds - \int\limits_0^\tau \sum_{j=1}^{n} \langle v_j(s), e^{-A_j^* s} e_j \rangle ds \tag{15}$$

$$\ge -\rho_p \int\limits_0^\tau \sqrt{\sum_{j=1}^{n} |e^{-A_j^* s} e_j|^2 ds} + \sigma \int\limits_0^\tau \sqrt{\sum_{j=1}^{n} |e^{-A_j^* s} e_j|^2 ds} \ge 0 \tag{16}$$

because $\rho_p \le \sigma$. Thus, we have showed that $\langle \Psi_p(\tau), e \rangle \ge 0$. By our assumption $\Psi_p(\tau) = 0$ and hence $\langle \Psi_p(\tau), e \rangle = 0$.

However, in the inequality (15), the equality sign holds if and only if $\rho_p = \sigma$, and $u_{pl}(t)$, $l = 1, 2, \dots, n$, $0 \le t \le \tau$, are defined by (14), and $\sum\limits_{j=1}^{n} (z_{i_0 j}, e_j) = 0$ meaning that the equality sign holds in (9). Comparing (7) and (14), we then have

$$u_{pl}(t) = v_l(t), \ \ l = 1, 2, \dots, n, \ \ 0 \le t \le \tau.$$

Substituting this into equation (5) yields

$$\Psi_{pl}(\tau) = z_{pl}^0, \ \ l = 1, 2, \dots, n.$$

By the choice of the number n, we obtain the following inequality

$$\Psi_p(\tau) = (\psi_{p1}(\tau), \dots, \psi_{pn}(\tau)) = (z_{p1}^0, \dots, z_{pn}^0) \ne 0.$$

On the other side, by assumption $\Psi_p(\tau) = 0$, contradiction. Thus, $\Psi_i(t) \ne 0$, for all $t \in [0, T]$ and $i = 1, 2, \dots, m$. The proof of the theorem is complete.
□

4. Conclusions

For an infinite system of binary differential equations in the space l_2, we have studied an evasion differential game problem. In [42], it has been shown that if the pursuer control set contains the evader control set, i.e., if $\rho > \sigma$, then the pursuit can be completed for a finite time. Consequently, if in game (3) $\rho_k > \sigma$ for some $k \in \{1, 2, \dots, m\}$, then the pursuers can capture the evader. For this reason, in the present work, we have studied differential game (3) under the conditions that $\rho_i \le \sigma$ for all $i = 1, 2, \dots, m$. In the case of a finite number of pursuers, we have proved a theorem of evasion. In addition, we have built an evasion strategy that ensures evasion.

As a future work, it is recommended to study the differential game described by equation (2) with countably many pursuers x_i, $i = 1, 2, \dots$, and one evader y whose control parameters satisfy the conditions $||u_i|| \le 1$, $i = 1, 2, \dots$, $||v|| \le 1$. Similar to the work in [41], it is required to find the set of all the initial positions $(y_0, x_{10}, x_{20}, \dots)$ from which the pursuit can be completed.

Author Contributions: Conceptualization, G.I., R.K. and B.A.P.; methodology, G.I., R.K. and B.A.P.; validation G.I., R.K. and B.A.P.; formal analysis, G.I., R.K. and B.A.P.; investigation, G.I., R.K. and B.A.P.; writing—original draft preparation, G.I., R.K. and B.A.P.; writing—review and editing, G.I., R.K. and B.A.P.; visualization, G.I.; supervision, G.I.; project administration, R.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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