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SOME CONSTRUCTIONS OF LEVEL-DEPENDENT HERMITE SUBDIVISION OPERATORS

Abstract. The aim of this short note is to present a strategy for constructing Hermite subdivision operators preserving elements in the space of polynomials and exponentials. The masks associated to such operators are suitable for processing data of Hermite type, where such data exhibit trascendental features, and can be used for the realization of novel wavelet systems.

1. Introduction

Hermite subdivision schemes have been a popular topic of research for many years. Such schemes have been firstly investigated by Merrien [14] and by Dyn and Levin [10], and successively studied by many authors ([3, 4, 8, 9, 12, 15, 18, 19] just to mention a few of them).

We recall that an Hermite scheme is an iterative procedure which acts on vector-valued data sequences $c \in \ell^{d+1}(\mathbb{Z})$, interpreting the j-th components as a j-th derivative. Starting from a vector of control points $c^{[0]} = \{c_k^{[0]}\}_{k\in\mathbb{Z}}$, with $c_k^{[0]} \in \mathbb{R}^{d+1}$ representing function values and derivatives up to the d-th order, it produces sequences of sequences by means of the following rule:

$$(D^{n+1}c^{[n+1]})_j = \sum_{k\in\mathbb{Z}} A_{j-2k}^{[n]} D^n c_k^{[n]}, \quad j \in \mathbb{Z}, \quad n = 0,1,2,\ldots, \tag{1.1}$$

where D is the diagonal matrix with elements $1, 1/2, \ldots, 1/2^d$.

In a more compact notation, the above formulas read as

$$D^{n+1}c^{[n+1]} = \mathcal{S}_{A^{[n]}} D^n c^{[n]}, \quad n = 0,1,2,\ldots,$$

where $\mathcal{S}_{A^{[n]}}$ is the *Hermite subdivision operator* at level n. These operators are, in general, level-dependent, for they depend on $(d+1)\times(d+1)$ matrix *masks* $\{A_k^{[n]}\}_{k\in\mathbb{Z}}$ which possibly vary at each iteration n of the subdivision process.

In particular, such level-dependency is unavoidable if the operator is required to preserve polynomial and exponential functions, that is elements in the space

$$V_{d,\Lambda} = \text{span}\{1, x, \ldots, x^p, e^{\lambda_1 x}, \ldots, e^{\lambda_r x}\},$$

where we have set $\Lambda = \{\lambda_1, \ldots, \lambda_r\} \subset \mathcal{C}\backslash\{0\}$, $\lambda_i \neq \lambda_j$, $i \neq j$, and $d = p + \#\Lambda = p + r$.

The use of subdivision schemes with this property is required when, for example, one wishes to process data that possess transcendental, rather then just polynomial, characteristics. Schemes of this type have been studied and realized, for example, in [11, 17], in the non-Hermite case and in [1, 2, 5, 6, 7, 13, 16], in the Hermite situation.

In our framework, such preservation property is expressed by the so-called $V_{d,\Lambda}$-*spectral condition* (see [1]) according to which

$$\mathcal{S}_{A^{[n]}} v^{[n]}(f) = v^{[n+1]}(f), \qquad f \in V_{p,\Lambda}, n \geqslant 0, \tag{1.2}$$

where, for $f \in C^d(\mathbb{R})$, we denote by $v^{[n]}(f)$ the vector sequence with elements

$$v_k^{[n]}(f) := [f(2^{-n}k), 2^{-n}f'(2^{-n}k), \ldots, 2^{-nd}f^{(d)}(2^{-n}k)]^T, \quad k \in \mathbb{Z}.$$

A thorough study of these operators and associated schemes has recently been carried out in [1, 2, 7, 6]. In some of such papers, examples of schemes have been given, all relying on the presence of double frequencies (i.e. $\pm\lambda$) in $V_{p,r}$ and all being interpolatory.

In this short note, inspired by the examples of Hermite subdivision schemes proposed by Dubuc and Merrien in [9], we would like to illustrate a different construction, based on the results given in [1], but relaxing the double frequency constraint.

Like in the above-mentioned papers, we assume that the size $(d+1)\times(d+1)$ of the matrices in the masks correspond to the order of the derivatives $0, 1, \ldots, d$, involved.

It turns out that the obtained operators produce, in the limit, those of the standard Hermite schemes realized in [9], which reproduce polynomial data only, since they satisfy the polynomial spectral condition, obtained as a particular case of (1.2) when the function f is in the space of polynomials of degree less than or equal d.

2. Construction of Hermite subdivision masks

We construct two classes of masks, respectively interpolatory and non-interpolatory. We recall that a Hermite subdivision scheme is called *interpolatory* if $c_{2j}^{[n+1]} = c_j^{[n]}$, $j \in \mathbb{Z}$, for any $n \in \mathbb{N}$. In this case, the mask sequence $(A^{[n]} : n \geqslant 0)$ has the property $A_j^{[n]} = D\delta_{j0}$, $j \in \mathbb{Z}$, for each $n \in \mathbb{N}$ (δ_{j0} denotes the generic element of the Dirac δ-sequence).

Our starting point is the solution to the *two-point Hermite interpolation problem*, on the interval $[0,1]$:
Given two vectors $y^\varepsilon = (y_0^\varepsilon, y_1^\varepsilon, \ldots, y_d^\varepsilon)$, $\varepsilon \in \{0,1\}$, the element

$$p \in V' = \text{span}\{1, x, \ldots, x^p, e^{\lambda_1 x}, \ldots, e^{\lambda_r x}, x^{d+1}, \ldots, x^{2d+1}\},$$

which satisfies:

$$p^{(i)}(0) = y_i^0, \; p^{(i)}(1) = y_i^1, \quad i = 0, 1, \ldots, d,$$

is given by:

$$p(x) = \sum_{j=0}^{d} \left(y_j^0 h_{j;\Lambda}^0(x) + y_j^1 h_j^1(x) \right), \tag{2.3}$$

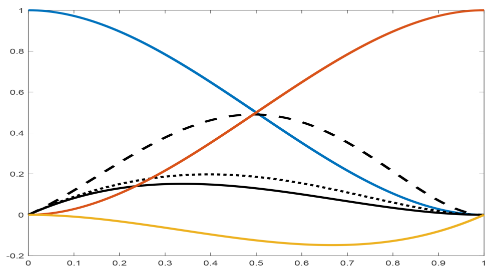

Figure 1: Plots of $h^0_0(x)$ (blue), $h^0_{1;1}(x)$ (black line), $h^0_{1;2}(x)$ (black dotted line), $h^0_{1;3}(x)$ (black dashed line), $h^1_0(x)$ (red), $h^1_1(x)$ (yellow). Observe that only $h^0_{1,\Lambda}$ is actually dependent on the parameter λ, and it is showed for three values of it.

where

$$h^1_j(x) = \frac{x^{d+1}}{j!}\sum_{k=0}^{d-j}\binom{d+k}{k}(-1)^k(x-1)^{j+k}$$

and

$$h^0_{j;\Lambda}(x) = g_{j;\Lambda}(x) - \sum_{k=0}^{d}\left(g_{j;\Lambda}\right)^{(k)}(1)\,h^1_k(x),$$

with $g_{j;\Lambda}$ the j-th component of the vector

$$g_\Lambda(x) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p! & 0 & \cdots & 0 \\ 1 & \lambda_1 & \cdots & \lambda_1^p & \lambda_1^{p+1} & \cdots & \lambda_1^d \\ 1 & \lambda_2 & \cdots & \lambda_2^p & \lambda_2^{p+1} & \cdots & \lambda_2^d \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_r & \cdots & \lambda_r^p & \lambda_r^{p+1} & \cdots & \lambda_r^d \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^p \\ e^{\lambda_1 x} \\ e^{\lambda_2 x} \\ \vdots \\ e^{\lambda_r x} \end{bmatrix}.$$

The non-singularity of the Vandermonde matrix above is guaranteed by the fact that the frequencies are all different.

This result extends the one given in [1] (Lemma 7), where the special case of exponentials with double frequencies (of the type $\pm\lambda$) has been treated.

For the local interpolant over a generic interval dyadic $[k/2^n, (k+1)/2^n]$, $k \in \mathbb{Z}$, substituting $t = 2^n(x - k/2^n)$ in (2.3) and differentiating, one gets:

(2.4) $$D^n P\left(\frac{t+k}{2^n}\right) = H^0_{2^{-n}\Lambda}(t)D^n y^0 + H^1(t)D^n y^1$$

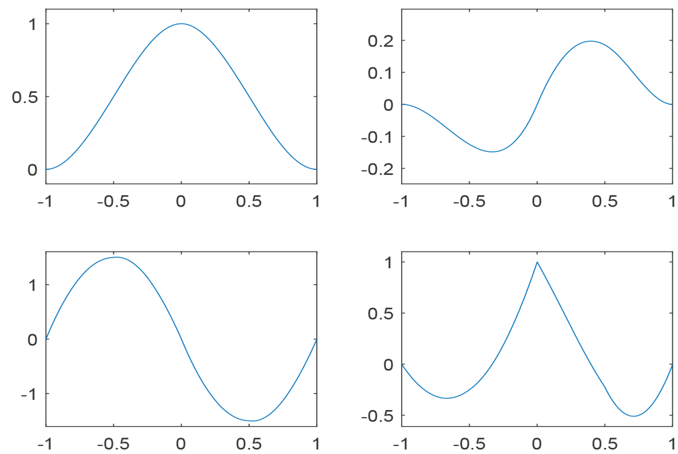

Figure 2: Basic limit function associated with the proposed interpolatory subdivision scheme, when $\lambda = 2$.

where $P = (p, p', \ldots, p^{(d)})$, and

$$H^0_{2^{-n}\Lambda}(t) = \left[(h^0_{j;2^{-n}\Lambda})^{(i)}(t) \,:\, i,j = 0,\ldots,d\right], \quad H^1(t) = \left[(h^1_j)^{(i)}(t) \,:\, i,j = 0,\ldots,d\right],$$

satisfying the property:

$$H^0_{2^{-n}\Lambda}(k) = I\delta_{k,0}, H^1_{2^{-n}\Lambda}(k) = I\delta_{k,1}, \quad k \in \mathbb{Z}.$$

We underline that, after the variable transformation, the use of the dilated frequencies $2^{-n}\Lambda$ is necessary in order for the same space $V_{d,\Lambda}$ to be spanned.

Formula (2.4) can now be used to define both interpolatory and non-interpolatory schemes, as follows.

At a generic step of the scheme, we use the vector sequence $c^{[n]} = \{c^{[n]}_k\}_{k\in\mathbb{Z}}$, with $c^{[n]}_k : \mathbb{Z} \to \mathbb{R}^{d+1}$, in (2.4) to produce the new sequence $c^{[n+1]}$, by evaluating the local interpolants for suitable values of t.

If we set $y^0 = c^{[n]}_k$ and $y^1 = c^{[n]}_{k+1}$ in (2.4) and evaluate it for $t = 0$ and $t = 1/2$, we get

$$D^n P\left(\frac{k}{2^n}\right) = D^n c^{[n]}_k,$$

$$D^n P\left(\frac{k+1}{2^n}\right) = H^0_{2^{-n}\Lambda}\left(\frac{1}{2}\right) D^n c^{[n]}_k + H^1\left(\frac{1}{2}\right) D^n c^{[n]}_{k+1},$$

which, by left-multiplying each side by D and interpreting the values of the refined vector sequence as:

$$c^{[n+1]}_k = P\left(\frac{k}{2^{n+1}}\right),$$

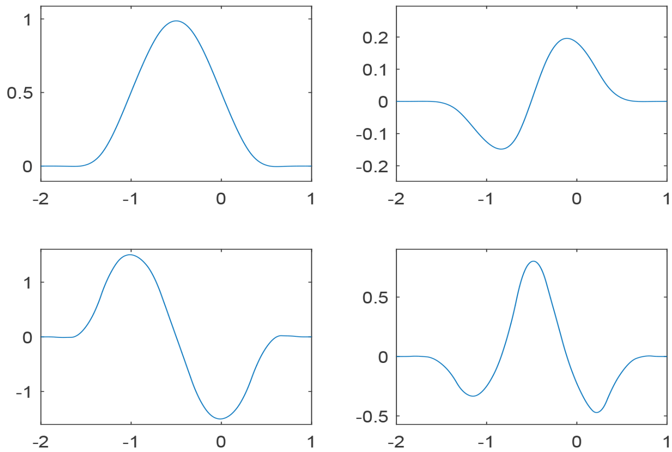

Figure 3: Basic limit function associated with the proposed non-interpolatory subdivision scheme, when $\lambda = 2$.

produce the following (interpolatory) subdivision rules

$$D^{n+1} c_{2k}^{[n+1]} = A_0^{[n]} D^n c_k^{[n]},$$

$$D^{n+1} c_{2k+1}^{[n+1]} = A_1^{[n]} D^n c_k^{[n]} + A_{-1}^{[n]} D^n c_{k+1}^{[n]},$$

with $A_{-1}^{[n]} = DH^1(1/2)$, $A_0^{[n]} = D$, $A_1^{[n]} = DH^0_{2^{-n}\Lambda}(1/2)$.

We limit ourselves to display the mask taps in the case $p = 0$, $r = 1$:

$$A_{-1}^{[n]} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{bmatrix}, A_0^{[n]} = D, A_1^{[n]} = \begin{bmatrix} \frac{1}{2} & \frac{\lambda_n \mathrm{e}^{\lambda_n} - 4\mathrm{e}^{\lambda_n} + 8\mathrm{e}^{\lambda_n/2} - 4}{8\lambda_n} \\ -\frac{3}{4} & \frac{\lambda_n \mathrm{e}^{\lambda_n} - 6\mathrm{e}^{\lambda_n} + 4\mathrm{e}^{\lambda_n/2}\lambda_n + 6}{8\lambda_n} \end{bmatrix}.$$

A second, non-interpolatory, scheme can be derived by setting, as before, $y^0 = c_k^{[n]}$ and $y^1 = c_{k+1}^{[n]}$ in (2.4), but evaluating it now for $t = 1/4$ and $t = 3/4$. We obtain

$$D^n P\left(\frac{k}{2^n} + \frac{1}{2^{n+2}}\right) = H^0_{2^{-n}\Lambda}\left(\frac{1}{4}\right) D^n c_k^{[n]} + H^1\left(\frac{1}{4}\right) D^n c_{k+1}^{[n]},$$

$$D^n P\left(\frac{k}{2^n} + \frac{3}{2^{n+2}}\right) = H^0_{2^{-n}\Lambda}\left(\frac{3}{4}\right) D^n c_k^{[n]} + H^1\left(\frac{3}{4}\right) D^n c_{k+1}^{[n]},$$

which, by left-multiplying each side by D and interpreting the values of the refined vector sequence as:

$$c_k^{[n+1]} = P\left(\frac{k}{2^{n+1}} + \frac{1}{2^{n+2}}\right),$$

produce the following subdivision rules

$$D^{n+1} c_{2k}^{[n+1]} = A_0^{[n]} D^n c_k^{[n]} + A_{-2}^{[n]} D^n c_{k+1}^{[n]},$$

$$D^{n+1} c_{2k+1}^{[n+1]} = A_1^{[n]} D^n c_k^{[n]} + A_{-1}^{[n]} D^n c_{k+1}^{[n]},$$

with $A_{-2}^{[n]} = DH^1(1/4)$, $A_{-1}^{[n]} = DH^1(3/4)$, $A_0^{[n]} = DH^0_{2^{-n}\Lambda}(1/4)$, $A_1^{[n]} = DH^0_{2^{-n}\Lambda}(3/4)$.

In the case $p = 0$, $r = 1$, this gives:

$$A_{-2}^{[n]} = \begin{bmatrix} \frac{5}{32} & -\frac{3}{64} \\ \frac{9}{16} & -\frac{5}{32} \end{bmatrix}, A_{-1}^{[n]} = \begin{bmatrix} \frac{27}{32} & -\frac{9}{64} \\ \frac{9}{16} & \frac{3}{32} \end{bmatrix},$$

$$A_0^{[n]} = \begin{bmatrix} \frac{27}{32} & \frac{3\lambda_n e^{\lambda_n} - 10 e^{\lambda_n} + 64 e^{\lambda_n/4} - 54}{64\lambda_n} \\ -\frac{9}{16} & \frac{5\lambda_n e^{\lambda_n} - 18 e^{\lambda_n} + 16 e^{\lambda_n/4}\lambda_n + 18}{32\lambda_n} \end{bmatrix}, A_1^{[n]} = \begin{bmatrix} \frac{5}{32} & \frac{9\lambda_n e^{\lambda_n} + 64 e^{3/4\lambda_n} - 54 e^{\lambda_n} - 10}{64\lambda_n} \\ -\frac{9}{16} & \frac{16 e^{3/4\lambda_n}\lambda_n - 3\lambda_n e^{\lambda_n} - 18 e^{\lambda_n} + 18}{32\lambda_n} \end{bmatrix}.$$

We conclude the section by mentioning that, using the convergence results given in [2], it is possible to verify that each of the above constructions gives rise to a C^d-convergent subdivision scheme, that is, starting from any input vector data c, the sequence of refinements generated by (1.1) converges to a vector consisting of some C^d function (depending on c) and all its derivatives up to the order d. This implies the existence, for each scheme, of the so-called basic limit matrix function. In particular, for the case $p = 0$, $r = 1$, it corresponds to the matrix $\begin{bmatrix} \phi_0 & \phi_1 \\ \phi_0' & \phi_1' \end{bmatrix}$, where the columns are the limit of the subdivision process respectively applied to the delta vector sequences $\delta[1,0]^T$, $\delta[0,1]^T$. Such limit functions are shown in figs. 2, 3, for a particular choice of the parameter λ.

As a final remark, it is worthwhile to underline that the above classes of masks include, as limit cases, as the frequencies tend to zero, the standard (that is, polynomial reproducing) Hermite operators realized in [9].

Conclusion

In this short communication, we have illustrated a construction of level-dependent subdivision masks, for realizing Hermite subdivision preserving elements in the space of exponentials and polynomials. The practical implementation of such level-dependent subdivision recursions in the context of data reconstruction for testing their effectiveness with respect to standard Hermite subdivision goes beyond the scope of this note. It is currently being investigated, together with the realization of associated wavelet systems, in order to extend the range of application, for example to differential problems, and will be the topic of an upcoming paper.

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