Hindawi
Journal of Control Science and Engineering
Volume 2017, Article ID 9360430, 9 pages
https://doi.org/10.1155/2017/9360430

Research Article

Stability and Hopf Bifurcation for a Delayed Computer Virus Model with Antidote in Vulnerable System

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Received 6 March 2017; Revised 21 May 2017; Accepted 25 May 2017; Published 20 June 2017

Academic Editor: Petko Petkov

A delayed computer virus model with antidote in vulnerable system is investigated. Local stability of the endemic equilibrium and existence of Hopf bifurcation are discussed by analyzing the associated characteristic equation. Further, direction of the Hopf bifurcation and stability of the bifurcating periodic solutions are investigated by using the normal form theory and the center manifold theorem. Finally, numerical simulations are presented to show consistency with the obtained results.

1. Introduction

Applications based on computer networks are becoming more and more popular in our daily life. While bringing convenience to us, computer networks are exposed to various threats [1, 2]. Therefore, it is urgent to explore the spreading law of computer viruses in networks. To this end, many dynamical models describing propagation of computer viruses have been established by scholars at home and abroad. Particularly the classic epidemic models, such as SIRS [3–5] model, SEIRS model [6], and SEIQRS model [7, 8], are used to investigate the spreading law of computer viruses due to the common feature between the computer virus and the biological virus. Some computer virus models with infectivity in both seizing and latent computers have been also proposed by Yang et al. [9–13].

Recently, Khanh and Huy [14] proposed the following computer virus model with different antidote rates of nodes in vulnerable system considering the immunizations ways and the vulnerabilities of the operating system:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t)L(t) + \varpi I(t) - \varepsilon R(t) - \mu S(t),\\
\frac{dL(t)}{dt} &= \beta S(t)L(t) - (\delta + \gamma + \mu)L(t),\\
\frac{dI(t)}{dt} &= \gamma L(t) - (\alpha + \mu + \varpi)I(t),\\
\frac{dR(t)}{dt} &= \delta L(t) + \alpha I(t) - (\mu - \varepsilon)R(t),
\end{aligned} \tag{1}$$

where $S(t)$, $L(t)$, $I(t)$, and $R(t)$ are the sizes of susceptible, exposed, infectious, and recovered nodes at time t, respectively; A is the constant recruitment of the susceptible nodes; μ is the same rate at which every node in the states $S(t)$, $L(t)$, $I(t)$, and $R(t)$ disconnects from the network; ε is the constant rate at which every susceptible node acquires temporary immunity due to antidote and Khanh and Huy [14] assume that $\varepsilon < \mu$ taking system vulnerability into account; α, β, δ, and ϖ are the other state transition rates of system (1). Khanh and Huy [14] studied stability of system (1) and suggested some effective strategies for eliminating viruses.

It is well known that time delays of one type or another have been incorporated into computer virus models due to latent period of virus, temporary immunization period of nodes, or other reasons. Computer virus models with time delay have been investigated extensively in recent years [15–18]. Time delays can play a complicated role in the dynamics of dynamical systems, especially that they can cause Hopf bifurcation phenomenon of the models. In reality, the occurrence of Hopf bifurcation means that the state of computer virus spreading changes from an equilibrium point to a limit cycle, which is not welcomed in networks. Motivated by the work above and considering that it needs a period to reinstall system for the infected nodes in the network, we propose the following system with time delay:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t)L(t) + \varpi I(t-\tau) - \varepsilon R(t) - \mu S(t),\\
\frac{dL(t)}{dt} &= \beta S(t)L(t) - (\delta+\gamma+\mu)L(t),\\
\frac{dI(t)}{dt} &= \gamma L(t) - (\alpha+\mu)I(t) - \varpi I(t-\tau),\\
\frac{dR(t)}{dt} &= \delta L(t) + \alpha I(t) - (\mu-\varepsilon)R(t),
\end{aligned} \tag{2}$$

where τ is the time delay due to the period that the infected nodes use to reinstall system.

The rest of the paper is organized as follows. Section 2 obtains the basic reproduction number of system (2) and discusses local stability of the endemic equilibrium and existence of Hopf bifurcation; Section 3 determines direction of the Hopf bifurcation and stability of the bifurcated periodic solutions; Section 4 covers numerical analysis and simulations. Finally, Section 5 summarizes this work.

2. Existence of Hopf Bifurcation

By a direct computation, we get that if $R_0 = \beta A/(\mu(\delta+\gamma+\mu)) > 1$, then system (2) has a unique endemic equilibrium $P_*(S_*, L_*, I_*, R_*)$, where $S_* = (\delta+\gamma+\mu)/\beta$, $L_* = (\mu-\varepsilon)(\alpha+\mu+\varpi)G/(\beta F)$, $I_* = \gamma(\mu-\varepsilon)G/(\beta F)$, and $(\alpha\delta+\alpha\gamma+\delta\mu+\delta\varpi)G/(\beta F)$ with

$$\begin{aligned}
F &= \mu\left[\alpha(\delta+\gamma+\mu-\varepsilon) + \delta(\mu+\varpi)\right.\\
&\quad \left.+ (\mu-\varepsilon)(\gamma+\mu+\varpi)\right],\\
G &= \beta A - \mu(\delta+\gamma+\mu),
\end{aligned} \tag{3}$$

and R_0 is the basic reproduction number of system (2).

Based on the leading matrix of system (2) at the endemic equilibrium $P_*(S_*, L_*, I_*, R_*)$, the characteristic equation can be obtained as the following form:

$$\begin{vmatrix}
\lambda-\alpha_{11} & -\alpha_{12} & -\beta_{13}e^{-\lambda\tau} & -\alpha_{14}\\
-\alpha_{21} & \lambda-\alpha_{22} & 0 & 0\\
0 & -\alpha_{32} & \lambda-\alpha_{33}-\beta_{33}e^{-\lambda\tau} & 0\\
0 & -\alpha_{42} & -\alpha_{43} & \lambda-\alpha_{44}
\end{vmatrix} = 0, \tag{4}$$

which derives that

$$M(\lambda) + N(\lambda)e^{-\lambda\tau} = 0, \tag{5}$$

where

$$M(\lambda) = \lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0, \tag{6}$$

$$N(\lambda) = \beta_3\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0, \tag{7}$$

with

$$\begin{aligned}
\alpha_0 &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\alpha_{33}\alpha_{44}\\
&\quad + (\alpha_{33}\alpha_{42} - \alpha_{32}\alpha_{43})\alpha_{14}\alpha_{21},\\
\alpha_1 &= (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22})(\alpha_{33}+\alpha_{44})\\
&\quad - (\alpha_{11}+\alpha_{22})\alpha_{33}\alpha_{44} - \alpha_{14}\alpha_{21}\alpha_{42},\\
\alpha_2 &= \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + \alpha_{33}\alpha_{44}\\
&\quad + (\alpha_{11}+\alpha_{22})(\alpha_{33}+\alpha_{44}),\\
\alpha_3 &= -(\alpha_{11}+\alpha_{22}+\alpha_{33}+\alpha_{44}),\\
\beta_0 &= (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})\alpha_{44}\beta_{33}\\
&\quad + \alpha_{21}(\alpha_{14}\alpha_{42}\beta_{33} + \alpha_{32}\alpha_{44}\beta_{13}),\\
\beta_1 &= \beta_{33}(\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}) - \alpha_{44}\beta_{33}(\alpha_{11}+\alpha_{22})\\
&\quad - \alpha_{21}\alpha_{32}\beta_{13},\\
\beta_2 &= \beta_{33}(\alpha_{11}+\alpha_{22}+\alpha_{44}),\\
\beta_3 &= -\beta_{33},\\
\alpha_{11} &= -\beta L_* - \mu,\\
\alpha_{12} &= -\beta S_*,\\
\alpha_{14} &= \varpi,\\
\alpha_{21} &= \beta L_*,\\
\alpha_{22} &= \beta S_* - (\delta+\gamma+\mu),\\
\alpha_{32} &= \gamma,\\
\alpha_{33} &= -(\alpha+\mu),\\
\alpha_{42} &= \delta,\\
\alpha_{43} &= \alpha,\\
\alpha_{44} &= -(\mu-\varepsilon),\\
\beta_{13} &= \varpi,\\
\beta_{33} &= -\varpi.
\end{aligned} \tag{8}$$

When $\tau = 0$, (5) takes the following form:

$$\begin{aligned}
&\lambda^4 + (\alpha_3+\beta_3)\lambda^3 + (\alpha_2+\beta_2)\lambda^2 + (\alpha_1+\beta_1)\lambda + \alpha_0\\
&\quad + \beta_0 = 0.
\end{aligned} \tag{9}$$

According to the expressions of α_i and β_i $(i = 0, 1, 2, 3)$, we can get

$$\begin{aligned}
\alpha_0 + \beta_0 &= \mu(\mu-\varepsilon)(\alpha+\mu+\varpi)\left[\alpha(\gamma+\mu+\delta-\varepsilon)\right.\\
&\quad \left.+ \delta(\mu+\varpi) + (\mu-\varepsilon)(\gamma+\mu+\varpi)\right]Q,\\
\alpha_1 + \beta_1 &= (\mu-\varepsilon)(\alpha+\mu+\varpi)\left[\mu + 2\mu(\mu-\varepsilon+\varpi)\right.\\
&\quad \left.+ (2\mu-\varepsilon)(\alpha+\gamma+\varpi) + \alpha(\delta+\gamma) + \delta\varpi\right]Q,
\end{aligned}$$

$$\begin{aligned}\alpha_2+\beta_2 &= \alpha(2\mu-\varepsilon)+(\mu-\varepsilon)(2\mu+\varpi)+\mu(\mu+\varpi)\\ &\quad +(\mu-\varepsilon)(\alpha+\mu+\varpi)(\alpha+\delta-\varepsilon+\gamma+3\mu+\varpi)Q,\\ \alpha_3+\beta_3 &= \alpha+\varpi+3\mu-\varepsilon+(\mu-\varepsilon)(\alpha+\mu+\varpi)Q,\end{aligned} \tag{10}$$

with $Q = G/F$ and F, G are specified by (6) and (7), respectively. According to the assumption $\varepsilon < \mu$ considering system vulnerability, we know that $\alpha_0 + \beta_0 > 0$, $\alpha_1 + \beta_1 > 0$, $\alpha_2 + \beta_2 > 0$, and $\alpha_3 + \beta_3 > 0$. In addition, one can be conclude that

$$\begin{aligned}&(\alpha_1+\beta_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)-(\alpha_1+\beta_1)^2\\ &\quad -(\alpha_0+\beta_0)(\alpha_3+\beta_3)^2>0\end{aligned} \tag{11}$$

based on the analysis by Khanh and Huy in [14] when $R_0 > 1$. Therefore, we can conclude that the endemic equilibrium $P_*(S_*, L_*, I_*, R_*)$ is locally stable for $R_0 > 1$ according to the Routh-Hurwitz criterion.

The time delay is always positive in our physical system. It follows that if instability occurs for a particular value of the delay $\tau > 0$, a characteristic root of (5) must intersect the imaginary axis according to Corollary 2.4 in [19]. Assume that (5) has a purely imaginary root $i\omega$ ($\omega > 0$). Then, the following identity must be true

$$\begin{aligned}&(\beta_1\omega-\beta_3\omega^3)\sin\tau\omega+(\beta_0-\beta_2\omega^2)\cos\tau\omega\\ &\quad =\alpha_2\omega^2-\omega^4-\alpha_0,\\ &(\beta_1\omega-\beta_3\omega^3)\cos\tau\omega-(\beta_0-\beta_2\omega^2)\sin\tau\omega\\ &\quad =\alpha_3\omega^3-\alpha_1\omega,\end{aligned} \tag{12}$$

which yields the following equation:

$$\omega^8+p_3\omega^6+p_2\omega^4+p_1\omega^2+p_0=0, \tag{13}$$

where

$$\begin{aligned}p_0 &= \alpha_0^2-\beta_0^2,\\ p_1 &= \alpha_1^2-2\alpha_0\alpha_2-\beta_1^2+2\beta_0\beta_2,\\ p_2 &= \alpha_2^2+2\alpha_0-2\alpha_1\alpha_3+2\beta_1\beta_3-\beta_2^2,\\ p_3 &= \alpha_3^2-\beta_3^2-2\alpha_2.\end{aligned} \tag{14}$$

Denote $\omega^2 = v$; (13) becomes

$$v^4+p_3v^3+p_2v^2+p_1v+p_0=0. \tag{15}$$

Define

$$f(v)=v^4+p_3v^3+p_2v^2+p_1v+p_0. \tag{16}$$

Thus,

$$f'(v)=4v^3+3p_3v^2+2p_2v+p_1. \tag{17}$$

Set

$$4v^3+3p_3v^2+2p_2v+p_1=0. \tag{18}$$

Let $y = v + 3p_3/4$. Then, (18) becomes

$$y^3+r_1y+s_1=0, \tag{19}$$

where

$$\begin{aligned}r_1 &= \frac{p_2}{2}-\frac{3}{16}p_3^2,\\ s_1 &= \frac{p_3^3}{32}-\frac{p_2p_3}{8}+p_1.\end{aligned} \tag{20}$$

Denote

$$\begin{aligned}D &= \left(\frac{s_1}{2}\right)^2+\left(\frac{r_1}{3}\right)^3,\\ \sigma &= \frac{-1+\sqrt{3}i}{2},\\ y_1 &= \sqrt[3]{-\frac{s_1}{2}+\sqrt{D}}+\sqrt[3]{-\frac{s_1}{2}-\sqrt{D}},\\ y_2 &= \sqrt[3]{-\frac{s_1}{2}+\sqrt{D}}\sigma+\sqrt[3]{-\frac{s_1}{2}-\sqrt{D}}\sigma^2,\\ y_3 &= \sqrt[3]{-\frac{s_1}{2}+\sqrt{D}}\sigma^2+\sqrt[3]{-\frac{s_1}{2}-\sqrt{D}}\sigma,\\ v_i &= y_i-\frac{3p_3}{4},\quad i=1,2,3.\end{aligned} \tag{21}$$

Discussion about the distribution of the roots of (15) is similar to that in [20]. Thus, we have the following lemma.

Lemma 1. *For (15), one has the following:*

(i) *If $p_0 < 0$, (15) has at least one positive root.*

(ii) *If $p_0 \geq 0$ and $D \geq 0$, (15) has positive roots if and only if $v_1 > 0$ and $f(v_1) < 0$.*

(iii) *If $p_0 \geq 0$ and $D < 0$, (15) has positive roots if and only if there exists at least one $v_* \in \{v_1, v_2, v_3\}$ such that $v_* > 0$ and $f(v_*) \leq 0$.*

In what follows, we assume that (H_1)and the coefficients in $f(v)$ satisfy one of the following conditions in (a)–(c).

(a) $p_0 < 0$.

(b) $p_0 \geq 0$, $D \geq 0$, $v_1 > 0$, and $f(v_1) < 0$.

(c) $p_0 \geq 0$ and $D < 0$, and there exists at least one $v_* \in \{v_1, v_2, v_3\}$ such that $v_* > 0$ and $f(v_*) \leq 0$.

Suppose that the condition (H_1) holds; then (15) has at least one positive root v_0 such that (13) has a positive root $\omega_0 = \sqrt{v_0}$. Further, we can get

$$\tau_0 = \frac{1}{\omega_0} \arccos \left\{ \frac{(\beta_2 - \alpha_3\beta_3)\,\omega_0^6 + (\alpha_1\beta_3 + \alpha_3\beta_1 - \alpha_2\beta_2)\,\omega_0^4 + (\alpha_0\beta_2 + \alpha_2\beta_0 - \alpha_1\beta_1)\,\omega_0^2 - \alpha_0\beta_0}{(\beta_1\omega_0 - \beta_3\omega_0^3)^2 + (\beta_0 - \beta_2\omega_0^2)^2} \right\}. \tag{22}$$

Differentiating both sides of (5) with respect to τ yields

$$\begin{aligned} \left[\frac{d\lambda}{d\tau}\right]^{-1} &= -\frac{4\lambda^3 + 3\alpha_3\lambda^2 + 2\alpha_2\lambda + \alpha_1}{\lambda\,(\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0)} \\ &\quad + \frac{3\beta_3\lambda^2 + 2\beta_2\lambda + \beta_1}{\lambda\,(\beta_3\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0)} - \frac{\tau}{\lambda}. \end{aligned} \tag{23}$$

Further, we have

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{f'(v_{**})}{(\beta_1\omega_0 - \beta_3\omega_0^3)^2 + (\beta_0 - \beta_2\omega_0^2)^2}, \tag{24}$$

where $f(v) = v^4 + p_3v^3 + p_2v^2 + p_1v + p_0$ and $v_{**} = \omega_0^2$.

Obviously, if the condition (H_2), $f'(\omega_0^2) \neq 0$, holds, then $\operatorname{Re}[d\lambda/d\tau]^{-1}_{\tau=\tau_0} \neq 0$. Therefore, the transversality condition is satisfied if (H_2), $f'(\omega_0^2) \neq 0$, holds. According to the Hopf bifurcation theorem in [21], we can obtain the following results.

Theorem 2. *Suppose that conditions (H_1) and (H_2) hold for system (2). The endemic equilibrium $P_*(S_*, L_*, I_*, R_*)$ is locally asymptotically stable when $\tau \in [0, \tau_0)$ and a Hopf bifurcation occurs at the endemic equilibrium $P_*(S_*, L_*, I_*, R_*)$ when $\tau = \tau_0$.*

3. Properties of the Hopf Bifurcation

Let $\tau = \tau_0 + \mu$, $\mu \in R$. By the transformation, $x_1(t) = S(t) - S_*$, $x_2(t) = L(t) - L_*$, $x_3(t) = I(t) - I_*$, $x_4(t) = R(t) - R_*$, $t \to (t/\tau)$, and system (2) is equivalent to the following in $C = C([-1, 0], R^4)$:

$$\dot{x}(t) = L_\mu x_t + F(\mu, x_t), \tag{25}$$

where $x_t = (x_1(t), x_2(t), x_3(t), x_4(t))^T = (S, L, I, R)^T \in R^4$, $x_t(\theta) = x(t + \theta) \in C$, and L_μ, $F(\mu, x_t)$ are defined as follows, respectively:

$$\begin{gathered} L_\mu\phi = (\tau_0 + \mu)\,(M\phi(0) + N\phi(-1)), \\ F(\mu, \phi) = (\tau_0 + \mu) \begin{pmatrix} -\beta\phi_1(0)\,\phi_2(0) \\ \beta\phi_1(0)\,\phi_2(0) \\ 0 \\ 0 \end{pmatrix}, \end{gathered} \tag{26}$$

with

$$\begin{gathered} M = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{33} & 0 \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}, \\ N = \begin{pmatrix} 0 & 0 & \beta_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{27}$$

Thus, there exists a 4×4 matrix function whose components are functions $\eta(\theta)$ of the bounded variation in $\theta \in [-1, 0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\,\phi(\theta), \quad \phi \in C. \tag{28}$$

In fact, we choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)\,(M\delta(\theta) + N\delta(\theta + 1)). \tag{29}$$

For $\phi \in C([-1, 0], R^4)$, define

$$\begin{gathered} A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\,\phi(\theta), & \theta = 0, \end{cases} \\ R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \end{gathered} \tag{30}$$

Then system (25) is equivalent to

$$\dot{x}(t) = A(\mu)\,u_t + R(\mu)\,u_t. \tag{31}$$

For $\varphi \in C^1([0, 1], (R^4)^*)$, we define the adjoint operator A^* of $A(0)$

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s, 0)\,\varphi(-s), & s = 0, \end{cases} \tag{32}$$

and the following bilinear inner product

$$\begin{aligned}\langle \varphi(s), \phi(\theta)\rangle &= \overline{\varphi}(0)\,\phi(0)\\ &\quad - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\varphi}(\xi-\theta)\, d\eta(\theta)\,\phi(\xi)\, d\xi,\end{aligned} \tag{33}$$

where $\eta(\theta) = \eta(\theta, 0)$.

Let $q(\theta) = (1, q_2, q_3, q_4)^T e^{i\omega_0\tau_0\theta}$ be the eigenvector of $A(0)$ belonging to $+i\omega_0\tau_0$ and $q^*(s) = V(1, q_2^*, q_3^*, q_4^*)e^{i\omega_0\tau_0 s}$ be the eigenvector of $A^*(0)$ belonging to $-i\omega_0\tau_0$. It is not difficult to show that

$$\begin{aligned} q_2 &= \frac{\alpha_{21}}{i\omega_0 - \alpha_{22}},\\ q_3 &= \frac{\alpha_{32} q_2}{i\omega_0 - \alpha_{33} - \beta_{33} e^{-i\tau_0\omega_0}},\\ q_4 &= \frac{i\omega_0 - \alpha_{11} - \alpha_{12} q_2 - \beta_{13} e^{-i\tau_0\omega_0} q_3}{\alpha_{14}},\\ q_2^* &= -\frac{i\omega_0 + \alpha_{11}}{\alpha_{21}},\\ q_3^* &= -\frac{\beta_{13} e^{i\tau_0\omega_0} + \alpha_{43} q_3}{i\omega_0 + \alpha_{33} + \beta_{33} e^{i\tau_0\omega_0}},\\ q_4^* &= -\frac{\alpha_{14}}{i\omega_0 + \alpha_{44}}. \end{aligned} \tag{34}$$

From (33), we can get

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \overline{V}\left[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^*\right.\\ &\quad \left.+ \tau_0 e^{-i\tau_0\omega_0} q_3\left(\beta_{13} + \beta_{33}\overline{q}_3^*\right)\right].\end{aligned} \tag{35}$$

Then we choose

$$\begin{aligned}\overline{V} &= \left[1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^*\right.\\ &\quad \left.+ \tau_0 e^{-i\tau_0\omega_0} q_3\left(\beta_{13} + \beta_{33}\overline{q}_3^*\right)\right]^{-1}.\end{aligned} \tag{36}$$

such that $\langle q^*, q\rangle = 1$.

Next, we compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solutions of (31) when $\mu = 0$. Define

$$\begin{aligned} z(t) &= \langle q^*, x_t\rangle,\\ W(t,\theta) &= x_t(\theta) - 2\operatorname{Re}\{z(t)\,q(\theta)\}.\end{aligned} \tag{37}$$

On the center manifold C_0, we have

$$\begin{aligned} W(t,\theta) &= W(z,\overline{z},\theta)\\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)\,z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2}\\ &\quad + \cdots,\end{aligned} \tag{38}$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q}^*$. We now consider only the real solution $x_t \in C_0$ of (31), which gives

$$\begin{aligned}\dot{z} &= i\omega_0\tau_0 z + \overline{q}^* F\left(0, W(z,\overline{z},0) + 2\operatorname{Re}\{zq(\theta)\}\right)\\ &= i\omega_0\tau_0 z + g(z,\overline{z}),\end{aligned} \tag{39}$$

where

$$\begin{aligned} g(z,\overline{z}) &= \overline{q}^*(0)\,F_0(z,\overline{z})\\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots.\end{aligned} \tag{40}$$

Thus,

$$\begin{aligned} &g(z,\overline{z})\\ &\quad = \overline{V}\left(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*\right)\left(F_1(0,x_t), F_2(0,x_t), 0, 0\right)^T,\end{aligned} \tag{41}$$

where

$$\begin{aligned} F_1(0,x_t) &= -\beta\tau_0\left[\phi_1(0)\,\phi_2(0)\right],\\ F_2(0,x_t) &= \beta\tau_0\left[\phi_1(0)\,\phi_2(0)\right].\end{aligned} \tag{42}$$

Since

$$\begin{aligned} x_t &= W(z,\overline{z},\theta) + zq(\theta) + \overline{z}\,\overline{q}(\theta),\\ q(\theta) &= \left(1, q_2, q_3, q_4\right)^T e^{i\omega_0\tau_0\theta},\end{aligned} \tag{43}$$

we have

$$\begin{aligned} x_t &= \begin{bmatrix} W^{(1)}(t+\theta)\\ W^{(2)}(t+\theta)\\ 0\\ 0\end{bmatrix} + z\begin{bmatrix}1\\ q_2\\ q_3\\ q_4\end{bmatrix} e^{i\omega_0\tau_0\theta}\\ &\quad + \overline{z}\begin{bmatrix}1\\ \overline{q}_2\\ \overline{q}_3\\ \overline{q}_4\end{bmatrix} e^{-i\omega_0\tau_0\theta},\\ \phi_1(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)\,z\overline{z} + W_{02}^{(1)}\frac{\overline{z}^2}{2}\\ &\quad + \cdots,\\ \phi_2(0) &= zq_2 + \overline{z}\,\overline{q}_2 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)\,z\overline{z}\\ &\quad + W_{02}^{(2)}\frac{\overline{z}^2}{2} + \cdots,\\ \phi_3(0) &= zq_3 + \overline{z}\,\overline{q}_3 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)\,z\overline{z}\\ &\quad + W_{02}^{(3)}\frac{\overline{z}^2}{2} + \cdots,\end{aligned}$$

$$\phi_4(0) = zq_4 + \overline{z}\,\overline{q}_4 + W_{20}^{(4)}(0)\frac{z^2}{2} + W_{11}^{(4)}(0)\,z\overline{z} + W_{02}^{(4)}\frac{\overline{z}^2}{2} + \cdots. \tag{44}$$

From (40)–(43), we have

$$g(z,\overline{z}) = \overline{V}\left(1, \overline{q}_2^*, \overline{q}_3^*, \overline{q}_4^*\right) \times \begin{bmatrix} M_{11}z^2 + M_{12}z\overline{z} + M_{13}\overline{z}^2 + M_{14}z^2\overline{z} \\ M_{21}z^2 + M_{22}z\overline{z} + M_{23}\overline{z}^2 + M_{24}z^2\overline{z} \\ 0 \\ 0 \end{bmatrix} + \cdots, \tag{45}$$

with

$$\begin{aligned} M_{11} &= -\beta q_2, \\ M_{12} &= -\beta\left(q_2 + \overline{q}_2\right), \\ M_{13} &= -\beta\overline{q}_2, \\ M_{14} &= -\beta\Big(W_{11}^{(1)}(0)\,q_2 + \frac{1}{2}W_{20}^{(1)}(0)\,\overline{q}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big), \\ M_{21} &= \beta q_2, \\ M_{22} &= \beta\left(q_2 + \overline{q}_2\right), \\ M_{23} &= \beta\overline{q}_2, \\ M_{24} &= \beta\Big(W_{11}^{(1)}(0)\,q_2 + \frac{1}{2}W_{20}^{(1)}(0)\,\overline{q}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big). \end{aligned} \tag{46}$$

Therefore,

$$g(z,\overline{z}) = \overline{V}\left[\left(M_{11} + \overline{q}_2^* M_{21}\right)z^2 + \left(M_{12} + \overline{q}_2^* M_{22}\right)z\overline{z} + \left(M_{13} + \overline{q}_2^* M_{23}\right)\overline{z}^2 + \left(M_{14} + \overline{q}_2^* M_{24}\right)z^2\overline{z}\right] + \cdots. \tag{47}$$

Thus, from (40) and (47), we can obtain

$$\begin{aligned} g_{20} &= 2\tau_0\overline{V}\beta q_2\left(\overline{q}_2^* - 1\right), \\ g_{11} &= \tau_0\overline{V}\left(\overline{q}_2^* - 1\right)\beta\left(q_2 + \overline{q}_2\right), \\ g_{02} &= 2\tau_0\overline{V}\beta\overline{q}_2\left(\overline{q}_2^* - 1\right), \\ g_{21} &= 2\beta\tau_0\overline{V}\left(\overline{q}_2^* - 1\right)\beta\Big(W_{11}^{(1)}(0)\,q_2 + \frac{1}{2}W_{20}^{(1)}(0)\,\overline{q}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\Big). \end{aligned} \tag{48}$$

In order to compute g_{21}, we need to compute W_{20} and W_{11}. From (37)-(40), we have

$$\begin{aligned} \dot{W} &= \begin{cases} AW - 2\operatorname{Re}\left\{\overline{q}^*(0)\,F_0 q(\theta)\right\}, & -1 \le \theta < 0, \\ AW - 2\operatorname{Re}\left\{\overline{q}^*(0)\,F_0 q(\theta)\right\} + F_0, & \theta = 0, \end{cases} \\ &= AW + H(z,\overline{z},\theta), \end{aligned} \tag{49}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)\frac{(z\overline{z})}{2} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{50}$$

From (38), (39), (49), and (50), one can obtain

$$\left(2i\omega_0\tau_0 - A\right)W_{20}(\theta) = H_{20}(\theta), \tag{51}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{52}$$

Now for $\theta \in [-1, 0)$,

$$\begin{aligned} H(z,\overline{z},\theta) &= -2\operatorname{Re}\left\{\overline{q}^*(0)\,F_0 q(\theta)\right\} \\ &= -\left(g_{20}q(\theta) + \overline{g}_{11}\overline{q}(\theta)\right)\frac{z^2}{2} - \left(g_{11}q(\theta) + \overline{g}_{02}\overline{q}(\theta)\right)z\overline{z} + \cdots, \end{aligned} \tag{53}$$

which on comparing the coefficients with (50) gives

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \tag{54}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{55}$$

From (51), (54), and the definition of A, we have

$$\begin{aligned} \dot{W}_{20}(\theta) &= 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta), \\ \dot{W}_{11}(\theta) &= g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta). \end{aligned} \tag{56}$$

Thus,

$$W_{20}(\theta) = \frac{ig_{20}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_1 e^{2i\tau_0\omega_0\theta}, \tag{57}$$

$$W_{11}(\theta) = -\frac{ig_{11}q(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_2, \tag{58}$$

where E_1 and E_2 are constant vectors, to be determined. It follows from the definition of A, (54), and (55) that

$$\int_{-1}^{0} d\eta(\theta)\,W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) - H_{20}(0), \tag{59}$$

$$\int_{-1}^{0} d\eta(\theta)\,W_{11}(\theta) = -H_{11}(0), \tag{60}$$

where $\eta(\theta) = \eta(0,\theta)$. From (51) and (52), we obtain

$$H_{20}(0) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(0) + (M_{11}, M_{21}, 0, 0)^T, \quad (61)$$

$$H_{11}(0) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(0) + (M_{12}, M_{22}, 0, 0)^T. \quad (62)$$

Noticing that

$$\begin{gathered}\left(i\omega_0\tau_0 I - \int_{-1}^{0} d\eta(\theta)\, e^{i\omega_0\tau_0}\right) q(0) = 0,\\ \left(-i\omega_0\tau_0 I - \int_{-1}^{0} d\eta(\theta)\, e^{-i\omega_0\tau_0}\right) \overline{q}(0) = 0,\end{gathered} \quad (63)$$

and substituting (57) and (61) into (59), we obtain

$$\left(2i\omega_0\tau_0 I - \int_{-1}^{0} d\eta(\theta)\, e^{2i\omega_0\tau_0\theta}\right) E_1 = \begin{pmatrix} -\beta q_2 \\ \beta q_2 \\ 0 \\ 0 \end{pmatrix}. \quad (64)$$

Therefore, we can obtain

$$E_1 = \begin{pmatrix} 2i\omega_0 - \alpha_{11} & -\alpha_{12} & -\beta_{13}e^{-2i\tau_0\omega_0} & -\alpha_{14} \\ -\alpha_{21} & 2i\omega_0 - \alpha_{22} & 0 & 0 \\ 0 & -\alpha_{32} & 2i\omega_0 - \alpha_{33} - \beta_{33}e^{-2i\tau_0\omega_0} & 0 \\ 0 & -\alpha_{42} & -\alpha_{43} & 2i\omega_0 - \alpha_{44} \end{pmatrix}^{-1} \times \begin{pmatrix} -\beta q_2 \\ \beta q_2 \\ 0 \\ 0 \end{pmatrix}. \quad (65)$$

Similarly, we have

$$E_2 = -\begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ 0 & \alpha_{32} & \alpha_{33} + \beta_{33} & 0 \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}^{-1} \times \begin{pmatrix} -2\beta \operatorname{Re}\{q_2\} \\ 2\beta \operatorname{Re}\{q_2\} \\ 0 \\ 0 \end{pmatrix}. \quad (66)$$

Then, we have

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \\ \rho_2 &= 2\operatorname{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0}. \end{aligned} \quad (67)$$

Thus, the properties of the Hopf bifurcation of system (2) can be stated as follows.

Theorem 3. *Direction of the Hopf bifurcation is determined by* μ_2*: if* $\mu_2 > 0$ $(\mu_2 < 0)$*, then the Hopf bifurcation is supercritical (subcritical). Stability of the bifurcating periodic solutions is determined by* ρ_2*: if* $\rho_2 > 0$ $(\rho_2 < 0)$*, then the bifurcating periodic solutions are stable (unstable). Period of the bifurcating periodic solutions is determined by* T_2*: if* $T_2 > 0$ $(T_2 < 0)$*, then the period of the bifurcating periodic solutions increases (decreases).*

4. Numerical Simulations

In this section, numerical simulations are presented by taking partial parameters from numerical simulations in [14] and they are as follows: $A = 1$, $\beta = 0.35$, $\varpi = 0.8$, $\varepsilon = 0.15$, $\mu = 0.35$, $\gamma = 0.05$, $\alpha = 0.35$, and $\delta = 0.15$. Then, the following system can be obtained:

$$\begin{aligned} \frac{dS(t)}{dt} &= 1 - 0.35S(t)L(t) + 0.8I(t-\tau) - 0.15R(t) \\ &\quad - 0.35S(t), \\ \frac{dL(t)}{dt} &= 0.35S(t)L(t) - 0.55L(t), \\ \frac{dI(t)}{dt} &= 0.05L(t) - 0.7I(t) - 0.8I(t-\tau), \\ \frac{dR(t)}{dt} &= 0.15L(t) + 0.35I(t) - 0.2R(t). \end{aligned} \quad (68)$$

By virtue of the chosen values of parameters, we can get $R_0 = 1.8182$ and the unique endemic equilibrium $P_*(1.5714, 0.6712, 0.0224, 0.5427)$. Further, we can verify that the conditions indicated in Theorem 2 are satisfied. In this case we can obtain $\omega_0 = 0.5209$, $\tau_0 = 5.3442$, and $f'(\omega_0^2) = 1.9227 \neq 0$.

Therefore, $P_*(1.5714, 0.6712, 0.0224, 0.5427)$ is asymptotically stable when $0 < \tau < \tau_0 = 5.3442$ according to Theorem 2, which can be shown as the numerical simulation in Figure 1. In this case, propagation of the computer viruses can be controlled easily. However, system (68) undergoes a

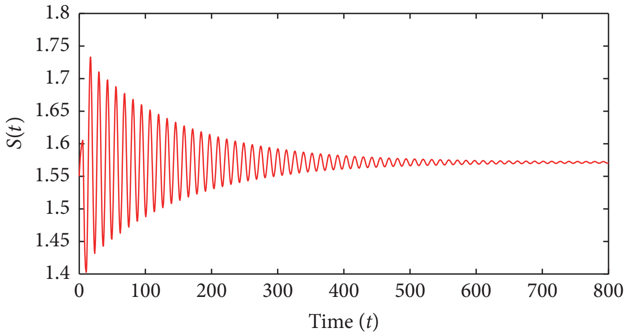

Figure 1: P_* is locally asymptotically stable when $\tau = 5.2482 < \tau_0 = 5.3442$ with initial functions “1.55, 0.79, 0.136, 0.75.”

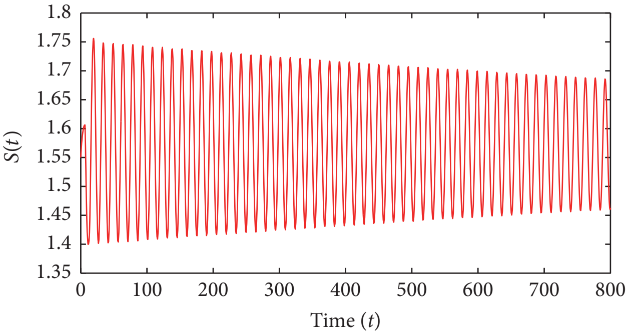

Figure 2: P_* lose its stability when $\tau = 6.1752 > \tau_0 = 5.3442$ with initial functions “1.55, 0.79, 0.136, 0.75.”

Hopf bifurcation at $P_*(1.5714, 0.6712, 0.0224, 0.5427)$ when τ passes through the critical value τ_0. This property can be illustrated by the numerical simulation in Figure 2 and propagation of the computer viruses will be out of control. In addition, according to (67), we have $\mu_2 = 0.8048 > 0$, $\rho_2 = -1.2498 < 0$, and $T_2 = -0.7991 < 0$. Thus, we know that the direction of the Hopf bifurcation at τ_0 is supercritical; the period of the bifurcating periodic solutions decreases; and the bifurcating periodic solutions are stable. From the viewpoint of biology, if the periodic solutions bifurcating from the Hopf bifurcation are stable, then the susceptible, exposed, infectious, and recovered nodes in system (68) may coexist in an oscillatory mode. This phenomenon is not welcome in a real network. We regret to say that only supercritical Hopf bifurcation is identified in our numerical case study. However, it should be pointed out that subcritical Hopf bifurcations are quite usual in dynamical systems with time delay, and the existence of an unstable periodic solution makes the dynamics even more intricate, which has been discussed in population dynamics in the literature [22].

It should be also pointed out that we know that onset of the Hopf bifurcation can be postponed by properly increasing values of the parameters δ, α, and μ based on numerical simulations. In reality, cure rate of the exposed node δ and cure rate of the infectious nodes can be properly enhanced by updating of antivirus software on nodes; disconnecting rate of every node μ can be properly increased by the managers of a real network. Therefore, propagation of the computer viruses in system (2) can be controlled by timely updating of antivirus software on nodes and timely disconnecting nodes from a real network when the connections are unnecessary.

5. Conclusions

A delayed computer virus model is proposed based on the literature [14] considering that the outbreak of computer virus usually lags. By theoretical analysis, the critical value of the delay τ_0 where the Hopf bifurcation occurs is obtained and our results show that the time delay plays an important role in the stability of the considered computer virus model in the present paper.

When the value of the delay $\tau < \tau_0$, the endemic equilibrium of the model is asymptotically stable and the propagation of the computer virus is divinable. However, when the value of the delay $\tau > \tau_0$, the endemic equilibrium of the model will lose its stability and a Hopf bifurcation occurs. In this case, propagation of the computer viruses will be out of control. Therefore, we should postpone occurrence of the Hopf bifurcation by taking some effective measures, such as timely updating of antivirus software on nodes and timely disconnecting nodes from the network when the connections are unnecessary. In addition, properties of the Hopf bifurcation are also investigated by using the normal form theory and center manifold theorem.

Of course, various factors in a real network can affect propagation of the computer viruses. Among them, time delay is important and the present paper thus concentrates on analyzing it. Other network impacts of computer viruses such as network topology, network eigenvalue, and patch forwarding cannot be neglected, and they will be a major emphasis of our future research. Also, stability of the endemic equilibrium and existence of the periodic solutions are investigated only at the critical value τ_0. As stated in the literature [22], it is an open question yet whether there exists stable endemic equilibrium for even larger time delay. This is the other interesting future research direction for us.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by Natural Science Foundation of Anhui Province (no. 1608085QF151, no. 1608085QF145, and no. 1708085MA17).

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