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HÖLDER CONTINUITY UP TO THE BOUNDARY OF SOLUTIONS TO NONLINEAR FOURTH–ORDER ELLIPTIC EQUATIONS WITH NATURAL GROWTH TERMS

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Abstract. In a bounded open set $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, we consider the nonlinear fourth-order partial differential equation $\sum_{|\alpha|=1,2}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x,u,Du,D^2u)+B(x,u,Du,D^2u)=0$. It is assumed that the principal coefficients $\{A_{\alpha}\}_{|\alpha|=1,2}$ satisfy the growth and coercivity conditions suitable for the energy space $\mathring{W}^{1,q}_{2,p}(\Omega)=\mathring{W}^{1,q}(\Omega)\cap\mathring{W}^{2,p}(\Omega)$, $1<p<n/2$, $2p<q<n$. The lower-order term $B(x,u,Du,D^2u)$ behaves as $b(u)\{|Du|^q+|D^2u|^p\}+g(x)$ where $g\in L^{\tau}(\Omega)$, $\tau>n/q$. We establish the Hölder continuity up to the boundary of any solution $u\in\mathring{W}^{1,q}_{2,p}(\Omega)\cap L^{\infty}(\Omega)$ by using the measure density condition on $\partial\Omega$, an interior local result and a modified Moser method with special test function.

1. Introduction

In this paper, we shall deal with nonlinear fourth-order elliptic equations in the divergent form

$$\sum_{|\alpha|=1,2}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x,u,\nabla_2 u)+B(x,u,\nabla_2 u)=0 \ \text{ in } \Omega, \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n\geqslant 3$, $\alpha=(\alpha_1,\ldots,\alpha_n)$ is an n-dimensional multiindex with nonnegative integer components α_i, $i=1,\ldots,n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $D^{\alpha}=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$ and $\nabla_2 u=\{D^{\alpha}u:|\alpha|=1,2\}$.

The coefficients A_{α} and B for almost every $x\in\Omega$ and for every $s\in\mathbb{R}$ and $\xi=\{\xi_{\alpha}\in\mathbb{R}:|\alpha|=1,2\}$ satisfy the following strengthened coercivity condition:

$$\sum_{|\alpha|=1,2}A_{\alpha}(x,s,\xi)\xi_{\alpha}\geqslant C\Big\{\sum_{|\alpha|=1}|\xi_{\alpha}|^q+\sum_{|\alpha|=2}|\xi_{\alpha}|^p\Big\}-f_1(x) \tag{1.2}$$

and the natural (q,p)-growth condition:

$$|B(x,s,\xi)|\leqslant b(|s|)\Big\{\sum_{|\alpha|=1}|\xi_{\alpha}|^q+\sum_{|\alpha|=2}|\xi_{\alpha}|^p\Big\}+f_2(x) \tag{1.3}$$

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where $C > 0$, $p \in (1, n/2)$, $q \in (2p, n)$, $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function and f_i, $i = 1, 2$ are nonnegative functions from $L^\tau(\Omega)$, $\tau > n/q$.

Note that in the case $n > q > mp$, $m \geqslant 2$, nonlinear elliptic equations in the divergence form

$$\sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \ldots, D^m u) = 0 \quad \text{in } \Omega, \tag{1.4}$$

with the condition

$$\sum_{1 \leqslant |\alpha| \leqslant m} A_\alpha(x, \xi)\xi_\alpha \geqslant c_1 \Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=m} |\xi_\alpha|^p \Big\} - c_2 \sum_{1<|\alpha|<m} |\xi_\alpha|^{p_\alpha} - c_2|\xi_0|^q - f(x) \tag{1.5}$$

where c_1, c_2 are positive constants, $f \in L^{t_*}(\Omega)$ with $t_* > 1$, $f \geqslant 0$, $\{p_\alpha\}$ is a set of exponents and $\xi = \{\xi_\alpha : |\alpha| \leqslant m\}$, have been introduced in [36], where the boundedness and Hölder continuity have been established for arbitrary weak solutions from

$$W^{1,q}_{m,p}(\Omega) = W^{1,q}(\Omega) \cap W^{m,p}(\Omega) \quad (\mathring{W}^{1,q}_{m,p}(\Omega) = \mathring{W}^{1,q}(\Omega) \cap \mathring{W}^{m,p}(\Omega)).$$

In particular, for $m = 2$, the structure of equation (1.4) with condition (1.5) is determined by the inequality of the form (1.2).

A regularity condition at the boundary (like the Wiener condition) for solutions of equations in the form (1.4), (1.5) was established in [38]. However, equations with the natural growth condition were not considered in [36], [38].

One of the decisive factors in [36] and in subsequent investigations of equations with strengthened coercivity is the fact that, due to the inequality $q > mp$, the chain rule for weak differentiation in $W^{1,q}_{m,p}(\Omega)$ $(\mathring{W}^{1,q}_{m,p}(\Omega))$ is valid (see, e.g., [27, Lemma 2.2], [28, Lemma 3.5], [40, Lemmas 3.1, 3.2], Lemma 3.4). Due to this fact, in the case $n > q > mp$, there is an analogy with the known methods in the theory of second-order equations, when the superposition $h(u)$ of the solution u and some specially selected function h is used as a test function. However, the use of this analogy for high order equations encounters a number of significant difficulties. First, the function h must be smoother than a Lipschitz function. Secondly, it is necessary to appropriately take into account the terms associated with the derivatives $h^{(i)}(u)$, $i \geqslant 2$ in the corresponding integral identities.

The noted mathematical difficulties are further amplified for equations with the natural growth condition, like (1.3). Because in this case, preliminary regularization of the lower-order term is required in order to obtain existence results and the function h should be chosen so as to neutralize the influence of the lower-order term for further considerations. All this is a reason for a separate consideration of equations with natural growing terms. In this regard we refer to [1], [2], [3], [33], [34] for second order equations and, especially, to [39]–[41], where the existence of bounded generalized solutions of the homogeneous Dirichlet problem for equations, like (1.1)–(1.3), is proved. Next, following the approach of [36], appropriately modified, in [42] has been established the interior local result on Hölder continuity of bounded generalized solutions of the same equation. For existence results of unbounded solutions to fourth-order equations with natural growth condition and L^1-data see [12]. We also cite papers [11], [14], [15], [16], [17], [18] on the regularity of solutions of second-order equations with L^1-data.

In this paper, we establish Hölder continuity up to the boundary of any solution $u \in \mathring{W}^{1,q}_{m,p}(\Omega) \cap L^\infty(\Omega)$ to equation (1.1) under the conditions (1.2), (1.3). The proof follows [42] with additional modifications of test functions near the boundary $\partial\Omega$, which are suggested by the papers [8], [29] and [42]. In particular, due to the chain rule (see below Lemma 3.4), we can use (see below Lemma 4.1)

$$v = \zeta^q(\exp(\lambda|u|) - 1)\,\mathrm{sign}\,u, \quad \zeta \in C_0^\infty(B_\rho), \quad B_\rho \cap \partial\Omega \neq \emptyset$$

as a test function to obtain the estimate

$$\int_{\Omega\cap B_\rho} \Big(\sum_{|\alpha|=1} |D^\alpha u|^q + \sum_{|\alpha|=2} |D^\alpha u|^p \Big) \zeta^q\, dx \leqslant c\rho^n \Big(1 + \rho^{-q} + \max_{B_\rho} \Big\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \Big\} \Big),$$

which is the key to the proof. In this case, a suitable choice of λ makes it possible to neutralize the lower order term.

We remark that the Hölder regularity in particular subsets of Ω for solutions of nonlinear fourth-order elliptic equations with the strengthened coercivity condition and L^1-right-hand sides was studied in [7]. In addition, the integral functionals in the form

$$I(u) = \int_\Omega \{A(x, \nabla_2 u) + A_0(x,u)\}\, dx$$

were considered in [6], [9]. It was assumed that $A(x,\xi)$ is convex respect to ξ and satisfies the following growth condition: for almost every $x \in \Omega$ and for every $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1,2\}$,

$$\tilde c_1 \Big\{ \sum_{|\alpha|=1} \nu(x)|\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x)|\xi_\alpha|^p \Big\} - \tilde f(x) \leqslant A(x,\xi) \leqslant \tilde c_2 \Big\{ \sum_{|\alpha|=1} \nu(x)|\xi_\alpha|^q + \sum_{|\alpha|=2} \mu(x)|\xi_\alpha|^p \Big\} + \tilde f(x),$$

where $\tilde c_1, \tilde c_2$ are positive constants, $\tilde f$ is a nonnegative function belonging to suitable Lebesgue space and ν, μ are positive measurable functions. The boundedness and Hölder regularity of minimizers to such functionals was obtained by using a modified Moser method with a special test function.

Finally, principal results on continuity at boundary points of solutions to nonlinear second-order elliptic equations were established in [22], [23], [26], [32], [37]. For the local interior results see, for example, [10], [13], [30], [31], [33], [34] in the case of second-order equations and systems, as well as [19], [20], [21] for high-order elliptic equations with strengthened coercivity. Moreover, the papers [4], [5], [10], [13], [22], [23], [30], [33], [34] cover the second-order equations with natural growth conditions.

The present paper is organized as follows. In Section 2 we formulate the hypotheses, we state our problem and the main result. Section 3 consists of preliminary Lemmas needed to prove the main result. In Section 4 we prove the main result on the Hölder continuity up to the boundary of solutions of the Dirichlet problem for equation (1.1) assuming that the lower-order term has the natural (q,p)-growth.

2. Main result

Let $n\in\mathbb{N}$, $n>2$, and let Ω be a bounded open set of $\mathbb{R}^n$. We shall use the following notation: $\mathbb{R}_+=[0,+\infty)$; $\partial\Omega$ is the boundary of Ω, $\overline{\Omega}=\Omega\cup\partial\Omega$ is the closure of Ω; $B_\rho(y):=\{x\in\mathbb{R}^n : |x-y|<\rho\}$ is the open ball with center y and radius $\rho>0$; when not important, we shall omit denoting the center as follows: $B_\rho\equiv B_\rho(y)$; Λ is the set of all n-dimensional multi-indices α such that $|\alpha|=1$ or $|\alpha|=2$; $\mathbb{R}^{n,2}$ is the space of all mappings $\xi:\Lambda\to\mathbb{R}$; if $u\in W^{2,1}(\Omega)$, then $\nabla_2 u:\Omega\to\mathbb{R}^{n,2}$, and for every $x\in\Omega$ and for every $\alpha\in\Lambda$, $(\nabla_2 u(x))_\alpha=D^\alpha u(x)$. If $E\subset\mathbb{R}^n$ is a measurable set, then $|E|$ is the n-dimensional Lebesgue measure of the set E. If $\tau\in[1,+\infty]$, then $\|\cdot\|_{L^\tau(E)}$ is the norm in the usual Lebesgue space $L^\tau(E)$.

Let $p\in(1,n/2)$ and $q\in(2p,n)$. We denote by $W^{1,q}_{2,p}(\Omega)$ the set of all functions in $W^{1,q}(\Omega)$ that have the second-order generalized derivatives in $L^p(\Omega)$. The set $W^{1,q}_{2,p}(\Omega)$ is a Banach space with the norm

$$\|u\|=\|u\|_{W^{1,q}(\Omega)}+\Big(\sum_{|\alpha|=2}\int_\Omega |D^\alpha u|^p\,dx\Big)^{1/p}.$$

We denote by $\mathring{W}^{1,q}_{2,p}(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,q}_{2,p}(\Omega)$.

We consider the equation

$$\sum_{\alpha\in\Lambda}(-1)^{|\alpha|}D^\alpha A_\alpha(x,u,\nabla_2 u)+B(x,u,\nabla_2 u)=0 \quad \text{in } \Omega \tag{2.1}$$

under the following assumptions:

(A1) For every $\alpha\in\Lambda$, $A_\alpha:\Omega\times\mathbb{R}\times\mathbb{R}^{n,2}\to\mathbb{R}$ and $B:\Omega\times\mathbb{R}\times\mathbb{R}^{n,2}\to\mathbb{R}$ are Carathéodory functions, i.e. for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^{n,2}$, the functions $A_\alpha(\cdot,s,\xi)$ and $B(\cdot,s,\xi)$ are measurable on Ω and, for almost every $x\in\Omega$, the functions $A_\alpha(x,\cdot,\cdot)$ and $B(x,\cdot,\cdot)$ are continuous in $\mathbb{R}\times\mathbb{R}^{n,2}$.

(A2) For almost every $x\in\Omega$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^{n,2}$ the following inequalities hold:

$$\sum_{\alpha\in\Lambda}A_\alpha(x,s,\xi)\xi_\alpha\geqslant a(|s|)\Big\{\sum_{|\alpha|=1}|\xi_\alpha|^q+\sum_{|\alpha|=2}|\xi_\alpha|^p\Big\}-g_0(x), \tag{2.2}$$

$$\sum_{|\alpha|=1}|A_\alpha(x,s,\xi)|^{q/(q-1)}\leqslant a_1(|s|)\Big\{\sum_{|\alpha|=1}|\xi_\alpha|^q+\sum_{|\alpha|=2}|\xi_\alpha|^p\Big\}+g_1(x), \tag{2.3}$$

$$\sum_{|\alpha|=2}|A_\alpha(x,s,\xi)|^{p/(p-1)}\leqslant a_2(|s|)\Big\{\sum_{|\alpha|=1}|\xi_\alpha|^q+\sum_{|\alpha|=2}|\xi_\alpha|^p\Big\}+g_2(x), \tag{2.4}$$

$$|B(x,s,\xi)| \leqslant b(|s|)\Big\{ \sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big\} + g_3(x), \tag{2.5}$$

where $a: \mathbb{R}_+ \to (0,+\infty)$ is a continuous nonincreasing function, $a_1, a_2, b: \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, g_0, g_1, g_2, g_3 are nonnegative summable functions on Ω.

DEFINITION 1. A generalized solution of (2.1) is a function $u \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ such that for every function $v \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x,u,\nabla_2 u) D^\alpha v + B(x,u,\nabla_2 u)\, v \Big\}\, dx = 0. \tag{2.6}$$

The existence of a solution $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ for equation (2.1) is proved in [39], [40] under additional assumptions on the coefficients $\{A_\alpha\}_{\alpha \in \Lambda}$ and B and on the functions g_0, g_1, g_2, g_3. In particular, it was assumed that

(A3) $$g_0,\ g_1,\ g_2,\ g_3 \in L^\tau(\Omega), \quad \tau > n/q.$$

The local Hölder continuity in Ω of every solution $u \in W^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ to (2.1) under the assumptions (A1), (A2), (A3) is proved in [42].

Examples illustrating the fulfilment of the assumptions (A1) and (A2) are given in [42, Examples 2.4–2.6].

The main result of the present article is a theorem on the Hölder continuity up to the boundary of any generalized solution $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ to equation (2.1) under assumptions (A1), (A2), (A3) and the following assumption on $\partial\Omega$:

(A4) there exist c_*, $R_* > 0$ such that for every $y \in \partial\Omega$ and $R \in (0, R_*]$,

$$|B_R(y) \setminus \Omega| \geqslant c_* |B_R(y)|. \tag{2.7}$$

THEOREM 1. *Assume that conditions* (A1), (A2), (A3) *and* (A4) *are satisfied. Let* $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ *be a generalized solution of equation* (2.1) *and* $\mathrm{M} = \|u\|_{L^\infty(\Omega)}$. *Then there exists a function* $\tilde{u}: \overline{\Omega} \to \mathbb{R}$ *such that* $\tilde{u} = u$ *a.e. in* Ω *and for every* x, $y \in \overline{\Omega}$,

$$|\tilde{u}(x) - \tilde{u}(y)| \leqslant C|x-y|^\varepsilon,$$

where the positive constants C *and* ε *depend only on* n, p, q, τ, R_*, $|\Omega|$, M, $a(\mathrm{M})$, $a_1(\mathrm{M})$, $a_2(\mathrm{M})$, $b(\mathrm{M})$ *and* $\max_{0 \leqslant i \leqslant 3} \|g_i\|_\tau$.

3. Auxiliary results

The following is the well-known Sobolev inequality for functions in $\mathring{W}^{1,q}(\mathscr{O})$, $\mathscr{O} \subset \mathbb{R}^n$; see for example [24, Theorem 7.10].

LEMMA 3.1. *Set $q^* = nq/(n-q)$. Let $\mathscr{O}$ be a bounded open set in $\mathbb{R}^n$. Then $\mathring{W}^{1,q}(\mathscr{O}) \subset L^{q*}(\mathscr{O})$. Furthermore, there exists a positive constant $c_{n,q}$ depending only on n and q such that, for every function $u \in \mathring{W}^{1,q}(\mathscr{O})$,*

$$\Big(\int_{\mathscr{O}} |u|^{q^*} dx\Big)^{1/q^*} \leqslant c_{n,q}\Big(\sum_{|\alpha|=1} \int_{\mathscr{O}} |D^\alpha u|^q \, dx\Big)^{1/q}. \tag{3.1}$$

The proof of the following lemma is given in [35, Chapter 1, §2, Lemma 4].

LEMMA 3.2. *Let $f \in W^{1,q}(B_\rho)$. Suppose there exists a measurable subset $G \subset B_\rho$ and positive constants C' and C'' such that*

$$|G| \geqslant C'\rho^n, \quad \max_G |f| \leqslant C''.$$

Then

$$\int_{B_\rho} |f|^q \, dx \leqslant C\rho^q\Big(\sum_{|\alpha|=1} \int_{B_\rho} |D^\alpha f|^q \, dx + \rho^{n-q}\Big)$$

where C is a positive constant depending only on n, q, C', C''.

The following lemma is due to John and Nirenberg [25] (see also [24, Theorem 7.21]).

LEMMA 3.3. *Let $f \in W^{1,1}(\mathscr{O})$ where $\mathscr{O}$ is a convex domain in $\mathbb{R}^n$. Suppose there exists a positive constant K such that*

$$\sum_{|\alpha|=1} \int_{\mathscr{O}\cap B_\rho} |D^\alpha f| \, dx \leqslant K\rho^{n-1} \quad \textit{for all balls } B_\rho.$$

Then there exist positive constants σ_0 and C depending only on n such that

$$\int_{\mathscr{O}} \exp\Big(\frac{\sigma}{K}|f-(f)_{\mathscr{O}}|\Big)\, dx \leqslant C\,(\operatorname{diam}\mathscr{O})^n$$

where $\sigma = \sigma_0|\mathscr{O}|\,(\operatorname{diam}\mathscr{O})^{-n}$, $(f)_{\mathscr{O}} = \frac{1}{|\mathscr{O}|}\int_{\mathscr{O}} f\, dx$.

LEMMA 3.4. *Let h be an odd function on $\mathbb{R}$ such that $h \in C^1(\mathbb{R})$, $h \in C^2(\mathbb{R} \setminus \{0\})$ and h'' has a discontinuity of the first kind at the origin. Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. Then $h(u) \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:*

(i) for every n-dimensional multi-index α, $|\alpha| = 1$,

$$D^\alpha h(u) = h'(u) D^\alpha u \quad \textit{a.e. in } \Omega;$$

(ii) for every n-dimensional multi-index α, $|\alpha| = 2$,

$$D^{\alpha}h(u) = h'(u)D^{\alpha}u + h''(u)D^{\beta}uD^{\gamma}u\,\mathbb{I}_{\{u\neq 0\}} \quad \text{a.e. in } \Omega,$$

where $\alpha = \beta + \gamma$, $|\beta| = |\gamma| = 1$ and $\mathbb{I}_{\{u\neq 0\}}$ is the indicator of the set $\{u \neq 0\}$.

For the proof of this fact, see [40, Lemma 3.2].

The following result is discussed in [24, Lemma 8.23].

LEMMA 3.5. *Let ω be a non-decreasing function on an interval $(0, R_0]$ satisfying, for all $R \leqslant R_0$, the inequality*

$$\omega(\vartheta R) \leqslant \theta\,\omega(R) + \varphi(R)$$

where φ is also non-decreasing function and $0 < \vartheta, \theta < 1$. Then, for any $\delta \in (0,1)$ and $R \leqslant R_0$, we have

$$\omega(R) \leqslant C\Big(\Big(\frac{R}{R_0}\Big)^{\varepsilon}\omega(R_0) + \varphi(R^{\delta}R_0^{1-\delta})\Big)$$

where $C = C(\vartheta, \theta)$ and $\varepsilon = \varepsilon(\vartheta, \theta, \delta)$ are positive constants.

4. Proof of Theorem 1

Suppose that the assumptions (A1)–(A4) are satisfied. Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^{\infty}(\Omega)$ be a generalized solution of equation (2.1). Denote by $\overline{u}$ the extension of u to 0 on $\mathbb{R}^n \setminus \Omega$. From the definition of $\mathring{W}^{1,q}_{2,p}(\Omega)$ it follows that for any open bounded set $\Omega' \supset \overline{\Omega}$, $\overline{u} \in \mathring{W}^{1,q}_{2,p}(\Omega')$ and for every $\alpha \in \Lambda$, $D^{\alpha}\overline{u} = D^{\alpha}u$ a.e. in Ω and $D^{\alpha}\overline{u} = 0$ a.e. in $\Omega' \setminus \Omega$. We set $\mathrm{M} = \|u\|_{L^{\infty}(\Omega)}$, thus

$$|\overline{u}| \leqslant \mathrm{M} < +\infty \quad \text{a.e. in } \mathbb{R}^n. \tag{4.1}$$

By c_i, $i = 0, 1, \ldots$, we shall denote positive constants depending only on

$$\texttt{data} \equiv \big(n, p, q, \tau, R_*, |\Omega|, \mathrm{M}, a(\mathrm{M}), a_1(\mathrm{M}), a_2(\mathrm{M}), b(\mathrm{M}), \max_{0\leqslant i\leqslant 3}\|g_i\|_{\tau}\big).$$

Fixing $x_0 \in \partial\Omega$ for every $R > 0$ we set $\Omega_R(x_0) = \Omega \cap B_R(x_0)$,

$$\mu(R) = \operatorname*{ess\,inf}_{B_R(x_0)} \overline{u}, \quad M(R) = \operatorname*{ess\,sup}_{B_R(x_0)} \overline{u}, \quad \omega(R) = M(R) - \mu(R).$$

By (2.7), it is obvious that $\mu(R) \leqslant 0$ and $M(R) \geqslant 0$ for every $R > 0$. We fix a positive number r such that

$$r < \min\Big\{1 - \frac{n}{q\tau}, \frac{q-2p}{q-p}\Big\}. \tag{4.2}$$

For every $R \in \big(0, \min\{1, \frac{1}{2}R_*\}\big]$ we shall establish the inequality

$$\omega(R) \leqslant c_0\,\omega(2R) + R^r \tag{4.3}$$

with $c_0 \in (0,1)$. This inequality, Lemma 3.5 and [42, Theorem 2.3] imply the validity of Theorem 1.

To prove (4.3), we fix $R \in \big(0, \min\{1, \frac{1}{2}R_*\}\big]$ and define a function $\overline{v}_0 : B_{2R}(x_0) \to \mathbb{R}$ as follows:

$$\overline{v}_0(x) = \begin{cases} \ln \dfrac{2e\omega(2R)}{M(2R) - \overline{u}(x) + R^r} & \text{if } M(2R) \geqslant \omega(2R)/2, \\ \ln \dfrac{2e\omega(2R)}{\overline{u}(x) - \mu(2R) + R^r} & \text{if } M(2R) < \omega(2R)/2. \end{cases} \tag{4.4}$$

We denote by v_0 the restriction of the function $\overline{v}_0$ on $\Omega_{2R}(x_0)$. It is easy to see that (4.3) follows from the estimate

$$\|\overline{v}_0\|_{L^\infty(B_R(x_0))} \leqslant c_1 \tag{4.5}$$

or, which is the same thing, from the inequality $\|v_0\|_{L^\infty(\Omega_R(x_0))} \leqslant c_1$. For definiteness we assume that the function $\overline{v}_0$ is defined by the first line in (4.4). We can also assume that

$$\omega(2R) \geqslant R^r, \tag{4.6}$$

and therefore, $\overline{v}_0 \geqslant 1$ a.e. in $B_{2R}(x_0)$, otherwise inequality (4.3) holds.

To derive inequality (4.5), we need some integral estimates of the function $\overline{u}$ and its derivatives in the balls B_ρ that intersect $\partial\Omega$. We set

$$\Phi = \sum_{|\alpha|=1} |D^\alpha \overline{u}|^q + \sum_{|\alpha|=2} |D^\alpha \overline{u}|^p.$$

LEMMA 4.1. *Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be a function such that*

$$\zeta = 0 \text{ in } \mathbb{R}^n \setminus B_\rho \quad \text{and} \quad 0 \leqslant \zeta \leqslant 1. \tag{4.7}$$

Then there exists a positive constant c_2 such that

$$\int_{B_\rho} \Phi \zeta^q \, dx \leqslant c_2 \rho^n \Big(1 + \rho^{-q} + \max_{B_\rho} \Big\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \Big\}\Big). \tag{4.8}$$

Proof. We assume that $B_\rho \cap \Omega \neq \emptyset$, otherwise $\Phi \equiv 0$ in B_ρ and (4.8) is trivial. We set $\lambda = b(\mathrm{M})/a(\mathrm{M})$ and define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(s) = (e^{\lambda|s|} - 1)\,\mathrm{sign}\, s, \quad s \in \mathbb{R}.$$

Elementary calculations show that

$$a(\mathrm{M})\, h' - b(\mathrm{M})\, |h| = b(\mathrm{M}) \quad \text{in } \mathbb{R}. \tag{4.9}$$

For every $x \in \Omega$ we set

$$v_1(x) = h(u(x))\, \zeta^q(x).$$

By Lemma 3.4, $h(u) \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. Therefore, $v_1 \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ and the following assertions hold:

(a) for every n-dimensional multi-index α, $|\alpha| = 1$,

$$D^\alpha v_1 = h'(u) D^\alpha u\, \zeta^q + q h(u)\, \zeta^{q-1} D^\alpha \zeta \quad \text{a.e. in } \Omega,$$

(b) for every n-dimensional multi-index α, $|\alpha| = 2$,

$$\begin{aligned} &|D^\alpha v_1 - h'(u) D^\alpha u\, \zeta^q| \\ &\leqslant |h''(u)| \Big\{ \sum_{|\beta|=1} |D^\beta u|^2 \Big\} \mathbb{I}_{\{u \neq 0\}}\, \zeta^q + 2q h'(u) \Big\{ \sum_{|\beta|=1} |D^\beta u| \Big\} \Big\{ \sum_{|\beta|=1} |D^\beta \zeta| \Big\}\, \zeta^{q-1} \\ &\quad + q(q-1)\, |h(u)| \Big\{ \sum_{|\beta|=1} |D^\beta \zeta|^2 \Big\}\, \zeta^{q-2} + q |h(u)|\, |D^\alpha \zeta|\, \zeta^{q-1} \quad \text{a.e. in } \Omega. \end{aligned}$$

By (2.6), we have

$$\int_\Omega \Big\{ \sum_{\alpha \in \Lambda} A_\alpha(x, u, \nabla_2 u) D^\alpha v_1 + B(x, u, \nabla_2 u)\, v_1 \Big\}\, dx = 0.$$

From this equality, using (2.2), (2.5), (4.1), (4.7), and assertions (a) and (b), we deduce that

$$\int_{\Omega_\rho} \Phi\big(a(\mathrm{M})\, h'(u) - b(\mathrm{M})\, |h(u)|\big)\, \zeta^q\, dx \leqslant \sum_{i=1}^{4} I_i + e^{\lambda \mathrm{M}} \int_{\Omega_\rho} (g_3 + \lambda g_0)\, \zeta^q\, dx$$

where $\Omega_\rho = \Omega \cap B_\rho$,

$$\begin{aligned} I_1 &= q \sum_{\alpha \in \Lambda} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)|\, |D^\alpha \zeta|\, |h(u)|\, \zeta^{q-1}\, dx, \\ I_2 &= \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)|\, |D^\beta u|^2\, |h''(u)|\, \mathbb{I}_{\{u \neq 0\}}\, \zeta^q\, dx, \\ I_3 &= q^2 \sum_{|\alpha|=2} \sum_{|\beta|=1} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)|\, |D^\beta \zeta|^2\, |h(u)|\, \zeta^{q-2}\, dx, \\ I_4 &= 2q \sum_{|\alpha|=2} \sum_{|\beta|=1} \sum_{|\gamma|=1} \int_{\Omega_\rho} |A_\alpha(x, u, \nabla_2 u)|\, |D^\beta u|\, |D^\gamma \zeta|\, h'(u)\, \zeta^{q-1}\, dx. \end{aligned}$$

From this and (4.9) it follows that

$$b(\mathrm{M}) \int_{\Omega_\rho} \Phi\, \zeta^q\, dx \leqslant I_1 + I_2 + I_3 + I_4 + e^{\lambda \mathrm{M}} \int_{\Omega_\rho} (g_3 + \lambda g_0)\, \zeta^q\, dx. \tag{4.10}$$

Further, to obtain suitable estimates for the terms on the right-hand side of (4.10), we argue similarly to the proof of [42, Lemma 4.1].

Estimate for I_1. Using the Young's inequality with the exponents $q/(q-1)$ and q, (2.3), (4.1) and (4.7), we obtain

$$q\sum_{|\alpha|=1}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,|h(u)|\,\zeta^{q-1}\,dx$$
$$\leqslant \frac{b(\mathrm{M})}{16}\int_{\Omega_\rho}\Phi\zeta^q\,dx + c_3\int_{\Omega_\rho}g_1\,dx + c_3\rho^n\max_{B_\rho}\sum_{|\alpha|=1}|D^\alpha\zeta|^q.$$

Using the Young's inequality with the exponents $p/(p-1)$ and p, (2.4), (4.1) and (4.7), we obtain

$$q\sum_{|\alpha|=2}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,|h(u)|\,\zeta^{q-1}\,dx$$
$$\leqslant \frac{b(\mathrm{M})}{16}\int_{\Omega_\rho}\Phi\zeta^q\,dx + c_4\int_{\Omega_\rho}g_2\,dx + c_4\rho^n\max_{B_\rho}\sum_{|\alpha|=2}|D^\alpha\zeta|^p.$$

Summing the last two inequalities, we obtain

$$I_1 \leqslant \frac{b(\mathrm{M})}{8}\int_{\Omega_\rho}\Phi\zeta^q\,dx + c_5\int_{\Omega_\rho}(g_1+g_2)\,dx + c_5\rho^n\max_{B_\rho}\Phi_\zeta \tag{4.11}$$

where

$$\Phi_\zeta = \sum_{|\alpha|=1}|D^\alpha\zeta|^q + \sum_{|\alpha|=2}|D^\alpha\zeta|^p.$$

Estimates for I_2, I_3 and I_4. It is obvious that

$$\frac{p-1}{p}+\frac{2}{q}+\frac{q-2p}{qp}=1,\quad q-1=(p-1)\frac{q}{p}+\Big(\frac{q}{p}-1\Big). \tag{4.12}$$

Using this equalities, the Young's inequality, (2.4), (4.1) and (4.7), we obtain

$$I_2 \leqslant \frac{b(\mathrm{M})}{8}\int_{\Omega_\rho}\Phi\zeta^q\,dx + c_6\int_{\Omega_\rho}g_2\,dx + c_6\rho^n, \tag{4.13}$$

$$I_3 \leqslant \frac{b(\mathrm{M})}{8}\int_{\Omega_\rho}\Phi\zeta^q\,dx + c_7\int_{\Omega_\rho}g_2\,dx + c_7\rho^n\max_{B_\rho}\Phi_\zeta + c_7\rho^n, \tag{4.14}$$

$$I_4 \leqslant \frac{b(\mathrm{M})}{8}\int_{\Omega_\rho}\Phi\zeta^q\,dx + c_8\int_{\Omega_\rho}g_2\,dx + c_8\rho^n\max_{B_\rho}\Phi_\zeta + c_8\rho^n. \tag{4.15}$$

From (4.10), (4.11), (4.13)–(4.15) it follows that

$$\frac{b(\mathrm{M})}{2}\int_{\Omega_\rho}\Phi\zeta^q\,dx \leqslant c_9\Big(\int_{\Omega_\rho}g\,dx + \rho^n\max_{B_\rho}\Phi_\zeta + \rho^n\Big)$$

where $g = g_0+g_1+g_2+g_3$. By Hölder's inequality and the inequality $\tau > n/q$ we have

$$\int_{\Omega_\rho}g\,dx \leqslant \|g\|_{L^\tau(\Omega)}\,|B_\rho|^{(\tau-1)/\tau} \leqslant c_{10}\rho^{n-q}.$$

The last two inequalities and (4.1) imply inequality (4.8). The proof is complete. □

LEMMA 4.2. *Let $B_\rho \subset B_{2R}(x_0)$ and let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be a function such that condition (4.7) be satisfied. Then there exists a positive constant c_{11} such that*

$$\begin{aligned}
&\int_{B_\rho} \frac{\Phi\,\zeta^q\,dx}{(M(2R)-\overline{u}+R^r)^q} \\
&\leqslant c_{11}\,\rho^n\Big(\rho^{-q}+\max_{B_\rho}\Big\{\sum_{|\alpha|=1}|D^\alpha\zeta|^q+\sum_{|\alpha|=2}|D^\alpha\zeta|^p\Big\} \\
&\quad +\rho^{2p-q}\max_{B_\rho}\sum_{|\alpha|=2}|D^\alpha\zeta|^p+\rho^{-q(q-2p)/(q-p)}\max_{B_\rho}\sum_{|\alpha|=1}|D^\alpha\zeta|^{qp/(q-p)}\Big).
\end{aligned} \tag{4.16}$$

In particular, if $|D^\alpha\zeta|\leqslant K\rho^{-|\alpha|}$ for any $\alpha\in\Lambda$, then

$$\int_{B_\rho} \frac{\Phi\,\zeta^q\,dx}{(M(2R)-\overline{u}+R^r)^q} \leqslant c_{12}\,(K+1)\,\rho^{n-q}. \tag{4.17}$$

Proof. We assume that $B_\rho\cap\Omega\neq\emptyset$, otherwise $\Phi\equiv 0$ in B_ρ and (4.16) is trivial. For every $x\in B_{2R}(x_0)$, we set $U(x)=M(2R)-\overline{u}(x)+R^r$. Note that due to the inequality $M(2R)\geqslant\omega(2R)/2$ we have

$$|U^{1-q}-(M(2R)+R^r)^{1-q}|\leqslant 2^{q-1}U^{1-q} \quad \text{a.e. in } \Omega_{2R}(x_0). \tag{4.18}$$

We define the function $v_2:\Omega\to\mathbb{R}$ by

$$v_2(x)=\begin{cases}[U^{1-q}(x)-(M(2R)+R^r)^{1-q}]\,\zeta^q(x) & \text{if } x\in\Omega_{2R}(x_0),\\ 0 & \text{if } x\in\Omega\setminus B_{2R}(x_0).\end{cases}$$

Simple calculations show that $v_2\in\mathring{W}^{1,q}_{2,p}(\Omega)\cap L^\infty(\Omega)$ and due to (4.7) and (4.18) the following assertions hold:

(c) for every n-dimensional multi-index α, $|\alpha|=1$,

$$|D^\alpha v_2-(q-1)\,U^{-q}\,\zeta^q D^\alpha u|\leqslant q2^{q-1}U^{1-q}\,\zeta^{q-1}|D^\alpha\zeta| \quad \text{a.e. in } \Omega\cap B_\rho,$$

(d) for every n-dimensional multi-index α, $|\alpha|=2$,

$$\begin{aligned}
&\Big|D^\alpha v_2-(q-1)\,U^{-q}\,\zeta^q D^\alpha u\Big| \\
&\leqslant q2^{q-1}\,U^{1-q}\,\zeta^{q-1}|D^\alpha\zeta|+q(q-1)\,2^{q-1}\,U^{1-q}\,\zeta^{q-2}\sum_{|\beta|=1}|D^\beta\zeta|^2 \\
&\quad +2q(q-1)\,U^{-q}\,\zeta^{q-1}\Big\{\sum_{|\beta|=1}|D^\beta u|\Big\}\Big\{\sum_{|\beta|=1}|D^\beta\zeta|\Big\} \\
&\quad +q(q-1)\,U^{-1-q}\,\zeta^q\sum_{|\beta|=1}|D^\beta u|^2 \quad \text{a.e. in } \Omega\cap B_\rho.
\end{aligned}$$

In virtue of (2.6) we have

$$\int_\Omega\Big\{\sum_{\alpha\in\Lambda}A_\alpha(x,u,\nabla_2 u)D^\alpha v_2+B(x,u,\nabla_2 u)\,v_2\Big\}\,dx=0.$$

Hence by using (2.2), (2.5), (4.1), (4.7), (4.18) and assertions (c) and (d), we obtain

$$a(\mathrm{M})\int_{\Omega_\rho}\Phi U^{-q}\zeta^q\,dx\leqslant c_{13}\Big(\sum_{i=1}^{5}I_i'+\int_{\Omega_\rho}(g_0+g_3)U^{-q}dx\Big) \tag{4.19}$$

where $\Omega_\rho=\Omega\cap B_\rho$,

$$I_1'=\sum_{\alpha\in\Lambda}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,U^{1-q}\zeta^{q-1}\,dx,$$

$$I_2'=\sum_{|\alpha|=2}\sum_{|\beta|=1}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\beta u|^2\,U^{-1-q}\zeta^q\,dx,$$

$$I_3'=\sum_{|\alpha|=2}\sum_{|\beta|=1}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\beta\zeta|^2\,U^{1-q}\zeta^{q-2}\,dx,$$

$$I_4'=\sum_{|\alpha|=2}\sum_{|\beta|=1}\sum_{|\gamma|=1}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\beta u|\,|D^\gamma\zeta|\,U^{-q}\zeta^{q-1}\,dx,$$

$$I_5'=\int_{\Omega_\rho}\Phi U^{1-q}\zeta^q\,dx.$$

Further, to obtain suitable estimates for the terms on the right-hand side of (4.19), we argue similarly to the proof of [42, Lemma 4.2].

Estimate for I_1'. Using the Young's inequality with the exponents $q/(q-1)$ and q, (2.3), (4.1) and (4.7), we obtain

$$\begin{aligned}&\sum_{|\alpha|=1}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,U^{1-q}\zeta^{q-1}\,dx\\&\leqslant\frac{a(\mathrm{M})}{20c_{13}}\int_{\Omega_\rho}\Phi U^{-q}\zeta^q\,dx+c_{14}\int_{\Omega_\rho}g_1U^{-q}\,dx+c_{14}\rho^n\max_{B_\rho}\sum_{|\alpha|=1}|D^\alpha\zeta|^q.\end{aligned} \tag{4.20}$$

We use the Young's inequality with the exponents $p/(p-1)$ and p, (2.4), (4.1) and (4.7) to obtain

$$\begin{aligned}&\sum_{|\alpha|=2}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,U^{1-q}\zeta^{q-1}\,dx\\&\leqslant\frac{a(\mathrm{M})}{20c_{13}}\int_{\Omega_\rho}\Phi U^{-q}\zeta^q\,dx+c_{15}\int_{\Omega_\rho}g_2U^{-q}\,dx+c_{15}\sum_{|\alpha|=2}\int_{\Omega_\rho}|D^\alpha\zeta|^p\,U^{p-q}\zeta^{q-p}\,dx,\end{aligned}$$

whence, taking into account the inequalities $U\geqslant R^r$, $\rho/2<R<1$ and (4.2), we derive

$$\begin{aligned}&\sum_{|\alpha|=2}\int_{\Omega_\rho}|A_\alpha(x,u,\nabla_2 u)|\,|D^\alpha\zeta|\,U^{1-q}\zeta^{q-1}\,dx\\&\leqslant\frac{a(\mathrm{M})}{20c_{13}}\int_{\Omega_\rho}\Phi U^{-q}\zeta^q\,dx+c_{15}\int_{\Omega_\rho}g_2U^{-q}\,dx+c_{16}\rho^{n-q+2p}\max_{B_\rho}\sum_{|\alpha|=2}|D^\alpha\zeta|^p.\end{aligned} \tag{4.21}$$

Summing inequalities (4.20) and (4.21), we obtain

$$\begin{aligned} I_1' \leqslant \frac{a(\mathrm{M})}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q\, dx + c_{17} \int_{\Omega_\rho} (g_1+g_2) U^{-q}\, dx \\ + c_{14}\rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^q + c_{16}\rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p. \end{aligned} \tag{4.22}$$

Estimate for I_2'. We use (4.1), the first equality in (4.12), Young's inequality, (2.4) and (4.7) to obtain

$$I_2' \leqslant \frac{a(\mathrm{M})}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q\, dx + c_{18} \int_{\Omega_\rho} g_2 U^{-q}\, dx + c_{18} \int_{\Omega_\rho} U^{-q(q-p)/(q-2p)} \zeta^q\, dx.$$

Estimating the last integral in this inequality by means of the inequalities $U \geqslant R^r$, $\rho/2 < R < 1$ and (4.2), we obtain

$$I_2' \leqslant \frac{a(\mathrm{M})}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q\, dx + c_{18} \int_{\Omega_\rho} g_2 U^{-q}\, dx + c_{19}\rho^{n-q}. \tag{4.23}$$

Estimates for I_3' and I_4'. Using the reasoning similar the proof of (4.23), we obtain

$$\begin{aligned} I_3' \leqslant \frac{a(\mathrm{M})}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q\, dx + c_{20} \int_{\Omega_\rho} g_2 U^{-q}\, dx \\ + c_{20}\rho^n \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^q + c_{20}\rho^{n-q}, \end{aligned} \tag{4.24}$$

$$\begin{aligned} I_4' \leqslant \frac{a(\mathrm{M})}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q\, dx + c_{21} \int_{\Omega_\rho} g_2 U^{-q}\, dx \\ + c_{21}\rho^{n-q(q-2p)/(q-p)} \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^{qp/(q-p)}. \end{aligned} \tag{4.25}$$

Estimate for I_5'. We use Young's inequality and Lemma 4.1, to obtain

$$I_5' \leqslant \frac{a(\mathrm{M})}{10c_{13}} \int_{\Omega_\rho} \Phi U^{-q} \zeta^q\, dx + c_{22}\rho^n \Big(\rho^{-q} + \max_{B_\rho} \Big\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \Big\} \Big).$$

Collecting (4.19), (4.22)–(4.25) and the above inequality, we obtain

$$\begin{aligned} &\frac{a(\mathrm{M})}{2} \int_{B_\rho} \Phi U^{-q} \zeta^q\, dx \\ &\leqslant c_{23} \Big(\rho^{n-q} + \rho^n \max_{B_\rho} \Big\{ \sum_{|\alpha|=1} |D^\alpha \zeta|^q + \sum_{|\alpha|=2} |D^\alpha \zeta|^p \Big\} \\ &\quad + \rho^{n-q+2p} \max_{B_\rho} \sum_{|\alpha|=2} |D^\alpha \zeta|^p + \rho^{n-q(q-2p)/(q-p)} \max_{B_\rho} \sum_{|\alpha|=1} |D^\alpha \zeta|^{qp/(q-p)} \\ &\quad + \int_{\Omega_\rho} g U^{-q}\, dx \Big) \end{aligned}$$

where $g = g_0 + g_1 + g_2 + g_3$. Finally, to obtain (4.16), we estimate the last integral in this inequality by means of Holder's inequality and the relations (4.2), $U \geqslant R^r$ and $\rho/2 < R < 1$. Inequality (4.17) is a trivial consequence of (4.16). The proof is complete. □

LEMMA 4.3. *For every* $\kappa \geqslant 1$ *there is a positive constant* $c = c(\mathtt{data}, \kappa)$ *such that* $\lim_{\kappa\to+\infty} c(\mathtt{data}, \kappa) = +\infty$ *and*

$$\int_{\Omega_{3R/2}(x_0)} v_0^{\kappa}\, dx \leqslant cR^n. \tag{4.26}$$

Proof. First, we estimate from above the average integral

$$(\overline{v}_0)_{B_{3R/2}(x_0)} = \frac{1}{|B_{3R/2}(x_0)|} \int_{B_{3R/2}(x_0)} \overline{v}_0\, dx$$

by a constant depending only on data. For this purpose we choose a function $\zeta_1 \in C_0^{\infty}(\mathbb{R}^n)$ such that

$$0 \leqslant \zeta_1 \leqslant 1 \text{ in } \mathbb{R}^n, \quad \zeta_1 = 1 \text{ in } B_{3R/2}(x_0), \quad \zeta_1 = 0 \text{ in } \mathbb{R}^n \setminus B_{7R/4}(x_0),$$
$$|D^{\alpha}\zeta_1| \leqslant K_1 R^{-|\alpha|} \quad \text{for } |\alpha| = 1,2,$$

where K_1 is an absolute constant, not depending on R. We have

$$\overline{v}_0 \in W^{1,q}(B_{3R/2}(x_0)).$$

Moreover by (2.7)

$$|B_{3R/2}(x_0) \setminus \Omega| \geqslant c_{24} R^n$$

and by the inequalities (4.6) and $M(2R) \geqslant \omega(2R)/2$,

$$1 \leqslant \overline{v}_0 \leqslant 1 + \ln 4 \text{ on } B_{3R/2}(x_0) \setminus \Omega.$$

Taking into account these facts and using Holder's inequality, Lemmas 3.2 and 4.2 and the properties of the function ζ_1, we obtain

$$\begin{aligned}(\overline{v}_0)_{B_{3R/2}(x_0)} &\leqslant c_{25} R^{-n/q} \Big(\int_{B_{3R/2}(x_0)} \overline{v}_0^{\,q}\, dx \Big)^{1/q} \\ &\leqslant c_{26} R^{1-n/q} \Big(\int_{B_{7R/4}(x_0)} \frac{\Phi \zeta_1^q\, dx}{(M(2R) - \overline{u} + R^r)^q} + R^{n-q} \Big)^{1/q} \leqslant c_{27}.\end{aligned} \tag{4.27}$$

Next, let $B_{2\rho} \subset B_{2R}(x_0)$, and let $\zeta_2 \in C_0^{\infty}(\mathbb{R}^n)$ be a function such that

$$0 \leqslant \zeta_2 \leqslant 1 \text{ in } \mathbb{R}^n, \quad \zeta_2 = 1 \text{ in } B_{\rho}, \quad \zeta_2 = 0 \text{ in } \mathbb{R}^n \setminus B_{2\rho},$$
$$|D^{\alpha}\zeta_2| \leqslant K_2 \rho^{-|\alpha|} \quad \text{for } |\alpha| = 1,2,$$

where K_2 is an absolute constant, not depending on ρ. Using Holder's inequality, Lemma 4.2 and the properties of the function ζ_2, we derive that

$$\sum_{|\alpha|=1}\int_{B_\rho}|D^\alpha \overline{v}_0|\,dx \leqslant c_{28}\,\rho^{n-n/q}\left(\sum_{|\alpha|=1}\int_{B_\rho}|D^\alpha \overline{v}_0|^q\,dx\right)^{1/q}$$
$$\leqslant c_{28}\,\rho^{n-n/q}\left(\int_{B_{2\rho}}\frac{\Phi\,\zeta_2^q\,dx}{(M(2R)-\overline{u}+R^r)^q}\right)^{1/q}\leqslant c_{29}\,\rho^{n-1}.$$

Hence, by Lemma 3.3, we have

$$\int_{B_{3R/2}(x_0)}\exp\left(c_{30}|\overline{v}_0-(\overline{v}_0)_{B_{3R/2}(x_0)}|\right)dx\leqslant c_{31}R^n. \tag{4.28}$$

Now let $\kappa\geqslant 1$. Then inequalities (4.27) and (4.28) imply (4.26). The proof is complete. □

Now we are ready to prove inequality (4.5).

LEMMA 4.4. *There exists a positive constant c_1 such that inequality (4.5) holds.*

Proof. The proof is based on obtaining the estimate $R^{-n/k}\|\overline{v}_0\|_{L^k(B_R(x_0))}\leqslant c_1$ at $k\to+\infty$ by adapting the Moser's iterative technique. We divide the proof in four steps.

Step 1. Let us introduce some auxiliary functions.

We fix a function $\psi_0\in C_0^\infty(\mathbb{R})$ such that

$$0\leqslant\psi_0\leqslant 1 \text{ on } \mathbb{R},\quad \psi_0=1 \text{ in } [-1,1],\quad \psi_0=0 \text{ in } \mathbb{R}\setminus(-3/2,3/2).$$

We define the function $\psi:\mathbb{R}^n\to\mathbb{R}$ by $\psi(x)=\psi_0\big(\frac{|x-x_0|}{R}\big)$. Let $E\subset\Omega_{2R}(x_0)$ be a set such that $|E|=0$ and for every $x\in\Omega_{2R}(x_0)\setminus E$,

$$\mu(2R)\leqslant u(x)\leqslant M(2R). \tag{4.29}$$

Let $\mathscr{U}:\Omega\to\mathbb{R}$ be the function such that

$$\mathscr{U}(x)=\begin{cases}M(2R)-u(x)+R^r & \text{if } x\in\Omega_{2R}(x_0)\setminus E,\\ 2\omega(2R) & \text{if } x\in(\Omega\setminus B_{2R}(x_0))\cup E.\end{cases}$$

Let now

$$\varkappa=2qp/(q-2p), \tag{4.30}$$
$$k\geqslant\overline{k}:=\max\{q,12\mathrm{M}b(\mathrm{M})/a(\mathrm{M})\}, \tag{4.31}$$
$$l:=\max\{q,\varkappa\}<t\leqslant C_0k, \tag{4.32}$$

and $C_0=C_0(n,p,q,\tau)>1$ is a constant that will be specified below. Define

$$G=\max\{\mathscr{U}^{(1-q)/\varkappa}-(M(2R)+R^r)^{(1-q)/\varkappa},0\},$$

$$v_0 = \ln(2e\omega(2R)\,\mathscr{U}^{-1}), \quad w = G^{\varkappa} v_0^k \psi^t.$$

Due to (4.6), (4.29), $2p < q$ we have $w \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$,

$$|D^\alpha w - (q-1)\, v_0^k \psi^t G^{\varkappa-1}\, \mathscr{U}^{(1-q)/\varkappa-1} D^\alpha u - k\, v_0^{k-1} \psi^t G^{\varkappa}\, \mathscr{U}^{-1} D^\alpha u| \\ \leqslant c_{32}\, k R^{-1} G^{\varkappa} v_0^k \psi^{t-1} \quad \text{if } |\alpha| = 1, \tag{4.33}$$

$$|D^\alpha w - (q-1)\, v_0^k \psi^t G^{\varkappa-1}\, \mathscr{U}^{(1-q)/\varkappa-1} D^\alpha u - k v_0^{k-1} \psi^t G^{\varkappa}\, \mathscr{U}^{-1} D^\alpha u| \\ \leqslant c_{32}\, k^2 v_0^k \psi^{t-2} G^{\varkappa-2}\, \mathscr{U}^{2(1-q)/\varkappa} \Big\{ R^{-2} + \mathscr{U}^{-2} \sum_{|\beta|=1} |D^\beta u|^2 \Big\} \quad \text{if } |\alpha| = 2. \tag{4.34}$$

Putting the function w in (2.6) instead of v and using (2.2), (2.5), (4.33), (4.34) and the inequalities $G \leqslant \mathscr{U}^{(1-q)/\varkappa}$, $\mathscr{U} \leqslant 2\mathrm{M}+1$ we deduce that

$$(q-1)\, a(\mathrm{M}) \int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa-1}\, \mathscr{U}^{(1-q)/\varkappa-1} v_0^k \psi^t \, dx \\ + k\, a(\mathrm{M}) \int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa}\, \mathscr{U}^{-1} v_0^{k-1} \psi^t \, dx \\ \leqslant b(\mathrm{M}) \int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa} v_0^k \psi^t \, dx + \int_{\Omega_{2R}(x_0)} k\, g_4 v_0^k \psi^t\, \mathscr{U}^{-q} \, dx + \mathscr{I}_1 + \mathscr{I}_2, \tag{4.35}$$

where $g_4 = 2g_0 + (2\mathrm{M}+1) g_3$,

$$\mathscr{I}_1 = \frac{c_{32}\, k}{R} \sum_{|\alpha|=1} \int_{\Omega_{2R}(x_0)} |A_\alpha(x,u,\nabla_2 u)|\, G^{\varkappa} v_0^k \psi^{t-1} \, dx, \tag{4.36}$$

$$\mathscr{I}_2 = c_{32}\, k^2 \sum_{|\alpha|=2} \int_{\Omega_{2R}(x_0)} |A_\alpha(x,u,\nabla_2 u)| \\ \times G^{\varkappa-2}\, \mathscr{U}^{2(1-q)/\varkappa} \Big\{ R^{-2} + \mathscr{U}^{-2} \sum_{|\beta|=1} |D^\beta u|^2 \Big\} v_0^k \psi^{t-2} \, dx. \tag{4.37}$$

Step 2. We show that the first term on the right-hand side of inequality (4.35) is absorbed by the second term in its left-hand side. For this we need the inequality

$$\mathscr{U} v_0 \leqslant 12\,\mathrm{M} \quad \text{a.e. in } \Omega_{2R}(x_0), \tag{4.38}$$

which follows from (4.1) and from the fact that $\ln s < s$ for every $s > 0$.

Using (4.38), the first term on the right-hand side of inequality (4.35) is estimated in the following way

$$b(\mathrm{M}) \int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa} v_0^k \psi^t \, dx \leqslant 12\,\mathrm{M}\, b(\mathrm{M}) \int_{\Omega_{2R}(x_0)} \Phi G^{\varkappa}\, \mathscr{U}^{-1} v_0^{k-1} \psi^t \, dx. \tag{4.39}$$

Now (4.31), (4.35) and (4.39) imply the inequality

$$(q-1)\,a(\mathrm{M})\int_{\Omega_{2R}(x_0)}\Phi\, G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}v_0^k\psi^t\,dx \\ \leqslant k\int_{\Omega_{2R}(x_0)} g_4 v_0^k\psi^t\,\mathscr{U}^{-q}\,dx+\mathscr{I}_1+\mathscr{I}_2. \quad (4.40)$$

Step 3. Let us estimate from above the quantities $\mathscr{I}_1$ and $\mathscr{I}_2$, which are defined by (4.36) and (4.37) respectively.

Estimate for $\mathscr{I}_1$. We use (2.3), the inequality $G\leqslant\mathscr{U}^{(1-q)/\varkappa}$ and Young's inequality $|yz|\leqslant\varepsilon|y|^{q/(q-1)}+\varepsilon^{1-q}|z|^q$, where

$$y=|A_\alpha(x,u,\nabla_2 u)|\,(G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}\psi^t)^{(q-1)/q},\quad |\alpha|=1,$$

$$z=kR^{-1}G^{(\varkappa+q-1)/q}\,\mathscr{U}^{((q-1)/\varkappa+1)(q-1)/q}\,\psi^{t/q-1}$$

and ε is an appropriate positive number, to obtain

$$\begin{aligned}\mathscr{I}_1\leqslant{}&\frac{(q-1)\,a(\mathrm{M})}{4}\int_{\Omega_{2R}(x_0)}\Phi\, G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}v_0^k\psi^t\,dx\\&+c_{33}\int_{\Omega_{2R}(x_0)} g_1 v_0^k\psi^t\,\mathscr{U}^{-q}\,dx+\frac{c_{33}\,k^q}{R^q}\int_{\Omega_{2R}(x_0)} v_0^k\psi^{t-q}\,dx.\end{aligned}\quad (4.41)$$

Estimate for $\mathscr{I}_2$. Using the first equality in (4.12), Young's inequality, the inequality $G\leqslant\mathscr{U}^{(1-q)/\varkappa}$, (4.30) and (4.2), we establish that if $\varepsilon>0$, α, $\beta\in\Lambda$, $|\alpha|=2$ and $|\beta|=1$, then

$$\begin{aligned}&k^2R^{-2}|A_\alpha(x,u,\nabla_2 u)|\,G^{\varkappa-2}\,\mathscr{U}^{2(1-q)/\varkappa}\psi^{t-2}\\&=|A_\alpha(x,u,\nabla_2 u)|\,(G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}\psi^t)^{(p-1)/p}k^2R^{-2}\psi^{2t/q}\\&\quad\times G^{(\varkappa-p-1)/p}\,\mathscr{U}^{(p-1)/p-(q-1)(p+1)/\varkappa p}\,\psi^{2t/\varkappa-2}\\&\leqslant\varepsilon|A_\alpha(x,u,\nabla_2 u)|^{p/(p-1)}G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}\psi^t\\&\quad+\varepsilon(1+\varepsilon^{-\varkappa/2})\,k^qR^{-q}\psi^{t-\varkappa}\qquad\text{a.e. in }\Omega_{2R}(x_0),\end{aligned}\quad (4.42)$$

$$\begin{aligned}&k^2|A_\alpha(x,u,\nabla_2 u)|\,G^{\varkappa-2}\,\mathscr{U}^{2(1-q)/\varkappa-2}|D^\beta u|^2\,\psi^{t-2}\\&=|A_\alpha(x,u,\nabla_2 u)|\,(G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}\psi^t)^{(p-1)/p}\\&\quad\times|D^\beta u|^2\,(G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}\psi^t)^{2/q}\\&\quad\times k^2G^{(\varkappa-2)/\varkappa}\,\mathscr{U}^{(\varkappa+2)(1-q-\varkappa)/\varkappa^2}\,\psi^{2t/\varkappa-2}\\&\leqslant\varepsilon(|A_\alpha(x,u,\nabla_2 u)|^{p/(p-1)}+|D^\beta u|^q)\,G^{\varkappa-1}\,\mathscr{U}^{(1-q)/\varkappa-1}\psi^t\\&\quad+\varepsilon^{1-\varkappa/2}k^\varkappa R^{-q}\psi^{t-\varkappa}\qquad\text{a.e. in }\Omega_{2R}(x_0).\end{aligned}\quad (4.43)$$

From (2.4), (4.37), (4.42), (4.43), taking into account (4.32) and the suitable choice of

ε, we deduce the estimate

$$\begin{aligned}\mathscr{I}_2 \leqslant \frac{(q-1)\,a(\mathrm{M})}{4} \int_{\Omega_{2R}(x_0)} \Phi\, G^{\varkappa-1}\, \mathscr{U}^{(1-q)/\varkappa-1} v_0^k \psi^t\, dx \\ + c_{34} \int_{\Omega_{2R}(x_0)} g_2\, v_0^k \psi^t\, \mathscr{U}^{-q}\, dx + \frac{c_{34}\, k^l}{R^q} \int_{\Omega_{2R}(x_0)} v_0^k \psi^{t-l}\, dx.\end{aligned} \tag{4.44}$$

From (4.40), (4.41), (4.44), (4.32) and (4.2) it follows that

$$\begin{aligned}&\int_{\Omega_{2R}(x_0)} \Phi\, G^{\varkappa-1}\, \mathscr{U}^{(1-q)/\varkappa-1} v_0^k \psi^t\, dx \\ &\leqslant \frac{c_{35}\, k^l}{R^q} \int_{\Omega_{2R}(x_0)} v_0^k \psi^{t-l}\, dx + \frac{c_{35}\, k^l}{R^{q-n/\tau}} \int_{\Omega_{2R}(x_0)} (g_1+g_2+g_4)\, v_0^k \psi^{t-l}\, dx.\end{aligned}$$

Estimating the last two integrals by Hölder's inequality with the exponents τ and $\tau/(\tau-1)$ and taking into account (4.31) and (4.32), we obtain that for every $k \geqslant \overline{k}$ and $t \in (l, C_0 k]$ the following inequality holds:

$$\begin{aligned}&\int_{\Omega_{2R}(x_0)} \Phi\, G^{\varkappa-1}\, \mathscr{U}^{(1-q)/\varkappa-1} v_0^k \psi^t\, dx \\ &\qquad\leqslant \frac{c_{36}\, k^l}{R^{q-n/\tau}} \Big(\int_{\Omega_{2R}(x_0)} (v_0^k \psi^{t-l})^{\tau/(\tau-1)}\, dx \Big)^{(\tau-1)/\tau}.\end{aligned} \tag{4.45}$$

Define $E_R(x_0) = \{x \in \Omega_{3R/2}(x_0) : \mathscr{U}(x) \leqslant 2^{\varkappa/(1-q)}(M(2R)+R^r)\}$.
We shall suppose that

$$|E_R(x_0)| \neq 0. \tag{4.46}$$

If (4.46) is not true, then (4.5) is a simple consequence of the inequalities

$$M(2R) \geqslant \omega(2R)/2 \quad \text{and} \quad \mathscr{U}(x) > 2^{\varkappa/(1-q)}(M(2R)+R^r) \quad \text{a.e. in } \Omega_{3R/2}(x_0).$$

For every $x \in E_R(x_0)$ we have $G(x) \geqslant \frac{1}{2}[\mathscr{U}(x)]^{(1-q)/\varkappa}$. Using this fact from (4.45), we deduce that

$$\int_{E_R(x_0)} \Phi\, \mathscr{U}^{-q} v_0^k \psi^t\, dx \leqslant \frac{c_{37}\, k^l}{R^{q-n/\tau}} \Big(\int_{\Omega_{2R}(x_0)} (v_0^k \psi^{t-l})^{\tau/(\tau-1)}\, dx \Big)^{(\tau-1)/\tau}. \tag{4.47}$$

Step 4. At this step, using inequality (4.47), we organize an iterative Moser-type process for the functions $v_0^k \psi^t$ in $\Omega_{2R}(x_0)$, which completes the proof of the lemma. We set

$$L = \ln \frac{2^{\varkappa/(q-1)+1} e\,\omega(2R)}{M(2R)+R^r},$$

$$J(k,t) = \frac{1}{R^n} \int_{\Omega_{2R}(x_0)} v_0^k \psi^t\, dx + L^k, \quad k>0,\ t>0, \tag{4.48}$$

$$\theta = \frac{\tau}{\tau-1}\cdot\frac{q}{q^*}, \quad \tilde{l} = \frac{(q+l)q^*}{q}. \tag{4.49}$$

We now prove the following assertion: if $k \geqslant \overline{k}q^*/q$ and $\tilde{l} < t \leqslant C_0 k$, then

$$J(k,t) \leqslant c_{38}\, k^{\tilde{l}}[J(k\theta, t\theta - \tilde{l})]^{1/\theta}. \tag{4.50}$$

Let $k \geqslant \overline{k}q^*/q$ and $\tilde{l} < t \leqslant C_0 k$. We set $\overline{v} = \big(\max\{\overline{v}_0^k, L^k\}\big)^{1/q^*}\psi^{t/q^*}$. It is easy to see that

$$\int_{\Omega_{2R}(x_0)} v_0^k \psi^t\, dx \leqslant \int_{B_{2R}(x_0)} \overline{v}^{q^*}\, dx. \tag{4.51}$$

Applying inequality (3.1) to the function $\overline{v} \in \mathring{W}^{1,q}(B_{2R}(x_0))$ and taking into account (4.32) and the definitions of the functions $\overline{v}$ and ψ, we obtain

$$\int_{B_{2R}(x_0)} \overline{v}^{q^*}\, dx \leqslant c_{39} k^{q^*} L^k R^n + c_{39} k^{q^*}\Big(\int_{E_R(x_0)} \Phi\,\mathscr{U}^{-q} v_0^{kq/q^*}\psi^{tq/q^*-q}\, dx$$
$$+R^{-q}\int_{\Omega_{2R}(x_0)} v_0^{kq/q^*}\psi^{tq/q^*-q}\, dx\Big)^{q^*/q}.$$

From this inequality, estimating the first addend in the brackets by means of (4.47) and the second addend by means of Hölder's inequality and using (4.48) and (4.49), we deduce that

$$\int_{B_{2R}(x_0)} \overline{v}^{q^*}\, dx \leqslant c_{40} R^n\, k^{\tilde{l}}[J(k\theta, t\theta - \tilde{l})]^{1/\theta}.$$

The last inequality, (4.48) and (4.51) imply (4.50).

Now, we choose a number $i_0 \in \mathbb{N}$ such that $\theta^{-i_0} > \overline{k}\,q^*/q$ and set $C_0 = \tilde{l}/(1-\theta)$,

$$k_i = \theta^{-i_0-i}, \quad t_i = \frac{\tilde{l}\,(\theta^{-i_0-i}-1)}{1-\theta}, \quad J_i = J(k_i, t_i), \quad i = 0,1,2,\ldots.$$

Then (4.50) and the inequality $\theta < 1$ imply that for every $i = 0,1,2,\ldots$,

$$J_i^{1/k_i} \leqslant c_{41} J_0^{\theta^{i_0}}. \tag{4.52}$$

Due to Lemma 4.3 and to the inequality $M(2R) \geqslant \omega(2R)/2$ we have

$$J_0^{\theta^{i_0}} \leqslant c_{42}. \tag{4.53}$$

From (4.52) and (4.53) it follows that

$$\|\overline{v}_0\|_{L^\infty(B_R(x_0))} = \lim_{i\to\infty}\Big(\frac{1}{R^n}\int_{B_R(x_0)} \overline{v}_0^{k_i}\, dx\Big)^{1/k_i} \leqslant \limsup_{i\to\infty} J_i^{1/k_i} \leqslant c_1.$$

The proof is complete. □

Thus, the validity of (4.3) is established. Then using Lemma 3.5 and the interior regularity result of [42, Theorem 2.3], we come to the conclusion of Theorem 1. The proof of Theorem 1 is complete.

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