# A variational approach to multiplicity results for boundary-value problems on the real line 

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#### Abstract

We study the existence and multiplicity of solutions for a parametric equation driven by the $p$-Laplacian operator on unbounded intervals. Precisely, by using a recent local minimum theorem we prove the existence of a non-trivial non-negative solution to an equation on the real line, without assuming any asymptotic condition neither at 0 nor at $\infty$ on the nonlinear term. As a special case, we note the existence of a non-trivial solution for the problem when the nonlinear term is sublinear at 0 . Moreover, under a suitable superlinear growth at $\infty$ on the nonlinearity we prove a multiplicity result for such a problem. 


## 1. Introduction

Boundary-value problems (briefly BVPs) on infinite intervals model many problems arising from physical phenomena, such as the flow of a gas through a semi-infinite porous medium or non-Newtonian fluid flows (see [20] and references therein), and, as a result, they are widely studied (see, for example, $[15,18,25]$ ). More generally, ellipti uations on the whole space were investigated and we refer the der to [3, and [24, ch. 6.4] for an overview on this subject; see also [12] the non-smooth case.

The aim of this paper is to investigate elliptic problems on the real line. To be precise, we are interested in the existence and multiplicity of non-negative solutions to the following problem. Find $u \in W^{1, p}(\mathbb{R})$ satisfying
$\left(P_{\lambda}\right)\left(\left|u^{\prime}(x)\right|^{p-2} u^{\prime}(x)\right)^{\prime}+B|u(x)|^{p-2} u(x)=\lambda \alpha(x) g(u(x))$ for almost every (a.e.) $x$ in $\mathbb{R}$,
where $\lambda$ is a real positive parameter, $B$ is a real positive number, and $\alpha, g: \mathbb{R} \rightarrow \mathbb{R}$ are two functions such that

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$$
\alpha \in L^{1}(\mathbb{R}), \quad \alpha(x) \geqslant 0 \text { for a.e. } x \in \mathbb{R}, \quad \alpha \not \equiv 0
$$

and $g$ is continuous and non-negative. Many authors studied BVPs (parametric or otherwise) on unbounded $\pi$ rvals and approached the problem using different techniques (see, for example, $15,17-20,23,25]$ ). In particular, in [18] the authors studied the existence and uniqueness of positive solutions of a one-parameter family of logistic equations of the type

$$
u^{\prime \prime}+a f(x) u-b(x) u^{p}=0 \quad \text { in } \mathbb{R} \text { or in } \mathbb{R}_{+} .
$$

They obtained the solution as a minimum point of the energy functional associated with the previous equation in $\mathcal{D}^{1,2}(\mathbb{R})$ and $\mathcal{D}_{0}^{1,2}\left(\mathbb{R}_{+}\right)$, respectively, with $a \in\left(\lambda_{1}, \lambda_{*}\right)$ and they showed the non-existence of solutions for $a \geqslant \lambda_{*}$.

The method of upper and lower solutions was used in [15, 25] for two SturmLiouville value problems in $[0,+\infty[$. In [25] the authors looked for positive unbounded solutions. They gave necessary and sufficient conditions for the existence of positive solutions, with a sublinear growth assumption on the nonlinear term. Using a particular cone and a fixed-point theorem they also discussed the multiplicity. The method of unbounded upper and lower solutions of [25] was generalized in [15], where the authors used the Schauder fixed-point theorem to show the existence of a positive solution to their problem.

In our paper the structure of the problem, as well as the assumptions on the nonlinear term, are not comparable with the papers cited above. Our primary tool in proving the main result of this paper is a local minimum theorem recently established in [5] and, in order to obtain multiple solutions, we use a two critical points theorem presented in [6]. Our main result (theorem 3.1) ensures the existence of a non-trivial solution without requiring any asymptotic condition on $g$ either at 0 or at $\infty$. Moreover, as a consequence, we point out a result where only the sublinearity of $g$ at 0 is required in order to obtain the existence of a non-trivial solution (see corollary 3.4). Finally, we present a result where two non-trivial solutions are guaranteed under a suitable growth of $g$ at $\infty$ (see theorem 3.10 and remark 3.11). Such (arowth type was introduced and developed by Ambrosetti and Rabinowitz in [1 ${ }^{m m d}$ it is worth noting that such an assumption is usually accompanied by the superlinearity of $g$ at 0 to ensure the existence of only one non-trivial solution.

As an example, here we point out the following special cases of our results.
Theorem 1.1. Assume that

$$
\int_{0}^{11} g(t) \mathrm{d} t<11 \int_{0}^{1} g(t) \mathrm{d} t
$$

Then, for each

$$
\lambda \in] \frac{11}{\pi} \frac{1}{\int_{0}^{1} g(t) \mathrm{d} t}, \frac{11}{\pi} \frac{11}{\int_{0}^{11} g(t) \mathrm{d} t}[
$$

the problem

$$
\begin{gathered}
-u^{\prime \prime}+u=\lambda \frac{g(u)}{1+x^{2}}, \quad x \in \mathbb{R} \\
u(-\infty)=u(+\infty)=0
\end{gathered}
$$

admits at least one non-trivial and non-negative classical solution $u_{0, \lambda}$ such that

$$
\left|u_{0, \lambda}(x)\right|<11
$$

for all $x \in \mathbb{R}$.
Theorem 1.2. Assume that $\alpha$ is continuous in $\mathbb{R}, g(0)>0$ and

$$
0<\mu \int_{0}^{\xi} g(s) \mathrm{d} s \leqslant \xi g(\xi)
$$

for all $\xi \geqslant s$ and for some $s>0$ and $\mu>p$.
Then there exists $\lambda^{*}>0$ such that, for each $\left.\lambda \in\right] 0, \lambda^{*}[$, the problem

$$
\begin{gathered}
-\left(\left|u^{\prime}(x)\right|^{p-2} u^{\prime}(x)\right)^{\prime}+|u(x)|^{p-2} u(x)=\lambda \alpha(x) g(u(x)), \quad x \in \mathbb{R} \\
u(-\infty)=u(+\infty)=0
\end{gathered}
$$

admits at least two non-trivial and non-negative classical solutions.
In theorem 1.1, no asymptotic conditions either at 0 or at $\infty$ are required, while theorem 1.2 ensures two non-trivial solutions under a suitable condition at $\infty$ of Ambrosetti-Rabinowitz type.

The paper is arranged as follows. In $\S 2$ we establish all the preliminary results that we need, and in $\S 3$ we present our main results.

## 2. Mathematical background

Let $(E,|\cdot|)$ be a real Banach space. We denote by $E^{*}$ the dual space of $E$, while $\langle\cdot, \cdot\rangle$ stands for the duality pairing between $E^{*}$ and $E$.

We denote by $|\cdot|$ and by $|\cdot|_{t}$ the usual norms on $\mathbb{R}$ and on $L^{t}(\mathbb{R})$, for all $t \in[1,+\infty]$, while $W^{1, p}(\mathbb{R})$ indicates the closure of $C_{0}^{\infty}(\mathbb{R})$ with respect to the norm

$$
\|u\|_{1, p}:=\left(\left|u^{\prime}\right|_{p}^{p}+|u|_{p}^{p}\right)^{1 / p} .
$$

When $p=2$ the norm is induced by the scalar product

$$
(u, v)=\left(u^{\prime}, v^{\prime}\right)_{L^{2}}+(u, v)_{L^{2}}
$$

It is well known that $W^{1, p}(\mathbb{R}) \equiv W_{0}^{1, p}(\mathbb{R})$ and $W^{1, p}(\mathbb{R})$ is embedded in $L^{t}(\mathbb{R})$ for any $t \in[p,+\infty]$.

REmARK 2.1. If $\left\{u_{n}\right\}_{n \in \mathbb{N}}$ is a bounded sequence in $W^{1, p}(\mathbb{R})$, then it has a subsequence that pointwise converges to some $u \in W^{1, p}(\mathbb{R})$ and also weakly converges in $L^{\infty}(\mathbb{R})$. Indeed, it can be inferred from the compact embedding $W^{1, p}(\mathbb{R}) \hookrightarrow$ $C([-R, R]), R>0$, and the continuity of $W^{1, p}(\mathbb{R}) \rightarrow L^{\infty}(\mathbb{R})$.

In the following, we consider $W^{1, p}(\mathbb{R})$ endowed by the norm


$$
\|u\|=\left(\int_{\mathbb{R}}\left(\left|u^{\prime}(x)\right|^{p}+B|u(x)|^{p}\right) \mathrm{d} x\right)^{1 / p}
$$

which is equivalent to the usual one. We have the following proposition.

Proposition 2.2. One has

$$
\begin{equation*}
|u|_{\infty} \leqslant c_{B}\|u\| \tag{2.1}
\end{equation*}
$$

for all $u \in W^{1, p}(\mathbb{R})$, where

$$
\begin{equation*}
c_{B}=2^{(p-2) / p}\left(\frac{p-1}{p}\right)^{(p-1) / p}\left(\frac{1}{B}\right)^{(p-1) / p^{2}} \tag{2.2}
\end{equation*}
$$

Proof. We follow the argument in [10, theorem 4, p. 138, formula (4.61)], taking the equivalence of the norms into account. For clarity, we give a sketch of the proof. Let $v \in W^{1,1}(\mathbb{R})$. From

$$
v(z)-v(w)=\int_{w}^{z} v^{\prime}(t) \mathrm{d} t
$$

taking into account that $\lim _{|y| \rightarrow+\infty} v(y)=0$, one has

$$
v(x)=\int_{-\infty}^{x} v^{\prime}(t) \mathrm{d} t \quad \text { and } \quad-v(x)=\int_{x}^{\infty} v^{\prime}(t) \mathrm{d} t
$$

Hence,

$$
2|v(x)| \leqslant \int_{-\infty}^{\infty}\left|v^{\prime}(t)\right| \mathrm{d} t
$$

that is,

$$
|v(x)| \leqslant \frac{1}{2} \int_{\mathbb{R}}\left|v^{\prime}(t)\right| \mathrm{d} t
$$

for all $v \in W^{1, p}(\mathbb{R})$. Now, let $u \in W^{1, p}(\mathbb{R})$. By choosing $v(x)=B^{(p-1) / p}|u(x)|^{p}$ for all $x \in \mathbb{R}$, one has

$$
B^{(p-1) / p}|u(x)|^{p} \leqslant \frac{1}{2} \int_{\mathbb{R}} B^{(p-1) / p} p|u(t)|^{p-1}\left|u^{\prime}(t)\right| \mathrm{d} t
$$

From Hölder inequality one has

$$
B^{(p-1) / p}|u(x)|^{p} \leqslant \frac{p}{2}\left(B^{1 / p}|u|_{p}\right)^{p-1}\left|u^{\prime}\right|_{p}
$$

that is,

$$
|u(x)| \leqslant\left(\frac{1}{B}\right)^{(p-1) / p^{2}}\left(\frac{p}{2}\right)^{1 / p}\left(B^{1 / p}|u|_{p}\right)^{(p-1) / p}\left|u^{\prime}\right|_{p}^{1 / p}
$$

Noting that $x^{\alpha} y^{1-\alpha} \leqslant \alpha^{\alpha}(1-\alpha)^{(1-\alpha)}(x+y), x, y \geqslant 0,0<\alpha<1$ (see [10, p. 130, formula (4.47)]), one has

$$
\begin{aligned}
&|u(x)| \leqslant\left(\frac{1}{B}\right)^{(p-1) / p^{2}}\left(\frac{p}{2}\right)^{1 / p}\left(\frac{p-1}{p}\right)^{(p-1) / p}\left(\frac{1}{p}\right)^{1 / p} \\
& \times\left[\left(\int_{\mathbb{R}}\left|u^{\prime}(x)\right|^{p} \mathrm{~d} x\right)^{1 / p}+\left(\int_{\mathbb{R}} B|u(x)|^{p} \mathrm{~d} x\right)^{1 / p}\right]
\end{aligned}
$$

Therefore, taking into account the classical inequality

$$
a^{1 / p}+b^{1 / p} \leqslant 2^{(p-1) / p}(a+b)^{1 / p}
$$

one has

$$
\begin{aligned}
&|u(x)| \leqslant\left(\frac{1}{B}\right)^{(p-1) / p^{2}}\left(\frac{1}{2}\right)^{1 / p}\left(\frac{p-1}{p}\right)^{(p-1) / p} 2^{(p-1) / p} \\
& \times\left[\left(\int_{\mathbb{R}}\left|u^{\prime}(x)\right|^{p} \mathrm{~d} x\right)+\left(\int_{\mathbb{R}} B|u(x)|^{p} \mathrm{~d} x\right)\right]^{1 / p}
\end{aligned}
$$

that is,

$$
|u|_{\infty} \leqslant c_{B}\|u\|
$$

and the proof is complete.
We set

$$
\begin{equation*}
G(t)=\int_{0}^{t} g(\xi) \mathrm{d} \xi \quad \text { for all } t \in \mathbb{R} \tag{2.3}
\end{equation*}
$$

Our hypotheses on $g$ guarantee that $G \in C^{1}(\mathbb{R})$ and $G^{\prime}(t)=g(t) \geqslant 0$ for all $t \in \mathbb{R}$, so $G$ is non-decreasing.

Now, we put

$$
\begin{equation*}
\Phi(u)=\frac{1}{p}\|u\|^{p} \tag{2.4}
\end{equation*}
$$

and we define $\Psi: W^{1, p}(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
\Psi(u)=\int_{\mathbb{R}} \alpha(x) G(u(x)) \mathrm{d} x=\int_{\mathbb{R}} \alpha(x)\left(\int_{0}^{u(x)} g(\xi) \mathrm{d} \xi\right) \mathrm{d} x, \quad \forall u \in W^{1, p}(\mathbb{R}) \tag{2.5}
\end{equation*}
$$

It is clear that the assumptions on $\alpha$ and $g$ guarantee that the functional $\Psi$ is well defined. In fact, one sees that the following inequality holds for any $u \in W^{1, p}(\mathbb{R})$.

$$
\begin{aligned}
&|\Psi(u)| \\
& \leqslant \int_{\mathbb{R}} \alpha(x)|G(u(x))| \mathrm{d} x \leqslant \int_{\mathbb{R}} \alpha(x) \max _{x \in \mathbb{R}}|G(u(x))| \mathrm{d} x \leqslant \int_{\mathbb{R}} \alpha(x) \max _{|\xi| \leqslant|u|_{\infty}}|G(\xi)| \mathrm{d} x \\
&=\int_{\mathbb{R}} \alpha(x) \max \left\{-G\left(-|u|_{\infty}\right), G\left(|u|_{\infty}\right)\right\} \mathrm{d} x \\
&=M_{u}|\alpha|_{1}
\end{aligned}
$$

Our main tool is a local minimum theorem proved in [5] (see [5, theorem 3.1]). Here, we use the version as given in [6] (see theorem 2.6; see also [8,22]). Before stating it, we give some definitions. Let $E$ be a real Banach space and let $\Phi, \Psi: E \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals, put $I_{\lambda}=\Phi-\lambda \Psi, \lambda>0$, and fix $r \in]-\infty,+\infty]$.

Definition 2.3. We say that a functional $I_{\lambda}$ verifies the Palais-Smale condition

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 cut-off upper at $r$ (in short, the (PS) ${ }^{[r]}$-condition) if any sequence $\left\{u_{n}\right\}$ such that

- $I_{\lambda}\left(u_{n}\right)$ is bounded,
- $\lim _{n \rightarrow+\infty}\left\|I_{\lambda}^{\prime}\left(u_{n}\right)\right\|_{*}=0$,
- $\Phi\left(u_{n}\right)<r$
has a convergent subsequence.

When $r=+\infty$ the previous definition recovers the classical definition of the Palais-Smale condition given below.

Definition 2.4. We say that the functional $I_{\lambda}$ verifies the Palais-Smale condition Journal style is to not introduce definition (or
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with a fragment so I have completed this sentence: OK? (in short, the (PS)-condition) if any sequence $\left\{u_{n}\right\}$ such that

- $I_{\lambda}\left(u_{n}\right)$ is bounded,
- $\lim _{n \rightarrow+\infty}\left\|I_{\lambda}^{\prime}\left(u_{n}\right)\right\|_{*}=0$
has a convergent subsequence.
Definition 2.5. We say that $u \in E$ is a critical point of $I_{\lambda}$ when $I_{\lambda}^{\prime}(u)=0_{E^{*}}$, that is, $I_{\lambda}^{\prime}(u)(v)=0$ for all $v \in E$.

Theorem 2.6 (Bonanno [6, theorem 2.2]). Let E be a real Banach space and let $\Phi, \Psi: E \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf _{E} \Phi=\Phi(0)=\Psi(0)=0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in E$ with $0<\Phi(\tilde{u})<r$ such that

$$
\begin{equation*}
\frac{\sup _{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r}<\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.6}
\end{equation*}
$$

and, for each

$$
\lambda \in] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup _{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}[
$$

the functional $I_{\lambda}=\Phi-\lambda \Psi$ satisfies the $(P S)^{[r]}$-condition. Then, for each

$$
\lambda \in] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup _{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}[
$$

there is a $u_{\lambda} \in \Phi^{-1}(] 0, r[)$ (hence, $u_{\lambda} \neq 0$ ) such that $I_{\lambda}\left(u_{\lambda}\right) \leqslant I_{\lambda}(u)$ for all $u \in$ $\Phi^{-1}(] 0, r[)$ and $I_{\lambda}^{\prime}\left(u_{\lambda}\right)=0$.

We explicitly observe that, contrary to [22, theorem 2.5], in theorem 2.6 the sequential weak lower semi-continuity of $I_{\lambda}$ is not required and, in addition, the local minimum is non-trivial.

Now we recall a multiple critical points result obtained in [6] that is based on the theorem of the local minimum [5, theorem 3.1] and on the classical theorem of Ambrosetti-Rabinowitz in [1].

Theorem 2.7 (Bonanno [6, theorem 3.2]). Let $E$ be a real Banach space and let $\Phi, \Psi: E \rightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\Phi$ is bounded from below and $\Phi(0)=\Psi(0)=0$. Fix $r>0$ and assume that, for each

$$
\lambda \in] 0, \frac{r}{\sup _{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}[
$$

the functional $I_{\lambda}=\Phi-\lambda \Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each

$$
\lambda \in] 0, \frac{r}{\sup _{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}[
$$

the functional $I_{\lambda}$ admits two distinct critical points.
In our situation, the space $E$ coincides with $W^{1, p}(\mathbb{R})$, while $I_{\lambda}: W^{1, p}(\mathbb{R}) \rightarrow \mathbb{R}$ is the energy functional related to $\left(P_{\lambda}\right)$, and is defined as

$$
I_{\lambda}(u)=\Phi(u)-\lambda \Psi(u)
$$

where $\Phi, \Psi$ are given in (2.4) and (2.5). It is well know that $\Phi, \Psi$ are continuously Gâteaux differentiable. If $u$ is a critical point of $I_{\lambda}$, then $I_{\lambda}^{\prime}(u) \equiv 0$, that is,

$$
\int_{\mathbb{R}}\left(\left|u^{\prime}(x)\right|^{p-2} u^{\prime}(x) v^{\prime}(x)+B|u(x)|^{p-2} u(x) v(x)-\lambda \alpha(x) g(u(x)) v(x)\right) \mathrm{d} x=0
$$

for all $v \in W^{1, p}(\mathbb{R})$, so $u$ is a (weak) solution to $\left(P_{\lambda}\right)$. Moreover, when $\alpha$ is, in addition, a continuous function on $\mathbb{R}$, the (weak) solutions of $\left(P_{\lambda}\right)$ are actually classical, as standard computations show.

Lemma 2.8. Let $\Phi$ and $\Psi$ be defined as above and fix $\lambda>0$. Then, $I_{\lambda}=\Phi-\lambda \Psi$ satisfies the $(P S)^{[r]}$-condition for any $r>0$.

Proof. Let $\left\{u_{n}\right\} \subseteq W^{1, p}(\mathbb{R})$ be a sequence such that $\left\{I_{\lambda}\left(u_{n}\right)\right\}$ is bounded,

$$
\lim _{n \rightarrow+\infty}\left\|I_{\lambda}\left(u_{n}\right)\right\|_{W^{1, p}(\mathbb{R})^{*}}=0
$$

and $\Phi\left(u_{n}\right)<r$ for all $n \in \mathbb{N}$.
From $\Phi\left(u_{n}\right)<r$, taking into account that $\Phi$ is coercive, $\left\{u_{n}\right\}$ is bounded $W^{1, p}(\mathbb{R})$. Therefore, up to a subsequence, $u_{n}(x) \rightarrow u(x), x \in \mathbb{R}$, and $\left\{u_{n}\right\}$ weakly converges to $u$ in $L^{\infty}(\mathbb{R})$ (see remark 2.1).

Now, taking into account that $\left\{u_{n}\right\}$ is bounded in $L^{\infty}(\mathbb{R})$ (being weakly convergent in $\left.L^{\infty}(\mathbb{R})\right)$, there is an $M>0$ such that $\left|u_{n}(x)\right| \leqslant M$ for all $n \in \mathbb{N}$, for a.e. $x \in \mathbb{R}$. It follows that $g\left(u_{n}(x)\right) \leqslant \max _{|\xi| \leqslant M} g(\xi)$ for which $\alpha g\left(u_{n}\right) \in L^{1}(\mathbb{R})$ for all $n \in \mathbb{N}$. Since $g\left(u_{n}(x)\right) \rightarrow g(u(x))$ for a.e. $x \in \mathbb{R}(g$ is a continuous function), from Lebesgue's theorem one has that $\left\{\alpha g\left(u_{n}\right)\right\}$ is strongly converging to $\alpha g(u)$ in $L^{1}(\mathbb{R})$. Since $u_{n} \rightharpoonup u$ in $L^{\infty}(\mathbb{R}), \alpha g\left(u_{n}\right), \alpha g(u) \in L^{1}(\mathbb{R}) \subseteq\left(L^{\infty}(\mathbb{R})\right)^{*}$ (see [9, p. 102]) and $\alpha g\left(u_{n}\right) \rightarrow \alpha g(u)$ in $L^{1}(\mathbb{R})$, the definition of weak convergence and [9, proposition III.5(iv)] leads to

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \int_{\mathbb{R}} \alpha(x) g\left(u_{n}(x)\right)\left(u_{n}(x)-u(x)\right) \mathrm{d} x=0 \tag{2.7}
\end{equation*}
$$

Now, from $\lim _{n \rightarrow+\infty}\left\|I_{\lambda}\left(u_{n}\right)\right\|_{W^{1, p}(\mathbb{R})^{*}}=0$, there exists a sequence $\left\{\varepsilon_{n}\right\}$, with $\varepsilon_{n} \rightarrow 0^{+}$, such that

$$
\left|\int_{\mathbb{R}}\left(\left|u_{n}^{\prime}\right|^{p-2} u_{n}^{\prime} v^{\prime}+B\left|u_{n}\right|^{p-2} u_{n} v-\lambda \alpha g\left(u_{n}\right) v\right) \mathrm{d} x\right| \leqslant \varepsilon_{n}
$$

for all $n \in \mathbb{N}$, for all $v \in W^{1, p}(\mathbb{R})$ such that $\|v\| \leqslant 1$.

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$W^{1, p}(\mathbb{R})$ '?

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Setting $v=\left(u_{n}-u\right) /\left\|u_{n}-u\right\|$, one has

$$
\begin{align*}
& \int_{\mathbb{R}}\left(\left|u_{n}^{\prime}\right|^{p-2} u_{n}^{\prime}\left(u_{n}^{\prime}-u^{\prime}\right)+B\left|u_{n}\right|^{p-2} u_{n}\left(u_{n}-u\right)\right. \\
&\left.-\lambda \alpha g\left(u_{n}\right)\left(u_{n}-u\right)\right) \mathrm{d} x \leqslant \varepsilon_{n}\left\|u_{n}-u\right\| \tag{2.8}
\end{align*}
$$

for all $n \in \mathbb{N}$.
Noting that

$$
|a|^{p-1}|b| \leqslant \frac{p-1}{p}|a|^{p}+\frac{1}{p}|b|^{p}
$$

one has

$$
\begin{aligned}
\int_{\mathbb{R}}\left(\left|u_{n}^{\prime}\right|^{p-2} u_{n}^{\prime}\right. & \left.\left(u_{n}^{\prime}-u^{\prime}\right)+B\left|u_{n}\right|^{p-2} u_{n}\left(u_{n}-u\right)\right) \mathrm{d} x \\
& =\int_{\mathbb{R}}\left(\left|u_{n}^{\prime}\right|^{p}+B\left|u_{n}\right|^{p}\right) \mathrm{d} x-\int_{\mathbb{R}}\left(\left|u_{n}^{\prime}\right|^{p-2} u_{n}^{\prime} u^{\prime}+B\left|u_{n}\right|^{p-2} u_{n} u\right) \mathrm{d} x \\
& \geqslant\left\|u_{n}\right\|^{p}-\int_{\mathbb{R}}\left(\frac{p-1}{p}\left|u_{n}^{\prime}\right|^{p}+\frac{1}{p}\left|u^{\prime}\right|^{p}+B \frac{p-1}{p}\left|u_{n}\right|^{p}+B \frac{1}{p}|u|^{p}\right) \mathrm{d} x \\
& =\left\|u_{n}\right\|^{p}-\frac{p-1}{p}\left\|u_{n}\right\|^{p}-\frac{1}{p}\|u\|^{p} \\
& =\frac{1}{p}\left\|u_{n}\right\|^{p}-\frac{1}{p}\|u\|^{p} .
\end{aligned}
$$

Thus, from (2.8), one has

$$
\frac{1}{p}\left\|u_{n}\right\|^{p}-\frac{1}{p}\|u\|^{p} \leqslant \lambda \int_{\mathbb{R}} \alpha g\left(u_{n}\right)\left(u_{n}-u\right) \mathrm{d} x+\varepsilon_{n}\left\|u_{n}-u\right\|,
$$

that is,

$$
\begin{equation*}
-\varepsilon_{n}\left\|u_{n}-u\right\|+\frac{1}{p}\left\|u_{n}\right\|^{p} \leqslant \lambda \int_{\mathbb{R}} \alpha g\left(u_{n}\right)\left(u_{n}-u\right) \mathrm{d} x+\frac{1}{p}\|u\|^{p} \tag{2.9}
\end{equation*}
$$

Taking into account (2.7), from (2.9) one has

$$
\limsup _{n \rightarrow+\infty} \frac{1}{p}\left\|u_{n}\right\|^{p} \leqslant \frac{1}{p}\|u\|^{p}
$$

Hence, [9, proposition III.30] ensures that $\left\{u_{n}\right\}$ strongly converges to $u \in W^{1, p}(\mathbb{R})$ and the proof is complete.

Now, if we assume in addition that $g$ satisfies an Ambrosetti-Rabinowitz-type condition at $\infty$, then $I_{\lambda}$ satisfies the classical (PS)-condition. To be precise, we have the following result.

Lemma 2.9. Assume that
(AR) there are $s>0$ and $\mu>p$ such that $0<\mu G(\xi) \leqslant \xi g(\xi)$ for all $\xi \geqslant s$.
Then, $I_{\lambda}$ satisfies the (PS)-condition and it is unbounded from below.

Proof. Let $\left\{u_{n}\right\}$ be a sequence such that

$$
\begin{equation*}
\left|I_{\lambda}\left(u_{n}\right)\right| \leqslant M \quad \text { for some } M>0 \text { for all } n \in \mathbb{N} \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{\lambda}^{\prime}\left(u_{n}\right) \rightarrow 0 \quad \text { in } W^{1, p}(\mathbb{R})^{*} \text { as } n \rightarrow \infty \tag{2.11}
\end{equation*}
$$

First, we claim that there is a $K \geqslant 0$ such that

$$
\begin{equation*}
u_{n}(x) \geqslant-K \tag{2.12}
\end{equation*}
$$

for a.e. $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. To this end, setting $u_{n}^{-}$in the usual way, we prove that $\left\{u_{n}^{-}\right\}$is bounded in $W^{1, p}(\mathbb{R})$. From (2.11) one has $\left|I_{\lambda}^{\prime}\left(u_{n}\right)(v)\right| \leqslant \varepsilon_{n}\|v\|$ for all $v \in W^{1, p}(\mathbb{R})$ with $\varepsilon_{n} \rightarrow 0^{+}$. Thus, in particular, $\left|I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}^{-}\right)\right| \leqslant \varepsilon_{n}\left\|u_{n}^{-}\right\|$, that is,

$$
\left|\int_{\mathbb{R}}\left(\left|u_{n}^{\prime}\right|^{p-2} u_{n}^{\prime} u_{n}^{-\prime}+B\left|u_{n}\right|^{p-2} u_{n} u_{n}^{-}\right) \mathrm{d} x-\lambda \int_{\mathbb{R}} \alpha(x) g\left(u_{n}(x)\right) u_{n}^{-}(x) \mathrm{d} x\right| \leqslant \varepsilon_{n}\left\|u_{n}^{-}\right\|
$$

hence

$$
\left\|u_{n}^{-}\right\|^{p}+\lambda \int_{\mathbb{R}} \alpha(x) g\left(u_{n}(x)\right) u_{n}^{-}(x) \mathrm{d} x \leqslant \varepsilon_{n}\left\|u_{n}^{-}\right\|
$$

Therefore,

$$
0 \leqslant\left\|u_{n}^{-}\right\|^{p} \leqslant\left\|u_{n}^{-}\right\|^{p}+\lambda \int_{\mathbb{R}} \alpha(x) g\left(u_{n}(x)\right) u_{n}^{-}(x) \mathrm{d} x \leqslant \varepsilon_{n}\left\|u_{n}^{-}\right\| .
$$

Thus, $\left\{u_{n}^{-}\right\}$strongly converges to 0 in $W^{1, p}(\mathbb{R})$, so it is bounded in $W^{1, p}(\mathbb{R})$.
Thus, in particular, it is bounded in $L^{\infty}(\mathbb{R})$ (see (2.1)) and one has $0 \leqslant u_{n}^{-}(x) \leqslant$ $K$ for some $K \geqslant 0$ and for a.e. $x \in \mathbb{R}$, and our claim is proved.

Now, we prove that $\left\{u_{n}\right\}$ is bounded in $W^{1, p}(\mathbb{R})$. Again, from (2.11), one has $\left|I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}\right)\right| \leqslant \varepsilon_{n}\left\|u_{n}\right\|$. Then,

$$
\begin{equation*}
-I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}\right) \leqslant \varepsilon_{n}\left\|u_{n}\right\| \tag{2.13}
\end{equation*}
$$

for all $n \in \mathbb{N}$ and with $\varepsilon_{n} \rightarrow 0^{+}$.
On the other hand, one has

$$
I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}\right)=\left\|u_{n}\right\|^{p}-\lambda \int_{\mathbb{R}} \alpha(x) g\left(u_{n}(x)\right) u_{n}(x) \mathrm{d} x
$$

and

$$
\begin{aligned}
\frac{1}{\mu} I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}\right)= & \frac{1}{\mu}\left\|u_{n}\right\|^{p}-\frac{\lambda}{\mu} \int_{\mathbb{R}} \alpha(x) g\left(u_{n}(x)\right) u_{n}(x) \mathrm{d} x \\
= & \frac{1}{\mu}\left\|u_{n}\right\|^{p}-\frac{\lambda}{\mu} \int_{\mathbb{R}} \alpha(x)\left[g\left(u_{n}(x)\right) u_{n}(x)-\mu G\left(u_{n}(x)\right)\right] \mathrm{d} x \\
& \quad-\lambda \int_{\mathbb{R}} \alpha(x) G\left(u_{n}(x)\right) \mathrm{d} x
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I_{\lambda}\left(u_{n}\right)-\frac{1}{\mu} I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}\right)=\frac{1}{p} & \left\|u_{n}\right\|^{p}-\lambda \int_{\mathbb{R}} \alpha(x) G\left(u_{n}(x)\right) \mathrm{d} x-\frac{1}{\mu}\left\|u_{n}\right\|^{p} \\
& +\frac{\lambda}{\mu} \int_{\mathbb{R}} \alpha(x)\left[g\left(u_{n}(x)\right) u_{n}(x)-\mu G\left(u_{n}(x)\right)\right] \mathrm{d} x \\
& +\lambda \int_{\mathbb{R}} \alpha(x) G\left(u_{n}(x)\right) \mathrm{d} x
\end{aligned}
$$

that is,
$I_{\lambda}\left(u_{n}\right)-\frac{1}{\mu} I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}\right)=\left(\frac{1}{p}-\frac{1}{\mu}\right)\left\|u_{n}\right\|^{p}+\frac{\lambda}{\mu} \int_{\mathbb{R}} \alpha(x)\left[g\left(u_{n}(x)\right) u_{n}(x)-\mu G\left(u_{n}(x)\right)\right] \mathrm{d} x$.
Taking (AR) into account, one has

$$
\int_{u_{n}(x) \geqslant s} \alpha(x)\left[g\left(u_{n}(x)\right) u_{n}(x)-\mu G\left(u_{n}(x)\right)\right] \mathrm{d} x \geqslant 0 .
$$

Moreover, from (2.12), one has

$$
\begin{aligned}
&\left|\int_{-K \leqslant u_{n}(x)<s} \alpha(x)\left[g\left(u_{n}(x)\right) u_{n}(x)-\mu G\left(u_{n}(x)\right)\right] \mathrm{d} x\right| \\
& \leqslant \int_{-K \leqslant u_{n}(x)<s} \alpha(x) \max _{\xi \in[-K, s]}|g(\xi) \xi-\mu G(\xi)| \mathrm{d} x \\
& \leqslant|\alpha|_{1} T
\end{aligned}
$$

where $T=\max _{\xi \in[-K, s]}|g(\xi) \xi-\mu G(\xi)|$. Hence,

$$
\begin{aligned}
I_{\lambda}\left(u_{n}\right)-\frac{1}{\mu} I_{\lambda}^{\prime}\left(u_{n}\right)\left(u_{n}\right) \geqslant & \left(\frac{1}{p}-\frac{1}{\mu}\right)\left\|u_{n}\right\|^{p} \\
& +\frac{\lambda}{\mu} \int_{K \leqslant u_{n}(x)<s} \alpha(x)\left[g\left(u_{n}(x)\right) u_{n}(x)-\mu G\left(u_{n}(x)\right)\right] \mathrm{d} x \\
\geqslant & \left(\frac{1}{p}-\frac{1}{\mu}\right)\left\|u_{n}\right\|^{p}-\frac{\lambda}{\mu}|\alpha|_{1} T
\end{aligned}
$$

From (2.10) and (2.13), it follows that

$$
\left(\frac{1}{p}-\frac{1}{\mu}\right)\left\|u_{n}\right\|^{p}-\frac{\lambda}{\mu}|\alpha|_{1} T \leqslant M+\frac{1}{\mu} \varepsilon_{n}\left\|u_{n}\right\|
$$

that is,

$$
\begin{equation*}
\left(\frac{1}{p}-\frac{1}{\mu}\right)\left\|u_{n}\right\|^{p} \leqslant M+\frac{1}{\mu} \varepsilon_{n}\left\|u_{n}\right\|+\frac{\lambda}{\mu}|\alpha|_{1} T . \tag{2.14}
\end{equation*}
$$

Hence, (2.14) ensures that $\left\{u_{n}\right\}$ is bounded in $W^{1, p}(\mathbb{R})$.
Now, arguing exactly as in the proof of lemma $2.8,\left\{u_{n}\right\}$ admits a convergent subsequence, so $I_{\lambda}$ satisfies (PS).

Finally, standard computations show that (AR) implies that

$$
G(\xi) \geqslant a_{1} \xi^{\mu}-a_{2}
$$

for all $\xi \geqslant 0$ and some positive constants $a_{1}$ and $a_{2}$, and hence $I_{\lambda}$ is unbounded from below. The proof is complete.

## 3. Main results

Throughout this section we adopt the following notation for some constants that will appear often in the following. Put

$$
\left.\begin{array}{rl}
A & =\frac{\int_{-1}^{1} \alpha(x) \mathrm{d} x}{|\alpha|_{1}}=\frac{\alpha_{0}}{|\alpha|_{1}} \\
l & =c_{B}\left(2^{2 p-1}+\frac{B}{2(p+1)}+2 B\right)^{1 / p}  \tag{3.1}\\
R & =\frac{A}{l^{p}}
\end{array}\right\}
$$

where $c_{B}$ is given in proposition 2.2
We observe that if, for example, $p=2, B=1$ and $\alpha(x)=1 /\left(1+x^{2}\right)$, then $l=\left(\frac{61}{12}\right)^{1 / 2}$ and $R=\frac{6}{61}$.

Our main result is the following.
Theorem 3.1. Assume that there exist two positive constants $\gamma$, $\kappa$, with $\kappa<\gamma$, such that

$$
\begin{equation*}
\frac{G(\gamma)}{\gamma^{p}}<R \frac{G(\kappa)}{\kappa^{p}} \tag{3.2}
\end{equation*}
$$

Then, for each

$$
\lambda \in] \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{1}{R} \frac{\kappa^{p}}{G(\kappa)}, \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{\gamma^{p}}{G(\gamma)}[
$$

problem $\left(P_{\lambda}\right)$ admits at least one non-trivial and non-negative solution $u_{0, \lambda}$ such that $\left|u_{0, \lambda}\right|_{\infty}<\gamma$.

Proof. Our aim is to apply theorem 2.6. To this end, we take $E=W^{1, p}(\mathbb{R})$, and $\Phi$, $\Psi, I_{\lambda}$ are as in $\S 2$. All of the assumptions on regularity required on $\Phi$ and $\Psi$ are established and, from lemma 2.8, the functional $I_{\lambda}$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r>0$. It is enough to prove (2.6). To this end, choose $r=\left(1 / p c_{B}^{p}\right) \gamma^{p}$ and

Perhaps 'all of the
assumptions on the necessary regularity on...'?
Change OK?

$$
\tilde{u}(x)= \begin{cases}4 \kappa(x+1)+\kappa & \text { if } x \in[-5 / 4,-1[ \\ \kappa & \text { if } x \in[-1,1] \\ 4 \kappa(1-x)+\kappa & \text { if } x \in] 1,5 / 4] \\ 0 & \text { otherwise }\end{cases}
$$

Clearly, $\tilde{u} \in W^{1, p}(\mathbb{R})$. Moreover, one has

$$
\begin{aligned}
\Phi(\tilde{u}) & =\frac{1}{p}\|\tilde{u}\|^{p} \\
& =\frac{1}{p}\left(\int_{\mathbb{R}}\left|\tilde{u}^{\prime}(x)\right|^{p} \mathrm{~d} x+B \int_{\mathbb{R}}|\tilde{u}(x)|^{p} \mathrm{~d} x\right) \\
& =\frac{1}{p}\left(\frac{(4 \kappa)^{p}}{2}+B\left(\frac{1}{2(p+1)}+2\right) \kappa^{p}\right) \\
& =\frac{\kappa^{p}}{p}\left(2^{2 p-1}+\frac{B}{2(p+1)}+2 B\right) \\
& =\kappa^{p} \frac{1}{p} \frac{l^{p}}{c_{B}^{p}}
\end{aligned}
$$

and

$$
\Psi(\tilde{u})=\int_{-5 / 4}^{5 / 4} \alpha(x) G(\tilde{u}(x)) \mathrm{d} x \geqslant \int_{-1}^{1} \alpha(x) G(\tilde{u}(x)) \mathrm{d} x=\alpha_{0} G(\kappa)
$$

Hence,

$$
\begin{equation*}
\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geqslant|\alpha|_{1} p c_{B}^{p} \frac{A}{l^{p}} \frac{G(\kappa)}{\kappa^{p}} . \tag{3.3}
\end{equation*}
$$

Moreover, from $\kappa<\gamma$ one has $\kappa l<\gamma$. In fact, arguing by contradiction, if we assume that $\kappa<\gamma \leqslant l \kappa$, one has $R\left(G(\kappa) / \kappa^{p}\right)=A\left(G(\kappa) / l^{p} \kappa^{p}\right) \leqslant A\left(G(\kappa) / \gamma^{p}\right) \leqslant$ $G(\kappa) / \gamma^{p} \leqslant G(\gamma) / \gamma^{p}$, which contradicts (3.2). Thus, from $\kappa l<\gamma$, one has

$$
\Phi(\tilde{u})=\kappa^{p} \frac{1}{p} \frac{l^{p}}{c_{B}^{p}}<\frac{1}{p c_{B}^{p}} \gamma^{p}=r
$$

so $0<\Phi(\tilde{u})<r$.
Moreover, for all $u \in W^{1, p}(\mathbb{R})$ such that $\|u\|<(p r)^{1 / p}$, taking proposition 2.2 into account, one has

$$
\begin{equation*}
|u|_{\infty} \leqslant c_{B}\|u\|<c_{B}(p r)^{1 / p}=\gamma \tag{3.4}
\end{equation*}
$$

Hence,

$$
\begin{aligned}
\sup _{\Phi(u)<r} \Psi(u) & =\sup _{\|u\|<(p r)^{1 / p}} \int_{\mathbb{R}} \alpha(x) G(u(x)) \mathrm{d} x \\
& \leqslant \int_{\mathbb{R}} \alpha(x) \sup _{|\xi|<\gamma} G(\xi) \mathrm{d} x \\
& \leqslant|\alpha|_{1} G(\gamma)
\end{aligned}
$$

From this we deduce that

$$
\begin{equation*}
\frac{\sup _{\Phi(u)<r} \Psi(u)}{r} \leqslant \frac{|\alpha|_{1} G(\gamma)}{\left(1 / p c_{B}^{p}\right) \gamma^{p}}=|\alpha|_{1} p c_{B}^{p} \frac{G(\gamma)}{\gamma^{p}} \tag{3.5}
\end{equation*}
$$

Hence, from assumption (3.2), owing to (3.3) and (3.5), one has

$$
\frac{\sup _{\Phi(u)<r} \Psi(u)}{r}<\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}
$$

and (2.6) is proved. Moreover, taking into account that, again owing to (3.3) and (3.5), one has

$$
] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup _{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}[\supseteq] \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{1}{R} \frac{\kappa^{p}}{G(\kappa)}, \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{\gamma^{p}}{G(\gamma)}[.
$$

Therefore, theorem 2.6 ensures that, for all

$$
\lambda \in] \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{1}{R} \frac{\kappa^{p}}{G(\kappa)}, \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{\gamma^{p}}{G(\gamma)}[
$$

there is a $u_{0, \lambda} \in \Phi^{-1}(] 0, r[)$ (hence, $\left.u_{0, \lambda} \neq 0\right)$ such that $I_{\lambda}\left(u_{0, \lambda}\right) \leqslant I_{\lambda}(u)$ for all $u \in \Phi^{-1}(] 0, r[)$ and $I_{\lambda}^{\prime}\left(u_{0, \lambda}\right)=0$. It follows that $u_{0, \lambda}$ is a non-zero solution of problem ( $P_{\lambda}$ ) and, from (3.4), one has $\left|u_{0, \lambda}\right|_{\infty}<\gamma$.

Finally, by standard computations, we have $u_{0, \lambda} \geqslant 0$. In fact, from $I_{\lambda}^{\prime}\left(u_{0, \lambda}\right)(v)=$

Changes to sentence OK?
hence $\left\|u_{0, \lambda}^{-}\right\|=0$. The proof is complete.
Remark 3.2. Theorem 3.1 ensures the existence of one non-trivial solution without requiring asymptotic conditions either at 0 or at $\infty$. Theorem 1.1 is an immediate consequence.

Example 3.3. Put

$$
\tilde{g}(u)= \begin{cases}u^{2} & \text { if } u \leqslant 1 \\ \frac{1}{u} & \text { if } 1<u<11 \\ h(u) & \text { if } u \geqslant 11\end{cases}
$$

where $h: \mathbb{R} \rightarrow \mathbb{R}$ is a completely arbitrary function. From theorem 1.1 the problem

$$
\begin{gathered}
-u^{\prime \prime}+u=\frac{4 \pi}{1+x^{2}} \tilde{g}(u), \quad x \in \mathbb{R} \\
u(-\infty)=u(+\infty)=0
\end{gathered}
$$

admits at least one non-trivial and non-negative classical solution. Indeed, it is enough to apply theorem 1.1 to the continuous function

$$
g^{*}(u)= \begin{cases}u^{2} & \text { if } u \leqslant 1 \\ \frac{1}{u} & \text { if } 1<u<11 \\ \frac{1}{11} & \text { if } u \geqslant 11\end{cases}
$$

so that the solution $\bar{u}$ relative to $g^{*}$, with $|\bar{u}|_{\infty}<11$, is also a solution to our problem; we note that

$$
\int_{0}^{11} g^{*}(t) \mathrm{d} t=\frac{1}{3}+\ln 11<\frac{11}{3}=11 \int_{0}^{1} g^{*}(t) \mathrm{d} t
$$

and

$$
\frac{11}{\pi} \frac{1}{\int_{0}^{1} g^{*}(t) \mathrm{d} t}=\frac{33}{\pi}<4 \pi<\frac{121}{\pi} \frac{1}{(1 / 3)+\ln 11}=\frac{11}{\pi} \frac{11}{\int_{0}^{11} g^{*}(t) \mathrm{d} t}
$$

We now point out some consequences of theorem 3.1.
Corollary 3.4. Assume that

$$
\lim _{t \rightarrow 0^{+}} \frac{g(t)}{t^{p-1}}=+\infty
$$

Then, for each $\gamma>0$ and for each

$$
\lambda \in] 0, \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{\gamma^{p}}{G(\gamma)}[
$$

problem $\left(P_{\lambda}\right)$ admits at least one non-trivial and non-negative solution $u_{0, \lambda}$ such that $\left|u_{0, \lambda}\right|_{\infty}<\gamma$.

Proof. Let $\gamma$ be an arbitrary positive real number and let

$$
\lambda \in] 0, \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{\gamma^{p}}{G(\gamma)}[
$$

From our assumption, one has $\lim _{t \rightarrow 0^{+}} p c_{B}^{p}|\alpha|_{1} R\left(G(t) / t^{p}\right)=+\infty$. Thus, corresponding to $M>1 / \lambda$ there exists $\kappa^{*}>0$ such that for any $\left.\kappa \in\right] 0, \kappa^{*}[$ one has $p c_{B}^{p}|\alpha|_{1} R\left(G(\kappa) / \kappa^{p}\right)>M$. Therefore, by choosing $\kappa<\min \left\{\kappa^{*}, \gamma\right\}$, we can apply theorem 3.1 and we obtain the conclusion.

Remark 3.5. We explicitly observe that corollary 3.4 ensures the existence of a non-trivial solution under the condition that $g$ is sublinear at 0 , without requiring any condition at $\infty$. Easy examples that satisfy this assumption can be constructed, for example, $g(u)=\sqrt{|u|}$. We recall that in order to apply the mountain pass theorem, the superlinearity of $g$ at 0 must be required as well as a suitable condition at $\infty$.

Corollary 3.6. Assume that

$$
\lim _{t \rightarrow+\infty} \frac{g(t)}{t^{p-1}}=0
$$

Then, for each $\kappa>0$ and for each

$$
\lambda \in] \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{1}{R} \frac{\kappa^{p}}{G(\kappa)},+\infty[,
$$

the problem $\left(P_{\lambda}\right)$ admits at least one non-trivial and non-negative solution $u_{0, \lambda}$.

Proof. Let $\kappa$ be an arbitrary positive real number and let

$$
\lambda \in] \frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{1}{R} \frac{\kappa^{p}}{G(\kappa)},+\infty[
$$

From our assumption one has $\lim _{t \rightarrow+\infty} p c_{B}^{p}|\alpha|_{1}\left(G(t) / t^{p}\right)=0$. Thus, corresponding to $\varepsilon>0$ such that $\varepsilon<1 / \lambda$, there exists $\gamma^{*}>0$ such that for any $\gamma>\gamma^{*}$ one has $p c_{B}^{p}|\alpha|_{1}\left(G(\gamma) / \gamma^{p}\right)<\varepsilon$. Therefore, by choosing $\gamma>\max \left\{\gamma^{*}, \kappa\right\}$, we can apply theorem 3.1 and the conclusion follows.

Remark 3.7. We explicitly observe that the solution guaranteed by corollary 3.6 is non-trivial.

Corollary 3.8. Assume that

$$
\lim _{t \rightarrow 0^{+}} \frac{g(t)}{t^{p-1}}=+\infty \quad \text { and } \quad \lim _{t \rightarrow+\infty} \frac{g(t)}{t^{p-1}}=0
$$

Then, for any $\lambda>0,\left(P_{\lambda}\right)$ admits at least one non-trivial and non-negative solution $u_{0, \lambda}$.

Proof. It follows by arguing as in the proofs of corollaries 3.4 and 3.6.
Remark 3.9. Clearly, the conclusion of corollary 3.4 holds under the assumption that

$$
\limsup _{t \rightarrow 0^{+}} \frac{G(t)}{t^{p}}=+\infty
$$

and corollary 3.6 holds under the assumption

$$
\liminf _{t \rightarrow+\infty} \frac{G(t)}{t^{p}}=0
$$

Now we point out a multiplicity result, where only a condition at $\infty$ on $g$ is required.

Theorem 3.10. Assume that
(AR) there are $s>0$ and $\mu>p$ such that $0<\mu G(\xi) \leqslant \xi g(\xi)$ for all $\xi \geqslant s$.
Then, for each

$$
\lambda \in] 0, \frac{1}{p c_{B}^{p}|\alpha|_{1}} \sup _{\gamma>0} \frac{\gamma^{p}}{G(\gamma)}[,
$$

problem $\left(P_{\lambda}\right)$ admits at least two distinct non-negative solutions $u_{0, \lambda}$ and $u_{1, \lambda}$.
Proof. Our aim is to apply theorem 2.7. To this end, we take $E=W^{1, p}(\mathbb{R})$, and $\Phi, \Psi, I_{\lambda}$ are as in $\S 2$. All the assumptions on regularity required on $\Phi$ and $\Psi$ are established and, from lemma 2.9, the functional $I_{\lambda}$ satisfies the (PS)-condition and it is unbounded from below. Moreover, for a fixed $\lambda$ as in the conclusion and $\gamma$ such that

$$
\lambda<\frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{\gamma^{p}}{G(\gamma)},
$$

See earlier comment
regarding this.

I'm unsure what conclusion this refers
clarify.
arguing as in the proof of theorem 3.1 (see (3.5)), one has

$$
\frac{1}{p c_{B}^{p}|\alpha|_{1}} \frac{\gamma^{p}}{G(\gamma)} \leqslant \frac{r}{\sup _{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}
$$

Hence, from theorem 2.7, the functional $I_{\lambda}$ admits at least two distinct critical points, which are, as seen in the proof of theorem 3.1, non-negative solutions of $\left(P_{\lambda}\right)$, and the conclusion follows.

REMARK 3.11. If $g(0) \neq 0$, both the solutions guaranteed by theorem 3.10 are nontrivial. It follows that theorem 1.2 is an immediate consequence of theorem 3.10.

Example 3.12. From theorem 3.10, the problem

$$
\begin{aligned}
& -u^{\prime \prime}+u=\frac{1}{4} \frac{1+u^{4}}{1+x^{2}} \\
& u(-\infty)=u(+\infty)=0
\end{aligned}
$$

admits at least two non-trivial and non-negative classical solutions. Indeed, it is enough to verify that $0<3\left(\xi+\left(\xi^{5} / 5\right)\right) \leqslant \xi\left(1+\xi^{4}\right)$ for all $\xi \geqslant 5^{1 / 4}$ and

$$
\frac{1}{4}<\frac{1}{2 c_{B}^{2}|\alpha|_{1}} \frac{1}{G(1)}
$$

## References

1 A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. J. Funct. Analysis 14 (1973), 349-381.
2 G. Barletta. Existence results for semilinear elliptical hemivariational inequalities. Nonlin. Analysis 68 (2008), 2417-2430.
3 T. Bartsch and Z.-Q. Wang. Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{N}$. Commun. PDEs 20 (1995), 1725-1741.
4 T. Bartsch, A. Pankov and Z.-Q. Wang. Nonlinear Schrödinger equations with steep potential well. Commun. Contemp. Math. 4 (2001), 549-569.
5 G. Bonanno. A critical point theorem via the Ekeland variational principle. Nonlin. Analysis 75 (2012), 2992-3007.
6 G. Bonanno. Relations between the mountain pass theorem and local minima. Adv. Nonlin. Analysis 1 (2012), 205-220.
7 G. Bonanno and D. O'Regan. A boundary value problem on the half-line via critical point methods. Dynam. Syst. Applic. 15 (2006), 395-408.
8 G. Bonanno and A. Sciammetta. An existence result of one nontrivial solution for two point boundary value problems. Bull. Austral. Math. Soc. 84 (2011), 288-299.
9 H. Brézis. Analyse fonctionnelle: théorie et applications (Paris: Masson, 1983).
10 V. I. Burenkov. Sobolev spaces on domains, vol. 137 (Leipzig: Teubner, 1998).
11 F. H. Glarke. Optimization and nonsmooth analysis, Classies in Applied Mathematies, vol. 5 (Philadelphia, PA: SIAM, 1990).
12 F. Gazzola and V. Radulescu. A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbb{R}^{N}$. Diff. Integ. Eqns 13 (2000), 47-60.
13 N. Ghoussoub. Duality and perturbation methods in critical point theory, Cambridge Tracts in Mathematics, vol. 107 (Cambridge University Press, 1993).
14 A. Kristály. A double eigenvalue problem for Schrödinger equations involving sublinear nonlinearities at infinity. Electron. J. Diff. Eqns 42 (2007), 1-11.
15 H. Lian, P. Wang and W. Ge. Unbounded upper and lower solutions method for SturmLiouville boundary value problem on infinite intervals. Nonlin. Analysis 70 (2009), 26272633.


16 R. Livrea and S. A. Marano. Existence and classification of critical points for non-differentiable functions. Adv. Diff. Eqns 9 (2004), 961-978.
17 R. Ma. Existence of positive solutions for second-order boundary value problems on infinity intervals. Appl. Math. Lett. 16 (2003), 33-39.
18 L. Ma and X. Xu. Positive solutions of a logistic equation on unbounded intervals. Proc. Am. Math. Soc. 130 (2002), 2947-2958.
19 T. Moussaoui and K. Szymanska-Dȩbowska. Resonant problem for a class of BVPs on the Qalf-line. Electron. J. Qual. Theory Diff. Eqns 53 (2009), 1-10.
R P. Agarwal, O. G. Mustafa and Y. V. Rogovchenko. Existence and asymptotic behavior of solutions of a boundary value problem on an infinite interval. Math. Computer Modelling 41 (2005), 135-157.
$21 \quad$ P. H. Rabinowitz. Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conference Series in Mathematics, vol. 65 (Providence, RI: American Mathematical Society, 1986).
22 B. Ricceri. A general variational principle and some of its applications. J. Computat. Appl. Math. 113 (2000), 401-410.
23 K. Szymanska. Resonant problem for some second-order differential equation on the halfline. Electron. J. Diff. Eqns 160 (2007), 1-9.
24 M. Willem. Minimax theorems (Birkhäuser, 1996).
25 B. Yan, D. O'Regan and R. P. Agarwal. Unbounded solutions for singular boundary value problems on the semi-infinite interval: upper and lower solutions and multiplicity. J. Computat. Appl. Math. 197 (2006), 365-386.

