On the regularity of the residual scheme

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Abstract. In this paper, we deal with the Castelnuovo-Mumford regularity of the residual scheme $\mathrm{res}_Y X$ of X with respect to Y, where X and Y are closed subschemes of the n-dimensional projective space $\mathbb{P}^n$ over an algebraically closed field of arbitrary characteristic, moreover, we characterize it by studying its hyperplane section scheme. In addition, we investigate the case when $\mathrm{res}_Y X$ consists of points in uniform position, in particular we offer a method of constructing a set of points of a given projective space in uniform position.

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1 Introduction

Let X and Y be closed subschemes of the n-dimensional projective space $\mathbb{P}^n_K$ over a fixed algebraically closed field K, where n is a positive integer. The residual scheme $\mathrm{res}_Y X$ of X with respect to Y is the closed subscheme of $\mathbb{P}^n_K$ whose ideal sheaf is defined by the division ideal sheaf $\mathcal{I}_{\mathrm{res}_Y X} = (\mathcal{I}_X : \mathcal{I}_Y)$, where $\mathcal{I}_X$ and $\mathcal{I}_Y$ are the ideal sheaves of X and Y respectively.

An interesting problem is to understand the relationship between the general hypersurface section of the residual scheme in the projective space $\mathbb{P}^n_K$ and the general hypersurface sections of the schemes that define such residual scheme. In this direction, many interesting and fundamental results may be found in [2],[3], and [10]. In [6], the second author proved a nicely simple and fundamental statement: a sufficiently general hypersurface section commutes with the residual scheme.

The aim of this paper is to describe and to compare some geometrical properties of the residual scheme $\mathrm{res}_Y X$ concerning the Castelnuovo-Mumford regularity, and the "Uniform Position Principle" with those of $(\mathrm{res}_Y X) \cap H$ and $\mathrm{res}_{Y\cap H}(X \cap H)$ for any hyperplane section H. To be precise, in Section 2, we recall some definitions and results on the residual scheme whose the most relevant property is $(\mathrm{res}_Y X) \cap F = \mathrm{res}_{Y\cap F}(X\cap F)$, where F is a general hypersurface. In the first part of Section 3, under some reasonable hypotheses, we prove that the Castelnuovo-Mumford regularity of the residual scheme is equal to the regularity of the residual scheme between the hyperplane sections of the defining schemes, as stated below:

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Theorem 1.1. *Let X and Y be closed subschemes of the n-dimensional projective space $\mathbb{P}_K^n$ over a fixed algebraically closed field K. If $H \subseteq \mathbb{P}_K^n$ is a general hyperplane such that the ideals $I_X + I_H$ and $(I_X : I_Y) + I_H$ are saturated (where I_Z stands for the saturated ideal of the given closed subscheme Z of $\mathbb{P}_K^n$), then the following equality holds:*

$$\operatorname{reg}(\operatorname{res}_{Y\cap H}(X \cap H)) = \operatorname{reg}(\operatorname{res}_Y X).$$

Proof. See item 2) of Theorem 3.5 below. □

In the second part of Section 3, we recall the geometrical definition of the concept of "set of points in uniform position" (see for example [1], and [7]). Furthermore, for a linear space $V \subseteq \mathbb{P}_K^n$ of codimension r composed by general hyperplanes with respect to projective varieties X and Y of $\mathbb{P}_K^n$, we prove the next result which gives a way of constructing sets of points in uniform position:

Theorem 1.2. *Let X and Y be irreducible closed subschemes of the n-dimensional projective space $\mathbb{P}_K^n$ over a fixed algebraically closed field K of arbitrary characteristic. Let $V \subseteq \mathbb{P}_K^n$ be a linear space of codimension r composed by hyperplanes in general position with respect to X and Y. If $\operatorname{res}_Y X$ is irreducible of dimension r, then the closed subscheme $\operatorname{res}_{Y\cap V}(X \cap V)$ of $\mathbb{P}_K^n$ is a set of points in uniform position.*

Proof. See Corollary 3.10 below. □

Remark 1.1. Our results confirm the basic idea of using hyperplane sections as a faithful method to understand fundamental statements about schemes, such idea may be founded in many text books, see for example [8], and [11].

2 Notation and preliminaries

Hereafter, K denotes a fixed algebraically closed field of arbitrary characteristic, and $\mathbb{P}^n$ the n-dimensional projective space over K whose homogeneous coordinate ring is $K[x_0, \dots, x_n]$, that is, $\mathbb{P}^n = \operatorname{Proj}(K[x_0, \dots, x_n])$. Here, n is a positive integer and x_i is homogeneous of degree one for every $i \in \{0, \dots, n\}$.
The ideal generated by $x_0, \dots, x_n$ in $K[x_0, \dots, x_n]$ is usually called the irrelevant ideal, and we denote it by $\mathfrak{m}$. Recall that for any homogeneous ideal I of $K[x_0, \dots, x_n]$, the saturation $\bar{I}$ of I is the set $\{f \in K[x_0, \dots, x_n] : f\mathfrak{m}^\ell \subseteq I, \text{ for some positive integer } \ell\}$, which has obviously a structure of a homogeneous ideal of $K[x_0, \dots, x_n]$, and do not have the irrelevant ideal as an associated prime ideal. It is well-known that for any homogeneous ideal J of $K[x_0, \dots, x_n]$, one has that $J_t = (\bar{J})_t$ for all $t \gg 0$, where J_t and $(\bar{J})_t$ are the t-graded components of J and $\bar{J}$ respectively.

If $X \subseteq \mathbb{P}^n$ is a closed subscheme, we denote by $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^n}$ the ideal sheaf of X determined by the saturated homogeneous ideal $I_X \subseteq K[x_0, \dots, x_n]$ of X, where $\mathcal{O}_{\mathbb{P}^n}$ is the structure sheaf of $\mathbb{P}^n$. A variety will always be irreducible.

Furthermore, for any homogeneous element f of $K[x_0, \dots, x_n]$ of positive degree, we denote by $K[x_0, \dots, x_n, f^{-1}]_0$ the ring of elements of degree 0 in the graded ring $K[x_0, \dots, x_n, f^{-1}]$.

If $X \subseteq \mathbb{P}^n$ is a closed subscheme and F is a hypersurface in $\mathbb{P}^n$, then the hypersurface section of X with respect to F is the subscheme $X \cap F \subseteq \mathbb{P}^n$ such that

$$\mathcal{I}_{X\cap F} = \mathcal{I}_X + \mathcal{I}_F.$$

Definition 2.1. Let X and Y be closed subschemes of $\mathbb{P}^n$, with ideal sheaves respectively $\mathcal{I}_X$ and $\mathcal{I}_Y$, and saturated ideals respectively I_X and I_Y. The residual scheme $\text{res}_Y X$ of X with respect to Y is the closed subscheme of $\mathbb{P}^n$ given by the ideal sheaf $\mathcal{I}_{\text{res}_Y X} = (\mathcal{I}_X : \mathcal{I}_Y)$. Such sheaf is defined on the affine standard open $D_+(x_i)$ as follows:

$$(\mathcal{I}_X : \mathcal{I}_Y)(D_+(x_i)) := (I_X : I_Y)K[x_0, \dots, x_n, x_i^{-1}] \cap K[x_0, \dots, x_n, x_i^{-1}]_0,$$

for every $i = 0, ..., n$.

It is worth nothing that we obtain $(\mathcal{I}_X : \mathcal{I}_Y)(D_+(x_i))$ as a division between the ideals of X and Y restricted to the open sets $(D_+(x_i))$ for every $i = 0, \dots, n$. In fact, it occurs that $(I_X : I_Y)K[x_0, \dots, x_n, x_i^{-1}] \cap K[x_0, \dots, x_n, x_i^{-1}]_0$ is equal to

$$(I_X K[x_0, \dots, x_n, x_i^{-1}] \cap K[x_0, \dots, x_n, x_i^{-1}]_0 : I_Y K[x_0, \dots, x_n, x_i^{-1}] \cap K[x_0, \dots, x_n, x_i^{-1}]_0).$$

Remark 2.2. Let X and Y be closed subschemes of $\mathbb{P}^n$. The residual scheme of X with respect to Y is the closed subscheme $\text{res}_Y X$ of $\mathbb{P}^n$ such that

$$I_{\text{res}_Y X} = (I_X : I_Y).$$

Below, we describe some known relations among the residual scheme $\text{res}_Y X$ of X with respect to Y and the general hypersurface sections of the schemes X, Y, and $\text{res}_Y X$ for any closed subschemes X and Y of $\mathbb{P}^n$. The following lemma is helpful for our results.

Lemma 2.1. ([*6*], *Lemma* 3.1) *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{t}$ *be ideals in a noetherian commutative ring* A *with unit, and let* $\mathfrak{t} = (t)$ *be a principal ideal in* A. *Put* $\bar{A} = A/\mathfrak{a}, \bar{t} = t + \mathfrak{a}$, *we suppose that:*

1. $\bar{t}$ *is* $A/(\mathfrak{a} : \mathfrak{b})-$*regular*

2. *the sequence*

$$0 \to (\mathfrak{a} : \mathfrak{b})/\mathfrak{a} \xrightarrow{\bar{t}} (\mathfrak{a} : \mathfrak{b})/\mathfrak{a} \to ((\mathfrak{a} + \mathfrak{t}) : (\mathfrak{b} + \mathfrak{t}))/(\mathfrak{a} + \mathfrak{t}) \to 0$$

is exact, where the first map is induced by the multiplication by the element $\bar{t} \in \bar{A}$. *Then we have*

$$((\mathfrak{a} + \mathfrak{t}) : (\mathfrak{b} + \mathfrak{t})) = (\mathfrak{a} : \mathfrak{b}) + \mathfrak{t}.$$

Remark 2.3. The associated prime ideals of the quotient $(\mathfrak{a} : \mathfrak{b})$ are among the associated prime ideals of $\mathfrak{a}$, that is $\text{Ass}(\mathfrak{a} : \mathfrak{b}) \subseteq \text{Ass}\, \mathfrak{a}$ (see Chapter three of [9]).

Theorem 2.2. ([*6*],*Theorem 3.3*) *Let* $X, Y \subseteq \mathbb{P}^n$ *be two closed subschemes. Then, for a general hypersurface* $F \subseteq \mathbb{P}^n$ *of degree* d, *we have*

$$(\text{res}_Y X) \cap F = \text{res}_{Y \cap F}(X \cap F) \tag{2.1}$$

that means, in the sheaf language:

$$(\mathcal{I}_X : \mathcal{I}_Y) + \mathcal{I}_F = ((\mathcal{I}_X + \mathcal{I}_F) : (\mathcal{I}_Y + \mathcal{I}_F)).$$

Corollary 2.3. *With notation and assumptions as in Theorem 2.2. Let I_X, I_Y and I_F be the saturated homogeneous ideals associated of the closed schemes X, Y and F of $\mathbb{P}^n$ respectively. It follows that*

$$\overline{(I_X : I_Y) + I_F} = \overline{((I_X + I_F) : (I_Y + I_F))}.$$

Remark 2.4. The previous equality fails if we consider only homogeneous ideals, not saturated. On the other hand, such equality is always true if $B/I_X, B/I_Y$ and $B/(I_X : I_Y)$ are arithmetically Cohen-Macaulay (aCM), where B is equal to $K[x_0, \dots, x_n]$ (see for example [10, Corollary 3.8]).

We conclude this section by recalling the following classic result that we do not find an explicit proof anywhere.

Proposition 2.4. *Let I be a homogeneous ideal of $K[x_0, \dots, x_n]$. For every nonnegative integer d, there exists a sheaf morphism $e_d : I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}$ such that for a sufficiently large integer d, its image is constant. Here, I_d (respectively, K) means the constant sheaf on $\mathbb{P}^n$ with coefficients in I_d (respectively, in K).*

Proof. Let d be a nonnegative integer. By the universal property of the associated sheaf, construct the sheaf morphism $e_d : I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}$ is equivalent to construct a presheaf morphism $e_d^- : (I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d))^- \to \mathcal{O}_{\mathbb{P}^n}$, where $(I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d))^-$ is the presheaf defined by the following way: for every open set U of $\mathbb{P}^n$, $(I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d))^-(U) = I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d)(U)$, and for every open sets V and W of $\mathbb{P}^n$ such that $W \subseteq V$, we have that $\rho_{(I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d))^-}{}_W^V = \rho_{I_d}{}_W^V \otimes \rho_{\mathcal{O}_{\mathbb{P}^n}(-d)}{}_W^V$, where $\rho_{\mathcal{F}}{}_W^V$ denotes the restriction map of the presheaf $\mathcal{F}$ from $\mathcal{F}(V)$ to $\mathcal{F}(W)$ for any presheaf $\mathcal{F}$ on $\mathbb{P}^n$. Let U be a nonempty open set of $\mathbb{P}^n$. Consider the following application:

$$\begin{array}{rcl} \sigma_U : I_d \times \mathcal{O}_{\mathbb{P}^n}(-d)(U) & \to & \mathcal{O}_{\mathbb{P}^n}(U) \\ (\lambda, s) & \mapsto & \sigma_U(\lambda, s) \end{array}$$

where $\sigma_U(\lambda, s) : U \to \prod_{p \in U} K[x_0, \dots, x_n]_{(p)}$ is such that $\sigma_U(\lambda, s)(q) = \lambda(q)s(q)$ for every $q \in U$. It is not difficult to prove that σ_U is a K-bilinear application, therefore there exists the K-linear application $e_{d_U}^- : I_d \otimes_K (\mathcal{O}_{\mathbb{P}^n}(-d)(U)) \to \mathcal{O}_{\mathbb{P}^n}(U)$ such that for every $\lambda \in I_d$ and for every $s \in \mathcal{O}_{\mathbb{P}^n}(-d)(U)$ we have that $e_{d_U}^-(\lambda \otimes s) = \sigma_U(\lambda, s)$. Moreover, we obtain the presheaf morphism $e_d{}^-$ from $(I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d))^-$ to $\mathcal{O}_{\mathbb{P}^n}$, henceforth, by the universal property of the associated sheaf, there exists a morphism $e_d : I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}$ of $\mathcal{O}_{\mathbb{P}^n}$-modules (see [8, Proposition-Definition 1.2, page 64]).

Now, what is left is to show that for a sufficiently large integer d, the image of the morphism e_d is constant. Indeed, we may assume that the homogeneous ideal I is generated by homogeneous elements $a_1, \dots, a_r$ for some $r \in \mathbb{N}$. Fix a nonnegative integer d such that d is greater than or equal to the degree of a_i for any $i \in \{1, \dots, r\}$. Next, we prove that the image $im\, e_d$ of e_d is isomorphic to the $\mathcal{O}_{\mathbb{P}^n}$-module $\widetilde{I}$. To this end, it is enough to construct a presheaf morphism between $im^-\, e_d$ and $\widetilde{I}$. It is worth noting that by the construction of $e_d{}^-$, the image $e_{d_U}^-$ is contained in $\widetilde{I}(U)$ for every open set U of $\mathbb{P}^n$, and therefore, $im^-\, e_d$ is an $\mathcal{O}_{\mathbb{P}^n}$-submodule of $\widetilde{I}$. This gives rise to the inclusion morphism $\iota^- : im^-\, e_d \to \widetilde{I}$, which induces obviously an injective application $\iota_p^- : (im^- e_d)_p \to \widetilde{I}_p$ for every $p \in \mathbb{P}^n$. Henceforth, the induced morphism $\iota : im\, e_d \to \widetilde{I}$ is injective. Recall that $\mathbb{P}^n$ has an open covering by the affine standard open sets $(D_+(x_i))_{i=0,\ldots,n}$. Fix $i \in \{0,\ldots,n\}$, our first aim is to prove that $\iota^-_{D_+(x_i)}$ is surjective. Indeed, let s be an element of $\widetilde{I}(D_+(x_i))$, using the fact that $\widetilde{I}(D_+(x_i))$ is isomorphic to $I_{(x_i)}$ (see [8, Proposition 5.11, page 116]), there exists a nonnegative integer m and a homogeneous element λ of I of degree m such that s is equal to the element $\widetilde{\frac{\lambda}{x_i^m}}$ of $\widetilde{I}(D_+(x_i))$. On the other hand, there exists homogeneous elements $\mu_1,\ldots,\mu_r \in K[x_0,x_1,\ldots,x_n]$ such that $\lambda = \sum_{j=1}^r \mu_j a_j$, where $deg(\mu_j) = m - deg(a_j)$ for every $j \in \{1,\ldots,r\}$. Consider the following element of $I_d \otimes_K (\mathcal{O}_{\mathbb{P}^n}(-d)(D_+(x_i)))$:

$$f = \sum_{j=1}^{r} [x_i^{d-deg(a_j)} a_j \otimes \widetilde{\frac{\mu_j}{x_i^{m-deg(a_j)+d}}}],$$

where $\widetilde{\frac{\mu_j}{x_i^{m-deg(a_j)+d}}}$ is the element of $\widetilde{I}(D_+(x_i))$ associated to $\frac{\mu_j}{x_i^{m-deg(a_j)+d}}$ for every $j \in \{1,\ldots,r\}$. Consequently, $\theta_{D_+(x_i)}(f)$ belongs to $(I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d))(D_+(x_i))$, where θ is the canonical morphism between $(I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d))^-$ and $I_d \otimes_K \mathcal{O}_{\mathbb{P}^n}(-d)$. Furthermore, we have that $e_{dD_+(x_i)}(\theta_{D_+(x_i}(f)) = s$. Since the sheaf $\widetilde{I}$ is flasque, or flabby (see [8, Exercise 1.16, page 67] or [12, Section 6.1, page 111]), we get that $\iota^-_{D_+(x_i)\cap W}$ is surjective for every nonempty open set W of $\mathbb{P}^n$. This proves the surjectivity of ι_p^- for every $p \in \mathbb{P}^n$. Finally, we conclude that ι is an isomorphism between $im\, e_d$ and $\widetilde{I}$, and we are done.

3 Main results

The Castelnuovo-Mumford regularity is a very interesting geometric invariant for schemes and there are important conjectures involving the regularity that explore purely algebraic approaches to discover new properties of a projective variety (e.g. [4], and [8]). In order to state the main result of this section on the regularity of the hyperplane section of a residual scheme, we give some definitions and results. Remind that for any homogeneous ideal I of $K[x_0,\ldots,x_n]$ and for any positive integer d, let I_d be the $K-$vector space generated by all forms of degree d of I.

From the last section, we know that for every homogeneous ideal I of $K[x_0,\ldots,x_n]$, and for a sufficiently large integer d, the image of the canonical sheaf morphism $e_d : I_d \otimes \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}$ is constant, and it is usually called the sheafification $\widetilde{I}$ of I.

Proposition 3.1. *With the above notation, the sheaf $\widetilde{I}$ is a coherent sheaf of $\mathbb{P}^n$.*

Proof. See [8, Proposition 5.11, page 116]. □

For any nonnegative integer i and for any sheaf $\mathcal{F}$ on $\mathbb{P}^n$, let $H^i(\mathbb{P}^n,\mathcal{F})$ be the i^{th}-cohomology group of $\mathcal{F}$ (see [8, Chapter three, Section two, page 206]). If m is an integer, we denote by $\mathcal{F}(m)$ the sheaf $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(m)$.

Definition 3.1. Let m be an integer. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is m-regular if $H^q(\mathbb{P}^n,\mathcal{F}(m-q)) = \{0\}$ for every positive integer q.

Here comes the concept of Castelnuovo-Mumford regularity of a given coherent sheaf.

Definition 3.2. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. The Castelnuovo-Mumford regularity of $\mathcal{F}$ is the smallest m for which $\mathcal{F}$ is m-regular, it is denoted by $\text{reg}(\mathcal{F})$. Furthermore, for every closed subscheme X of $\mathbb{P}^n$, the regularity of X is the regularity of the coherent sheaf $\widetilde{R/I_X}$, where I_X is the unique saturated homogeneous ideal associated to X, and R is equal to $K[x_0, \dots, x_n]$.

Definition 3.3. Let m be an integer. A homogeneous ideal I of $K[x_0, \dots, x_n]$ is m-satured if $I_d = (\bar{I})_d$, for every $d \geq m$. The satiety, known also as the saturation index, of I is the smallest integer m for which I is $m-$saturated, and it is usually denoted by $\text{sat}(I)$.

Proposition 3.2. ([7], *Proposition* 2.6) *An ideal I is m-regular if and only if I is m-saturated and its sheafification $\widetilde{I}$ is m-regular.*

An useful algebraic version of the definition of the regularity is due to Eisenbud-Goto (see [5]). Here we review briefly such version. Let k be a field, $R = k[x_0, \dots, x_n]$ the polynomial ring over k, and let M be a finitely generated graded R-module. Then as an R-module, M admits a finite minimal graded free resolution:

$$0 \to \oplus_j R(-j)^{b_{pj}} \to \dots \to \oplus_j R(-j)^{b_{0j}} \to M \to 0.$$

Definition 3.4. ([5]) With the above notation, the Castelnuovo-Mumford regularity of M is the integer

$$\text{reg}(M) = \max \{j - p,\, b_{pj} \neq 0\}.$$

The following theorem is crucial to use the definition of regularity of a finitely generated R-module, in particular of an ideal of R, from the algebraic point of view.

Theorem 3.3. ([7], *Proposition* 2.6) *Let I be a homogeneous saturated ideal of R. Then the regularity of I and the regularity of its sheafification are equals.*

Also, we need the following result:

Theorem 3.4.

1) Let I be a homogeneous ideal of $R = k[x_0, \dots, x_n]$, and let H be a principal ideal generated by a linear form h of R. If $h \notin I$ and $I + H \neq (x_0, \dots, x_n)$, then

$$\text{reg}(I) = \text{reg}(I + H). \tag{3.1}$$

2) Let X be a closed subscheme of $\mathbb{P}^n$ and let H be a general hyperplane of $\mathbb{P}^n$. If $I_X + I_H$ is a saturated ideal of $K[x_0, \dots, x_n]$, then the following equality occurs:

$$\text{reg}(X) = \text{reg}(X \cap H). \tag{3.2}$$

Proof. 1) Let $E. := 0 \to E_{n-1} \to \dots \to E_1 \to E_0 \to R \to R/I \to 0$ be the minimal graded resolution of R/I, where

$$E_p = \oplus_j R(-j)^{b_{pj}},$$

for every $p \in \{0, \ldots, n-1\}$. Recall that by definition $\operatorname{reg}(I) = max\{j - p, b_{pj} \neq 0\}$. Consider the tensor product of the resolution $E.$ for R/H. If $Tor_1(R/I, R/H)$ is null, then $E. \otimes_R (R/H)$ is a minimal graded resolution of $R/(I+H)$ and

$$\operatorname{reg}(I) = \operatorname{reg}(R/I) - 1 = \operatorname{reg}(R/(I+H)) - 1 = \operatorname{reg}(I+H).$$

Since $H \notin \operatorname{Ass}(I)$, it implies that the localization $Tor_1(R/I, R/H)_P$ of $Tor_1(R/I, R/H)$ at the prime ideal P is null for every $P \in \operatorname{Ass}(I)$, so we are done.

2) The fact that H is a general hyperplane implies that $I_H = (h)$ for some general linear form $h \in K[x_0, \ldots, x_n]$, and $h \notin I_X$. Using Theorem 3.3 and the previous item 1), we obtain the following equalities:

$$\operatorname{reg}(X \cap H) = \operatorname{reg}(R/I_{X\cap H}) = \operatorname{reg}(\widetilde{I_X + I_H}) - 1 = \operatorname{reg}(I_X + I_H) - 1 = \operatorname{reg}(I_X) - 1.$$

Hence, we conclude that

$$\operatorname{reg}(X \cap H) = \operatorname{reg}(R/I_{X\cap H}) = \operatorname{reg}(R/I_X) = \operatorname{reg}(X).$$

□

Example 3.5. Here, we provide an example to show that the saturation hypothesis in the item 2) of the previous theorem is necessary. Let $R = K[x_0, x_1]$. Consider the closed subscheme X of $\mathbb{P}^1$ defined by the ideal (x_0), and the hyperplane H defined by the ideal $(x_0 + x_1)$. Note that by construction, H is general. It is clear that $I_X = (x_0)$ and $I_H = (x_0 + x_1)$ are saturated ideals, however, the ideal $I_X + I_H = (x_0, x_0 + x_1) = (x_0, x_1)$ is not saturated. Now, the fact that $\operatorname{reg}(X) = \operatorname{reg}(R/I_X) = \operatorname{reg}(R/(X_0)) = 0$ and $X \cap H = \varnothing$ implies that the Equation (3.2) is not true. On the other hand, note that $\operatorname{reg}(I_X) = \operatorname{reg}((x_0)) = 1$ and $\operatorname{reg}(I_H) = \operatorname{reg}((x_0 + x_1)) = 1$, this implies that the Equation (3.1) is true in spite of the hypothesis $I_X + I_H \neq (x_0, x_1)$ is not satisfied.

Now we are able to prove the result stated in the introduction.

Theorem 3.5.

1) *Let I and J be homogeneous ideals of $R = k[x_0, \ldots, x_n]$ and let H be a principal ideal generated by a linear form h of R. If $H \notin \operatorname{Ass}(I)$ and $[(I : J) + H] \neq (x_0, \ldots, x_n)$, then*

$$\operatorname{reg}((I+H) : (J+H)) = \operatorname{reg}(I : J). \tag{3.3}$$

2) *Let X and Y be closed subschemes of $\mathbb{P}^n$. If $H \subseteq \mathbb{P}^n$ is a general hyperplane such that the ideals $I_X + I_H$ and $(I_X : I_Y) + I_H$ are saturated, then the following equality holds:*

$$\operatorname{reg}(\operatorname{res}_{Y\cap H}(X \cap H)) = \operatorname{reg}(\operatorname{res}_Y X). \tag{3.4}$$

Proof. 1) By hypothesis and Remark 2.3, we have that h is general with respect to the ideals I and $(I : J)$. Hence, the equality $(I : J) + H = ((I+H) : (J+H))$ holds true from Lemma 2.1. By item 1) of Theorem 3.4 we get

$$\operatorname{reg}((I : J) + H) = \operatorname{reg}(I : J).$$

2) Observe that the equality $(\mathrm{res}_Y X) \cap H = \mathrm{res}_{Y\cap H}(X \cap H)$ is achieved by using the Equation (2.1) of Theorem 2.2. Hence, by hypothesis we have:

$$\overline{(I_X : I_Y) + I_H} = (I_X : I_Y) + I_H = \overline{((I_X + I_H) : (I_Y + I_H))}.$$

On the other hand, the fact that the ideal $I_X + I_H$ is saturated implies that the ideal $((I_X+I_H) : (I_Y+I_H))$ is saturated. Indeed, $\mathrm{Ass}((I_X+I_H) : (I_Y+I_H)) \subseteq \mathrm{Ass}(I_X+I_H)$, so $\mathfrak{m} \notin \mathrm{Ass}(I_X + I_H)$. Therefore, $\mathfrak{m} \notin \mathrm{Ass}((I_X + I_H) : (I_Y + I_H))$ and consequently $((I_X + I_H) : (I_Y + I_H))$ is saturated. Then by the item 1), it follows that

$$\mathrm{reg}(I_X : I_Y) = \mathrm{reg}((I_X : I_Y) + I_H) = \mathrm{reg}((I_X + I_H) : (I_Y + I_H)).$$

This proves that $\mathrm{reg}(\mathrm{res}_Y X) = \mathrm{reg}(\mathrm{res}_{Y\cap H}(X \cap H))$. □

Example 3.6. Here, we present an example to show that the saturation hypothesis in the item 2) of the previous theorem is necessary. Let R be the homogeneous coordinate ring $K[x_0, x_1]$ of the projective line $\mathbb{P}^1$ defined over a field K. Consider the closed subschemes X and Y of $\mathbb{P}^1$ defined by the ideals (x_0x_1) and (x_0) respectively, the hyperplane H defined by the ideal $(x_0 + x_1)$, and we denote $Z = \mathrm{res}_Y X$. Note that H is general by construction. We have that the ideals $I_X = (x_0x_1)$, $I_Y = (x_0)$, and $I_H = (x_0 + x_1)$ are saturated, however, the ideal

$$(I_X : I_Y) + I_H = (x_1) + (x_0 + x_1) = (x_0, x_1)$$

is not saturated. The fact that $\mathrm{reg}(Z) = \mathrm{reg}(\widetilde{(x_1)}) - 1 = \mathrm{reg}((x_1)) - 1 = 0$ and that $Z \cap H$ is empty implies that $\mathrm{reg}(Z) \neq \mathrm{reg}(Z \cap H)$, and consequently the Equation (3.4) is not true. On the other hand, the equality

$$\begin{aligned} ((I_X + I_H) : (I_Y + I_H)) &= ((x_0x_1, x_0 + x_1) : (x_0, x_0 + x_1)) \\ &= ((x_0x_1, x_0 + x_1) : (x_0, x_1)) \\ &= (x_0, x_1) \end{aligned}$$

give us that $\mathrm{reg}((I_X + I_H) : (I_Y + I_H)) = 1$. So, the Equation (3.3) holds because $\mathrm{reg}(I_X : I_Y) = \mathrm{reg}((x_1)) = 1$, in spite of the hypothesis $(I_X : I_Y) + I_H \neq (x_0, x_1)$ is not satisfied.

Remark 3.7. If I is not saturated, we have the following definition of regularity:

$$\mathrm{reg}(I) = max\{\mathrm{reg}(\bar{I}), \mathrm{sat}(I)\}.$$

Remark 3.8. Let $<$ be a term order introduced on the monomials of the polynomial ring $R = K[x_0, \ldots, x_n]$ and let $\mathrm{in}_<(I)$ be the initial ideal of an homogeneous ideal I of R. If the ideal I is saturated, it may happen that $\mathrm{in}_<(I)$ is not saturated, however, the inequality $\mathrm{reg}(I) \leq \mathrm{reg}(\mathrm{in}_<(I))$ still true (see [7, Corollary 2.12]). The latter could be not true for the regularity of schemes. Indeed, let X be an algebraic variety of $\mathbb{P}^n$, henceforth, $\mathrm{reg}(X) = \mathrm{reg}(R/I_X)$,where $X = \mathrm{Proj}(R/I_X)$ and I_X is saturated. Despite of the fact that $\mathrm{reg}(I_X) \leq \mathrm{reg}(\mathrm{in}_<(I_X))$ holds true, the ideal $\mathrm{in}_<(I_X)$ is not longer saturated in general, so we cannot speak of $\mathrm{Proj}(R/\,\mathrm{in}_<(I_X))$.

Now we focus on the second topic that we want to investigate. We recall that the "Uniform Position Principle" for a set of points Γ of the projective space $\mathbb{P}^n$ is formulated in terms of the Hilbert function of the scheme Γ. To be precise, we have:

Definition 3.9. A set Γ of points of $\mathbb{P}^n$ is in uniform position if every pair of subsets of Γ having the same number of points have the same Hilbert function.

The relevance of the concept comes from the following known result:

Theorem 3.6 (Uniform Position Principle)**.** *Let X be a variety of the projective space $\mathbb{P}^n$ of dimension r. If V is a general linear space of $\mathbb{P}^n$ of codimension r, then $X \cap V$ is a set of points in uniform position.*

Proof. See pages 109-113 of [1]. □

To state the most relevant geometric consequence in our context, we have the following concept:

Definition 3.10. Let X be a variety of the projective space $\mathbb{P}^n$ whose defining ideal is I_X. A linear space V of $\mathbb{P}^n$ is general with respect to X if every associated prime ideal of its defining ideal I_V is not an associated prime ideal of I_X.

Proposition 3.7. *A linear space V of a projective space $\mathbb{P}^n$ of codimension r is a variety of $\mathbb{P}^n$.*

Proof. The ideal I_V can be generated by $n-r$ circuits (see [11, Example 1.5]), hence I_V can be generated by a regular sequence of $n-r$ linear forms of $K[x_0, \ldots, x_n]$, then I_V is a prime ideal. □

Proposition 3.8. *Let X be a variety of a projective space $\mathbb{P}^n$. If $V \subseteq \mathbb{P}^n$ is a linear space consisting of hyperplanes in general position with respect to X, then V is general with respect to X.*

Proof. Assume that for some positive integer s we have $I_V = (h_1, \ldots, h_s)$, which is composed by hyperplanes defined by $H_i = (h_i)$ where $h_1, \ldots, h_s$ are linear forms of $K[x_0, \ldots, x_n]$, and such that each H_i is general with respect to X. This implies that $H_i \notin \mathrm{Ass}(I_X) = \{I_X\}$, and consequently $H_i \neq I_X$, for each $i = 1, \ldots, s$. If $I_V = I_X$, then $h_i \in I_X$, but this contradicts the fact that $H_i \neq I_X$. Therefore, $I_V \notin \mathrm{Ass}(I_X)$ and V is general with respect to X. □

Remark 3.11. The converse of Proposition 3.8 is not true. Suppose that V is general with respect to X. Since $I_V \notin \mathrm{Ass}(I_X)$, we have that $I_V \neq I_X$. Hence, $(h_i) \notin \mathrm{Ass}(I_X)$ for some $i \in \{1, \ldots, s\}$, and the hyperplane generated by H_i is general with respect to X. Nevertheless, we can have some hyperplane in V that is not general with respect to X.

Corollary 3.9. *Let X and Y be closed subschemes of $\mathbb{P}^n$ such that $\mathrm{res}_Y X$ is irreducible of dimension one. If $H \subseteq \mathbb{P}^n$ is a general hyperplane with respect to X, then $\mathrm{res}_{Y \cap H}(X \cap H)$ is a set of points in uniform position.*

Proof. The hyperplane H is a linear space of codimension $n-1$, then $(\mathrm{res}_Y X)\cap H$ is a set of points in uniform position. The assertion follows by the next equality which is given in Theorem 2.2:

$$(\mathrm{res}_Y X) \cap H = \mathrm{res}_{Y\cap H}(X \cap H).$$

The corollary below gives a way of constructing sets of points in uniform position.

Corollary 3.10. *Let X and Y be varieties of a projective space $\mathbb{P}^n$ such that $\mathrm{res}_Y X$ is irreducible of dimension r. If V is a linear space of $\mathbb{P}^n$ of codimension r that consists of hyperplanes in general position with respect to X and Y, then the variety $\mathrm{res}_{Y\cap V}(X \cap V)$ is a set of points in uniform position.*

Proof. By Proposition 3.8, we obtain that V is general with respect to X, Y and $\mathrm{res}_Y X$. Set $H_1, \ldots, H_{n-r}$ the hyperplanes that defines I_V (that is, $I_V = \sum_{i=1}^{n-r} H_i$). So, we infer that:

$$\begin{aligned}(I_X : I_Y) + I_V &= (I_X : I_Y) + \sum_{i=1}^{n-r} H_i \\ &= ((I_X + H_1) : (I_Y + H_1)) + \sum_{i=2}^{n-r} H_i \\ &\vdots \\ &= ((I_X + I_V) : (I_Y + I_V)).\end{aligned}$$

Hence $(\mathrm{res}_Y X)\cap V = \mathrm{res}_{Y\cap V}(X\cap V)$. Since $(\mathrm{res}_Y X)\cap V$ is a set of points in uniform position, the assertion follows. □

Below, we need the following result:

Lemma 3.11. *Let I and J be homogeneous ideals of $K[x_0, \ldots, x_n]$. If I and J are prime ideals, then $(I : J) = I$ or $(I : J) = K[x_0, \ldots, x_n]$.*

Proof. Let $J = (g_1, \ldots, g_s)$ where s is a positive integer and $g_1, \ldots, g_s$ are irreducible elements of $K[x_0, \ldots, x_n]$. It is known that $(I : J) = \bigcap_{i=1}^{s}(I : g_i)$. Two cases may occur. If $J \subseteq I$, then $(I : J) = K[x_0, \ldots, x_n]$. Otherwise, $(I : J) = I$. Indeed, one may find some $i \in \{1, \ldots, s\}$ such that $g_i \notin I$. Now, take a homogeneous element h of $K[x_0, \ldots, x_n]$ belonging to the ideal $(I : g_i)$. Since I is prime, it follows that $h \in I$, and therefore $(I : g_i) \subseteq I$. This proves that $(I : J) \subseteq I$, and we are done. □

We conclude the paper with the following natural question:

Problem: Let Γ and Γ' be sets of points of $\mathbb{P}^n$ in uniform position with sheaf ideals $\mathcal{I}_\Gamma$ and $\mathcal{I}_{\Gamma'}$. Does the quotient sheaf $\mathcal{I}_\Gamma : \mathcal{I}_{\Gamma'}$ define a scheme of points in uniform position?

Here, we give a partial answer to the above question:

Theorem 3.12. *Let X and Y be irreducible closed subschemes of dimension r of a projective space $\mathbb{P}^n$ such that $\operatorname{res}_Y X$ is irreducible of dimension r. If $V \subseteq \mathbb{P}^n$ is a linear space of codimension r which is general with respect to X and Y, then the closed subschemes $X \cap V$ and $Y \cap V$ of $\mathbb{P}^n$ are sets of points in uniform position, and either*

1. *the equality $\operatorname{res}_{Y \cap V}(X \cap V) = X \cap V$ holds, or*
2. *both $\operatorname{res}_Y X$ and $\operatorname{res}_{Y \cap V}(X \cap V)$ are empty.*

Proof. It is obvious that I_X and I_Y are homogeneous prime ideals. So, by Theorem 2.2 and Lemma 3.11, the statement *1.* corresponds to the case $(I_X : I_Y) = I_X$, and the statement *2.* corresponds to the case $(I_X : I_Y) = K[x_0, \ldots, x_n]$. □

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