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Structural symmetry within nonlocal integral elasticity: theoretical issues and computational strategies

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Abstract: The structural symmetry and the appropriate definition of a reduced (symmetric) mechanical/numerical model is discussed within a nonlocal elasticity context. In particular, reference is made to an integral model of Eringen-type. The paper highlights how the classical, *i.e.* local, concepts of structural symmetry have to be rephrased through the definition of an *enlarged symmetric model* of the analyzed structure. This enlarged model, endowed with apposite *nonlocal boundary conditions* enforced in an iterative fashion, is proved to be able to recover the nonlocal effects that the neglected portion of the structure exerts on the portion chosen for the analysis. It is shown how the mirrored symmetric solution exactly matches the complete one. Theoretical issues and computational strategies referred to a nonlocal version of the finite element method are discussed with reference to the analysis of a case-study.

Keywords: Structural symmetry; Nonlocal integral elasticity; Nonlocal finite element method; Enlarged symmetric model; Nonlocal boundary conditions.

1 Theoretical background and introductory remarks

Many boundary value problems (BVP) of solid mechanics are characterized by diffusive phenomena which arise at a scale that can be defined micro and/or nano, but whose influence is significant in the behavior of the structure at a macroscopic scale (see *e.g.* [1, 2, 3]). In such context a wide literature promotes the so-called *nonlocal continuum approaches* whose main peculiarity is the introduction of an *internal length material scale*. The latter, entering the constitutive equations, allows to catch the phenomena arising at the micro/nano structure, but preserving the hypothesis of continuity. The nonlocal approaches can be framed inside the so-called microcontinuum field theories which differ from each other for the way they introduce the internal length (see *e.g.* Eringen [3]) giving rise to a multitude of theories and methods already in the restricted realm of *nonlocal elasticity* hereafter addressed. In the relevant literature, can be cited, among others, some well assessed formulations, such as: the peridynamic model [4], the continualization procedures [5], the integral approach [6], the gradient approach [7], however the analysis of such formulations is not here discussed because is out of the aim of the present work. Indeed, this paper refers to a specific nonlocal elastic formulation which assumes a nonlocal elastic constitutive model of integral type, known as strain-difference model [8], which has been implemented by the authors (see *e.g.* [9] and references therein) in a non local finite element code used to solve the nonlocal structure analyzed in the following.

On the other hand, it also well known that many problems of solid/structural mechanics are characterized by *structural symmetry*, which is given by the presence, in the domain of definition of the tackled BVP, of subdomains, having identic shapes and delimited by lines or planes of symmetry. As a consequence, the solution computed in one of such subdomains, once mirrored, furnishes the correct solution of the entire structure with noticeable benefit from a computational point of view.

To this concern, it is worth noting that many numerical simulations of experimental tests on nano-structures or, among others, numerical simulations of experiments carried on nanocomposites and oriented to identification procedure of the internal length scale make use of structural symmetry. Moreover, the structural symmetry is often employed in other mechanical problems, for which the solving numerical procedures require a nonlocal treatment,

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such as in the solution of benchmark problems of fracture mechanics (mode I fracture for example).

The main goal of the present paper is then to investigate on the applicability of the theoretical framework of the *structural symmetry*, that, as already outlined, reduces the computational burdens of the analyzed problem, when a nonlocal elasticity of integral type is considered. Nevertheless, the remarks carried on seem to be of general validity in a wider context of nonlocal approaches. It is shown that, in order to obtain the correct solution, it is not sufficient to apply the concept of symmetry axis straightforward as in local elasticity; moreover both static or kinematic standard boundary conditions, acting along the symmetry lines or planes, need a nonlocal reinterpretation to consistently analyze the reduced symmetric structural portion. The paper, which is an enhanced version of a very recent contribution by the present authors on the topic [10], proposes an iterative numerical procedure to apply the above mentioned boundary conditions of nonlocal type, a point that in the above quoted paper, was left open to discussion. The effectiveness of the approach and the validity of the promoted rationale are proved by analyzing a simple example. Throughout the paper the displacements are assumed infinitesimal, while the applied loads are considered quasi-static. A brief description of the utilized nonlocal constitutive model, as well as of the related nonlocal finite element method (NL-FEM), for which more details can be found in [9], are given next for rendering the paper self-contained.

2 Constitutive assumptions: an Eringen-type integral model

Let's consider an elastic, isotropic, nonhomogeneous nonlocal material, occupying a volume V of the $3D$ Euclidean space and for which the following nonlocal constitutive relation is hypothesized [8]:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) - \alpha \int_V \mathcal{J}(\boldsymbol{x}, \boldsymbol{x}') : \left[\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})\right] \, \mathrm{d}\boldsymbol{x}' \quad (1)$$

$$\forall (\boldsymbol{x}, \boldsymbol{x}') \in V.$$

The stress field $\boldsymbol{\sigma}(\boldsymbol{x})$ given by equation (1) presents a first term which is of *local nature* beside a second *integral* term which possesses a *nonlocal nature*. The local stress term is governed by the standard elastic moduli tensor $\boldsymbol{D}(\boldsymbol{x})$ which is symmetric, positive definite and variable in space to manage also nonhomogeneous materials. The nonlocal stress term depends on the nonlocal operator $\mathcal{J}(\boldsymbol{x}, \boldsymbol{x}')$, which is also symmetric and positive definite, as well as on the strain difference field $\boldsymbol{\varepsilon}(\boldsymbol{x}')-\boldsymbol{\varepsilon}(\boldsymbol{x})$. A material parameter α, to be calibrated by identification procedures, multiplies the nonlocal term and controls the proportion of the nonlocal addition in the constitutive law.

The model given in (1) can be regarded as a generalization of the model proposed by Eringen in [6], but, unlike the latter, furnishes a uniform stress for any considered uniform strain field. Furthermore, the strain-difference formulation allows to remove undesired numerical instabilities close to the domain boundaries, circumstance that matches some experimental findings given in [11]. As shown in [12], in the context of nonlocal damage so referring to a damage-variable-difference field instead of a strain-difference field as in (1), the strain (or damage)-difference format is equivalent to assume a "symmetric nonlocal weight operator" as kernel of the integral (nonlocal) addition. Moreover, (1) can be viewed as a counterpart of the gradient formulations where the local field is enriched by a spatial gradient term which vanishes when there is no spatial variation.

In Eq.(1) $\mathcal{J}(\boldsymbol{x}, \boldsymbol{x}')$ is a nonlocal tensor defined as:

$$\mathcal{J}(\boldsymbol{x}, \boldsymbol{x}') := \left[\gamma(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x}) + \gamma(\boldsymbol{x}')\boldsymbol{D}(\boldsymbol{x}')\right] g(\boldsymbol{x}, \boldsymbol{x}') - \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x}') \quad (2)$$

$$\forall (\boldsymbol{x}, \boldsymbol{x}') \in V,$$

with:

$$\gamma(\boldsymbol{x}) := \int_V g(\boldsymbol{x}, \boldsymbol{x}') \, \mathrm{d}\,V'; \quad (3a)$$

$$\boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x}') := \int_V g(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{x}', \boldsymbol{y}) \boldsymbol{D}(\boldsymbol{y}) \, \mathrm{d}\,V^{y}. \quad (3b)$$

It is worth noting that the nonlocal operators defined in the expressions (2-3) arise from a thermodynamical consistent formulation (see *e.g.* Polizzotto *et al.* [8] for more insight). Moreover, in the above equations $g(\boldsymbol{x}, \boldsymbol{x}')$ is a positive scalar attenuation function depending on the Euclidean distance between points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in V and on an internal length material scale, say ℓ. This function plays an important role inside the nonlocal model and in practice it assigns a *weight* to the nonlocal effects induced at the field point $\boldsymbol{x}$ by a phenomenon acting at the source point $\boldsymbol{x}'$.

In the following the assumed attenuation function has a bi-exponential form that is:

$$g(\boldsymbol{x}, \boldsymbol{x}') := \lambda \exp\left(-|\boldsymbol{x} - \boldsymbol{x}'|/\ell\right), \quad (4)$$

where λ is a factor arising from the fulfilment of the normalization condition, that is:

$$\int_{V_\infty} g(\boldsymbol{x}, \boldsymbol{x}') \, \mathrm{d}\,V' = 1. \quad (5)$$

For ℓ approaching 0, the attenuation function degenerates into a Dirac delta allowing to recover the case of a local elastic material; moreover it rapidly decreases with increasing distance between $\boldsymbol{x}$ and $\boldsymbol{x}'$, eventually vanishing beyond the so-called *influence distance*, say L_R, the latter being a multiple of the internal length ℓ.

The analytical choice of $g(\boldsymbol{x}, \boldsymbol{x}')$, as well as of ℓ and, consequently, of L_R define the "extent of nonlocality" in the continuum model, allowing the connection between the macroscopic scale (nonlocal continuum model) and the real material scale (atomistic model). The above choices depends on the tackled nonlocal material see *e.g.* [13] or even on the analyzed mechanical problem [14].

3 Numerical tool: the Nonlocal Finite Element Method

The nonlocal (integral) formulation of the FEM, hereafter referred as NL-FEM, was originally conceived in [15], it was then rephrased with reference to the above discussed strain-difference model in [8] and finally, very recently, implemented in an effective numerical tool in [9]. Only the essentials are briefly reported in the following starting with the equations defining the nonlocal boundary value problem (NL-BVP), which can be given the shape:

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}) &= \boldsymbol{0} \quad \text{in } V; \\ \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) &= \boldsymbol{t}(\boldsymbol{x}) \quad \text{on } S_t; \end{aligned} \tag{6}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \nabla^s \boldsymbol{u}(\boldsymbol{x}) \quad \text{in } V; \qquad \boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) \quad \text{on } S_u; \tag{7}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) - \alpha \int_V \mathcal{J}(\boldsymbol{x}, \boldsymbol{x}') : \left[\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})\right] \mathrm{d}x' \tag{8}$$

in V.

In the above equations are easily recognizable the field and boundary equilibrium equations, (6), and the field and boundary compatibility equations, (7). They appear in their standard (local) format and are referred to a solid body/structure of volume V having boundary surface S. $\boldsymbol{b}(\boldsymbol{x})$ are the body forces acting in V, $\boldsymbol{t}(\boldsymbol{x})$ denote the surface tractions on S_t, $\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x})$ are the kinematic boundary conditions specified on $S_u = S - S_t$. The constituent material is, by hypothesis, the nonlocal elastic one satisfying Eq.(1) here rewritten as Eq. (8) for completeness.

Recalling the hypotheses of infinitesimal displacements and quasi-static loads, Equations (6–8) can be obtained as the optimality conditions of the following *nonlocal total potential energy functional*:

$$\begin{aligned} \Pi\left[\boldsymbol{u}(\boldsymbol{x})\right] := {} & \frac{1}{2} \int_V \nabla \boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{D}(\boldsymbol{x}) : \nabla \boldsymbol{u}(\boldsymbol{x}) \,\mathrm{d}V + \\ & + \frac{\alpha}{2} \int_V \nabla \boldsymbol{u}(\boldsymbol{x}) : \gamma^2(\boldsymbol{x}) \boldsymbol{D}(\boldsymbol{x}) : \nabla \boldsymbol{u}(\boldsymbol{x}) \,\mathrm{d}V + \\ & - \frac{\alpha}{2} \int_V \int_V \nabla \boldsymbol{u}(\boldsymbol{x}) : \mathcal{J}(\boldsymbol{x}, \boldsymbol{x}') : \nabla \boldsymbol{u}(\boldsymbol{x}') \,\mathrm{d}V' \,\mathrm{d}V + \\ & - \int_V \boldsymbol{b}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) \,\mathrm{d}V - \int_{S_t} \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{u}(\boldsymbol{x}) \,\mathrm{d}S. \end{aligned} \tag{9}$$

The functional expressed by Eq. (9) consists of a standard (local) part given by the first addend representing the *local elastic energy* and by the the forth and fifth addends which express the *energy potential of the external loads*. Beside this local part there are then two more terms, namely the second and the third ones which furnish the *nonlocal elastic energy*.

The NL-FEM formulation can be derived by discretizing the functional given by Eq. (9). Making use of the standard formalism of the FEM can be written:

$$\begin{aligned} \Pi\left[\boldsymbol{d_n}\right] = {} & \frac{1}{2} \sum_{n=1}^{N_e} \boldsymbol{d}_n^T \boldsymbol{k}_n^{loc} \boldsymbol{d}_n + \frac{\alpha}{2} \sum_{n=1}^{N_e} \boldsymbol{d}_n^T \boldsymbol{k}_n^{nonloc} \boldsymbol{d}_n + \\ & - \frac{\alpha}{2} \sum_{n=1}^{N_e} \sum_{m=1}^{N_e} \boldsymbol{d}_n^T \boldsymbol{k}_{nm}^{nonloc} \boldsymbol{d}_m - \sum_{n=1}^{N_e} \boldsymbol{d}_n^T \boldsymbol{f}_n \end{aligned} \tag{10}$$

where $\boldsymbol{f}_n$ is the vector of the equivalent nodal forces, $\boldsymbol{d}_n$ and $\boldsymbol{d}_m$ are the vectors collecting the nodal displacements and referred to the elements #n and #m respectively, while N_e gives the total number of FE in the whole mesh. Moreover, the following positions have been made:

$$\boldsymbol{k}_n^{loc} := \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x}) \, \boldsymbol{D}(\boldsymbol{x}) \, \boldsymbol{B}_n(\boldsymbol{x}) \,\mathrm{d}V_n; \tag{11}$$

$$\boldsymbol{k}_n^{nonloc} := \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x}) \, \gamma^2(\boldsymbol{x}) \, \boldsymbol{D}(\boldsymbol{x}) \, \boldsymbol{B}_n(\boldsymbol{x}) \,\mathrm{d}V_n; \tag{12}$$

$$\boldsymbol{k}_{nm}^{nonloc} := \int_{V_n} \int_{V_m} \boldsymbol{B}_n^T(\boldsymbol{x}) \, \mathcal{J}(\boldsymbol{x}, \boldsymbol{x}') \, \boldsymbol{B}_m(\boldsymbol{x}') \,\mathrm{d}V_m \,\mathrm{d}V_n; \tag{13}$$

$$\boldsymbol{f}_n := \int_{V_n} \boldsymbol{N}_n^T(\boldsymbol{x}) \, \boldsymbol{b}(\boldsymbol{x}) \,\mathrm{d}V_n + \int_{S_{t(n)}} \boldsymbol{N}_n^T(\boldsymbol{x}) \, \boldsymbol{t}(\boldsymbol{x}) \,\mathrm{d}S_n, \tag{14}$$

with $\boldsymbol{N}_n(\boldsymbol{x})$ being the suitable matrix of the shape functions referred to the element $n - th$ and $\boldsymbol{B}_n(\boldsymbol{x})$ being the matrix containing the Cartesian derivatives of $\boldsymbol{N}_n(\boldsymbol{x})$. V_n is the volume of element #n, while S_n is its surface boundary.

All the operators given by Eqs. (11–14) are defined with reference to a current element $\#n$. Moreover, it is easy to recognize in Eq. (11) the classical (local) stiffness matrix of the element and in Eq. (14) the classical equivalent nodal force vector, while Eq. (12) defines a *nonlocal stiffness* matrix and Eq. (13) gives a *set* of *nonlocal self* matrices (when $\#n \equiv \#m$) and nonlocal *cross stiffness* matrices (when $\#n \neq \#m$) .

The nonlocal nature of $\boldsymbol{k}_n^{nonloc}$ is due to the presence of the nonlocal operator $\gamma(\boldsymbol{x})$, which accounts for the influence exerted on the current element by the nonlocal diffusive processes over the whole domain. On the other hand, the nonlocal nature of $\boldsymbol{k}_{nm}^{nonloc}$, is witnessed by the presence, beside $\mathcal{J}(\boldsymbol{x}, \boldsymbol{x}')$, of the matrices $\boldsymbol{B}_n$ and $\boldsymbol{B}_m$ pertaining to the elements $\#n$ and $\#m$, respectively, such that, for each $\#n$, $\#m$ ranges, theoretically, over all the other elements in the mesh. Indeed, it is worth nothing that the nonlocal operators, such as $\gamma(\boldsymbol{x})$ and $\mathcal{J}(\boldsymbol{x}, \boldsymbol{x}')$ depending on the attenuation function $g(\boldsymbol{x}, \boldsymbol{x}')$ vanishes, together with the latter, beyond the influence distance L_R. As a consequence the cross contributions to be taken into account to build $\boldsymbol{k}_{nm}^{nonloc}$ are reduced to the elements falling inside a neighboring area around the current element, whose wideness depends on the influence distance L_R.

A solving global linear equation system could be obtained by minimizing the functional of Eq. (10) with respect to the global DOFs once they have been introduced by a standard rationale. The main peculiarity of the formulation concerns in the presence of a *symmetric positive definite nonlocal global stiffness* matrix which collects all the self– and cross–contributions of the FEs in the mesh. Further details are given in [10] but are in this paper omitted for brevity. The summarized NL-FEM is the numerical tool referred in the following to handle the FE analysis of a nonlocal elastic structure exhibiting structural symmetry.

4 Structural symmetry: the nonlocal symmetric mechanical model

As mentioned in Section 1, several engineering problems are characterized by structural symmetry; the solution of such problems appears identic within identical subdomains (symmetrical portions) in which the structure can be divided through lines or planes of symmetry. In the following it is assumed that the constituent material of the structure is isotropic and elastic; it is also assumed a unique solution for the pertinent (well posed) BVP.

It is important to remind that the definition of a symmetrical problem is subjected to the fulfillment of some peculiar properties that are: the geometrical shape of the structure must be symmetric as well as the boundary conditions and the applied loads, moreover all the material properties and coefficients entering the constitutive governing equation must posses symmetry too.

If such circumstances are met, the analysis of the structure can be reduced to that of only a symmetric portion of it, with evident computational and time gains. The portion to select is simply obtained by cutting the domain in correspondence to the symmetry lines and by removing the remaining portions of the structure. The cuts, that define the symmetric portion chosen for the analysis, create new boundaries which must be endowed with appropriate boundary conditions since they were lines or planes belonging to the internal part of the original domain.

The mechanical problem hereafter considered is defined in plane and, for simplicity, shows only one line of symmetry. In such circumstance the portion delimited by one side (with regard to material properties, geometry, loading and boundary conditions) as well as the pertinent solution computed on it (in terms of displacements, strains, stresses, etc.) are *mirror reflection* of the ones on the other side. It is well known that, in the context of local elasticity, the symmetry together with the smoothness of the exact solution, in terms of function and their derivatives, eventually furnish conditions of continuity along the new boundary or cut and then the appropriate (standard) boundary conditions to be enforced on it.

If the tackled problem is of *nonlocal nature*, *i.e.* the problem is governed by the set of equations (6)–(8) reported in Section 3, in order to classify it as symmetric the same requirements listed above on geometry, loading conditions, boundary conditions and material properties have to be satisfied. Nevertheless, as already said, for the peculiarities of the nonlocal diffusive processes, it is not possible to neglect the nonlocal influence that the removed portion of the structure exerts on the portion selected for the analysis. Moreover, having in mind the constitutive relation given by Eq. (1), the role played by the attenuation function, and, consequently the meaning of the influence distance L_R, it can be shown that the portion of the neglected structure that affects the solution of the analyzed one is adjacent to the symmetry line. Indeed, such portion is a *boundary zone* of wideness L_R. If the influence of such boundary zone is not taken into account the complete solution, obtained by mirroring the one computed on the symmetric portion, deviates from the exact one as confirmed by the numerical findings shown in the following section.

To overcome the above drawback and, as a consequence of the argumentations given in this section, the following key idea has been employed: it has to be considered an *enlarged symmetric model* resulting by the addition to the standard (local) symmetric portion of a *symmetrical boundary zone* which originally is a part of the portion to be removed and whose wideness is L_R. This remedy seems sensible and intuitive, it is also quite easy to be realized. It will be enough to extend the standard reduced mechanical model and the FE mesh within the symmetrical boundary zone having known geometry. The latter depends on the geometry of the entire problem, on the values set for the material parameters, on the chosen analytical shape of the attenuation function. However, a further question arises, which was left unsolved in the already quoted paper by the Authors [9] and here addressed: what are the apposite boundary conditions to be applied this time not only along the line of symmetry but also within the symmetrical boundary zone? As in local elasticity, it is necessary to consider appropriate boundary conditions to be applied in the analysis of the symmetrical portion. The conditions guarantee the matching between the solution computed on the symmetric portion with (once mirrored) the complete solution. A first convincement is that the (standard) boundary conditions have to be applied unaltered along the symmetry axis, in addition a sort of, let's say, *nonlocal boundary conditions* must be considered within the symmetrical boundary zone. A second convincement is that such nonlocal conditions (either of static or kinematic nature) are unknown, being part of the nonlocal solution of the problem.

In order to solve the above posed problem an iterative numerical procedure is proposed. A first analysis is carried on the enlarged symmetric model in which are applied only the (standard) boundary conditions enforced along the symmetry line, which is now in the interior of the model, the cut is now the one embracing the symmetrical boundary zone. A second analysis is carried on, smearing within the symmetrical boundary zone the nodal displacements, equivalent nodal forces, etc., obtained from the solution at the previous analysis, and so on till the obtained results give the same solution to within a fixed tolerance. No theoretical convergence proof is given hereafter, the validity of the procedure grounding on numerical evidence. Nevertheless, for the analyzed case the complete solution (*i.e.* the one computed on the whole structure) is available and will be used as reference solution to check the reliability of the results obtained at convergence. Further insights might be necessary to this concern. In the next Section the above rationale and the pertinent numerical implications are discussed with the aid of a simple example analyzed by means of the NL-FEM summarized in Section 3.

5 Case-study: enlarged symmetric model and boundary conditions

The symmetric square plate of Fig.1a is analyzed. This problem, already tackled by the authors in the above quoted references, is here solved applying the novel iterative numerical procedure promoted to enforce the nonlocal boundary conditions on the enlarged symmetric model shown in Fig. 1c.

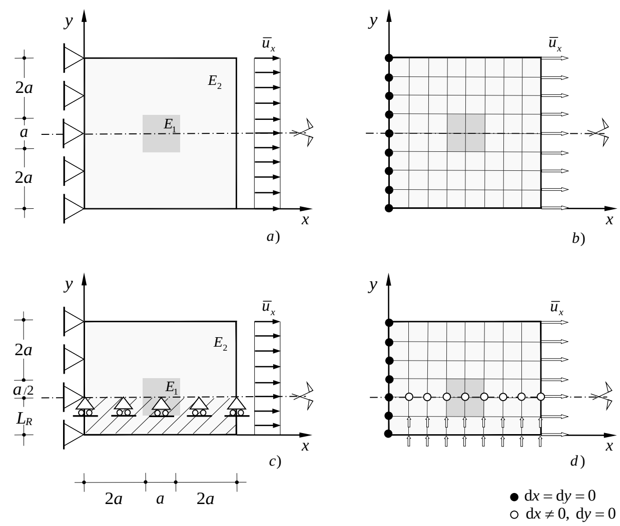

Figure 1: Nonlocal elastic symmetric square plate under tension with piecewise constant Young modulus. Mechanical model, boundary and loading conditions of: *a*) whole structure; *c*) enlarged symmetric model of the structure; *b*) and *d*) sketches of the NL-FE meshes adopted for models *a*) and *c*), respectively.

The symmetrical boundary zone, of wideness L_R, is put in evidence in Fig. 1c through the hatched area below the axis of symmetry. The related FE models are sketched in Figures 1b and 1d. A mesh of 400 isoparametric 8-nodes NL-FEs has been used for the complete model and of 320 NL-FEs for the enlarged one. Such number of NL-FEs it has be proved, in a previous study [9], being sufficient to ensure the accuracy of the results and their mesh independency. Beyond the standard boundary conditions, the nonlocal ones are also sketched in Fig. 1d, precisely the nodal vertical displacements smeared within the symmetrical boundary zone that will be enforced iteratively. Moreover, the plate has side length $a = 5$ cm and thickness equal to 0.5 cm. By hypothesis, the material constituent the plate is nonhomogeneous and nonlocal elastic. It has Young's modulus $E_2 = 260$ GPa in all the domain except for a central square inclusion of side $a = 1$ cm having Young's modulus $E_1 = 84$ GPa. The Poisson's coefficient ν is set equal to 0.2 and the nonlocal parameters assume the values $\alpha = 50$ and $\ell = 0.1$ cm, respectively. The adopted attenuation function is the one given by Eq. (5), that is the bi-exponential attenuation function, then the computational influence distance results $L_R = 11\ell$. The left edge of the

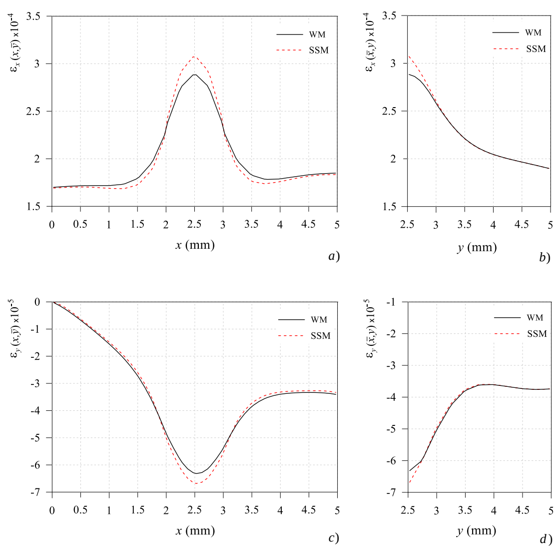

Figure 2: Nonlocal plate of figure 1. Strain components profiles ε_x and ε_y versus x at $y \backsimeq 2.5$ on the left side and versus y at $x \backsimeq 2.5$ on the right side. Solutions computed on the whole model (WM) of the plate (solid lines); solutions computed on the (half-structure) standard symmetric model (SSM) of the plate (dash lines).

plate is fixed, while the opposite (right) edge suffers uniform prescribed displacements of intensity $\bar{u}_x = 0.001$ cm, see Fig. 1a. The remaining edges, upper and lower, are free. The structure of Fig. 1a possesses an axis of symmetry coincident with the horizontal central axis.

If the the problem was of nonlocal nature, the half standard symmetric scheme obtained by considering one of the two halves above and below the symmetry horizontal axis, with their appropriate (standard) boundary conditions, could be used. However, if this standard model is used to compute the nonlocal solution the results are different from the expected ones and this difference increases with the nonlocal effects.

Figures 2a– 2d show such circumstance in terms of strain component profiles ε_x (along the loading direction) and ε_y (orthogonal to the loading direction) both plotted at mid-

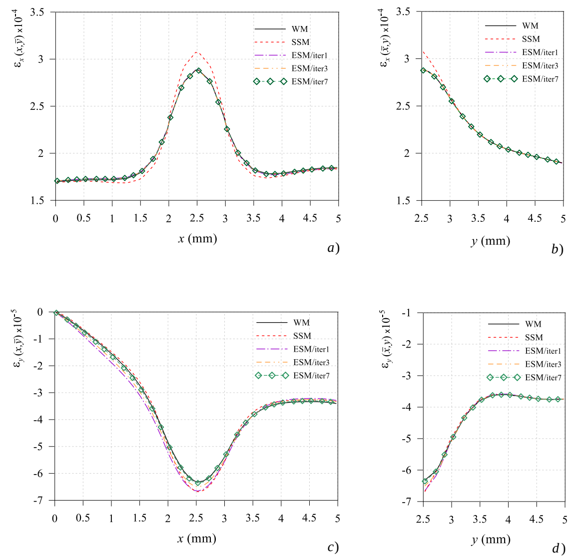

Figure 3: Nonlocal plate of figure 1. Strain components profiles ε_x and ε_y versus x at $y \backsimeq 2.5$ on the left side and versus y at $x \backsimeq 2.5$ on the right side. Solutions computed on the whole model (WM) of the plate (solid lines); solutions computed on the (half-structure) standard symmetric model (SSM) of the plate (dash lines); solutions computed on the enlarged symmetric model (ESM): at iteration 1 (dash-dot lines), at iteration 3 (double dash-double dot lines) and at iteration 7 (dash line with diamonds).

plate horizontal sections. By inspection of Figures 2a– 2d it is evident that the nonlocal solution computed on the standard symmetric model *deviates* from the complete one pertaining to the whole model. The nonlocal effects of the removed portion are missing. As already said, such nonlocal effects can be recovered by enlarging the standard symmetric scheme with the addition of a symmetrical boundary zone as the one sketched in Figures 1a– 1d. Figures 3a– 3d show the results obtained considering the enlarged symmetric model and the nonlocal boundary conditions enforced in an iterative fashion. It has been verified that the nodal vertical displacements turn out to be the appropriate nonlocal boundary conditions for the structure of Fig. 1c. The above displacements are zero at the symmetry axis and vary, increasing in absolute value, approaching the end of the symmetrical boundary zone. At the first iteration, the solution is computed on the enlarged symmetric model in which are applied only the (standard) boundary conditions enforced along the loaded/fixed edges and along the symmetry line. The vertical nodal displacements so computed within the symmetrical boundary zone, are then applied as nonlocal boundary conditions to carry on a second analysis (iteration) and this is repeated till two subsequent iterative solutions are close to each other. As shown in Figures 3a– 3d the exact solution (here assumed the one pertaining to the whole model) is attained in few iterations and the strain profiles are evaluated with a pretty good approximation.

6 Concluding remarks

The paper has focused on the analysis of symmetric structures within the context of nonlocal integral elasticity. It has been highlighted how the well known simplifications related to symmetry, if applied to structures made of nonlocal material, produce some incoherencies so that the solution evaluated on a symmetric portion of the structure deviates from the expected correct one.

The nonlocal elastic behavior, postulated for taking into account micro and nano phenomena, implies that what happens in a point of the domain is influenced by what happens in the neighboring points. Therefore, when the solution is computed making use of a symmetric portion of the structure, defined by removal of one, or more, parts the nonlocal effects exerted by the parts removed are lost.

To recover the claimed nonlocal effects the paper proposes the employment of an enlarged symmetric model of the structure. The latter is achieved by simply adding a symmetrical boundary zone, beyond the lines of symmetry, to the standard symmetric model. The wideness of such boundary zone is related to the internal length material scale.

Moreover, the paper proposes the application of appropriate boundary conditions on the symmetrical boundary zone, to be enforced in an iterative fashion, for guaranteeing the mechanical equivalence between the reduced model and the original structure.

The proposed remedies surely deserve further investigations but, at least for the case-study addressed, appear effective being able to remove the described inconveniences.

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