J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian

Mingqi Xiang [a], Binlin Zhang [b,c,*], Massimiliano Ferrara [d]

[a] College of Science, Civil Aviation University of China, Tianjin, 300300, PR China
[b] Chern Institute of Mathematics and LPMC, Nankai University, Tianjin, 300071, PR China
[c] Department of Mathematics, Heilongjiang Institute of Technology, Harbin, 150050, PR China
[d] Department of Law and Economics, University Mediterranea of Reggio Calabria, I-89127 Via dei Bianchi, 2-Palazzo Zani, Reggio Calabria, Italy

A R T I C L E I N F O

Article history:
Received 28 August 2014
Available online xxxx
Submitted by C. Gutierrez

Keywords:
Fractional p-Laplacian
Kirchhoff type problem
Integro-differential operator
Mountain Pass Theorem

A B S T R A C T

The purpose of this paper is to investigate the existence of weak solutions for a Kirchhoff type problem driven by a non-local integro-differential operator of elliptic type with homogeneous Dirichlet boundary conditions as follows:

$$\begin{cases} M\left(\int\limits_{\mathbb{R}^{2N}} |u(x)-u(y)|^p K(x-y)dxdy\right)\mathcal{L}_K^p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\mathcal{L}_K^p$ is a non-local operator with singular kernel K, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipshcitz boundary $\partial\Omega$, M is a continuous function and f is a Carathéodory function satisfying the Ambrosetti–Rabinowitz type condition. We discuss the above-mentioned problem in two cases: when f satisfies sublinear growth condition, the existence of nontrivial weak solutions is obtained by applying the direct method in variational methods; when f satisfies suplinear growth condition, the existence of two nontrivial weak solutions is obtained by using the Mountain Pass Theorem.

1. Introduction

Recently, a great attention has been focused on the study of problem involving fractional and non-local operators. This type of problem arises in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of

* Corresponding author at: Department of Mathematics, Heilongjiang Institute of Technology, Harbin, 150050, PR China.
E-mail addresses: xiangmqhit@gmail.com (M. Xiang), zhbinlin@gmail.com (B. Zhang), massimiliano.ferrara@unirc.it (M. Ferrara).

http://dx.doi.org/10.1016/j.jmaa.2014.11.055

stochastically stabilization of Lévy processes, see [3,7,24,27] and the references therein. The literature on non-local operators and their applications is very interesting and quite large, we refer the interested reader to [2,5,8,9,19,21,22,28,31,42] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the interested reader to [12].

In this paper we deal with the following Kirchhoff type problem:

$$\begin{cases} M\Big(\displaystyle\int_{\mathbb{R}^{2N}} \big|u(x)-u(y)\big|^p K(x-y)dx\,dy\Big)\,\mathcal{L}_K^p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $N > ps$ with $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $M : [0,\infty) \to (0,\infty)$ is a continuous function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\mathcal{L}_K^p$ is a non-local operator defined as follows:

$$\mathcal{L}_K^p u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \big|u(x)-u(y)\big|^{p-2}\big(u(x)-u(y)\big)K(x-y)dy, \quad x \in \mathbb{R}^N,$$

where $1 < p < \infty$ and $K : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ is a measurable function with the following property

$$\begin{cases} \gamma K \in L^1\big(\mathbb{R}^N\big), & \text{where } \gamma(x) = \min\big\{|x|^p, 1\big\}; \\ \text{there exists } k_0 > 0 \quad \text{such that} \quad K(x) \geq k_0|x|^{-(N+ps)} & \text{for any } x \in \mathbb{R}^N \setminus \{0\}; \\ K(x) = K(-x) & \text{for any } x \in \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{1.2}$$

A typical example for K is given by singular kernel $K(x) = |x|^{-(N+ps)}$. In this case, problem (1.1) becomes

$$\begin{cases} M\Big(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dx\,dy\Big)(-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $(-\Delta)_p^s$ is the fractional p-Laplace operator which (up to normalization factors) may be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,dy$$

for $x \in \mathbb{R}^N$, see [16,18–20] and the references therein for further details on the fractional p-Laplacian operator.

When $p = 2$ and $M = 1$, problem (1.3) reduces to the fractional Laplacian problem:

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

One typical feature of problem (1.4) is the nonlocality, in the sense that the value of $(-\Delta)^s u(x)$ at any point $x \in \Omega$ depends not only on Ω, but actually on the entire space $\mathbb{R}^N$. The functional framework that takes into account problem (1.4) with Dirichlet boundary condition was introduced in [38,39]. We refer also to [13,14,29,30,32,40,43,44] for further details on the functional framework and its applications to the existence of solutions for problem (1.4).

The Kirchhoff type equations arise in the description of nonlinear vibrations of an elastic string, see Kirchhoff [23]. In recent years, much interest has grown on p-Kirchhoff type problems with Dirichlet boundary data. In [11], the authors showed the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff type equations via variational methods. In [26], the author obtained the existence of infinite solutions to the p-Kirchhoff type quasilinear elliptic equations via the fountain theorem. In [10], the authors investigated higher order $p(x)$-Kirchhoff type equations via symmetric Mountain Pass Theorem, even in the degenerate case. However, they did not consider the existence of solutions for Kirchhoff type problems in the fractional setting. In the very recent paper [15], the authors first provided a detailed discussion about the physical meaning underlying the fractional Kirchhoff models and their applications, see [15, Appendix A] for further details. More precisely, the authors proposed a stationary Kirchhoff variational model, which takes into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. In [34], using a three critical points theorem for non-differential functionals, the author obtained the existence of three solutions for Kirchhoff type problems involving the nonlocal fractional Laplacian. In [35], the authors established the existence and multiplicity of nontrivial solutions for a Kirchhoff type eigenvalue problem in $\mathbb{R}^N$ involving critical nonlinearity and nonlocal fractional Laplacian, see also [4,36] for further details about this kind of nonlinearity.

Motivated by the above papers, the aim of this paper to study the existence of solutions for a Kirchhoff type problems involving the nonlocal fractional p-Laplacian. For this, we suppose that the Kirchhoff function $M : [0, \infty) \to (0, \infty)$ is a continuous function satisfying the following conditions:

$$\text{there exists } m_0 > 0 \quad \text{such that} \quad M(t) \geq m_0 \quad \text{for all } t \in [0, \infty); \tag{M1}$$

$$\text{there exists } \theta > 0 \quad \text{such that} \quad \widehat{M}(t) \geq \theta M(t)t \quad \text{for all } t \in [0, \infty), \tag{M2}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$.

A typical example for M is given by $M(t) = 1 + bt^m$ with $m > 0$, $b \geq 0$ for all $t \geq 0$. In [15,34,35], in order to obtain the existence of weak solutions the authors assume that M is a nondecreasing function on $[0, \infty)$. However, here we suppose that M satisfies (M2). Under assumption (M2), we can also deal with cases in which M is not monotone as $M(t) = (1 + t)^k + (1 + t)^{-1}$ with $0 < k < 1$ for all $t \geq 0$.

Also, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:

there exist $a > 0$ and $1 < q < p_s^*$ such that

$$\big|f(x, \xi)\big| \leq a\big(1 + |\xi|^{q-1}\big) \quad \text{a.e. } x \in \Omega,\ \xi \in \mathbb{R}; \tag{H1}$$

there exist $\mu > \frac{p}{\theta}$ and $r > 0$ such that for a.e. $x \in \Omega$ and $r \in \mathbb{R}$, $|\xi| \geq r$,

$$0 < \mu F(x, \xi) \leq \xi f(x, \xi), \tag{H2}$$

where $F(x, \xi) = \int_0^\xi f(x, \tau)d\tau$ and θ is given in assumption (M2);

$$\lim_{\xi \to 0} \frac{f(x, \xi)}{|\xi|^{p-1}} = 0 \quad \text{uniformly for a.e. } x \in \Omega; \tag{H3}$$

there exist $a_1 > 0$ and an open bounded set $\Omega_0 \subset \Omega$ such that

$$\big|f(x, \xi)\big| \geq a_1|\xi|^{q-1} \quad \text{for a.e. } x \in \Omega_0 \text{ and all } \xi \in \mathbb{R}. \tag{H4}$$

Note that assumption (H2) is not the usual Ambrosetti–Rabinowitz condition, since here we suppose that $\mu > p/\theta$. This difference is caused by the function M in problem (1.1).

Now, we give the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $u \in W_0$ is a weak solution of problem (1.1), if

$$M\big(\|u(x)\|_{W_0}^p\big) \int\limits_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p-2}\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)dxdy$$
$$= \int\limits_{\Omega} f\big(x, u(x)\big)\varphi(x)dx,$$

for any $\varphi \in W_0$, where space W_0 will be introduced in Section 2.

First, using the direct method in variational methods, we get the first main result.

Theorem 1.1. *Let $K : \mathbb{R}^N \backslash \{0\} \to (0, \infty)$ be a function satisfying (1.2). Suppose that M satisfies (M1) and f satisfies (H1) and (H4). If $1 < q < p$, then the problem (1.1) has a nontrivial weak solution in W_0.*

Then, using the Mountain Pass Theorem, we obtain the second main result.

Theorem 1.2. *Let $K : \mathbb{R}^N \backslash \{0\} \to (0, \infty)$ be a function satisfying (1.2). Suppose that M satisfies (M1) and (M2) and f satisfies (H1)–(H3). If $p < q < p_s^*$, then problem (1.1) has a nontrivial weak solution in W_0.*

This paper is organized as follows. In Section 2, we will present some necessary definitions and properties of space W_0. In Section 3, using variational methods, we obtain the existence of weak solutions for problem (1.1) in two cases: $1 < q \le p$ and $p < q < p_s^*$.

2. Variational framework

In this section, we first give some basic results that will be used in the next section. Let $0 < s < 1 < p < \infty$ be real numbers and the fractional critical exponent p_s^* be defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N \\ \infty & \text{if } sp \ge N. \end{cases}$$

In the following, we denote $Q = \mathbb{R}^N \backslash \mathcal{O}$, where

$$\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N},$$

and $\mathcal{C}(\Omega) = \mathbb{R}^N \backslash \Omega$. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\int\limits_{Q} |u(x) - u(y)|^p K(x-y)dxdy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \Big(\int\limits_{Q} |u(x) - u(y)|^p K(x-y)dxdy\Big)^{\frac{1}{p}}.$$

It is easy to prove that $\|\cdot\|_W$ is a norm on W. We shall work in the closed linear subspace

$$W_0 = \big\{u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\big\}.$$

Lemma 2.1. $C_0^\infty(\Omega) \subset W_0$.

Proof. Using the same arguments as in [38], this lemma can be proved. For completeness, we give its proof. For $v \in C_0^\infty(\Omega)$, we only need to check that $\int_{\mathbb{R}^{2N}} |v(x) - v(y)|^p K(x-y)dxdy < \infty$. Since $v = 0$ in $\mathbb{R}^N \setminus \Omega$, we have

$$\int_{\mathbb{R}^{2N}} \left|v(x) - v(y)\right|^p K(x-y)dxdy = \int_\Omega \int_\Omega \left|v(x) - v(y)\right|^p K(x-y)dxdy + 2\int_\Omega \int_{\mathbb{R}^N\setminus\Omega} \left|v(x) - v(y)\right|^p K(x-y)dxdy \le 2\int_\Omega \int_{\mathbb{R}^N} \left|v(x) - v(y)\right|^p K(x-y)dxdy.$$

Notice that $|v(x) - v(y)| \le \|\nabla v\|_{L^\infty(\mathbb{R}^N)}|x-y|$ and $|v(x) - v(y)| \le 2\|v\|_{L^\infty(\mathbb{R}^N)}$ for all $x, y \in \mathbb{R}^N$. Thus,

$$\left|v(x) - v(y)\right|^p \le \left(2\|v\|_{C^1(\mathbb{R}^N)}\right)^p \min\left\{|x-y|^p, 1\right\}.$$

Therefore, we obtain

$$\int_{\mathbb{R}^{2N}} \left|v(x) - v(y)\right|^p K(x-y)dxdy \le \left(2\|v\|_{C^1(\mathbb{R}^N)}\right)^p |\Omega| \int_{\mathbb{R}^N} \min\left\{|z|^p, 1\right\} K(z)dz.$$

Assumption (1.2) implies that $v \in W_0$. □

The Gagliardo seminorm is defined for all measurable function $u : \Omega \to \mathbb{R}$ by

$$[u]_{s,p} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy\right)^{\frac{1}{p}}.$$

The fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) = \left\{u \in L^p(\Omega) : [u]_{s,p} < \infty\right\},$$

endowed with the norm

$$\|u\|_{s,p} = \left(\|u\|_p^p + [u]_{s,p}^p\right)^{\frac{1}{p}}.$$

For a detailed account on the properties of $W^{s,p}(\Omega)$, we refer to [12]. We can define the space $W^{s,p}(\mathbb{R}^N)$ in the same way.

Lemma 2.2. *Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ satisfy assumption (1.2). Then the following assertions hold:*

(a) *if $v \in W$, then $v \in W^{s,p}(\Omega)$. Moreover,*

$$\|v\|_{W^{s,p}(\Omega)} \le \max\left\{1, k_0^{-\frac{1}{p}}\right\} \|v\|_W;$$

(b) *if* $v \in W_0$, *then* $v \in W^{s,p}(\mathbb{R}^N)$. *Moreover,*

$$\|v\|_{W^{s,p}(\Omega)} \le \|v\|_{W^{s,p}(\mathbb{R}^N)} \le \max\{1, k_0^{-\frac{1}{p}}\}\|v\|_W.$$

Proof. For $v \in W$, by (1.2), we have

$$\int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy \le \frac{1}{k_0} \int_Q \left|v(x)-v(y)\right|^p K(x-y)dxdy < \infty.$$

Thus, the first assertion is proved. For $v \in W$ and $v = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we have $\|v\|_{L^p(\mathbb{R}^N)} = \|v\|_{L^p(\Omega)} < \infty$ and

$$\begin{aligned}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy &= \int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy \\ &\le \frac{1}{k_0} \int_Q \left|v(x)-v(y)\right|^p K(x-y)dxdy < \infty.\end{aligned}$$

Thus, $v \in W^{s,p}(\mathbb{R}^N)$ and the estimate on the norm easily follows. □

Lemma 2.3. *Let* $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ *satisfy assumption* (1.2). *Then*

(1) *there exists a positive constant* $C_0 = C_0(N, p, s)$ *such that for any* $v \in W_0$ *and* $1 \le q \le p_s^*$

$$\begin{aligned}\|v\|^p_{L^q(\Omega)} &\le C_0 \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy \\ &\le \frac{C_0}{k_0} \int_Q \left|v(x)-v(y)\right|^p K(x-y)dxdy;\end{aligned}$$

(2) *there exists a constant* $\widetilde{C} = \widetilde{C}(N, p, s, k_0, \Omega)$ *such that for any* $v \in W_0$

$$\int_Q \left|v(x)-v(y)\right|^p K(x-y)dxdt \le \|v\|^p_W \le \widetilde{C} \int_Q \left|v(x)-v(y)\right|^p K(x-y)dxdy.$$

Proof. Let v be in W_0. By Lemma 2.2, we know that $v \in W^{s,p}(\Omega)$. Using Theorem 6.5 in [12], we obtain

$$\begin{aligned}\|v\|^p_{L^q(\Omega)} &\le C_0 \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy \\ &\le \frac{C_0}{k_0} \int_Q \left|v(x)-v(y)\right|^p K(x-y)dxdy,\end{aligned} \tag{2.1}$$

where C_0 is a positive constant depending only on N, s, p. Thus, we get the assertion (1). The assertion (2) easily follows by combining the definition of norm of W with (2.1). □

Remark 2.1. By (2), we get an equivalent norm on W_0 defined as

$$\|v\|_{W_0} = \left(\int_Q |v(x) - v(y)|^p K(x-y)dxdy \right)^{\frac{1}{p}}, \quad \text{for all } v \in W_0.$$

Indeed, it is enough to prove that if $\|v\|_{W_0} = 0$, then $v = 0$ a.e. in $\mathbb{R}^N$. By $\|v\|_{W_0} = 0$, we have

$$\int_Q |v(x) - v(y)|^p K(x-y)dxdy = 0.$$

Thus, $v(x) = v(y)$ a.e. $(x, y) \in Q$. Since $v = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$, we get $v = 0$ a.e. in $\mathbb{R}^N$.

Lemma 2.4. $(W_0, \|\cdot\|_{W_0})$ *is a uniformly convex Banach space.*

Proof. We first prove that W_0 is complete with respect to the norm $\|\cdot\|_{W_0}$. Let $\{u_n\}$ be a Cauchy sequence in W_0. Thus, for any $\varepsilon > 0$ there exists μ_ε such that if $n, m \geq \mu_\varepsilon$, then

$$\frac{k_0}{C_0}\|u_n - u_m\|^p_{L^p(\Omega)} \leq \|u_n - u_m\|^p_{W_0} < \varepsilon. \tag{2.2}$$

By the completeness of $L^p(\Omega)$, there exists $u \in L^p(\Omega)$ such that $u_n \to u$ strongly in $L^p(\Omega)$ as $n \to \infty$. Since $u_n = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$, we define $u = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$. Then $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$ as $n \to \infty$. So, there exists a subsequence u_{n_k} in W_0 such that $u_{n_k} \to u$ a.e. in $\mathbb{R}^N$ (see [6], Theorem IV.9). Therefore, by the Fatou Lemma and the second inequality in (2.2) with $\varepsilon = 1$, we have

$$\begin{aligned}\int_Q |u(x) - u(y)|^p K(x-y)dxdy &\leq \liminf_{k\to\infty} \int_Q |u_{n_k}(x) - u_{n_k}(y)|^p K(x-y)dxdy \\ &\leq \liminf_{k\to\infty} \left(\|u_{n_k} - u_{\mu_1}\|_{W_0} + \|u_{\mu_1}\|_{W_0}\right)^p \\ &\leq \left(1 + \|u_{\mu_1}\|_{W_0}\right)^p < \infty.\end{aligned}$$

Thus, $u \in W_0$. Let $n \geq \mu_\varepsilon$, by the second inequality in (2.2) and the Fatou Lemma, we get

$$\|u_n - u\|^p_{W_0} \leq \liminf_{k\to\infty} \|u_n - u_{n_k}\|^p_{W_0} \leq \varepsilon,$$

that is, $u_n \to u$ strongly in W_0 as $n \to \infty$.

Next, we prove that $(W_0, \|\cdot\|_{W_0})$ is uniformly convex. Now, let $u, v \in W_0$ satisfy $\|u\|_{W_0} = \|v\|_{W_0} = 1$ and $\|u - v\|_{W_0} \geq \varepsilon$, where $\varepsilon \in (0, 2)$.

Case $p \geq 2$. By the inequality (28) in [1], we have

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|^p_{W_0} + \left\|\frac{u-v}{2}\right\|^p_{W_0} &= \int_Q \left|\frac{u(x)+v(x)}{2} - \frac{u(y)+v(y)}{2}\right|^p K(x-y)dxdy \\ &\quad + \int_Q \left|\frac{u(x)-v(x)}{2} - \frac{u(y)-v(y)}{2}\right|^p K(x-y)dxdy \\ &\leq \frac{1}{2}\int_Q |u(x) - u(y)|^p K(x-y)dxdy + \frac{1}{2}\int_Q |v(x) - v(y)|^p K(x-y)dxdy \\ &= \frac{1}{2}\|u\|^p_{W_0} + \frac{1}{2}\|v\|^p_{W_0} = 1.\end{aligned} \tag{2.3}$$

It follows from (2.3) that $\|\frac{u+v}{2}\|_{W_0}^p \leq 1 - \varepsilon^p/2^p$. Taking $\delta = \delta(\varepsilon)$ such that $1-(\varepsilon/2)^p = (1-\delta)^p$, we obtain that $\|\frac{u+v}{2}\|_{W_0} \leq (1-\delta)$.

Case $1 < p < 2$. First, notice that

$$\|u\|_{W_0}^{p'} = \left[\int\limits_Q \left(\left(\left|u(x)-u(y)\right|K^{\frac{1}{p}}(x-y)\right)^{p'}\right)^{p-1} dxdy\right]^{\frac{1}{p-1}},$$

where $p' = p/p-1$. Using the reverse Minkowski inequality (see [1], Theorem 2.13) and the inequality (27) in [1], we get

$$\begin{aligned}
\left\|\frac{u+v}{2}\right\|_{W_0}^{p'} + \left\|\frac{u-v}{2}\right\|_{W_0}^{p'} &= \left\{\int\limits_Q \left[\left(\left|\frac{u(x)+v(x)}{2} - \frac{u(y)+v(y)}{2}\right|K^{\frac{1}{p}}(x-y)\right)^{p'}\right]^{p-1} dxdy\right\}^{\frac{1}{p-1}} \\
&\quad + \left\{\int\limits_Q \left[\left(\left|\frac{u(x)-v(x)}{2} - \frac{u(y)-v(y)}{2}\right|K^{\frac{1}{p}}(x-y)\right)^{p'}\right]^{p-1} dxdy\right\}^{\frac{1}{p-1}} \\
&\leq \left\{\int\limits_Q \left[\left(\left|\frac{u(x)-u(y)}{2} + \frac{v(x)-v(y)}{2}\right|K^{\frac{1}{p}}(x-y)\right)^{p'}\right.\right. \\
&\quad \left.\left. + \left(\left|\frac{u(x)-u(y)}{2} - \frac{v(x)-v(y)}{2}\right|K^{\frac{1}{p}}(x-y)\right)^{p'}\right]^{p-1} dxdy\right\}^{\frac{1}{p-1}} \\
&\leq \left(\frac{1}{2}\int\limits_Q \left|u(x)-u(y)\right|^p K(x-y)dxdy + \frac{1}{2}\int\limits_Q \left|v(x)-v(y)\right|^p K(x-y)dxdy\right)^{p'-1} \\
&= \left(\frac{1}{2}\|u\|_{W_0}^p + \frac{1}{2}\|v\|_{W_0}^p\right)^{p'-1} = 1. \qquad (2.4)
\end{aligned}$$

By (2.4), we have

$$\left\|\frac{u+v}{2}\right\|_{W_0}^{p'} \leq 1 - \frac{\varepsilon^{p'}}{2^{p'}}.$$

Taking $\delta = \delta(\varepsilon)$ such that $1-(\varepsilon/2)^{p'} = (1-\delta)^{p'}$, we get the desired claim. □

Remark 2.2. By Theorem 1.21 in [1], W_0 is a reflexive Banach space.

Lemma 2.5. *Let $K : \mathbb{R}^N \backslash \{0\} \to (0,\infty)$ satisfy assumptions (1.2) and let v_j be a bounded sequence in W_0. Then, there exists $v \in L^\nu(\mathbb{R}^N)$ with $v = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$ such that up to a subsequence,*

$$v_j \to v \quad \textit{strongly in } L^\nu(\Omega), \;\; \textit{as } j \to \infty,$$

for any $\nu \in [1, p_s^*)$.

Proof. Lemma 2.2-(b) implies that $v_j \in W^{s,p}(\mathbb{R}^N)$ and so $v_j \in W^{s,p}(\Omega)$. Moreover, by Lemma 2.2-(b), Lemma 2.3-(2) and the definition of W_0, we have

$$\|v_j\|_{W^{s,p}(\Omega)} \leq \|v_j\|_{W^{s,p}(\mathbb{R}^N)} \leq (\widetilde{C})^{\frac{1}{p}} \max\{1,\; k_0^{-\frac{1}{p}}\}\|v\|_{W_0}.$$

Hence v_j is bounded in $W^{s,p}(\Omega)$. By Corollary 7.2 in [12] and our assumptions on Ω, there exits $v \in L^\nu(\Omega)$ such that up to a subsequence, $v_j \to v$ strongly in L^ν as $j \to \infty$, for any $\nu \in [1, p_s^*)$. Since $v_j = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we can define $v = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. □

Remark 2.3. We notice that by a similar way, the space W is defined independently in [17] and was used to investigate the related problem. It is worth mentioning that our functional setting above is inspired by the pioneering works of Servadei and Valdinoci in [38,39], in which the corresponding functional framework was discussed as $p = 2$.

3. Proofs of Theorems 1.1 and 1.2

For $u \in W_0$, we define

$$J(u) = \frac{1}{p}\widehat{M}\left(\int_Q |u(x)-u(y)|^p K(x-y)\,dxdy\right), \qquad H(u) = \int_\Omega F(x,u)\,dx,$$

and

$$I(u) = J(u) - H(u).$$

Obviously, the energy functional $I : W_0 \to \mathbb{R}$ associated with problem (1.1) is well defined.

Lemma 3.1. *If f satisfies assumption (H1), then the functional $H \in C^1(W_0, \mathbb{R})$ and*

$$\langle H'(u), v\rangle = \int_\Omega f(x,u)v\,dx \quad \text{for all } u, v \in W_0.$$

Proof. (i) H is Gâteaux-differentiable in W_0.

Let $u, v \in W_0$. For each $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem, there exits $0 < \delta < 1$,

$$\begin{aligned}\frac{1}{t}\big(F(x,u+tv) - F(x,u)\big) &= \frac{1}{t}\int_0^{u+tv} f(x,s)\,ds - \frac{1}{t}\int_0^u f(x,s)\,ds \\ &= \frac{1}{t}\int_u^{u+tv} f(x,s)\,ds = f(x,u+\delta tv)v.\end{aligned}$$

Combing assumption (H1) with Young's inequality, we get

$$\begin{aligned}\big|f(x,u+\delta tv)v\big| &\le a\big(|v| + |u+\delta tv|^{q-1}|v|\big) \\ &\le a\big(2|v|^q + |u+\delta tv|^q + 1\big) \le a2^q\big(|v|^q + |u|^q + 1\big).\end{aligned}$$

Since $1 < q < p_s^*$, by Lemma 2.2 and Lemma 2.3 we have $u, v \in L^q(\Omega)$. Moreover, the Lebesgue Dominated Convergence Theorem implies

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\big(H(u+tv) - H(u)\big) &= \lim_{t\to 0}\int_\Omega f(x,u+\delta tv)v\,dx \\ &= \int_\Omega \lim_{t\to 0} f(x,u+\delta tv)v\,dx = \int_\Omega f(x,u)v\,dx.\end{aligned}$$

(ii) The continuity of Gateaux-derivative.

Let $\{u_n\}, u \in W_0$ such that $u_n \to u$ strongly in W_0 as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ a.e. in $\mathbb{R}^N$. By assumption (H1), for any measurable subset $U \subset \Omega$,

$$\int_U \left|f(x,u_n)\right|^{q'} dx \leq 2^{\frac{q+1}{q-1}} a^{\frac{q}{q-1}} \left(\int_U |u_n|^q dx + |U| \right),$$

where $|U|$ denotes the N dimensional Lebesgue measure of set U. Since $1 < q < p_s^*$, by Lemma 2.3 and Hölder's inequality, we have

$$\begin{aligned} \int_U \left|f(x,u_n)\right|^{q'} dx &\leq 2^{\frac{q+1}{q-1}} a^{\frac{q}{q-1}} \left(\left\| |u_n|^q \right\|_{L^{\frac{p_s^*}{q}}(U)} \|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(U)} + |U| \right) \\ &\leq C|U|^{\frac{p_s^*-q}{p_s^*}} + C|U|. \end{aligned} \tag{3.1}$$

It follows from (3.1) that the sequence $\{|f(x,u_n) - f(x,u)|^{q'}\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. The Vitali Convergence Theorem (see Rudin [37]) implies

$$\lim_{n\to\infty} \int_\Omega \left|f(x,u_n) - f(x,u)\right|^{q'} dx = 0.$$

Thus, by Hölder's inequality and Lemma 2.3-(1), we obtain

$$\begin{aligned} \left\|H'(u_n) - H'(u)\right\| &= \sup_{\varphi\in W_0, \|\varphi\|_{W_0}=1} \left| \int_\Omega \big(f(x,u_n) - f(x,u)\big)\varphi dx \right| \\ &\leq \left\|f(x,u_n) - f(x,u)\right\|_{L^{q'}(\Omega)} \|\varphi\|_{L^q(\Omega)} \\ &\leq \left(\frac{C_0}{k_0}\right)^{\frac{1}{p}} \left\|f(x,u_n) - f(x,u)\right\|_{L^{q'}(\Omega)} \\ &\longrightarrow 0, \end{aligned}$$

as $n \to \infty$. Hence, we complete the proof of Lemma 3.1. □

Using the same strategy as in Lemma 3.1, we have

Lemma 3.2. *Let (M1) hold. Then the functional $J \in C^1(W_0, \mathbb{R})$ and*

$$\langle J'(u), v \rangle = M\big(\|u\|_{W_0}^p\big) \int_Q \left|u(x) - u(y)\right|^{p-2} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) K(x-y) dxdy,$$

for all u, $v \in W_0$. Moreover, for each $u \in W_0$, $J'(u) \in W_0^*$, where W_0^* denotes the dual space of W_0.

Proof. First, it is easy to see that

$$\langle J'(u), v \rangle = M\big(\|u\|_{W_0}^p\big) \int_Q \left|u(x) - u(y)\right|^{p-2} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) K(x-y) dxdy, \tag{3.2}$$

for all u, $v \in W_0$. It follows from (3.2) that for each $u \in W_0$, $J'(u) \in W_0^*$.

Please cite this article in press as: M. Xiang et al., Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian, J. Math. Anal. Appl. (2015), http://dx.doi.org/10.1016/j.jmaa.2014.11.055

Next, we prove that $J \in C^1(W_0, \mathbb{R})$. Let $\{u_n\} \subset W_0$, $u \in W_0$ with $u_n \to u$ strongly in W_0 as $n \to \infty$. By Lemma 2.5 there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \to u$ a.e. in $\mathbb{R}^N$. Then the sequence

$$\left\{ \left|u_n(x) - u_n(y)\right|^{p-2} \left(u_n(x) - u_n(y)\right) \left(K(x-y)\right)^{1/p'} \right\}_n \quad \text{is bounded in } L^{p'}(Q),$$

as well as

$$\left|u_n(x) - u_n(y)\right|^{p-2} \left(u_n(x) - u_n(y)\right) \left(K(x-y)\right)^{1/p'} \to \left|u(x) - u(y)\right|^{p-2} \left(u(x) - u(y)\right) \left(K(x-y)\right)^{1/p'}$$

a.e. in Q. Thus, the Brézis–Lieb Lemma (see [6]) implies

$$\begin{aligned} &\lim_{n\to\infty} \int_Q \left| \left|u_n(x) - u_n(y)\right|^{p-2} \left(u_n(x) - u_n(y)\right) - \left|u(x) - u(y)\right|^{p-2} \left(u(x) - u(y)\right) \right|^{p'} K(x-y)\,dxdy \\ &= \lim_{n\to\infty} \int_Q \left(\left|u_n(x) - u_n(y)\right|^p K(x-y) - \left|u(x) - u(y)\right|^p K(x-y) \right) dxdy. \end{aligned} \tag{3.3}$$

The fact that $u_n \to u$ strongly in W_0 yields that

$$\lim_{n\to\infty} \int_Q \left(\left|u_n(x) - u_n(y)\right|^p K(x-y) - \left|u(x) - u(y)\right|^p K(x-y) \right) dxdy = 0.$$

Moreover, the continuity of M implies that

$$\lim_{n\to\infty} M\left(\|u_n\|_{W_0}^p\right) = M\left(\|u\|_{W_0}^p\right). \tag{3.4}$$

From (3.3) it follows that

$$\lim_{n\to\infty} \int_Q \left| \left|u_n(x) - u_n(y)\right|^{p-2} \left(u_n(x) - u_n(y)\right) - \left|u(x) - u(y)\right|^{p-2} \left(u(x) - u(y)\right) \right|^{p'} K(x-y)\,dxdy = 0. \tag{3.5}$$

Combining (3.4), (3.5) with the Hölder inequality, we have

$$\begin{aligned} \left\|J'(u_n) - J'(u)\right\| &= \sup_{v \in W_0,\, \|v\|_{W_0} \le 1} \left| \left\langle J'(u_n) - J'(u), v \right\rangle \right| \\ &\longrightarrow 0, \end{aligned}$$

as $n \to \infty$. □

Combining Lemma 3.1 and Lemma 3.2, we get that $I \in C^1(W_0, \mathbb{R})$ and

$$\begin{aligned} \left\langle I'(u), v \right\rangle = M\left(\|u\|_{W_0}^p\right) &\int_Q \left|u(x) - u(y)\right|^{p-2} \left(u(x) - u(y)\right) \left(v(x) - v(y)\right) K(x-y)\,dxdy \\ &- \int_\Omega f(x, u) v\,dx, \end{aligned}$$

for all $u, v \in W_0$.

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3.1. Case 1: $1 < q \le p$

In this subsection, we prove the existence of weak solutions of problem (1.1), where the growth exponent q of function f satisfies $1 < q \le p$.

Lemma 3.3. *Let (M1) and (H1) be satisfied. Then the functional $I \in C^1(W_0, \mathbb{R})$ is weakly lower semi-continuous.*

Proof. First, notice that the map $v \mapsto \|v\|_{W_0}^p$ is lower semi-continuous in the weak topology of W_0 and $\widehat{M}$ is a nondecreasing continuous function, so that $v \mapsto \widehat{M}(\|v\|_{W_0}^p)$ is lower semi-continuous in the weak topology of W_0. Indeed, we define a functional $\psi : W_0 \to \mathbb{R}$ as

$$\psi(v) = \int_Q \big|v(x) - v(y)\big|^p K(x-y)dxdy.$$

Similar to Lemma 3.2, we obtain $\psi \in C^1(W_0)$ and

$$\langle \psi'(w), v \rangle = p \int_Q \big|w(x) - w(y)\big|^{p-2} \big(w(x) - w(y)\big)\big(v(x) - v(y)\big) K(x-y)dxdy,$$

for all $w,\ v \in W_0$. Notice that

$$\begin{aligned}
\psi\left(\frac{w+v}{2}\right) &= \int_Q 2^{-p}\big|w(x) + v(x) - w(y) - v(y)\big|^p K(x-y)dxdy \\
&\le \int_Q 2^{-1}\big|w(x) - w(y)\big|^p K(x-y) + 2^{-1}\big|v(x) - v(y)\big|^p K(x-y)dxdy \\
&= \frac{1}{2}\psi(w) + \frac{1}{2}\psi(v).
\end{aligned}$$

Thus, ψ is a convex functional in W_0. Further, ψ is subdifferentiable and the subdifferential denoted by $\partial\psi$ satisfies $\partial\psi(u) = \{\psi'(u)\}$ for each $u \in W_0$ (see [33], Proposition 1.1). Now, let $\{v_n\} \subset W_0, v \in W_0$ with $v_n \rightharpoonup v$ weakly in W_0 as $n \to \infty$. Then it follows from the definition of subdifferential that

$$\psi(v_n) - \psi(v) \ge \langle \psi'(v), v_n - v \rangle.$$

Hence, we obtain $\psi(v) \le \liminf_{n\to\infty} \psi(v_n)$, that is, the map $v \mapsto \|v\|_{W_0}^p$ is weakly lower semi-continuous.

Let $u_n \rightharpoonup u$ weakly in W_0. By assumption (H1) and Lemma 2.5, up to a subsequence, $u_n \to u$ strongly in $L^q(\Omega)$. Without loss of generality, we assume that $u_n \to u$ a.e. in Ω. Assumption (H1) implies

$$F(x,t) \le a\big(|t| + q^{-1}|t|^q\big) \le 2a\big(|t|^q + 1\big).$$

Thus, for any measurable subset $U \subset \Omega$,

$$\int_U \big|F(x, u_n)\big|dx \le 2a \int_U |u_n|^q dx + 2a|U|.$$

By $1 < q < p_s^*$, Lemma 2.3 and Hölder's inequality, we have

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$$\int_U |F(x,u_n)|dx \le 2a\big\||u_n|^q\big\|_{L^{\frac{p_s^*}{q}}(U)} \|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(U)} + 2a|U|$$

$$\le 2aC\|u_n\|_{W_0}^q |U|^{\frac{p_s^*-q}{p_s^*}} + 2a|U|.$$

Similar to the proof of Lemma 3.1, we obtain

$$\lim_{n\to\infty} \int_\Omega F(x,u_n)dx = \int_\Omega F(x,u)dx.$$

Thus, the functional H is weakly continuous. Further, we get that I is weakly lower semi-continuous. □

Proof of Theorem 1.1. From assumptions (M1) and (H1), we have $|F(x,t)| \le 2a(|t|^p + 1)$ and $\widehat{M}(t) \ge m_0 t$. Thus, by Lemma 2.3, we get

$$I(u) \ge \frac{m_0}{p} \int_Q |u(x)-u(y)|^p K(x-y)dxdy - 2a\int_\Omega |u|^q dx - 2a|\Omega|$$

$$\ge \frac{m_0}{p}\|u\|_{W_0}^p - 2a\left(\frac{C_0}{k_0}\right)^{\frac{q}{p}} \|u\|_{W_0}^q - 2a|\Omega|.$$

Since $q < p$, we have $I(u) \to \infty$, as $\|u\|_{W_0} \to \infty$. By Lemma 3.3, I is weakly lower semi-continuous on W_0. So, functional I has a minimum point u_0 in W_0 (see [41], Theorem 1.2) and $u_0 \in W_0$ is a weak solution of problem (1.1).

Next we need to verify that u_0 is nontrivial. Let $x_0 \in \Omega_0, 0 < R < 1$ satisfy $B_{2R}(x_0) \subset \Omega_0$, where $B_{2R}(x_0)$ is the ball of radius $2R$ with center at the point x_0 in $\mathbb{R}^N$. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ satisfies $0 \le \phi \le 1$ and $\phi \equiv 1$ in $B_R(x_0)$. Lemma 3.1 implies that $\|\phi\|_{W_0} < \infty$. Then for $0 < t < 1$, by the mean value theorem and (H4), we have

$$I(t\phi) = \frac{1}{p}\widehat{M}\big(\|t\phi\|_{W_0}^p\big) - \int_\Omega F(x,t\phi)dx$$

$$\le \int_0^{\|t\phi\|_{W_0}^p} M(\tau)d\tau - \int_{\Omega_0} \frac{a_1}{q}|t\phi|^q dx$$

$$\le M(\upsilon)\|\phi\|_{W_0}^p t^p - \frac{a_1}{q}t^q \int_{\Omega_0} |\phi|^q dx$$

$$\le Ct^p - \frac{a_1}{q}t^q \int_{\Omega_0} |\phi|^q dx,$$

where $\upsilon \in [0, \|\phi\|_{W_0}^p)$ and C is a positive constant. Since $p > q$ and $\int_\Omega |\phi|^q dx > 0$, we have $I(t_0\phi) < 0$ for $t_0 \in (0,1)$ sufficiently small. Hence, the critical point u_0 of functional I satisfies $I(u_0) \le I(t_0\phi) < 0 = I(0)$, that is $u_0 \ne 0$. □

Now, we consider the following nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

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We already know that the first eigenvalue λ_1 of $(-\Delta)_p^s$ defined as

$$\lambda_1 = \inf_{u\in W_0\setminus\{0\}} \frac{\|u\|_{W_0}^p}{\|u\|_{L^p(\Omega)}^p}$$

lies in $(0,\infty)$, see [16,25]. Using the same method as in Theorem 1.1, we can get the following result:

Corollary 3.1. *Let $K : \mathbb{R}^N\setminus\{0\} \to (0,\infty)$ be a function satisfying (1.2) and suppose that M satisfies (M1) and f satisfies (H1). If $q = p$ and $a < (m_0k_0\lambda_1)/(2p)$, then the problem (1.1) admits a weak solution in W_0.*

Proof. In view of the proof of Theorem 1.1, we only need to check that $I(u) \to \infty$ as $\|u\|_{W_0} \to \infty$. Since $p = q$ and $a < (m_0k_0\lambda_1)/(2p)$, by assumption (H1) and the definition of first eigenvalue of $(-\Delta)_p^s$, we have

$$\begin{aligned} I(u) &\geq \frac{m_0}{p}\int_Q \left|u(x)-u(y)\right|^p K(x-y)dxdy - 2a\int_\Omega |u|^p dx - 2a|\Omega| \\ &\geq \frac{m_0}{p}\|u\|_{W_0}^p - \frac{2a}{k_0\lambda_1}\|u\|_{W_0}^p - 2a|\Omega| \\ &= \left(\frac{m_0}{p} - \frac{2a}{k_0\lambda_1}\right)\|u\|_{W_0}^p - 2a|\Omega| \\ &\longrightarrow \infty, \end{aligned}$$

as $\|u\|_{W_0} \to \infty$. $\square$

Remark 3.1. Evidently, if $f(x,0) \neq 0$ a.e. in Ω, then the weak solution obtained in Corollary 3.1 is nontrivial.

3.2. Case 2: $p < q < p_s^*$

In this subsection, we consider the case $p < q < p_s^*$.

Lemma 3.4. *Let $K : \mathbb{R}^N\setminus\{0\} \to (0,\infty)$ be a function satisfying (1.2) and suppose that M satisfies (M1) and (M2) and f satisfies (H1)–(H3). If $p < q < p_s^*$, then there exist $\rho > 0$ and $\alpha > 0$ such that*

$$I(u) \geq \alpha > 0,$$

for any $u \in W_0$ with $\|u\|_{W_0} = \rho$.

Proof. By assumptions (H1) and (H3), for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for any $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$, we have

$$\left|f(x,\xi)\right| \leq p\varepsilon|\xi|^{p-1} + qC(\varepsilon)|\xi|^{q-1}. \tag{3.6}$$

It follows from (3.6) that

$$\left|F(x,\xi)\right| \leq \varepsilon|\xi|^p + C(\varepsilon)|\xi|^q. \tag{3.7}$$

Let $u \in W_0$. By (3.7), (M1) and Lemma 2.3, we obtain

$$I(u) \geq \frac{1}{p}\widehat{M}\left(\int_Q |u(x)-u(y)|^p K(x-y)dxdy\right) - \varepsilon\int_\Omega |u(x)|^p dx - C(\varepsilon)\int_\Omega |u(x)|^q dx$$
$$\geq \frac{m_0}{p}\|u\|_{W_0}^p - \varepsilon C_0\|u\|_{W_0}^p - C(\varepsilon)\left(\frac{C_0}{k_0}\right)^{\frac{q}{p}}\|u\|_{W_0}^q. \tag{3.8}$$

Choosing $\varepsilon = m_0/(2pC_0)$, by (3.8), we have

$$I(u) \geq \frac{m_0}{2p}\|u\|_{W_0}^p - C\|u\|_{W_0}^q \geq \|u\|_{W_0}^p\left(\frac{m_0}{2p} - C\|u\|_{W_0}^{q-p}\right),$$

where C is a constant only depending on N, s, p, λ, k_0. Now, let $\|u\|_{W_0} = \rho > 0$. Since $q > p$, we can choose ρ sufficiently small such that $m_0/(2p) - C\rho^{q-p} > 0$, so that

$$I(u) \geq \rho^p\left(\frac{m_0}{2p} - C\rho^{q-p}\right) =: \alpha > 0.$$

Thus, the lemma is proved. □

Lemma 3.5. *Let $K : \mathbb{R}^N\backslash\{0\} \to (0,\infty)$ be a function satisfying (1.2) that let M satisfies (M1)–(M2) and f satisfies (H1)–(H3). If $p < q < p_s^*$, then there exists $e \in C_0^\infty(\Omega)$ such that $\|e\|_{W_0} \geq \rho$ and $I(\rho) < \alpha$, where ρ and α are given in Lemma 3.4.*

Proof. First, by assumption (M1), we get that

$$\widehat{M}(t) \leq \widehat{M}(1)t^{\frac{1}{\theta}}, \tag{3.9}$$

for any $t \geq 1$. From assumption (H2) it follows that

$$F(x,\xi) \geq r^{-\mu}\min\{F(x,r), F(x,-r)\}|\xi|^\mu, \tag{3.10}$$

for all $|\xi| > r$ and a.e. $x \in \Omega$. Thus, by (3.10) and $F(x,\xi) \leq \max_{|\xi|\leq r} F(x,\xi)$ for all $|\xi| \leq r$, we obtain

$$F(x,\xi) \geq r^{-\mu}\min\{F(x,r), F(x,-r)\}|\xi|^\mu - \max_{|\xi|\leq r} F(x,\xi) - \min\{F(x,r), F(x,-r)\}, \tag{3.11}$$

for any $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$.

By Lemma 2.1, we can fix $u_0 \in C_0^\infty(\Omega)$ such that $\|u_0\|_{W_0(\Omega)} = 1$. Now, let $t \geq 1$. Combining (3.9) with (3.11), we have

$$I(tu_0) = \frac{1}{p}\widehat{M}(\|tu_0\|_{W_0}^p) - \int_\Omega F(x, tu_0(x))dx$$
$$\leq \frac{1}{p}\widehat{M}(1)t^{\frac{p}{\theta}} - r^{-\mu}t^\mu\int_\Omega \min\{F(x,r), F(x,-r)\}|u_0(x)|^\mu dx$$
$$+ \int_\Omega \max_{|\xi|\leq r} F(x,\xi) + \min\{F(x,r), F(x,-r)\}dx.$$

From assumptions (H1) and (H2), we get that $0 < F(x,\xi) \leq a(|r| + |r|^q)$ for $|\xi| \leq r$ a.e. $x \in \Omega$. Thus, $0 < \min\{F(x,r), F(x,-r)\} < a(|r| + |r|^q)$ a.e. $x \in \Omega$. Since $\mu > p/\theta$ by assumption (H2), passing to the limit as $t \to \infty$, we obtain that $I(tu_0) \to -\infty$. Thus, the assertion follows by taking $e = Tu_0$ with T sufficiently large. □

Definition 3.1. We say that I satisfies (PS) condition in W_0, if for any sequence $\{u_n\} \subset W_0$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, there exists a convergent subsequence of $\{u_n\}$.

Lemma 3.6. *Let $K : \mathbb{R}^N \setminus \{0\} \to (0, \infty)$ be a function satisfying (1.2) and suppose that M satisfies (M1) and (M2) and f satisfies (H1)–(H3). If $p < q < p_s^*$, then the functional I satisfies* (PS) *condition.*

Proof. For any sequence $\{u_n\} \subset W_0$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, there exits $C > 0$ such that $|\langle I'(u_n), u_n\rangle| \le C\|u_n\|_{W_0}$ and $|I(u_n)| \le C$. By assumption (H1), we have

$$\left| \int\limits_{\Omega \bigcap \{|u_n| \le r\}} \big(F(x, u_n) - \mu^{-1} f(x, u_n)u_n\big)dx \right| \le \big(a + \mu^{-1}\big)\big(r + r^q\big)|\Omega| \le C, \tag{3.12}$$

where $\{|u_n| \le r\} = \{x \in \Omega : |u_n(x)| \le r\}$. Thus, by (M1), (M2), (H2) and (3.12), we get

$$\begin{aligned} C + C\|u_n\|_{W_0} &\ge I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &\ge \left(\frac{1}{p} - \frac{1}{\theta\mu}\right)\widehat{M}\big(\|u_n\|_{W_0}^p\big) - \int\limits_{\Omega \bigcap \{|u_n| \le r\}} \big(F(x, u_n) - \mu^{-1} f(x, u_n)u_n\big)dx \\ &\ge \left(\frac{1}{p} - \frac{1}{\theta\mu}\right)\widehat{M}\big(\|u_n\|_{W_0}^p\big) - C \\ &\ge m_0\left(\frac{1}{p} - \frac{1}{\theta\mu}\right)\|u_n\|_{W_0}^p - C, \end{aligned}$$

where C denote various positive constants. Hence, $\{u_n\}$ is bounded in W_0. Since W_0 is a reflexive Banach space, up to a subsequence, still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in W_0. Then $\langle I'(u_n), u_n - u\rangle \to 0$. Thus, we obtain

$$\begin{aligned} \big\langle I'(u_n), u_n - u\big\rangle \;=\; & M\left(\int\limits_Q \big|u_n(x) - u_n(y)\big|^p K(x - y)dxdy\right) \\ & \times \int\limits_Q \big|u_n(x) - u_n(y)\big|^{p-2}\big(u_n(x) - u_n(y)\big)\big(u_n(x) - u(x) - u_n(y) + u(y)\big)K(x - y)dxdy \\ & - \int\limits_\Omega f(x, u_n)(u_n - u)dx \\ \longrightarrow 0 & \end{aligned} \tag{3.13}$$

as $n \to \infty$. Moreover, by Lemma 2.5, up to a subsequence,

$$u_n \to u \quad \text{strongly in } L^q(\Omega) \text{ and a.e. in } \Omega.$$

Thus, $f(x, u_n)(u_n - u) \to 0$ a.e. in Ω as $n \to \infty$. It is easy to check that sequence $\{f(x, u_n)(u_n - u)\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. Hence, the Vitali Convergence Theorem (see Rudin [37]) implies

$$\lim_{n \to \infty} \int\limits_\Omega f(x, u_n)(u_n - u)dx = 0.$$

Therefore, by (3.13), we have

$$M\Big(\int_Q \big|u_n(x)-u_n(y)\big|^p K(x-y)dxdy\Big) \times \int_Q \big|u_n(x)-u_n(y)\big|^{p-2}\big(u_n(x)-u_n(y)\big)\big(u_n(x)-u(x)-u_n(y)+u(y)\big)K(x-y)dxdy \longrightarrow 0,$$

as $n\to\infty$. Further, assumption (M1) implies

$$\int_Q \big|u_n(x)-u_n(y)\big|^{p-2}\big(u_n(x)-u_n(y)\big)\big(u_n(x)-u(x)-u_n(y)+u(y)\big)K(x-y)dxdy \longrightarrow 0,$$

as $n\to\infty$. Thus, by the weak convergence of $\{u_n\}$ in W_0, we get

$$\int_Q \Big[\big|u_n(x)-u_n(y)\big|^{p-2}\big(u_n(x)-u_n(y)\big)-\big|u(x)-u(y)\big|^{p-2}\big(u(x)-u(y)\big)\Big] \times \big(u_n(x)-u(x)-u_n(y)+u(y)\big)K(x-y)dxdy \longrightarrow 0,$$

as $n\to\infty$. Using the well-known vector inequalities:

$$\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)\cdot(\xi-\eta)\ge C_p|\xi-\eta|^p,\quad p\ge 2;$$
$$\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)\cdot(\xi-\eta)\ge \widehat{C}_p\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}},\quad 1<p<2,$$

for all $\xi,\eta\in\mathbb{R}^N$, where $C_p,\widehat{C}_p$ are constants depending only on p. From which we obtain for $p>2$

$$\begin{aligned}\int_Q &\big|u_n(x)-u_n(y)-u(x)+u(y)\big|^p K(x-y)dxdy\\ &\le C_p^{-1}\int_Q \Big[\big|u_n(x)-u_n(y)\big|^{p-2}\big(u_n(x)-u_n(y)\big)-\big|u(x)-u(y)\big|^{p-2}\big(u(x)-u(y)\big)\Big]\\ &\quad\times\big(u_n(x)-u(x)-u_n(y)+u(y)\big)K(x-y)dxdy\\ &\longrightarrow 0,\end{aligned}\tag{3.14}$$

as $n\to\infty$. For $1<p<2$ we have

$$\begin{aligned}\int_Q &\big|u_n(x)-u_n(y)-u(x)+u(y)\big|^p K(x-y)dxdy\\ &\le \widehat{C}_p^{-\frac{p}{2}}\Big\{\int_Q \Big[\big|u_n(x)-u_n(y)\big|^{p-2}\big(u_n(x)-u_n(y)\big)-\big|u(x)-u(y)\big|^{p-2}\big(u(x)-u(y)\big)\Big]\\ &\quad\times\big(u_n(x)-u(x)-u_n(y)+u(y)\big)K(x-y)dxdy\Big\}^{p/2}\end{aligned}$$

$$\times\left\{\int_Q \left(|u_n(x)-u_n(y)|+|u(x)-u(y)|\right)^p K(x-y)dxdy\right\}^{(2-p)/2}$$

$$\leq C\left\{\int_Q \left[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y))\right]\right.$$

$$\left.\times\left(u_n(x)-u(x)-u_n(y)+u(y)\right)K(x-y)dxdy\right\}^{p/2}$$

$$\longrightarrow 0, \tag{3.15}$$

as $n\to\infty$. Combing (3.14) with (3.15), we get that $u_n\to u$ strongly in W_0 as $n\to\infty$. Therefore, I satisfies (PS) condition. □

Proof of Theorem 1.2. Since Lemma 3.4–Lemma 3.6 hold, the Mountain Pass Theorem (see [41], Theorem 6.1) gives that there exists a critical point $u\in W_0$ of I. Moreover,

$$I(u)\geq\alpha>0=I(0).$$

Thus, $u\neq 0$. □

Corollary 3.2. *Suppose that all the assumptions of Theorem 1.2 are satisfied. Then problem (1.1) at least exists two nontrivial weak solutions in which one is non-negative and another is non-positive.*

Proof. First, for $v\in W_0$, we have $v^+\in W_0$, where $v^+=\max\{v,0\}=\frac{|v|+v}{2}$. Indeed,

$$\begin{aligned}\int_Q |v^+(x)-v^+(y)|^p K(x-y)dxdy &= \int_Q \left|\frac{|v(x)|+v(x)}{2}-\frac{|v(y)|+v(y)}{2}\right|^p K(x-y)dxdy\\ &= \int_Q \left|\frac{|v(x)|-|v(y)|+v(x)-v(y)}{2}\right|^p K(x-y)dxdy\\ &\leq \int_Q |v(x)-v(y)|^p K(x-y)dxdy.\end{aligned}$$

Similarly, $v^-=\max\{-v,0\}\in W_0$. Thus, we can take $e\geq 0$ or $e\leq 0$ in Lemma 3.5.

Now, we define

$$F^{\pm}(x,\xi)=\int_0^{\xi} f^{\pm}(x,\tau)d\tau,$$

where

$$f^+(x,\xi)=\begin{cases} f(x,\xi) & \text{if } \xi\geq 0,\\ 0 & \text{if } \xi<0,\end{cases}\qquad f^-(x,s)=\begin{cases} f(x,\xi) & \text{if } \xi\leq 0,\\ 0 & \text{if } \xi>0.\end{cases}$$

It is easy to check that $f^{\pm}$ satisfies assumptions (H1)–(H3). We define functional $I^{\pm}:W_0\to\mathbb{R}$ as follows

$$I^{\pm}(u)=\frac{1}{p}\widehat{M}(\|u\|_{W_0}^p)-\int_\Omega F^{\pm}(x,u)dx.$$

Then, $I^{\pm}$ is well defined on W_0 and

$$\langle (I^{\pm}(u))', \varphi \rangle = M(\|u\|_{W_0}^p) \int_Q |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dxdy - \int_\Omega f^{\pm}(x,u)\varphi dx. \tag{3.16}$$

Obviously, $I^{\pm}$ satisfies Lemma 3.4, Lemma 3.6 and $I^{\pm}(0)=0$. For functional I^+, we can take $e \geq 0$ in Lemma 3.5. Then I^+ satisfies Lemma 3.5. Similarly, functional I^- satisfies Lemma 3.5 by taking $e \leq 0$. Therefore, by the Mountain Pass Theorem, there exist two nontrivial critical points u_+, $u_- \in W_0$ of I^+, I^-, respectively. Next, we prove that $u_+ \geq 0$ and $u_- \leq 0$ a.e. in $\mathbb{R}^N$. Since $u_+ \in W_0$, we have $(u_+)^- \in W_0$. Considering (3.16) with respect to I^+ and taking $\varphi = (u_+)^-$, we get

$$\begin{aligned}\langle (I^+(u_+))', (u_+)^- \rangle = M(\|u_+\|_{W_0}^p) \int_Q &|u_+(x)-u_+(y)|^{p-2}(u_+(x)-u_+(y)) \\ &\times ((u_+)^-(x)-(u_+)^-(y))K(x-y)dxdy \\ &- \int_\Omega f^+(x,u_+)(u_+)^- dx = 0.\end{aligned} \tag{3.17}$$

The definition of f^+ and (3.17) imply

$$M(\|u_+\|_{W_0}^p) \int_Q |u_+(x)-u_+(y)|^{p-2}(u_+(x)-u_+(y))((u_+)^-(x)-(u_+)^-(y))K(x-y)dxdy = 0.$$

Then the assumption (M1) yields

$$\int_Q |(u_+)^-(x)-(u_+)^-(y)|^p K(x-y)dxdy = 0.$$

Hence, $(u_+)^-(x) = (u_+)^-(y)$ a.e. $(x,y) \in Q$. Since $(u_+)^-(x) = 0$ a.e. in $\mathbb{R}^N \backslash \Omega$, we obtain that $(u_+)^-(x) = 0$ a.e. in $\mathbb{R}^N$, that is, $u_+ \geq 0$ a.e. in $\mathbb{R}^N$.

Using the same arguments, we get that $u_- \leq 0$ a.e. in $\mathbb{R}^N$. □

Now, we consider the following example which is a direct application of the main results.

Example 3.1. Let $0 < s < 1 < p < \infty$, $ps < N$ and Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary. We consider problem

$$\begin{cases} \left(a_0 + b_0\left(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)^m\right)(-\Delta)_p^s u = \lambda |u|^{q-2}u & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.18}$$

where a_0, b_0, m, λ are positive constants. It is easy to see that

$$M(t) = a_0 + b_0 t^m \geq a_0 > 0 \quad \text{for all } t \geq 0,$$

and

$$\widehat{M}(t) = \int_0^t M(\tau)d\tau \geq \frac{1}{m+1}M(t)t \quad \text{for all } t \geq 0.$$

Let $f(x,\xi) = \lambda|\xi|^{q-2}\xi$ and $F(x,\xi) = \frac{\lambda}{q}|\xi|^q$. Obviously, f satisfies (H1) and F satisfies

$$0 < qF(x,\xi) = f(x,\xi)\xi \quad \text{for all } x \in \Omega \text{ and } |\xi| > 0.$$

If $1 < q < p$, by Theorem 1.1, problem (3.18) admits a nontrivial weak solution in W_0.

When $p < q$, we have

$$\lim_{s\to 0} \frac{\lambda f(x,s)}{|s|^{p-1}} = \lim_{s\to 0} \frac{\lambda|s|^{q-1}s}{|s|^{p-1}} = 0 \quad \text{uniformly in } x \in \Omega.$$

If $(m+1)p < q < p_s^*$, then by Corollary 3.2, problem (3.18) exists two nontrivial weak solutions in which one is non-negative and another is non-positive.

Acknowledgments

The authors would like to thank the referees for constructive suggestions which help us in depth to improve the quality of the paper. Also, the authors would like to express their sincere thanks to Professor Patrizia Pucci for valuable suggestions on the topic of this paper. The first author was supported by Scientific Research Foundations of Civil Aviation University of China (No. 2014QD04X). The second author was supported by Natural Science Foundation of Heilongjiang Province of China (No. A201306) and Research Foundation of Heilongjiang Educational Committee (No. 12541667) and Doctoral Research Foundation of Heilongjiang Institute of Technology (No. 2013BJ15).

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