

## EXISTENCE OF SOLUTIONS FOR DIRICHLET QUASILINEAR SYSTEMS INCLUDING A NONLINEAR FUNCTION OF THE DERIVATIVE

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ABSTRACT. In this article we establish the existence of at least one non-trivial classical solution for Dirichlet quasilinear systems with a nonlinear dependence on the derivative. We use variational methods for smooth functionals defined on reflexive Banach spaces, and assumptions on the asymptotic behaviour of the nonlinear data.

### 1. INTRODUCTION

In this article we consider the quasilinear system

$$\begin{aligned} -(p_i - 1)|u_i'(x)|^{p_i-2}u_i''(x) &= \lambda F_{u_i}(x, u_1, \dots, u_n)h_i(x, u_i'), & x \in (a, b), \\ u_i(a) &= u_i(b) = 0 \end{aligned} \quad (1.1)$$

for  $1 \leq i \leq n$  where  $p_i > 1$  for  $1 \leq i \leq n$ ,  $\lambda > 0$ ,  $a, b \in \mathbb{R}$  with  $a < b$ ,  $F : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$  is measurable with respect to  $x$ , for every  $(t_1, \dots, t_n) \in \mathbb{R}^n$ , continuously differentiable in  $(t_1, \dots, t_n)$ , for almost every  $x \in [a, b]$ , and  $F_{t_i}(x, 0, \dots, 0) = 0$  for all  $x \in [a, b]$  and for  $1 \leq i \leq n$ ,  $h_i : [a, b] \times \mathbb{R} \rightarrow [0, +\infty[$  is a bounded and continuous function with  $m_i := \inf_{(x,t) \in [a,b] \times \mathbb{R}} h_i(x, t) > 0$  for  $1 \leq i \leq n$ . Here,  $F_{t_i}$  denotes the partial derivative of  $F$  with respect to  $t_i$ .

Owing to the importance of second-order differential equations with nonlinear derivative dependence in physics, in the last decade or so, many authors applied the variational method to study the existence of solutions of the problems of the form of (1.1) or its variations, see, for example, [2, 3, 5, 8, 9, 10, 11]. We note that the main tools in these cited papers are critical points theorems due to Ricceri and their variants.

Our goal is to establish some new criteria for system (1.1) to have at least one non-trivial classical solution by applying the following critical points theorem due to Ricceri [15, Theorem 2.1].

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2010 *Mathematics Subject Classification*. 34B15, 58E05.

*Key words and phrases*. Classical solution; Dirichlet quasilinear system; critical point theory; variational methods.

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Submitted November 17, 2015. Published February 25, 2016.

**Theorem 1.1.** *Let  $X$  be a reflexive real Banach space, let  $\Phi, \Psi : X \rightarrow \mathbb{R}$  be two Gâteaux differentiable functionals such that  $\Phi$  is sequentially weakly lower semicontinuous, strongly continuous and coercive in  $X$  and  $\Psi$  is sequentially weakly upper semicontinuous in  $X$ . Let  $I_\lambda$  be the functional defined as  $I_\lambda := \Phi - \lambda\Psi$ ,  $\lambda \in \mathbb{R}$ , and for every  $r > \inf_X \Phi$ , let  $\varphi$  be the function defined as*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

*Then, for every  $r > \inf_X \Phi$  and every  $\lambda \in (0, \frac{1}{\varphi(r)})$ , the restriction of the functional  $I_\lambda$  to  $\Phi^{-1}(-\infty, r)$  admits a global minimum, which is a critical point (precisely a local minimum) of  $I_\lambda$  in  $X$ .*

We refer the interested reader to the papers [1, 6, 7, 12, 13, 14] in which Theorem 1.1 has been used for the existence of at least one nontrivial solution for boundary value problem.

## 2. PRELIMINARIES

Let  $X$  be the Cartesian product of  $n$  Sobolev spaces  $W_0^{1,p_1}([a, b]), \dots$ , and  $W_0^{1,p_n}([a, b])$ , i.e.,  $X = W_0^{1,p_1}([a, b]) \times \dots \times W_0^{1,p_n}([a, b])$ , equipped with the norm

$$\|(u_1, \dots, u_n)\| = \sum_{i=1}^n \|u_i\|_{p_i}$$

where

$$\|u_i\|_{p_i} = \left( \int_a^b |u_i'(x)|^{p_i} dx \right)^{1/p_i}, \quad i = 1, \dots, n.$$

Since  $p_i > 1$  for  $i = 1, \dots, n$ ,  $X$  is compactly embedded in  $(C([a, b]))^n$ .

By a classical solution of (1.1), we mean a function  $u = (u_1, \dots, u_n)$  such that, for  $i = 1, \dots, n$ ,  $u_i \in C^1[a, b]$ ,  $u_i' \in AC[a, b]$ , and  $u_i(x)$  satisfies (1.1) a.e. on  $[a, b]$ . We say that a function  $u = (u_1, \dots, u_n) \in X$  is a weak solution of the system (1.1) if

$$\begin{aligned} & \sum_{i=1}^n \int_a^b \left( \int_0^{u_i'(x)} \frac{(p_i - 1)|\tau|^{p_i-2}}{h_i(x, \tau)} d\tau \right) v_i'(x) dx \\ & - \lambda \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx = 0 \end{aligned}$$

for every  $v = (v_1, \dots, v_n) \in X$ . Using standard methods, we see that a weak solution of system (1.1) is indeed a classical solution (see [8, Lemma 2.2]).

Let

$$\begin{aligned} \underline{p} &:= \min\{p_i : 1 \leq i \leq n\}, & \bar{p} &:= \max\{p_i : 1 \leq i \leq n\}, \\ m_i &:= \inf_{(x,t) \in [a,b] \times \mathbb{R}} h_i(x, t) > 0, & \text{for } 1 \leq i \leq n, \\ M_i &:= \sup_{(x,t) \in [a,b] \times \mathbb{R}} h_i(x, t), & \text{for } 1 \leq i \leq n, \\ \overline{M} &:= \max\{M_i : 1 \leq i \leq n\}, & \underline{M} &:= \min\{m_i : 1 \leq i \leq n\}. \end{aligned}$$

Then,  $\overline{M} \geq M_i \geq m_i \geq \underline{M} > 0$  for each  $i = 1, \dots, n$ . Put

$$H_i(x, t) = \int_0^t \left( \int_0^\tau \frac{(p_i - 1)|\delta|^{p_i-2}}{h_i(x, \delta)} d\delta \right) d\tau$$

for all  $(x, t) \in [a, b] \times \mathbb{R}$ ,  $1 \leq i \leq n$ .

### 3. MAIN RESULTS

We formulate our min result as follows. Put

$$p^* = \begin{cases} \overline{p} & \text{if } b - a \geq 1, \\ \underline{p} & \text{if } 0 < b - a < 1. \end{cases}$$

**Theorem 3.1.** *Assume that*

$$\sup_{r>0} \frac{r}{\int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)p^*-1\overline{p}Mr}{2^p})} F(x, t_1, \dots, t_n) dx} > 1 \tag{3.1}$$

where

$$Q\left(\frac{(b-a)p^*-1\overline{p}Mr}{2^p}\right) = \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \leq \frac{(b-a)p^*-1\overline{p}Mr}{2^p} \right\}$$

and there are a non-empty open set  $D \subseteq (a, b)$  and  $B \subset D$  of positive Lebesgue measure such that

$$\begin{aligned} \limsup_{(\xi_1, \dots, \xi_n) \rightarrow (0^+, \dots, 0^+)} \frac{\text{ess inf}_{x \in B} F(x, \xi_1, \dots, \xi_n)}{\sum_{i=1}^n |\xi_i|^p} &= +\infty, \\ \liminf_{(\xi_1, \dots, \xi_n) \rightarrow (0^+, \dots, 0^+)} \frac{\text{ess inf}_{x \in D} F(x, \xi_1, \dots, \xi_n)}{\sum_{i=1}^n |\xi_i|^p} &> -\infty. \end{aligned}$$

Then, for each

$$\lambda \in \Lambda = \left( 0, \sup_{r>0} \frac{r}{\int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)p^*-1\overline{p}Mr}{2^p})} F(x, t_1, \dots, t_n) dx} \right),$$

system (1.1) admits at least one non-trivial classical solution  $u_\lambda = (u_{1\lambda}, \dots, u_{n\lambda})$  in  $X$ . Moreover, we have

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \rightarrow \sum_{i=1}^n \int_a^b H_i(x, u'_{i\lambda}(x)) dx - \lambda \int_a^b F(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x)) dx$$

is negative and strictly decreasing in the open interval  $\Lambda$ .

*Proof.* Our aim is to apply Theorem 1.1 to system (1.1). Let the functionals  $\Phi, \Psi$  be defined by

$$\Phi(u) = \sum_{i=1}^n \int_a^b H_i(x, u'_i(x)) dx, \tag{3.2}$$

$$\Psi(u) = \int_a^b F(x, u_1(x), \dots, u_n(x)) dx \tag{3.3}$$

for  $u = (u_1, \dots, u_n) \in X$ , and put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for  $u \in X$ . Let us prove that the functionals  $\Phi$  and  $\Psi$  satisfy the required conditions in Theorem 1.1. It is well known that  $\Psi$  is a differentiable functional whose differential at the point  $u \in X$  is

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)dx$$

for every  $v = (v_1, \dots, v_n) \in X$ , and  $\Psi$  is sequentially weakly upper semicontinuous. Moreover, according to the definition  $\Phi$ , we see that  $\Phi$  is continuous. Since  $0 < \underline{M} \leq h_i(x, t) \leq \overline{M}$  for each  $(x, t) \in [a, b] \times \mathbb{R}$  and  $i = 1, \dots, n$ , from (3.2) we see that

$$\frac{1}{\overline{M}} \sum_{i=1}^n \frac{\|u'_i\|_{p_i}^{p_i}}{p_i} \leq \Phi(u) \leq \frac{1}{\underline{M}} \sum_{i=1}^n \frac{\|u'_i\|_{p_i}^{p_i}}{p_i} \quad \text{for all } u \in X. \tag{3.4}$$

From (3.4), it follows  $\lim_{\|u\| \rightarrow +\infty} \Phi(u) = +\infty$ , namely  $\Phi$  is coercive. Moreover,  $\Phi$  is continuously differentiable whose differential at the point  $u \in X$  is

$$\Phi'(u)(v) = \sum_{i=1}^n \int_a^b \left( \int_0^{u'_i(x)} \frac{(p_i - 1)|\tau|^{p_i-2}}{h_i(x, \tau)} d\tau \right) v'_i(x)dx$$

for every  $v \in X$ . Furthermore,  $\Phi$  is sequentially weakly lower semicontinuous (see [17, Proposition25.20]). Since for  $1 \leq i \leq n$ ,

$$\max_{x \in [a, b]} |u_i(x)| \leq \frac{(b - a)^{\frac{p_i-1}{p_i}}}{2} \|u'_i\|_{p_i}$$

for each  $u_i \in W_0^{1, p_i}([a, b])$  (see [16]), we have

$$\max_{x \in [a, b]} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq \frac{(b - a)^{p^*-1}}{2^p} \sum_{i=1}^n \frac{\|u'_i\|_{p_i}^{p_i}}{p_i}$$

for each  $u \in X$ . This, for each  $r > 0$ , along with (3.4), ensures that

$$\begin{aligned} & \Phi^{-1}(-\infty, r) \\ &= \{u \in X; \Phi(u) < r\} \\ &\subseteq \left\{ u \in X : \max \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq \frac{(b - a)^{p^*-1} \overline{p} \overline{M} r}{2^p} \text{ for each } x \in [a, b] \right\}. \end{aligned} \tag{3.5}$$

So,

$$\max_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) \leq \int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)^{p^*-1} \overline{p} \overline{M} r}{2^p})} F(x, t_1, \dots, t_n) dx. \tag{3.6}$$

From the definition of  $\varphi(r)$ , since  $0 \in \Phi^{-1}(-\infty, r)$  and  $\Phi(0) = \Psi(0) = 0$ , one has

$$\begin{aligned} \varphi(r) &= \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)} \\ &\leq \frac{\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)}{r} \\ &\leq \frac{\int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)^{p^*-1} \overline{p} \overline{M} r}{2^p})} F(x, t_1, \dots, t_n) dx}{r}. \end{aligned} \tag{3.7}$$

Hence, putting

$$\lambda^* = \sup_{r>0} \frac{r}{\int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)^{p^*-1} p M r}{2^p})} F(x, t_1, \dots, t_n) dx},$$

Theorem 1.1 ensures that for every  $\lambda \in (0, \lambda^*) \subseteq (0, \frac{1}{\varphi(r)})$ , the functional  $I_\lambda$  admits at least one critical point (local minima)  $u_\lambda \in \Phi^{-1}(-\infty, r)$ . Now for every fixed  $\lambda \in (0, \lambda^*)$  we show that  $u_\lambda = (u_{1\lambda}, \dots, u_{n\lambda}) \neq 0$  and the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda),$$

is negative. To this end, let us verify that

$$\limsup_{\|u\| \rightarrow 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.8}$$

Owing to our assumptions at zero, we can fix a sequence  $\{\xi^k = (\zeta^k, \dots, \zeta^k)\} \subset (\mathbb{R}^+)^n$  converging to zero and two constants  $\sigma, \kappa$  (with  $\sigma > 0$ ) such that

$$\begin{aligned} \lim_{k \rightarrow +\infty} \frac{\text{ess inf}_{x \in B} F(x, \xi^k)}{\sum_{i=1}^n |\zeta^k|^p} &= +\infty, \\ \text{ess inf}_{x \in D} F(x, \xi) &\geq \kappa \sum_{i=1}^n |\xi_i|^p \end{aligned}$$

where  $\xi = (\xi_1, \dots, \xi_n)$ ,  $\xi_i \in [0, \sigma]$ ,  $1 \leq i \leq n$ . Now, fix a set  $C \subset B$  of positive measure and a function  $v = (v_1, \dots, v_n) \in X$  such that:

- (i)  $v_i(x) \in [0, 1]$ , for  $1 \leq i \leq n$  and for every  $x \in [a, b]$ ,
- (ii)  $v(x) = (1, \dots, 1)$  for every  $x \in C$ ,
- (iii)  $v(x) = (0, \dots, 0)$  for every  $x \in (a, b) \setminus D$ .

Hence, fix  $M > 0$  and consider a real positive number  $\eta$  with

$$M < \frac{n\eta \text{meas}(C) + \kappa \int_{D \setminus C} \sum_{i=1}^n |v_i(x)|^p dx}{\frac{1}{pM} \sum_{i=1}^n \|v'_i\|_p^p}.$$

Then, there is  $k_0 \in \mathbb{N}$  such that  $\zeta^k < \sigma$  and

$$\text{ess inf}_{x \in B} F(x, \xi^k) \geq \eta \sum_{i=1}^n |\zeta^k|^p$$

for every  $k > k_0$ . Now, for every  $k > k_0$ , recalling the properties of the function  $v$  (that is  $0 \leq \zeta^k v_i(x) < \sigma$ ,  $1 \leq i \leq n$  for  $k$  sufficiently large), one has

$$\begin{aligned} \frac{\Psi(\xi^k v)}{\Phi(\xi^k v)} &= \frac{\int_C F(x, \zeta^k, \dots, \zeta^k) dx + \int_{D \setminus C} F(x, \zeta^k v_1(x), \dots, \zeta^k v_n(x)) dx}{\Phi(\xi^k v)} \\ &> \frac{n\eta \text{meas}(C) + \kappa \int_{D \setminus C} \sum_{i=1}^n |v_i(x)|^p dx}{\frac{1}{pM} \sum_{i=1}^n \|v'_i\|_p^p} > M \end{aligned}$$

where  $\xi^k v = (\zeta^k v_1, \dots, \zeta^k v_n)$ . Since  $M$  could be consider arbitrarily large, it follows that

$$\lim_{k \rightarrow \infty} \frac{\Psi(\xi^k v)}{\Phi(\xi^k v)} = +\infty,$$

from which (3.8) follows. Hence, there exists a sequence  $\{w_k = (w_{1k}, \dots, w_{nk})\} \subset X$  strongly converging to zero,  $w_k \in \Phi^{-1}(-\infty, r)$  and

$$I_\lambda(w_k) = \Phi(w_k) - \lambda\Psi(w_k) < 0.$$

Since  $u_\lambda = (u_{1\lambda}, \dots, u_{n\lambda})$  is a global minimum of the restriction of  $I_\lambda$  to the set  $\Phi^{-1}(-\infty, r)$ , we deduce that

$$I_\lambda(u_\lambda) < 0, \tag{3.9}$$

so that  $u_\lambda$  is not trivial. From (3.9) we easily see that the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.10}$$

is negative.

Now we prove that  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$ . Since  $\Phi$  is coercive and for  $\lambda \in (0, \lambda^*)$  the solution  $u_\lambda \in \Phi^{-1}(-\infty, r)$ , one has that there exists a positive constant  $L$  such that  $\|u_\lambda\| \leq L$  for every  $\lambda \in (0, \lambda^*)$ . Therefore, there exists a positive constant  $\mathcal{M}$  such that

$$\left| \int_a^b \sum_{i=1}^n F_{u_{\lambda_i}}(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x)) u_\lambda(x) dx \right| \leq \mathcal{M} \|u_\lambda\| \leq \mathcal{M}L \tag{3.11}$$

for every  $\lambda \in (0, \lambda^*)$ . Since  $u_\lambda$  is a critical point of  $I_\lambda$ , we have  $I'_\lambda(u_\lambda)(v) = 0$  for any  $v \in X$  and every  $\lambda \in (0, \lambda^*)$ . In particular  $I'_\lambda(u_\lambda)(u_\lambda) = 0$ ; that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_a^b \sum_{i=1}^n F_{u_{\lambda_i}}(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x)) u_\lambda(x) dx \tag{3.12}$$

for every  $\lambda \in (0, \lambda^*)$ . Then, since

$$0 \leq \frac{1}{M} \sum_{i=1}^n \|u'_i\|_{p_i}^{p_i} \leq \Phi'(u_\lambda)(u_\lambda)$$

by (3.12) it follows that

$$0 \leq \frac{1}{M} \sum_{i=1}^n \|u'_i\|_{p_i}^{p_i} \leq \lambda \int_a^b \sum_{i=1}^n F_{u_{\lambda_i}}(x, u_{1\lambda}(x), \dots, u_{n\lambda}(x)) u_\lambda(x) dx$$

for any  $\lambda \in (0, \lambda^*)$ . Letting  $\lambda \rightarrow 0^+$ , by (3.11) we have  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$ . Further, taking  $X \hookrightarrow (C([a, b]))^n$  into account, one has

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_\infty = 0. \tag{3.13}$$

Finally, we show that the map  $\lambda \mapsto I_\lambda(u_\lambda)$  is strictly decreasing in  $(0, \lambda^*)$ . For our aim we see that for any  $u \in X$ ,

$$I_\lambda(u) = \lambda \left( \frac{\Phi(u)}{\lambda} - \Psi(u) \right). \tag{3.14}$$

Now, let us fix  $0 < \lambda_1 < \lambda_2 < \lambda^*$  and let  $\bar{u}_{\lambda_1} = (u_{1\lambda_1}, \dots, u_{n\lambda_1})$  and  $\bar{u}_{\lambda_2} = (u_{1\lambda_2}, \dots, u_{n\lambda_2})$  be the global minimum of the functional  $I_{\lambda_i}$  restricted to  $\Phi(-\infty, r)$  for  $i = 1, 2$ . Also, let

$$m_{\lambda_i} = \left( \frac{\Phi(\bar{u}_{\lambda_i})}{\lambda_i} - \Psi(\bar{u}_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left( \frac{\Phi(v)}{\lambda_i} - \Psi(v) \right)$$

for  $i = 1, 2$ . Clearly, (3.10) and (3.14), since  $\lambda > 0$ , imply that

$$m_{\lambda_i} < 0, \quad \text{for } i = 1, 2. \tag{3.15}$$

Moreover, since  $0 < \lambda_1 < \lambda_2$ , we have

$$m_{\lambda_2} \leq m_{\lambda_1}. \tag{3.16}$$

Hence, from (3.14)-(3.16) and again since  $0 < \lambda_1 < \lambda_2$ , we obtain

$$I_{\lambda_2}(\bar{u}_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(\bar{u}_{\lambda_1}),$$

which means the map  $\lambda \mapsto I_\lambda(u_\lambda)$  is strictly decreasing in  $\lambda \in (0, \lambda^*)$ . Since  $\lambda < \lambda^*$  is arbitrary, we observe  $\lambda \mapsto I_\lambda(u_\lambda)$  is strictly decreasing in  $(0, \lambda^*)$ . The proof is complete.  $\square$

**Remark 3.2.** Here employing Ricceri’s variational principle we are looking for the existence of critical points of the functional  $I_\lambda$  naturally associated to system (1.1). We emphasize that by direct minimization, we can not argue, in general for finding the critical points of  $I_\lambda$ . Because, in general,  $I_\lambda$  can be unbounded from the following in  $X$ . Indeed, for example, in the case when

$$F(x, t_1, \dots, t_n) = \left( \sum_{i=1}^n (|t_i| + |t_i|^{q_i}) \right)$$

for all  $(x, t_1, \dots, t_n) \in [a, b] \times \mathbb{R}^n$  with  $q_i \in (\bar{p}, +\infty)$  for  $1 \leq i \leq n$ , for any fixed  $u \in X \setminus \{0\}$  and  $\iota \in \mathbb{R}$ , we obtain

$$\begin{aligned} I_\lambda(\iota u) &= \Phi(\iota u) - \lambda \int_a^b F(x, \iota u_1(x), \dots, \iota u_n(x)) dx \\ &\leq \frac{\iota^{\bar{p}}}{pM} \sum_{i=1}^n \|u_i\|_{p_i}^{p_i} - \lambda \iota \sum_{i=1}^n \|u_i\|_{L^1} - \lambda \sum_{i=1}^n \iota^{q_i} \|u_i\|_{L^{q_i}}^{q_i} \rightarrow -\infty \end{aligned}$$

as  $\iota \rightarrow +\infty$ .

**Remark 3.3.** We want to point out that the energy functional  $I_\lambda$  associated with system (1.1) is not coercive. Indeed, fix  $u \in X \setminus \{0\}$  and  $\iota \in \mathbb{R}$ , then, for  $F(x, t_1, \dots, t_n) = \sum_{i=1}^n |t_i|^{s_i}$  for all  $(x, t_1, \dots, t_n) \in [a, b] \times \mathbb{R}^n$  with  $s_i \in (\bar{p}, +\infty)$ . We have

$$\begin{aligned} I_\lambda(\iota u) &= \Phi(\iota u) - \lambda \int_a^b F(x, \iota u_1(x), \dots, \iota u_n(x)) dx \\ &\leq \frac{\iota^{\bar{p}}}{pM} \sum_{i=1}^n \|u_i\|_{p_i}^{p_i} - \lambda \sum_{i=1}^n \iota^{s_i} \|u_i\|_{L^{s_i}}^{s_i} \rightarrow -\infty \end{aligned}$$

as  $\iota \rightarrow +\infty$ .

**Remark 3.4.** If in Theorem 3.1,  $F_{t_i}(x, t_1, \dots, t_n) \geq 0$  for a.e.  $x \in [a, b]$ ,  $1 \leq i \leq n$ , the condition (3.1) becomes to the more simple and significative form

$$\sup_{r>0} \frac{r}{\int_a^b F(x, \sqrt[p]{\frac{(b-a)^{p^*-1} \bar{p}^2 M r}{2^p}}, \dots, \sqrt[p]{\frac{(b-a)^{p^*-1} \bar{p}^2 M r}{2^p}}) dx} > 1. \tag{3.17}$$

Moreover, if

$$\limsup_{r \rightarrow +\infty} \frac{r}{\int_a^b F(x, \sqrt[p]{\frac{(b-a)^{p^*-1} \bar{p}^2 M r}{2^p}}, \dots, \sqrt[p]{\frac{(b-a)^{p^*-1} \bar{p}^2 M r}{2^p}}) dx} > 1,$$

then, (3.17) automatically holds.

**Remark 3.5.** For fixed  $\bar{r} > 0$  let

$$\frac{\bar{r}}{\int_a^b \max_{(t_1, \dots, t_n) \in Q(\frac{(b-a)^{p^* - 1} \bar{p} \bar{M} \bar{r}}{2^p})} F(x, t_1, \dots, t_n) dx} > 1.$$

Then the result of Theorem 3.1 holds. In fact, it ensures the existence of least one non-trivial classical solution  $u_\lambda = (u_{1\lambda}, \dots, u_{n\lambda})$  in  $X$  such that

$$\max \sum_{i=1}^n \frac{|u_{i\lambda}(x)|^{p_i}}{p_i} \leq \frac{(b-a)^{p^* - 1} \bar{p} \bar{M} \bar{r}}{2^p} \quad \text{for each } x \in [a, b].$$

**Remark 3.6.** We observe that Theorem 3.1 is a bifurcation result in the sense that the pair  $(0, 0)$  belongs to the closure of the set  $\{(u_\lambda, \lambda) \in X \times (0, +\infty) : u_\lambda = (u_{1\lambda}, \dots, u_{n\lambda}) \text{ is a non-trivial classical solution of (1.1)}\}$  in  $X \times \mathbb{R}$ . Indeed, by Theorem 3.1 we have that

$$\|u_\lambda\| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0.$$

Hence, there exist two sequences  $\{u_j = (u_{1j}, \dots, u_{nj})\}$  in  $X$  and  $\{(\lambda_1, \dots, \lambda_n)\}$  in  $(\mathbb{R}^+)^n$  (here  $u_{ij} = u_{\lambda_i}$ ) such that for  $1 \leq i \leq n$

$$\lambda_i \rightarrow 0^+ \quad \text{and} \quad \|u_{ij}\| \rightarrow 0,$$

as  $j \rightarrow +\infty$ . Moreover, we want to emphasize that because the map  $(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$  is strictly decreasing, for every  $\lambda_1, \lambda_2 \in (0, \lambda^*)$ , with  $\lambda_1 \neq \lambda_2$ , the solutions  $\bar{u}_{\lambda_1}$  and  $\bar{u}_{\lambda_2}$  ensured by Theorem 3.1 are different.

We present the following example in which the hypotheses of Theorem 3.1 are satisfied.

**Example 3.7.** Consider the system

$$\begin{aligned} -u_1''(x) &= \lambda F_{u_1}(u_1, u_2) h_1(u_1'), & x \in (0, 1), \\ -u_2''(x) &= \lambda F_{u_2}(u_1, u_2) h_2(u_2'), & x \in (0, 1), \\ u_1(0) = u_1(1) &= 0, & u_2(0) = u_2(1) = 0 \end{aligned} \tag{3.18}$$

where  $F(t_1, t_2) = (t_1^2 + t_2^2) \sin^2(t_1) + \cos(t_2)$  for  $t_1, t_2 \in \mathbb{R}$ ,

$$h_1(t_1) = \begin{cases} 2, & \text{if } t_1 \in (-\infty, 0), \\ \frac{2}{1 + \sqrt{t_1}}, & \text{if } t_1 \in [0, 1], \\ 1, & \text{if } t_1 \in (1, +\infty) \end{cases}$$

and  $h_2(t_2) = \frac{4}{3 + \sin t_2}$  for  $t_2 \in \mathbb{R}$ . We observe that  $F$ ,  $h_1$  and  $h_2$  are continuous,  $h_1$  and  $h_2$  are bounded with  $m_1 = \inf h_1 = 1$ ,  $m_2 = \inf h_2 = 1$ ,  $M_1 = \sup h_1 = 2$  and  $M_2 = \sup h_2 = 2$ . By the definitions of  $h_1$  and  $h_2$  we have

$$H_1(t_1) = \begin{cases} \frac{1}{4} t_1^2, & \text{if } t_1 \in (-\infty, 0), \\ \frac{1}{4} t_1^2 + \frac{2}{15} t_1^{\frac{5}{2}}, & \text{if } t_1 \in [0, 1], \\ \frac{1}{2} t_1^2 - \frac{1}{4} t_1 + \frac{2}{15}, & \text{if } t_1 \in (1, +\infty) \end{cases}$$

and  $H_2(t_2) = \frac{3}{8} t_2^2 - \frac{\sin t_2}{4}$  for  $t_2 \in \mathbb{R}$ . It is easy to see that all assumptions of Theorem 3.1 are satisfied. By Theorem 3.1 for each  $\lambda \in (0, \frac{1}{4})$ , system (3.18) admits



at least one nontrivial classical solution  $u_\lambda = (u_{1\lambda}, u_{2\lambda})$  in  $X$ . Moreover, we have  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$  and the real function

$$\begin{aligned} \lambda \mapsto & \int_0^1 \left( H_1(u'_{1\lambda}(x)) + H_2(u'_{2\lambda}(x)) \right) dx \\ & - \lambda \int_0^1 \left( (u_{1\lambda}^2(x) + u_{2\lambda}^2(x)) \sin^2(u_{1\lambda}(x)) + \cos(u_{2\lambda}(x)) \right) dx \end{aligned}$$

is negative and strictly decreasing in  $(0, \frac{1}{4})$ .

As an application of Theorem 3.1, we consider the problem

$$\begin{aligned} -(p-1)|u'(x)|^{p-2}u''(x) &= \lambda f(x, u)h(u'), \quad x \in (a, b), \\ u(a) &= u(b) = 0 \end{aligned} \tag{3.19}$$

where  $p > 1$ ,  $\lambda > 0$ ,  $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  is an  $L^1$ -Carathéodory function such that  $f(x, 0) = 0$  for a.e.  $x \in [a, b]$ , and  $h : \mathbb{R} \rightarrow [0, +\infty)$  is a bounded and continuous function with  $m = \inf_{t \in \mathbb{R}} h(t) > 0$ .

Let  $M = \sup_{t \in \mathbb{R}} h(t)$ ,

$$\begin{aligned} F(x, t) &= \int_0^t f(x, \xi) d\xi \quad \text{for every } (x, t) \in [a, b] \times \mathbb{R}, \\ H(t) &= \int_0^t \left( \int_0^\tau \frac{(p-1)|\delta|^{p-2}}{h(\delta)} d\delta \right) d\tau \end{aligned}$$

for all  $t \in \mathbb{R}$ . For  $\gamma > 0$ , define

$$W(\gamma) = \{t \in \mathbb{R} : |t|^p \leq p\gamma\}.$$

We conclude this section by giving the following consequence of Theorem 3.1.

**Theorem 3.8.** *Assume that*

$$\sup_{r>0} \frac{r}{\int_a^b \max_{t \in W(\frac{(b-a)^{p-1}pMr}{2p})} F(x, t) dx} > 1$$

and there are a non-empty open sets  $D \subseteq (0, T)$  and  $B \subset D$  of positive Lebesgue measure such that

$$\begin{aligned} \limsup_{\xi \rightarrow 0^+} \frac{\text{ess inf}_{x \in B} F(x, \xi)}{\xi^p} &= +\infty, \\ \liminf_{\xi \rightarrow 0^+} \frac{\text{ess inf}_{x \in D} F(x, \xi)}{\xi^p} &> -\infty. \end{aligned}$$

Then, for each

$$\lambda \in \Lambda = \left( 0, \sup_{r>0} \frac{r}{\int_a^b \max_{t \in W(\frac{(b-a)^{p-1}pMr}{2p})} F(x, t) dx} \right),$$

problem (3.19) admits at least one nontrivial classical solution  $u_\lambda \in W_0^{1,p}([a, b])$  such that  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$  and the real function

$$\lambda \mapsto \int_a^b H(u'_\lambda(x)) dx - \lambda \int_a^b F(x, u_\lambda(x)) dx$$

is negative and strictly decreasing in the open interval  $\Lambda$ .

Now, we point out the following result, as a consequence of Theorem 3.8 in which the function  $f$  has separated variables.

**Theorem 3.9.** *Let  $\alpha \in L^\infty([a, b])$  with  $\text{ess inf}_{x \in [a, b]} \alpha(x) > 0$ . Further, let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function such that  $f(0) = 0$ . Put  $F(\xi) = \int_0^\xi f(t)dt$  for all  $\xi \in \mathbb{R}$ , and assume that*

$$\frac{1}{\int_a^b \alpha(x)dx} \sup_{r>0} \frac{r}{\max_{\xi \in W(\frac{(b-a)^2 Mr}{2})} F(\xi)} > 1,$$

$$\lim_{\xi \rightarrow 0^+} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for each

$$\lambda \in \Lambda = \left( 0, \frac{1}{\int_a^b \alpha(x)dx} \sup_{r>0} \frac{r}{\max_{\xi \in W(\frac{(b-a)^2 Mr}{2})} F(\xi)} \right),$$

the problem

$$\begin{aligned} -u'' &= \lambda \alpha(x) f(u) h(u'), & x \in (a, b), \\ u(a) &= u(b) = 0 \end{aligned} \tag{3.20}$$

admits at least one nontrivial classical solution  $u_\lambda \in W_0^{1,2}([a, b])$  such that

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \rightarrow \int_a^b H(u'_\lambda(x))dx - \lambda \int_a^b \alpha(x)F(u_\lambda(x))dx$$

is negative and strictly decreasing in  $\Lambda$ .

Next we given an example to illustrate Theorem 3.9.

**Example 3.10.** Consider the problem

$$\begin{aligned} -u'' &= \lambda(1 + e^x) f(u) h(u'), & x \in (0, 1), \\ u(0) &= u(1) = 0 \end{aligned} \tag{3.21}$$

where  $f(t) = t^2 \sin^2(t) - \sin(t)$  for  $t \in \mathbb{R}$  and

$$h(t) = \begin{cases} 1, & \text{if } t \in (-\infty, 0), \\ \frac{1}{1+t^2}, & \text{if } t \in [0, 1], \\ \frac{1}{2}, & \text{if } t \in (1, +\infty). \end{cases}$$

We observe that  $f$  and  $h$  are continuous and  $h$  is bounded with  $m = \inf_{t \in \mathbb{R}} h(t) = \frac{1}{2}$  and  $M = \sup_{t \in \mathbb{R}} h(t) = 1$ . By the expression of  $f$  and  $h$  we have

$$F(t) = \frac{1}{6}t^3 - \frac{1}{4} \left( t^2 \sin(2t) + t \cos(2t) - \frac{1}{2} \sin(2t) \right) + \cos(t) - 1$$

for every  $t \in \mathbb{R}$  and

$$H(t) = \begin{cases} \frac{1}{2}t^2, & \text{if } t \in (-\infty, 0), \\ \frac{1}{2}t^2 + \frac{1}{12}t^4, & \text{if } t \in [0, 1], \\ t^2 - \frac{1}{2}t + \frac{1}{12}, & \text{if } t \in (1, +\infty). \end{cases}$$

It is easy to see that all assumptions of Theorem 3.9 are satisfied. By Theorem 3.9 for each  $\lambda \in (0, \frac{1}{4e})$ , problem (3.21) admits at least one nontrivial classical solution  $u_\lambda$  in  $W_0^{1,2}([0, 1])$ . Moreover,  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$  and the real function

$$\lambda \mapsto \int_0^1 H(u'_\lambda(x))dx - \lambda \int_0^1 (1 + e^x)F(u_\lambda(x))dx$$

is negative and strictly decreasing in  $(0, 1/(4e))$ .

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