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Dirichlet problems for fully anisotropic elliptic equations

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The existence of a non-trivial bounded solution to the Dirichlet problem is established for a class of nonlinear elliptic equations involving a fully anisotropic partial differential operator. The relevant operator depends on the gradient of the unknown through the differential of a general convex function. This function need not be radial, nor have a polynomial-type growth. Besides providing genuinely new conclusions, our result recovers and embraces, in a unified framework, several contributions in the existing literature, and augments them in various special instances.

Keywords: anisotropic elliptic equations; minimax methods; critical-point methods; anisotropic Orlicz–Sobolev spaces

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1. Introduction

We are concerned with Dirichlet problems for elliptic equations of the form

$$\left.\begin{aligned} -\operatorname{div}(\Phi_\xi(\nabla u)) &= f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\;\; \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^n$, $n \geqslant 2$, with finite Lebesgue measure $|\Omega|$, the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and vanishes at 0, and $\Phi\colon \mathbb{R}^n \to [0,\infty)$ is an even, strictly convex function, vanishing at 0. The notation Φ_ξ stands for the gradient of Φ. Let us emphasize that $\Phi(\xi)$ neither necessarily depends on ξ through its length $|\xi|$, nor necessarily has a power-type behaviour.

The equation in (1.1) is the Euler equation of the functional

$$J_\Phi(u) = \int_\Omega (\Phi(\nabla u) - F(u))\,\mathrm{d}x, \tag{1.2}$$

where $F\colon \mathbb{R} \to \mathbb{R}$ is given by

$$F(t) = \int_0^t f(s)\, \mathrm{d}s \quad \text{for } t \in \mathbb{R}. \tag{1.3}$$

Clearly, the function $u = 0$ is a trivial solution to (1.1). The aim of this paper is to show that, under suitable assumptions on Φ and f, problem (1.1) also admits a non-trivial solution, which is a critical point of the functional (1.2).

Contributions on critical point methods for nonlinear elliptic boundary-value problems with lack of coercivity include [3] and the monographs [14, 16, 23, 26]. The literature on these topics is vast and we do not attempt even a partial list of them.

The existence of non-trivial solutions to elliptic equations associated with non-coercive functionals is well known to depend on a balance between the nonlinearity in the trial functions and the nonlinearity in their gradient. In particular, the behaviour near ∞ of the functions governing these nonlinearities is dictated by a Sobolev-type inequality. Clarifying this issue with regard to problem (1.1) is one of the main focuses of our research.

Prototypal results in this line of investigations deal with semilinear equations of the form

$$-\Delta u = f(u), \tag{1.4}$$

or with the more general p-Laplacian-type equations

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u). \tag{1.5}$$

Equation (1.5) is the Euler equation of (1.2) with $\Phi(\xi) = |\xi|^p/p$; (1.4) corresponds to the special case in which $p = 2$. Besides other assumptions, the existence of non-trivial solutions to Dirichlet problems associated with (1.5) is guaranteed if either $p > n$, or $1 < p < n$ and

$$\lim_{|t|\to\infty} \frac{tf(t)}{|t|^q} = 0 \quad \text{for some } q < p^*, \tag{1.6}$$

where $p^* = np/(n-p)$, the Sobolev conjugate of p. The threshold p^* for q in (1.6) is known to be sharp as a consequence of the Pohozhaev identity. In the borderline case when $p = n$, growths of $tf(t)$ slower than $\exp(t^{n'})$ are allowed, where $n' = n/(n-1)$. The function $\exp(t^{n'})$ appears in an embedding theorem in [15,25,27], which replaces the standard Sobolev embedding in this critical situation.

Equations associated with non-coercive functionals with not necessarily polynomial growth in the gradient have been investigated, for example, in [8] in an Orlicz–Sobolev-space setting (see also [9] for related problems). The relevant equations read

$$-\operatorname{div}\left(\frac{A'(|\nabla u|)}{|\nabla u|}\nabla u\right) = f(u), \tag{1.7}$$

where $A\colon [0,\infty) \to [0,\infty)$ is a continuously differentiable, strictly convex function vanishing at 0, and A' denotes its derivative. These equations are still isotropic in the sense that the coefficient of the differential operator, and the associated functional, just depend on the length of the gradient. Indeed, $\Phi(\xi) = A(|\xi|)$ in this case. The result of [8] requires that

$$\lim_{|t|\to\infty} \frac{tf(t)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0, \tag{1.8}$$

where B is a Young function introduced in a (non-sharp, in general) embedding theorem for Orlicz–Sobolev spaces of [10].

Genuinely anisotropic equations of the form

$$-\sum_{i=1}^{n} (|u_{x_i}|^{p_i-2} u_{x_i})_{x_i} = f(u), \tag{1.9}$$

where $p_i > 1$ for $i = 1, \dots, n$ and the subscript x_i denotes the partial derivative with respect to x_i, are the subject of [11]. They are the Euler equations of functionals whose integrand is endowed with a peculiar structure, and agrees with the sum of multiples of the powers p_i of the partial derivatives of trial functions. Precisely, $\Phi(\xi) = \sum_{i=1}^{n} |\xi_i|^{p_i}/p_i$. A basic role in discussing anisotropic equations of the form (1.9) is played by the harmonic average $\overline{p}$ of the exponents p_i, defined by

$$\frac{1}{\overline{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}. \tag{1.10}$$

In [11] it is assumed that either $\overline{p} > n$, or $1 < \overline{p} < n$ and

$$\lim_{|t|\to\infty} \frac{tf(t)}{|t|^q} = 0 \quad \text{for some } q < \overline{p}^*, \tag{1.11}$$

where $\overline{p}^*$ stands for the Sobolev conjugate of $\overline{p}$.

The novelty of our contribution is twofold. On one hand, it deals with general problems as in (1.1) involving a function Φ without any additional special structure. In particular, Dirichlet problems associated with equations of the form (1.4), (1.7) and (1.9) are encompassed as special instances. On the other hand, even in these special instances, our result enhances the available results in the literature in some respects.

The underlying functional framework of this paper is quite unconventional due to the general structure of the equations in question. This calls for the development of some new aspects of the theory of anisotropic Orlicz–Sobolev spaces, which provide a natural function space setting for the problems under consideration. A key role is played by a notion of subcritical growth for f near ∞, which depends on a sharp embedding theorem for anisotropic Orlicz–Sobolev spaces. Such an embedding involves a Young function Φ_n, which enters as an optimal Sobolev conjugate of Φ. A precise statement of our main result can be found in the next section. Here, we limit ourselves to mentioning that our requirement on f near ∞ amounts to

$$\lim_{|t|\to\infty} \frac{tf(t)}{\Phi_n(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0, \tag{1.12}$$

unless (a suitable average of) Φ grows so fast for every admissible function u to be automatically bounded, in which case no assumption on f near ∞ is needed. Let us emphasize that, not only are conditions (1.6), (1.8) and (1.11) included in (1.12), but they are also weakened by (1.12) in certain situations. For instance, even when Φ depends on ξ just through its length $|\xi|$, the function Φ_n may actually grow faster than the function B appearing in (1.8).

2. Main result

A formulation of our existence theorem requires some notation. Given any function $\Phi \in C^1(\mathbb{R}^n)$ as above, define the quantities

$$i_\Phi = \liminf_{|\xi|\to\infty} \frac{\xi \cdot \Phi_\xi(\xi)}{\Phi(\xi)}, \qquad s_\Phi = \limsup_{|\xi|\to\infty} \frac{\xi \cdot \Phi_\xi(\xi)}{\Phi(\xi)},$$

where '$\cdot$' denotes the scalar product in $\mathbb{R}^n$. Note that, owing to our assumptions on Φ, one has that $1 \leqslant i_\Phi \leqslant s_\Phi \leqslant \infty$.

By $\Phi_*\colon [0,\infty) \to [0,\infty)$ we denote the (convex) function obeying

$$|\{\xi \in \mathbb{R}^n : \Phi(\xi) \leqslant t\}| = |\{\xi \in \mathbb{R}^n : \Phi_*(|\xi|) \leqslant t\}| \quad \text{for } t \geqslant 0. \tag{2.1}$$

Observe that the function $\xi \mapsto \Phi_*(|\xi|)$ agrees with the spherically increasing symmetral of Φ, and can be regarded as a kind of 'average in measure' of Φ.

Next, we call $\Phi_n\colon [0,\infty) \to [0,\infty]$ the optimal Sobolev conjugate of Φ, defined as

$$\Phi_n(t) = \Phi_*(H^{-1}(t)) \quad \text{for } t \geqslant 0, \tag{2.2}$$

where $H\colon [0,\infty) \to [0,\infty)$ is given by

$$H(t) = \left(\int_0^t \left(\frac{\tau}{\Phi_*(\tau)} \right)^{1/(n-1)} \mathrm{d}\tau \right)^{(n-1)/n} \qquad \text{for } t \geqslant 0,$$

provided that the integral is convergent. Here, H^{-1} denotes the generalized left-continuous inverse of H. The function Φ_n was introduced in [6], where a sharp embedding theorem for anisotropic Orlicz–Sobolev spaces is presented (see also [4,5] for the isotropic case).

We are now ready to state and briefly comment on the assumptions of our main result, contained in theorem 2.1 below. To begin with, we require that

$$1 < i_\Phi \quad \text{and} \quad s_\Phi < \infty. \tag{2.3}$$

Condition (2.3) ensures, in particular, the reflexivity of the anisotropic Orlicz–Sobolev space associated with the function Φ (see proposition 3.1).

In the present setting, an Ambrosetti–Rabinowitz-type condition takes the form

$$\liminf_{t\to\pm\infty} F(t) > 0 \tag{2.4}$$

and

$$\liminf_{t\to\pm\infty} \frac{tf(t)}{F(t)} > s_\Phi. \tag{2.5}$$

A decay assumption for f at 0 depends on Φ only through Φ_*, and reads

$$\lim_{t\to 0} \frac{tf(t)}{\Phi_*(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0. \tag{2.6}$$

Finally, the subcritical growth condition on f at ∞ to which we alluded above comes into play. The growth condition in question is only needed when

$$\int^{\infty} \left(\frac{\tau}{\Phi_*(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau = \infty, \tag{2.7}$$

and amounts to requiring that

$$\int_0 \left(\frac{\tau}{\Phi_*(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau < \infty \tag{2.8}$$

and

$$\lim_{t\to\pm\infty} \frac{tf(t)}{\Phi_n(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0. \tag{2.9}$$

If, on the contrary,

$$\int^{\infty} \left(\frac{\tau}{\Phi_*(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau < \infty, \tag{2.10}$$

a condition that ensures that any function from the Orlicz–Sobolev space associated with Φ is bounded, then no further condition on f near ∞ has to be imposed.

Theorem 2.1. *Let Ω be an open set in $\mathbb{R}^n$, with $n \geqslant 2$, such that $|\Omega| < \infty$. Assume that $\Phi \in C^1(\mathbb{R}^n)$ is an even, strictly convex function, vanishing at 0. Let f be a continuous function. Assume that conditions (2.3)–(2.6) are fulfilled, and that either (2.7) holds and (2.8)–(2.9) are in force, or (2.10) holds. Then the Dirichlet problem (1.1) admits a non-trivial, bounded, weak solution u.*

Theorem 2.1 is adapted to a few special instances, including (1.4), (1.7) and (1.9), in section 5. Section 3 is devoted to some preliminary results on anisotropic Orlicz–Sobolev spaces built upon n-dimensional Young functions. The proof of theorem 2.1 is then accomplished in section 4.

3. Anisotropic Orlicz–Sobolev spaces and the Nemytskii operator

A function $A\colon [0,\infty) \to [0,\infty]$ is called a Young function if it is convex, vanishes at 0, and is neither identically equal to 0, nor to ∞. Definitions and properties concerning Young functions, as well as n-dimensional Young functions, to be used in what follows, are collected in the appendix.

Let G be a measurable set in $\mathbb{R}^N$, with $N \geqslant 1$. The Orlicz space $L^A(G)$ is the set of all measurable functions $u\colon G \to \mathbb{R}$ such that the Luxemburg norm

$$\|u\|_{L^A(G)} = \inf\left\{\lambda > 0\colon \int_G A\left(\frac{1}{\lambda}|u|\right) \mathrm{d}x \leqslant 1\right\}$$

is finite. The functional $\|\cdot\|_{L^A(G)}$ is a norm on $L^A(G)$, which makes the latter a Banach space. If $|G| < \infty$ and $A \in \Delta_2$ near ∞, then $\int_G A(|u|)\,\mathrm{d}x < \infty$ for every $u \in L^A(G)$.

The Hölder-type inequality

$$\int_G |uv|\,\mathrm{d}x \leqslant 2\|u\|_{L^A(G)}\|v\|_{L^{\widetilde{A}}(G)} \tag{3.1}$$

holds for every $u \in L^A(G)$ and $v \in L^{\widetilde{A}}(G)$. Here, $\widetilde{A}$ denotes the Young conjugate of A.

Assume that $|G| < \infty$. Let A and B be Young functions such that A dominates B near ∞. Then

$$L^A(G) \to L^B(G),$$

where '$\to$' stands for continuous embedding. In particular,

$$L^A(G) \to L^1(G)$$

for any Young function A.

Orlicz spaces of $\mathbb{R}^n$-valued measurable functions are built upon n-dimensional Young functions. Let $n \geqslant 1$. A function $\Phi\colon \mathbb{R}^n \to [0,\infty]$ is called an n-dimensional Young function if it is convex, $\Phi(0) = 0$, $\Phi(\xi) = \Phi(-\xi)$ for $\xi \in \mathbb{R}^n$, and for every $t > 0$, the set $\{\xi \in \mathbb{R}^n : \Phi(\xi) < t\}$ is bounded and contains an open neighbourhood of 0.

The Orlicz space $L^\Phi(G,\mathbb{R}^n)$ is the set of all measurable functions $U\colon G \to \mathbb{R}^n$ such that the norm

$$\|U\|_{L^\Phi(G,\mathbb{R}^n)} = \inf\left\{\lambda > 0\colon \int_G \Phi\left(\frac{1}{\lambda}U\right) \mathrm{d}x \leqslant 1\right\}$$

is finite. The space $L^\Phi(G,\mathbb{R}^n)$, equipped with this norm, is a Banach space.

The Hölder-type inequality

$$\int_G |U \cdot V| \,\mathrm{d}x \leqslant 2\|U\|_{L^\Phi(G,\mathbb{R}^n)}\|V\|_{L^{\tilde{\Phi}}(G,\mathbb{R}^n)} \tag{3.2}$$

holds for every $U \in L^\Phi(G,\mathbb{R}^n)$ and $V \in L^{\widetilde{\Phi}}(G,\mathbb{R}^n)$, where $\widetilde{\Phi}$ denotes the Young conjugate of Φ.

Assume that $|G| < \infty$. If $\Phi \in \Delta_2$ near ∞, then $\int_G \Phi(U)\,\mathrm{d}x < \infty$ for every $U \in L^\Phi(G,\mathbb{R}^n)$. By [20, corollary 7.2],

$$L^\Phi(G,\mathbb{R}^n) \text{ is reflexive if and only if } \Phi \in \Delta_2 \cap \nabla_2 \text{ near } \infty. \tag{3.3}$$

If Φ and Ψ are n-dimensional Young functions such that Φ dominates Ψ near ∞, then

$$L^\Phi(G,\mathbb{R}^n) \to L^\Psi(G,\mathbb{R}^n).$$

In particular,

$$L^\Phi(G,\mathbb{R}^n) \to L^1(G,\mathbb{R}^n) \tag{3.4}$$

for any n-dimensional Young function Φ.

Now, let Ω be an open set in $\mathbb{R}^n$, $n \geqslant 2$, such that $|\Omega| < \infty$. Given an n-dimensional Young function Φ, the anisotropic Orlicz–Sobolev space $W^{1,\Phi}_0(\Omega)$ is defined as

$$W^{1,\Phi}_0(\Omega) = \{u\colon \Omega \to \mathbb{R}\colon \text{the continuation of } u \text{ by 0 outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n, \text{ and } \nabla u \in L^\Phi(\Omega,\mathbb{R}^n)\}.$$

The isotropic Orlicz–Sobolev space $W^{1,A}_0(\Omega)$ associated with a Young function A is defined analogously on requiring that $|\nabla u| \in L^A(\Omega)$.

One has that $W^{1,\Phi}_0(\Omega)$, equipped with the norm

$$\|u\|_{W^{1,\Phi}_0(\Omega)} = \|\nabla u\|_{L^\Phi(\Omega,\mathbb{R}^n)},$$

is a Banach space. A proof of this fact relies upon standard properties of weak derivatives, and of n-dimensional Young functions.

PROPOSITION 3.1. *Let Φ be an n-dimensional Young function such that $\Phi \in \Delta_2 \cap \nabla_2$ near ∞. Then the Orlicz–Sobolev space $W^{1,\Phi}_0(\Omega)$ is reflexive.*

Proof. This is a consequence of property (3.3), and of the fact that $W^{1,\Phi}_0(\Omega)$ is isometrically isomorphic to a closed subspace of the Orlicz spaces $L^\Phi(\Omega,\mathbb{R}^n)$ via the map

$$W^{1,\Phi}_0(\Omega) \ni u \mapsto (u_{x_1},\dots,u_{x_n}) \in L^\Phi(\Omega,\mathbb{R}^n).$$

□

An anisotropic Poincaré-type inequality for functions in $W^{1,\Phi}_0(\Omega)$ is stated in the next proposition.

PROPOSITION 3.2. *Let Φ be an n-dimensional Young function and let Φ_* be the Young function given by (2.1). Then*

$$\int_\Omega \Phi_*(|u|)\,\mathrm{d}x \leqslant \int_\Omega \Phi(\omega_n^{-1/n}|\Omega|^{1/n}\nabla u)\,\mathrm{d}x, \tag{3.5}$$

and

$$\|u\|_{L^{\Phi_*}(\Omega)} \leqslant \omega_n^{-1/n}|\Omega|^{1/n}\|\nabla u\|_{L^\Phi(\Omega,\mathbb{R}^n)} \tag{3.6}$$

for every $u \in W^{1,\Phi}_0(\Omega)$. Here, $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$, the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Proof. Let us call $\Omega^\bigstar$ the open ball, centred at 0, with the same measure as Ω. Given any function $u \in W^{1,\Phi}_0(\Omega)$, denote by $u^\bigstar\colon \Omega^\bigstar \to [0,\infty)$ the spherical symmetral of u, namely, the radially decreasing function equimeasurable with u. An anisotropic version of the Polyá–Szegö principle tells us that $u^\bigstar \in W^{1,\Phi_*}_0(\Omega^\bigstar)$, and

$$\int_\Omega \Phi(\nabla u)\,\mathrm{d}x \geqslant \int_{\Omega^\bigstar} \Phi_*(|\nabla u^\bigstar|)\,\mathrm{d}x. \tag{3.7}$$

Inequality (3.7) is stated in [12]; a full proof can be found in [7, theorem 3.5]. On the other hand, an isotropic Poincaré-type inequality ensures that

$$\int_{\Omega^\bigstar} \Phi_*(|\nabla u^\bigstar|)\,\mathrm{d}x \geqslant \int_{\Omega^\bigstar} \Phi_*(\omega_n^{1/n}|\Omega|^{-1/n}u^\bigstar)\,\mathrm{d}x; \tag{3.8}$$

see [24, lemma 3]. Finally, since u and $u^\bigstar$ are equimeasurable,

$$\int_{\Omega^\bigstar} \Phi_*(\omega_n^{1/n}|\Omega|^{-1/n}u^\bigstar)\,\mathrm{d}x = \int_\Omega \Phi_*(\omega_n^{1/n}|\Omega|^{-1/n}|u|)\,\mathrm{d}x. \tag{3.9}$$

Inequality (3.5) is a consequence of (3.7)–(3.9). Inequality (3.6) follows on applying (3.5) with u replaced with $u/\|\nabla u\|_{L^\Phi(\Omega,\mathbb{R}^n)}$, via the very definition of the Luxemburg norm. □

A Sobolev–Poincaré inequality, with optimal Orlicz target norm, reads as follows. Assume that Φ is an n-dimensional Young function fulfilling (2.8), and let Φ_n be the Sobolev conjugate of Φ defined as in (2.2). By [6, theorem 1 and remark 1], there exists a constant $C = C(n)$ such that

$$\int_\Omega \Phi_n\left(\frac{|u|}{C(\int_\Omega \Phi(\nabla u)\,\mathrm{d}y)^{1/n}}\right)\mathrm{d}x \leqslant \int_\Omega \Phi(\nabla u)\,\mathrm{d}x \tag{3.10}$$

and

$$\|u\|_{L^{\Phi_n}(\Omega)} \leqslant C\|u\|_{W_0^{1,\Phi}(\Omega)} \tag{3.11}$$

for every $u \in W_0^{1,\Phi}(\Omega)$. Moreover, $L^{\Phi_n}(\Omega)$ is the optimal, i.e. smallest possible, Orlicz space that renders (3.11) true for all n-dimensional Young functions Φ with prescribed Φ_*.

In particular, if (2.10) holds, then $\Phi_n(t) = \infty$ for large t, and (3.11) yields

$$\|u\|_{L^\infty(\Omega)} \leqslant C\|u\|_{W_0^{1,\Phi}(\Omega)} \tag{3.12}$$

for every $u \in W_0^{1,\Phi}(\Omega)$.

Remark 3.3. Since we are assuming that $|\Omega| < \infty$, (3.11) and (3.12) continue to hold even if (2.8) fails, provided that Φ_n is defined with Φ_* replaced by another Young function that is equivalent near ∞, which renders (2.8) true. We shall adopt the convention that Φ_n is defined according to this procedure in what follows, whenever needed.

Let us notice that, under assumption (2.7),

$$\int_\Omega \Phi_n(c|u|)\,\mathrm{d}x < \infty \tag{3.13}$$

for every $u \in W_0^{1,\Phi}(\Omega)$ and every $c \geqslant 0$. This fact can be shown to follow from (3.10).

The embedding

$$W_0^{1,\Phi}(\Omega) \to L^1(\Omega)$$

is compact for any n-dimensional Young function Φ. Indeed, by (3.4), $W_0^{1,\Phi}(\Omega) \to W_0^{1,1}(\Omega)$, and the embedding $W_0^{1,1}(\Omega) \to L^1(\Omega)$ is compact.

We denote by $(W_0^{1,\Phi}(\Omega))^*$ the topological dual of $W_0^{1,\Phi}(\Omega)$, and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $((W_0^{1,\Phi}(\Omega))^*, W_0^{1,\Phi}(\Omega))$.

In proposition 3.6 we analyse properties of the Nemytskii operator, associated with a continuous function $f\colon \mathbb{R} \to \mathbb{R}$, in the anisotropic Orlicz–Sobolev space $W_0^{1,\Phi}(\Omega)$. In preparation for this, we need a few technical results contained in lemmas 3.4 and 3.5.

Let F be defined as in (1.3). We introduce the auxiliary functions $\overline{f}\colon \mathbb{R} \to [0,\infty)$, defined as

$$\overline{f}(t) = \max_{s\in[-|t|,|t|]} |f(s)| \quad \text{for } t \in \mathbb{R}, \tag{3.14}$$

and $\overline{F}\colon [0,\infty) \to [0,\infty)$, defined as

$$\overline{F}(t) = \int_0^t \overline{f}(\tau)\,\mathrm{d}\tau \quad \text{for } t \in [0,\infty). \tag{3.15}$$

Note that $\overline{f}$ is even, and is non-decreasing in $[0,\infty)$, and hence $\overline{F}$ is a Young function.

LEMMA 3.4. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuous function. Then*

$$|f(t)| \leqslant \frac{\overline{F}(2|t|)}{|t|} \quad \textit{for} \ \ t \neq 0, \tag{3.16}$$

and

$$|f(t)| \leqslant 2\widetilde{\overline{F}}^{-1}(\overline{F}(2|t|)) \quad \textit{for} \ \ t \in \mathbb{R}. \tag{3.17}$$

Proof. We have that

$$\overline{F}(2|t|) = \int_0^{2|t|} \overline{f}(\tau)\,\mathrm{d}\tau \geqslant \int_{|t|}^{2|t|} \overline{f}(\tau)\,\mathrm{d}\tau \geqslant \overline{f}(|t|)|t| \geqslant |f(t)|\,|t| \quad \text{for } t\in\mathbb{R},$$

namely, (3.16) holds. Inequality (3.17) follows from (3.16), via (A 2). □

LEMMA 3.5. *Assume that $|G|<\infty$. Let B and E be Young functions such that E increases essentially more slowly than B near ∞. Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function such that*

$$|f(t)| \leqslant c(1+\widetilde{E}^{-1}(E(c|t|))) \quad \textit{for } t\in\mathbb{R}, \tag{3.18}$$

and some constant $c>0$.

(i) *Let $u\in L^E(G)$ and let $\{u_k\}$ be a bounded sequence in $L^B(G)$ such that $u_k\to u$ in $L^E(G)$. Then*

$$\lim_{k\to\infty}\int_G f(u_k)(u_k-u)\,\mathrm{d}x = 0. \tag{3.19}$$

(ii) *Assume that $u\in L^B(G)$ and $\{u_k\}\subset L^B(G)$. If $u_k\to u$ in $L^B(G)$, then*

$$\lim_{k\to\infty}\|f(u_k)-f(u)\|_{L^{\widetilde{B}}(G)} = 0.$$

Proof. (i) By (3.18) and (3.2),

$$\begin{aligned}\left|\int_G f(u_k)(u_k-u)\,\mathrm{d}x\right| &\\ &\leqslant c\biggl(\|u_k-u\|_{L^1(G)} + \int_G \widetilde{E}^{-1}(E(c|u_k|))(u_k-u)\,\mathrm{d}x\biggr)\\ &\leqslant c(\|u_k-u\|_{L^1(G)} + 2\|\widetilde{E}^{-1}(E(c|u_k|))\|_{L^{\widetilde{E}}(G)}\|u_k-u\|_{L^E(G)}).\end{aligned} \tag{3.20}$$

Since the sequence $\{u_k\}$ is bounded in $L^B(G)$, and E increases essentially more slowly than B near ∞, there exists a constant $c'>0$ such that

$$\int_G \widetilde{E}(\widetilde{E}^{-1}(E(c|u_k|)))\,\mathrm{d}x = \int_G E(c|u_k|)\,\mathrm{d}x \leqslant c'$$

for $k\in\mathbb{N}$. Hence, by property (A 1), applied with $A=\widetilde{E}$,

$$1 \geqslant \frac{1}{\max\{1,c'\}}\int_G \widetilde{E}(\widetilde{E}^{-1}(E(c|u_k|)))\,\mathrm{d}x \geqslant \int_G \widetilde{E}\biggl(\frac{\widetilde{E}^{-1}(E(c|u_k|))}{\max\{1,c'\}}\biggr)\,\mathrm{d}x,$$

namely,

$$\|\widetilde{E}^{-1}(E(c|u_k|))\|_{L^{\widetilde{E}}(G)} \leqslant \max\{1, c'\}. \tag{3.21}$$

Since $u_k \to u$ in $L^E(G)$, and hence also in $L^1(G)$, (3.19) follows from (3.20) and (3.21).

(ii) By the definition of the Luxemburg norm it suffices to show that

$$\lim_{k\to\infty} \int_G \widetilde{B}\bigg(\frac{|f(u_k) - f(u)|}{\lambda}\bigg)\,\mathrm{d}x = 0 \quad \text{for every } \lambda > 0. \tag{3.22}$$

Since $u_k \to u$ in $L^B(G)$, there exists a subsequence, still denoted by $\{u_k\}$, and a function $v \in L^B(G)$ such that $u_k \to u$ almost everywhere in G, and $|u_k(x)| \leqslant v(x)$ for almost every (a.e.) $x \in G$, for every $k \in \mathbb{N}$. A proof of this fact follows along the same lines as in the classical special case when $L^B(G)$ is a Lebesgue space. The Fatou-type property of the norm $\|\cdot\|_{L^B(G)}$, which tells us that $\|w_k\|_{L^B(G)} \nearrow \|w\|_{L^B(G)}$ if $\{w_k\}$ is any sequence such that $0 \leqslant w_k \nearrow w$, plays a role here.

Hence,

$$\lim_{k\to\infty} f(u_k(x)) = f(u(x)) \quad \text{for a.e. } x \in G.$$

Equation (3.22) will thus follow if we prove that, for every $\lambda > 0$, there exists a function $v_\lambda \in L^1(G)$ such that

$$\widetilde{B}\bigg(\frac{|f(u_k(x)) - f(u(x))|}{\lambda}\bigg) \leqslant v_\lambda(x) \quad \text{for a.e. } x \in G, \tag{3.23}$$

for $k \in \mathbb{N}$. By the convexity of $\widetilde{B}$, and (3.18),

$$\begin{aligned}
&\widetilde{B}\bigg(\frac{|f(u_k(x)) - f(u(x))|}{\lambda}\bigg) \\
&\quad\leqslant \widetilde{B}\bigg(\frac{2|f(u_k(x))|}{\lambda}\bigg) + \widetilde{B}\bigg(\frac{2|f(u(x))|}{\lambda}\bigg) \\
&\quad\leqslant \widetilde{B}\bigg(\frac{2c}{\lambda} + \frac{2c}{\lambda}\widetilde{E}^{-1}(E(c|u_k(x)|))\bigg) + \widetilde{B}\bigg(\frac{2c}{\lambda} + \frac{2c}{\lambda}\widetilde{E}^{-1}(E(c|u(x)|))\bigg) \\
&\quad\leqslant \widetilde{B}\bigg(\frac{4c}{\lambda}\bigg) + \frac{1}{2}\bigg[\widetilde{B}\bigg(\frac{4c}{\lambda}\widetilde{E}^{-1}(E(c|u_k(x)|))\bigg) + \widetilde{B}\bigg(\frac{4c}{\lambda}\widetilde{E}^{-1}(E(c|u(x)|))\bigg)\bigg]
\end{aligned} \tag{3.24}$$

for a.e. $x \in G$, and for $k \in \mathbb{N}$. Thanks to (A 6), there exists $t_0 \geqslant 0$ such that

$$\widetilde{E}^{-1}(E(ct)) \leqslant \frac{\lambda}{4c}\widetilde{B}^{-1}(E(ct)) \quad \text{if } t > t_0.$$

Hence,

$$\begin{aligned}
\widetilde{B}\bigg(\frac{4c}{\lambda}\widetilde{E}^{-1}(E(c|u_k(x)|))\bigg) &\leqslant \widetilde{B}\bigg(\frac{4c}{\lambda}\widetilde{E}^{-1}(E(ct_0))\bigg) + \widetilde{B}\bigg(\frac{4c}{\lambda}\frac{\lambda}{4c}\widetilde{B}^{-1}(E(c|u_k(x)|))\bigg) \\
&= \widetilde{B}\bigg(\frac{4c}{\lambda}\widetilde{E}^{-1}(E(ct_0))\bigg) + E(c|u_k(x)|) \quad \text{for a.e. } x \in G.
\end{aligned}$$

An analogous estimate holds with u_k replaced by u. Altogether, from (3.24) we infer that

$$\widetilde{B}\left(\frac{|f(u_k(x)) - f(u(x))|}{\lambda}\right) \leqslant \widetilde{B}\left(\frac{4c}{\lambda}\right) + \widetilde{B}\left(\frac{4c}{\lambda}\widetilde{E}^{-1}(E(ct_0))\right) + \tfrac{1}{2}[E(cv(x)) + E(c|u(x)|)] \quad \text{for a.e. } x \in G, \tag{3.25}$$

for $k \in N$. Since $u, v \in L^B(G)$, and E increases essentially more slowly than B near ∞, the right-hand side of (3.25) is an integrable function in G. Inequality (3.23) follows. The proof is complete. □

PROPOSITION 3.6. *Let Φ be an n-dimensional Young function and let Φ_n be its Sobolev conjugate defined by (2.2) (according to the convention of remark 3.3). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function.*

(i) *Assume that Φ fulfils (2.7), and*

$$\limsup_{t\to\pm\infty} \frac{|tf(t)|}{\Phi_n(\lambda|t|)} < \infty \quad \textit{for some } \lambda > 0. \tag{3.26}$$

Then the operator $\mathcal{N}_f\colon W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$, given by

$$\langle \mathcal{N}_f(u), v\rangle = \int_\Omega f(u)v\,\mathrm{d}x$$

for $u, v \in W_0^{1,\Phi}(\Omega)$, is well defined.

Moreover, if (3.26) is strengthened by assuming (2.9), then the operator $\mathcal{N}_f$ is continuous.

(ii) *Assume that Φ fulfils (2.10). Then the operator $\mathcal{N}_f$ is well defined and continuous.*

Proof. (i) Assumption (3.26) implies that (and is in fact equivalent to)

$$|f(t)| \leqslant c\left(1 + \frac{\Phi_n(c|t|)}{|t|}\right) \quad \text{for } t \neq 0, \tag{3.27}$$

for some constant $c > 0$. We begin by proving that $f(u) \in L^{\widetilde{\Phi}_n}(\Omega)$ for every $u \in W_0^{1,\Phi}(\Omega)$. Here, $\widetilde{\Phi}_n$ stands for the Young conjugate of Φ_n. By (3.27), the convexity of $\widetilde{\Phi}_n$, and (A 2),

$$\begin{aligned}\widetilde{\Phi}_n\left(\frac{|f(u)|}{2c^2}\right) &\leqslant \widetilde{\Phi}_n\left(\frac{1}{2c}\left(1 + \frac{\Phi_n(c|u|)}{|u|}\right)\right)\\ &\leqslant \frac{1}{2}\widetilde{\Phi}_n\left(\frac{1}{c}\right) + \frac{1}{2}\widetilde{\Phi}_n\left(\frac{\Phi_n(c|u|)}{c|u|}\right)\\ &\leqslant \frac{1}{2}\widetilde{\Phi}_n\left(\frac{1}{c}\right) + \frac{1}{2}\widetilde{\Phi}_n(\widetilde{\Phi}_n^{-1}(\Phi_n(c|u|)))\\ &= \frac{1}{2}\widetilde{\Phi}_n\left(\frac{1}{c}\right) + \frac{1}{2}\Phi_n(c|u|) \quad \text{a.e. in } \Omega.\end{aligned} \tag{3.28}$$

From (3.28) and (3.13), we deduce that

$$\int_\Omega \widetilde{\Phi}_n\left(\frac{f(u)}{2c^2}\right)\mathrm{d}x < \infty,$$

whence $f(u) \in L^{\widetilde{\Phi}_n}(\Omega)$. Therefore, owing to (3.1) and (3.11), there exists a constant C such that

$$\begin{aligned}\int_\Omega |f(u)v|\,\mathrm{d}x &\leqslant 2\|f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)}\|v\|_{L^{\Phi_n}(\Omega)}\\ &\leqslant 2C\|f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)}\|v\|_{W_0^{1,\Phi}(\Omega)}\end{aligned}$$

for every $u, v \in W_0^{1,\Phi}(\Omega)$. This shows that $\mathcal{N}_f\colon W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ is well defined.

Assume now that (2.9) is fulfilled. In order to prove the continuity of $\mathcal{N}_f$, consider any function $u \in W_0^{1,\Phi}(\Omega)$ and any sequence $\{u_k\} \subset W_0^{1,\Phi}(\Omega)$ such that $u_k \to u$ in $W_0^{1,\Phi}(\Omega)$. By (3.1) and the Sobolev inequality (3.11), there exists a constant C such that

$$\begin{aligned}\left|\int_\Omega (f(u_k) - f(u))v\,\mathrm{d}x\right| &\leqslant 2\|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)}\|v\|_{L^{\Phi_n}(\Omega)}\\ &\leqslant C\|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)}\|v\|_{W_0^{1,\Phi}(\Omega)}\end{aligned}$$

for every $v \in W_0^{1,\Phi}(\Omega)$, and every $k \in \mathbb{N}$. Hence,

$$\begin{aligned}\|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{(W_0^{1,\Phi}(\Omega))^*} &= \sup_{\|v\|_{W_0^{1,\Phi}(\Omega)}\leqslant 1} |\langle \mathcal{N}_f(u_k), v\rangle - \langle \mathcal{N}_f(u), v\rangle|\\ &\leqslant C\|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)}. \qquad (3.29)\end{aligned}$$

On the other hand, the Sobolev inequality (3.11) again implies that $u_k \to u$ in $L^{\Phi_n}(\Omega)$. Assumption (2.9), via inequality (3.17), allows us to apply lemma 3.5(ii), with $c = 2$, $E = \overline{F}$, $B = \Phi_n$, and deduce that

$$\lim_{k\to\infty} \|f(u_k) - f(u)\|_{L^{\tilde{\Phi}_n}(\Omega)} = 0. \qquad (3.30)$$

The conclusion follows from (3.29) and (3.30).

(ii) The Sobolev inequality (3.12) holds. Thus, given any $u, v \in W_0^{1,\Phi}(\Omega)$, we have that $u, v \in L^\infty(\Omega)$, and, in particular, $f(u) \in L^1(\Omega)$. Consequently, by (3.12) again, there exists a constant C such that

$$\begin{aligned}\int_\Omega |f(u)v|\,\mathrm{d}x &\leqslant \|f(u)\|_{L^1(\Omega)}\|v\|_{L^\infty(\Omega)}\\ &\leqslant C\|f(u)\|_{L^1(\Omega)}\|v\|_{W_0^{1,\Phi}(\Omega)}\end{aligned}$$

for every $u, v \in W_0^{1,\Phi}(\Omega)$. Hence, $\mathcal{N}_f\colon W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ is well defined.

As for the continuity of $\mathcal{N}_f$, if $u \in W_0^{1,\Phi}(\Omega)$ and $\{u_k\} \subset W_0^{1,\Phi}(\Omega)$ are such that $u_k \to u$ in $W_0^{1,\Phi}(\Omega)$, then, by (3.12), $u_k \to u$ in $L^\infty(\Omega)$, and hence $f(u_k) \to f(u)$

in $L^1(\Omega)$. Therefore,

$$\begin{aligned}\|\mathcal{N}_f(u_k) - \mathcal{N}_f(u)\|_{(W_0^{1,\Phi}(\Omega))^*} &= \sup_{\|v\|_{W_0^{1,\Phi}(\Omega)} \leqslant 1} |\langle \mathcal{N}_f(u_k), v\rangle - \langle \mathcal{N}_f(u), v\rangle| \\ &\leqslant C\|f(u_k) - f(u)\|_{L^1(\Omega)},\end{aligned}$$

whence the conclusion follows. □

4. Proof of theorem 2.1

Assume throughout that Ω is an open set in $\mathbb{R}^n$, with $n \geqslant 2$, such that $|\Omega| < \infty$. Let Φ be an n-dimensional Young function, and let f be any continuous function such that $f(u)\varphi \in L^1(\Omega)$ for every $u, \varphi \in W_0^{1,\Phi}(\Omega)$. A function $u \in W_0^{1,\Phi}(\Omega)$ will be called a weak solution to problem (1.1) if

$$\int_\Omega \Phi_\xi(\nabla u) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f(u)\varphi \,\mathrm{d}x \tag{4.1}$$

for every test function $\varphi \in W_0^{1,\Phi}(\Omega)$.

The energy functional associated with (1.1) is the functional $J_\Phi \colon W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by (1.2). Any critical point of J_Φ satisfies (4.1), and is hence a solution to (1.1). In order to establish theorem 2.1 it will thus suffice to show that J_Φ has a non-trivial critical point. To this end, we shall make use of a version of the mountain pass theorem, stated below, for functionals defined on a Banach space X, and satisfying the Palais–Smale condition. Recall that a functional $I \colon X \to \mathbb{R}$ is said to satisfy the Palais–Smale condition if

$$\begin{aligned}&\text{any sequence } \{u_k\} \subset X \text{ such that } \{I(u_k)\} \text{ is bounded} \\ &\qquad \text{and } \lim_{k\to\infty} \|I'(u_k)\|_{X^*} = 0 \text{ has a convergent subsequence in } X.\end{aligned} \tag{4.2}$$

A sequence $\{u_k\}$ as in (4.2) will be called a Palais–Smale sequence for the functional I.

Mountain pass theorem [3]. *Let X be a real Banach space. Assume that the functional $I \colon X \to \mathbb{R}$ is of class C^1, satisfies the Palais–Smale condition (4.2), and fulfils the following properties:*

$$I(0) = 0, \tag{4.3}$$

$$\textit{there exist } \rho, \sigma > 0 \textit{ such that } \inf_{\|u\|_X = \rho} I(u) \geqslant \sigma, \tag{4.4}$$

$$\textit{there exists } \overline{u} \in X \textit{ such that } \|\overline{u}\|_X > \rho \textit{ and } I(\overline{u}) \leqslant 0. \tag{4.5}$$

Then I has a critical point u such that $I(u) = c \geqslant \sigma$, where

$$c = \inf_{\gamma\in\mathcal{G}} \max_{s\in[0,1]} I(\gamma(s))$$

and

$$\mathcal{G} = \{\gamma \in C^0([0,1], X) \colon \gamma(0) = 0,\ \gamma(1) = \overline{u}\}.$$

The continuous differentiability of the functional J_Φ is the subject of the following result.

Proposition 4.1. *Let $\Phi \in C^1(\mathbb{R}^n)$ be a strictly convex n-dimensional Young function satisfying (2.3). Let Φ_n be its Sobolev conjugate defined by (2.2) (according to the convention of remark 3.3). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that either (2.7) and (2.9) hold, or (2.10) holds. Then the functional J_Φ, defined by (1.2), is of class C^1.*

Proposition 4.1 is a consequence of the next two propositions.

Proposition 4.2. *Assume that $\Phi \in C^1(\mathbb{R}^n)$ is a strictly convex n-dimensional Young function satisfying (2.3). Then the functional $I_\Phi\colon W_0^{1,\Phi}(\Omega) \to \mathbb{R}$, defined as*

$$I_\Phi(u) = \int_\Omega \Phi(\nabla u)\,\mathrm{d}x$$

for $u \in W_0^{1,\Phi}(\Omega)$, is of class C^1.

Proof. It suffices to show that I_Φ is Gâteaux differentiable, and that its Gâteaux derivative $(I_\Phi)'_G$ is continuous. Let $u, \varphi \in W_0^{1,\Phi}(\Omega)$, and let $\mu \in (0,1)$. Since $\Phi \in C^1(\mathbb{R}^n)$,

$$\lim_{\mu \to 0^+} \Phi_\xi(\nabla u(x) + \mu \nabla\varphi(x)) \cdot \nabla\varphi(x) = \Phi_\xi(\nabla u(x)) \cdot \nabla\varphi(x) \quad \text{for a.e. } x \in \Omega. \tag{4.6}$$

Moreover, for a.e. $x \in \Omega$ there exists $\rho_{\mu,x} \in (0,1)$ such that

$$\frac{\Phi(\nabla u(x) + \mu\nabla\varphi(x)) - \Phi(\nabla u(x))}{\mu} = \Phi_\xi(\nabla u(x) + \mu\rho_{\mu,x}\nabla\varphi(x)) \cdot \nabla\varphi(x). \tag{4.7}$$

On the other hand, by (A 14), (A 23), the convexity of Φ and the fact that $\mu\rho_{\mu,x} \in (0,1)$, we deduce that

$$\begin{aligned}
&|\Phi_\xi(\nabla u(x) + \mu\rho_{\mu,x}\nabla\varphi(x)) \cdot \nabla\varphi(x)| \\
&\qquad \leqslant \widetilde{\Phi}(\Phi_\xi(\nabla u(x) + \mu\rho_{\mu,x}\nabla\varphi(x))) + \Phi(\nabla\varphi(x)) \\
&\qquad \leqslant \Phi(2\nabla u(x) + 2\mu\rho_{\mu,x}\nabla\varphi(x)) + \Phi(\nabla\varphi(x)) \\
&\qquad \leqslant \tfrac{1}{2}\Phi(4\nabla u(x)) + \tfrac{1}{2}\Phi(4\nabla\varphi(x)) + \Phi(\nabla\varphi(x)) \quad \text{for a.e. } x \in \Omega.
\end{aligned} \tag{4.8}$$

Since $\Phi \in \Delta_2$ near ∞, by (4.8) the right-hand side of (4.7) belongs to $L^1(\Omega)$. From (4.6) and (4.7), via the dominated convergence theorem, we obtain that

$$\langle (I_\Phi)'_G(u), \varphi \rangle = \lim_{\mu \to 0^+} \int_\Omega \frac{\Phi(\nabla u + \mu\nabla\varphi) - \Phi(\nabla u)}{\mu}\,\mathrm{d}x = \int_\Omega \Phi_\xi(\nabla u) \cdot \nabla\varphi\,\mathrm{d}x$$

for $u, \varphi \in W_0^{1,\Phi}(\Omega)$.

We next show that the operator $(I_\Phi)'_G\colon W_0^{1,\Phi}(\Omega) \to (W_0^{1,\Phi}(\Omega))^*$ is continuous. Let $\{u_k\}$ be any sequence in $W_0^{1,\Phi}(\Omega)$, converging to some function $u \in W_0^{1,\Phi}(\Omega)$. Then $\|\nabla u_k - \nabla u\|_{L^\Phi(\Omega)} \to 0$ as $k \to \infty$, and hence

$$\lim_{k \to \infty} \int_\Omega \Phi(\lambda(\nabla u_k - \nabla u))\,\mathrm{d}x = 0 \quad \text{for every } \lambda > 0. \tag{4.9}$$

Moreover, on passing, if necessary, to a subsequence, still denoted by $\{u_k\}$, we have that $\nabla u_k \to \nabla u$ almost everywhere in Ω. Hence,

$$\Phi(\nabla u_k) \to \Phi(\nabla u) \quad \text{and} \quad \Phi_\xi(\nabla u_k) \to \Phi_\xi(\nabla u) \quad \text{almost everywhere in } \Omega. \tag{4.10}$$

We have that

$$0 \leqslant \Phi(2\nabla u_k(x)) \leqslant \tfrac{1}{2}\Phi(4(\nabla u_k(x) - \nabla u(x))) + \tfrac{1}{2}\Phi(4\nabla u(x)) \quad \text{for a.e. } x \text{ in } \Omega, \tag{4.11}$$

for $k \in \mathbb{N}$. Equation (4.9) ensures that there exists $w \in L^1(\Omega)$ such that $\Phi(4(\nabla u_k - \nabla u)) \leqslant w$ almost everywhere in Ω, for $k \in \mathbb{N}$. Thus, inequality (4.11) implies that

$$\Phi(2\nabla u_k(x)) \leqslant \frac{w(x) + \Phi(4\nabla u(x))}{2} \quad \text{for a.e. } x \text{ in } \Omega, \tag{4.12}$$

the right-hand side of (4.12) being a function in $L^1(\Omega)$. By (2.3) and proposition A.5(ii), one has that $\Phi \in \nabla_2$ near ∞. This property is easily seen to imply that $\lim_{|\xi|\to\infty} \Phi(\xi)/|\xi| = \infty$. Proposition A.6 then yields $\widetilde{\Phi} \in \Delta_2$ near ∞. Consequently, there exist constants $C > 2$ and $M \geqslant 0$ such that

$$\widetilde{\Phi}(2\eta) \leqslant C\widetilde{\Phi}(\eta) \quad \text{if } |\eta| > M. \tag{4.13}$$

Finally, on making use of the fact that $\widetilde{\Phi}$ is an even convex function, and of (4.13), (A 23) and (4.12), one obtains that

$$\begin{aligned}
\widetilde{\Phi}(&\Phi_\xi(\nabla u_k(x)) - \Phi_\xi(\nabla u(x))) \\
&\leqslant \frac{\widetilde{\Phi}(2\Phi_\xi(\nabla u_k(x)))}{2} + \frac{\widetilde{\Phi}(-2\Phi_\xi(\nabla u(x)))}{2} \\
&\leqslant \max_{|\eta|\leqslant M} \widetilde{\Phi}(2\eta) + \tfrac{1}{2}C[\widetilde{\Phi}(\Phi_\xi(\nabla u_k(x))) + \widetilde{\Phi}(\Phi_\xi(\nabla u(x)))] \\
&\leqslant \max_{|\eta|\leqslant M} \widetilde{\Phi}(2\eta) + \tfrac{1}{2}C[\Phi(2\nabla u_k(x)) + \Phi(2\nabla u(x))] \\
&\leqslant \max_{|\eta|\leqslant M} \widetilde{\Phi}(2\eta) + \frac{C}{2}\left[\frac{w(x) + \Phi(4\nabla u(x))}{2} + \Phi(2\nabla u(x))\right] \quad \text{for a.e. } x \in \Omega.
\end{aligned} \tag{4.14}$$

Owing to (4.10) and (4.14), via the dominated convergence theorem one deduces that

$$\lim_{k\to\infty} \int_\Omega \widetilde{\Phi}(\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u))\,\mathrm{d}x = 0.$$

Since $\widetilde{\Phi} \in \Delta_2$ near ∞, this also implies that

$$\lim_{k\to\infty} \|\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega,\mathbb{R}^n)} = 0. \tag{4.15}$$

Clearly, the above argument applies to any subsequence of $\{u_k\}$. This ensures that (4.15) holds, in fact, for the whole sequence $\{u_k\}$.

Now, let $\varphi \in W_0^{1,\Phi}(\Omega)$. Thanks to (3.2),

$$
\begin{aligned}
|\langle (I_\Phi)'_G(u_k) - (I_\Phi)'_G(u), \varphi \rangle| &= \left| \int_\Omega (\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)) \cdot \nabla \varphi \, \mathrm{d}x \right| \\
&\leqslant 2 \|\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega,\mathbb{R}^n)} \|\nabla \varphi\|_{L^{\Phi}(\Omega,\mathbb{R}^n)}.
\end{aligned}
$$

Thereby, from (4.15) we infer that

$$
\lim_{k\to\infty} \|(I_\Phi)'_G(u_k) - (I_\Phi)'_G(u)\|_{(W_0^{1,\Phi}(\Omega))^*} \leqslant 2 \lim_{k\to\infty} \|\Phi_\xi(\nabla u_k) - \Phi_\xi(\nabla u)\|_{L^{\tilde{\Phi}}(\Omega,\mathbb{R}^n)} = 0.
$$

The continuity of $(I_\Phi)'_G$ is thus established. □

Proposition 4.3. *Let Φ be an n-dimensional Young function, and let Φ_n be its Sobolev conjugate defined by (2.2) (according to the convention of remark 3.3). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that either (2.7) and (2.9) hold, or (2.10) holds. Then the functional $L_f\colon W_0^{1,\Phi}(\Omega) \to \mathbb{R}$, defined by*

$$
L_f(u) = \int_\Omega F(u) \, \mathrm{d}x
$$

for $u \in W_0^{1,\Phi}(\Omega)$, is of class C^1.

The following lemma is needed in the proof of proposition 4.3. Its proof makes use of calculus arguments, and will be omitted for brevity.

Lemma 4.4. *Let B be a Young function and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Let F, $\overline{f}$ and $\overline{F}$ be the functions associated with f as in (1.3), (3.14) and (3.15), respectively.*

(i) *If*

$$
\lim_{t\to 0} \frac{tf(t)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0, \tag{4.16}
$$

then

$$
\lim_{t\to 0} \frac{t\overline{f}(t)}{B(\lambda|t|)} = \lim_{t\to 0} \frac{F(t)}{B(\lambda|t|)} = \lim_{t\to 0} \frac{\overline{F}(|t|)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0. \tag{4.17}
$$

(ii) *If*

$$
\lim_{t\to\infty} \frac{B(t)}{t} = \infty \tag{4.18}
$$

and

$$
\lim_{t\to\pm\infty} \frac{tf(t)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0, \tag{4.19}
$$

then

$$
\lim_{t\to\pm\infty} \frac{t\overline{f}(t)}{B(\lambda|t|)} = \lim_{t\to\pm\infty} \frac{F(t)}{B(\lambda|t|)} = \lim_{t\to\pm\infty} \frac{\overline{F}(|t|)}{B(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0. \tag{4.20}
$$

In particular, the function $\overline{F}$ increases essentially more slowly than B near ∞.

Proof of proposition 4.3. We shall demonstrate that L_f is Gâteaux differentiable in $W_0^{1,\Phi}(\Omega)$ and that its Gâteaux derivative $(L_f)'_G$ is continuous. To this end, fix any $u, \varphi \in W_0^{1,\Phi}(\Omega)$ and let $\mu \in (0,1)$. By the continuity of f,

$$\lim_{\mu\to 0} \frac{F(u(x)+\mu\varphi(x)) - F(u(x))}{\mu} = f(u(x))\varphi(x) \quad \text{for a.e. } x \in \Omega. \tag{4.21}$$

Moreover, for a.e. $x \in \Omega$, there exists $\theta_{\mu,x} \in (0,1)$ such that

$$\frac{F(u(x)+\mu\varphi(x)) - F(u(x))}{\mu} = f(u(x)+\mu\theta_{\mu,x}\varphi(x))\varphi(x). \tag{4.22}$$

By (3.16), and the fact that the function $(0,\infty) \mapsto \overline{F}(2s)/s$ is non-decreasing, we obtain that

$$\begin{aligned} |f(u(x)+\mu\theta_{\mu,x}\varphi(x))\varphi(x)| &\leqslant \frac{\overline{F}(2(|u(x)+\mu\theta_{\mu,x}\varphi(x)|))}{|u(x)+\mu\theta_{\mu,x}\varphi(x)|}|\varphi(x)| \\ &\leqslant \frac{\overline{F}(2(|u(x)|+|\varphi(x)|))}{|u(x)|+|\varphi(x)|}|\varphi(x)| \\ &\leqslant \overline{F}(2(|u(x)|+|\varphi(x)|)) \quad \text{for a.e. } x \in \Omega. \end{aligned} \tag{4.23}$$

Assume first that conditions (2.7) and (2.9) are in force. One can verify that the function $\Phi_n(t)$ dominates $t^{n/(n-1)}$ near ∞, whatever Φ is, and hence

$$\lim_{t\to\infty} \frac{\Phi_n(t)}{t} = \infty.$$

Thus, by lemma 4.4 applied with $B = \Phi_n$, the Young function $\overline{F}$ increases essentially more slowly than Φ_n near ∞. Owing to (3.13), the right-hand side of (4.23) belongs to $L^1(\Omega)$. If, instead, condition (2.10) holds, then the same assertion is true, owing to embedding (3.12). In any case, from (4.21)–(4.23) we obtain, via the dominated convergence theorem, that

$$\langle (L_f)'_G(u), \varphi\rangle = \int_\Omega f(u)\varphi\,\mathrm{d}x$$

for every $u, \varphi \in W_0^{1,\Phi}(\Omega)$.

The continuity of $(L_f)'_G$ is a straightforward consequence of proposition 3.6. □

Our next task consists in showing that the functional J_Φ satisfies the Palais–Smale condition. This is accomplished in the next proposition.

PROPOSITION 4.5. *Let $\Phi \in C^1(\mathbb{R}^n)$ be an n-dimensional Young function satisfying (2.3). Let Φ_n be its Sobolev conjugate defined by (2.2) (according to the convention of remark 3.3). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (2.5). Assume that either (2.7) and (2.9) hold, or (2.10) holds. Then the functional J_Φ satisfies the Palais–Smale condition (4.2).*

The proof of proposition 4.5 makes use of the next lemma. In what follows, '$\rightharpoonup$' denotes weak convergence.

Lemma 4.6. *Assume that $\varPhi \in C^1(\mathbb{R}^n)$ is an even, strictly convex, non-negative function, vanishing at 0, and satisfying (2.3). Then the operator $T\colon W_0^{1,\varPhi}(\varOmega) \to (W_0^{1,\varPhi}(\varOmega))^*$, defined as*

$$\langle Tu, v\rangle = \int_\varOmega \varPhi_\xi(\nabla u)\cdot \nabla v\, \mathrm{d}x$$

for $u, v \in W_0^{1,\varPhi}(\varOmega)$, is well defined. Moreover, if $u \in W_0^{1,\varPhi}(\varOmega)$, and $\{u_k\} \subset W_0^{1,\varPhi}(\varOmega)$ is a sequence such that

$$u_k \rightharpoonup u \quad \textit{in } W_0^{1,\varPhi}(\varOmega)$$

and

$$\limsup_{k\to\infty} \langle T(u_k), u_k - u\rangle \leqslant 0, \tag{4.24}$$

then $u_k \to u$ in $W_0^{1,\varPhi}(\varOmega)$.

Proof. Let us begin by showing that T is well defined. Owing to (A 23) and (3.2), if $u \in W_0^{1,\varPhi}(\varOmega)$, then $\varPhi_\xi(\nabla u) \in L^{\widetilde{\varPhi}}(\varOmega, \mathbb{R}^n)$. Moreover, if also $v \in W_0^{1,\varPhi}(\varOmega)$, then, by (3.2),

$$\int_\varOmega |\varPhi_\xi(\nabla u)\cdot \nabla v|\, \mathrm{d}x \leqslant 2\|\varPhi_\xi(\nabla u)\|_{L^{\tilde\varPhi}(\varOmega,\mathbb{R}^n)} \|\nabla v\|_{L^{\varPhi}(\varOmega,\mathbb{R}^n)}.$$

This guarantees that T is well defined, and that

$$\|T(u)\|_{(W_0^{1,\varPhi}(\varOmega))^*} \leqslant 2\|\varPhi_\xi(\nabla u)\|_{L^{\tilde\varPhi}(\varOmega,\mathbb{R}^n)}.$$

Now, let $\{u_k\}$ be a sequence as in the statement. Observe that

$$\lim_{k\to\infty} \int_\varOmega \varPhi_\xi(\nabla u)\cdot(\nabla u_k - \nabla u)\, \mathrm{d}x = 0. \tag{4.25}$$

By (4.24) and (4.25), for every $\sigma > 0$, there exists $k_\sigma \in \mathbb{N}$ such that

$$\int_\varOmega (\varPhi_\xi(\nabla u_k) - \varPhi_\xi(\nabla u))\cdot(\nabla u_k - \nabla u)\, \mathrm{d}x < \sigma \tag{4.26}$$

if $k > k_\sigma$. Given $t, \tau > 0$, set

$$l = \inf\{(\varPhi_\xi(\xi) - \varPhi_\xi(\eta))\cdot(\xi - \eta)\colon |\xi| \leqslant \tau,\ |\eta| \leqslant \tau,\ |\xi - \eta| > t\}.$$

Inasmuch as $\varPhi \in C^1(\mathbb{R}^n)$ and is strictly convex,

$$l > 0 \quad \text{for } t, \tau > 0.$$

Set

$$\varOmega_k = \{x \in \varOmega\colon |\nabla u_k(x)| \leqslant \tau,\ |\nabla u(x)| \leqslant \tau,\ |\nabla u_k(x) - \nabla u(x)| > t\}.$$

Owing to (4.26),

$$l|\varOmega_k| = \int_{\varOmega_k} l\, \mathrm{d}x \leqslant \int_\varOmega (\varPhi_\xi(\nabla u_k) - \varPhi_\xi(\nabla u))\cdot(\nabla u_k - \nabla u)\, \mathrm{d}x < \sigma$$

if $k > k_\sigma$. Thus,

$$|\Omega_k| < \frac{\sigma}{l} \quad \text{if } k > k_\sigma.$$

By the strict convexity of Φ, and property (A 7), there exists a constant $c > 0$ such that

$$\Phi(\xi) \geqslant c|\xi| \quad \text{if } |\xi| \geqslant 1.$$

Since $u_k \rightharpoonup u$, there exists $M > 0$ such that $\|u_k\|_{W^{1,\Phi}_0(\Omega)} \leqslant M$ for $k \in \mathbb{N}$, and $\|u\|_{W^{1,\Phi}_0(\Omega)} \leqslant M$. Fix $\tau > M$. Then,

$$\frac{c\tau}{M}|\{x \in \Omega\colon |\nabla u_k(x)| \geqslant \tau\}| \leqslant c\int_{\{|\nabla u_k| \geqslant \tau\}} \frac{|\nabla u_k|}{M}\,\mathrm{d}x \leqslant \int_{\{|\nabla u_k| \geqslant \tau\}} \Phi\bigg(\frac{\nabla u_k}{M}\bigg)\,\mathrm{d}x \leqslant 1$$

for $k \in \mathbb{N}$. Analogously,

$$\frac{c\tau}{M}|\{x \in \Omega\colon |\nabla u(x)| \geqslant \tau\}| \leqslant 1.$$

Hence,

$$|\{x \in \Omega\colon |\nabla u_k(x)| \geqslant \tau\}| \leqslant \frac{M}{c\tau} \quad \text{and} \quad |\{x \in \Omega\colon |\nabla u(x)| \geqslant \tau\} \leqslant \frac{M}{c\tau} \quad \text{for } \tau > M.$$

Fix $\varepsilon > 0$, and choose $\sigma = l\varepsilon/3$ and $\tau > \max\{M, 3M/c\varepsilon\}$. Then,

$$\begin{aligned}|\{x \in \Omega\colon |\nabla u_k(x) - \nabla u(x)| > t\}| &\leqslant |\Omega_k| + |\{x \in \Omega\colon |\nabla u_k(x)| \geqslant \tau\}| \\ &\quad + |\{x \in \Omega\colon |\nabla u(x)| \geqslant \tau\}| \\ &< \varepsilon \quad \text{if } k > k_\sigma.\end{aligned}$$

Thus, $\nabla u_k \to \nabla u$ in measure and, up to a subsequence still denoted by $\{u_k\}$, almost everywhere in Ω. Consequently, $\Phi(\nabla u_k) \to \Phi(\nabla u)$ almost everywhere in Ω, and by Fatou's theorem,

$$\liminf_{k\to\infty} \int_\Omega \Phi(\nabla u_k)\,\mathrm{d}x \geqslant \int_\Omega \Phi(\nabla u)\,\mathrm{d}x. \tag{4.27}$$

On the other hand, the convexity of Φ implies that

$$\Phi(\eta) \geqslant \Phi(\xi) + \Phi_\xi(\xi)(\eta - \xi) \quad \text{for } \xi, \eta \in \mathbb{R}^n.$$

Hence,

$$\int_\Omega \Phi_\xi(\nabla u_k)\cdot(\nabla u_k - \nabla u)\,\mathrm{d}x \geqslant \int_\Omega \Phi(\nabla u_k)\,\mathrm{d}x - \int_\Omega \Phi(\nabla u)\,\mathrm{d}x$$

for $k \in \mathbb{N}$, and, by (4.24),

$$\limsup_{k\to\infty} \int_\Omega \Phi(\nabla u_k)\,\mathrm{d}x \leqslant \int_\Omega \Phi(\nabla u)\,\mathrm{d}x. \tag{4.28}$$

Coupling (4.27) with (4.28) tells us that

$$\lim_{k\to\infty} \int_\Omega \Phi(\nabla u_k)\,\mathrm{d}x = \int_\Omega \Phi(\nabla u)\,\mathrm{d}x. \tag{4.29}$$

Since Φ is Δ_2 near ∞, the convergence of ∇u_k to ∇u almost everywhere and (4.29) imply that $\|\nabla u_k - \nabla u\|_{L^\Phi(\Omega)} \to 0$ as $k \to \infty$. This follows along the same lines as in the case when standard isotropic Orlicz norms are involved – see, for example, [17, ch. 3, theorem 12]. Inasmuch as the whole argument clearly applies to any subsequence of $\{u_k\}$, the conclusion follows. □

Proof of proposition 4.5. Let $\{u_k\} \subset W_0^{1,\Phi}(\Omega)$ be a Palais–Smale sequence for J_Φ. Since the sequence $\{J_\Phi(u_k)\}$ is bounded, there exists a subsequence, still denoted by $\{u_k\}$, and a number $c \in \mathbb{R}$ such that $\lim_{k\to\infty} J_\Phi(u_k) = c$. Thus, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$c - \varepsilon < J_\Phi(u_k) < c + \varepsilon \quad \text{if } k > k_\varepsilon. \tag{4.30}$$

On the other hand, since $\lim_{k\to\infty} \|J'_\Phi(u_k)\|_{(W_0^{1,\Phi}(\Omega))^*} = 0$, there exists a sequence $\{\varepsilon_k\}$ such that $\varepsilon_k \to 0^+$, and

$$-\varepsilon_k \|\varphi\|_{W_0^{1,\Phi}(\Omega)} \leqslant \int_\Omega \Phi_\xi(\nabla u_k) \cdot \nabla\varphi \, \mathrm{d}x - \int_\Omega f(u_k)\varphi \, \mathrm{d}x \leqslant \varepsilon_k \|\varphi\|_{W_0^{1,\Phi}(\Omega)} \tag{4.31}$$

for every $\varphi \in W_0^{1,\Phi}(\Omega)$.

Given any $\sigma > 0$, there exists $M \geqslant 0$ such that

$$(s_\Phi + \sigma)\Phi(\xi) - \Phi_\xi(\xi) \cdot \xi \geqslant 0 \quad \text{if } |\xi| \geqslant M.$$

On setting $\alpha = s_\Phi + 2\sigma$, the last inequality can be rewritten as

$$\sigma\Phi(\xi) \leqslant \alpha\Phi(\xi) - \Phi_\xi(\xi) \cdot \xi \quad \text{if } |\xi| \geqslant M. \tag{4.32}$$

By (2.5), if σ is sufficiently small, then

$$\alpha F(t) - f(t)t < 0 \quad \text{if } |t| \geqslant M, \tag{4.33}$$

provided that M is sufficiently large.

Now, choose $\varphi = u_k$ in (4.31), multiply through (4.30) by α, and add the resulting equations to obtain that

$$\begin{aligned} \int_\Omega (\alpha\Phi(\nabla u_k) - \Phi_\xi(\nabla u_k) \cdot \nabla u_k) \, \mathrm{d}x - \int_\Omega (\alpha F(u_k) - f(u_k)u_k) \, \mathrm{d}x \\ \leqslant \alpha(c + \varepsilon) + \varepsilon_k \|u_k\|_{W_0^{1,\Phi}(\Omega)} \end{aligned} \tag{4.34}$$

for $k > k_\varepsilon$. From (4.32), (4.34), (4.33) we deduce that

$$\begin{aligned} \sigma \int_\Omega \Phi(\nabla u_k) \, \mathrm{d}x &\leqslant \int_{\{|\nabla u_k| \geqslant M\}} (\alpha\Phi(\nabla u_k) - \Phi_\xi(\nabla u_k) \cdot \nabla u_k) \, \mathrm{d}x \\ &\quad + \sigma \int_{\{|\nabla u_k| < M\}} \Phi(\nabla u_k) \, \mathrm{d}x \\ &= \int_\Omega (\alpha\Phi(\nabla u_k) - \Phi_\xi(\nabla u_k) \cdot \nabla u_k) \, \mathrm{d}x \\ &\quad + \int_{\{|\nabla u_k| < M\}} (\sigma\Phi(\nabla u_k) - \alpha\Phi(\nabla u_k) + \Phi_\xi(\nabla u_k) \cdot \nabla u_k) \, \mathrm{d}x \end{aligned}$$

$$\leqslant \alpha(c+\varepsilon) + \varepsilon_k \|u_k\|_{W_0^{1,\Phi}(\Omega)} + \int_{|u_k|\leqslant M} (\alpha F(u_k) - f(u_k)u_k)\,\mathrm{d}x + C$$
$$\leqslant C' + \varepsilon_k \|u_k\|_{W_0^{1,\Phi}(\Omega)} \tag{4.35}$$

for $k > k_\varepsilon$, for some constants $C = C(M, \Phi, \alpha)$ and $C' = C'(M, \Phi, \alpha)$.

We claim that $\{u_k\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. To verify this claim, suppose, by contradiction, that $\{u_k\}$ is unbounded. In particular, on passing, if necessary, to a subsequence, we may assume that

$$\|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon > 1 \quad \text{for } k \in \mathbb{N}.$$

By the very definition of the Luxemburg norm, and property (A 7),

$$1 < \int_\Omega \Phi\biggl(\frac{\nabla u_k}{\|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon}\biggr)\,\mathrm{d}x \leqslant \int_\Omega \frac{\Phi(\nabla u_k)}{\|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon}\,\mathrm{d}x$$

for $k \in \mathbb{N}$. Hence,

$$\|u_k\|_{W_0^{1,\Phi}(\Omega)} - \varepsilon < \int_\Omega \Phi(\nabla u_k)\,\mathrm{d}x \tag{4.36}$$

for $k \in \mathbb{N}$. Coupling (4.35) with (4.36) yields

$$1 - \frac{\varepsilon}{\|u_k\|_{W_0^{1,\Phi}(\Omega)}} \leqslant \frac{C'}{\sigma \|u_k\|_{W_0^{1,\Phi}(\Omega)}} + \frac{\varepsilon_k}{\sigma} \tag{4.37}$$

for $k > k_\varepsilon$. Passing to the limit as $k \to \infty$ in (4.37) leads to a contradiction. Our claim is thus proved. Assumption (2.3) and lemma A.5 ensure, owing to proposition 3.1, that the space $W_0^{1,\Phi}(\Omega)$ is reflexive. Thus, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, that weakly converges to some function $u \in W_0^{1,\Phi}(\Omega)$. Now, if (2.7) and (2.9) hold, then by (3.17) and (4.20), we may apply lemma 3.5 with $E = \overline{F}$ and $B = \Phi_n$. Therefore, on choosing $\varphi = u - u_k$ in (4.31) and exploiting (3.19), we deduce that

$$\limsup_{k\to\infty} \int_\Omega \Phi_\xi(\nabla u_k) \cdot (\nabla u - \nabla u_k)\,\mathrm{d}x = 0. \tag{4.38}$$

Equation (4.38) continues to hold even if, instead, (2.10) is in force, since (3.19) trivially holds thanks to embedding (3.12). Equation (4.38), via lemma 4.6, implies that $u_k \to u$ in $W_0^{1,\Phi}(\Omega)$. □

Lemma 4.7. *Let Φ be an n-dimensional Young function and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that condition (2.6) is fulfilled, and that either (2.7) holds and (2.8)–(2.9) are in force, or (2.10) holds. Then,*

$$\lim_{\|u\|_{W_0^{1,\Phi}(\Omega)}\to 0} \frac{\int_\Omega F(u)\,\mathrm{d}x}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} = 0. \tag{4.39}$$

Proof. By lemma 4.4, for every $\varepsilon > 0$ there exists $t_\varepsilon \geqslant 0$ such that

$$|F(t)| < \varepsilon \Phi_*(\omega_n^{1/n} |\Omega|^{-1/n} |t|) \quad \text{if } |t| \leqslant t_\varepsilon. \tag{4.40}$$

From (4.40) and (3.5) we deduce that

$$\frac{|\int_{|u|\leqslant t_\varepsilon} F(u)\,\mathrm{d}x|}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} < \frac{\varepsilon \int_{|u|\leqslant t_\varepsilon} \Phi_*(\omega_n^{1/n}|\Omega|^{-1/n}|u|)\,\mathrm{d}x}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} < \frac{\varepsilon \int_\Omega \Phi(\nabla u)\,\mathrm{d}x}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} = \varepsilon. \quad (4.41)$$

Suppose first that (2.7)–(2.9) hold. Then we can apply lemma 4.4 with $B = \Phi_n$. In particular, $\overline{F}$ increases essentially more slowly than Φ_n. Thus, one can show that there exists $\lambda > 0$ such that

$$\overline{F}(|t|) \leqslant \varepsilon \Phi_n(\lambda |t|) \quad \text{if } |t| \geqslant t_\varepsilon. \quad (4.42)$$

Moreover, we can choose λ such that $C\lambda > 1$, where C denotes the constant appearing in (3.10). Fix $\delta < (C\lambda)^{-n}$. Since, in particular, $\delta < 1$, if $\|u\|_{W_0^{1,\Phi}(\Omega)} < \delta$, then $\int_\Omega \Phi(\nabla u)\,\mathrm{d}x < \delta$ as well. Thus,

$$\lambda < \frac{1}{C\delta^{1/n}} < \frac{1}{C(\int_\Omega \Phi(\nabla u)\,\mathrm{d}x)^{1/n}}.$$

Hence, by (4.42) and (3.10),

$$\begin{aligned} \frac{|\int_{|u|>t_\varepsilon} F(u)\,\mathrm{d}x|}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} &< \frac{\varepsilon \int_{|u|>t_\varepsilon} \Phi_n(\lambda|u|)\,\mathrm{d}x}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} \\ &< \frac{\varepsilon \int_\Omega \Phi_n(|u|/(C(\int_\Omega \Phi(\nabla u)\,\mathrm{d}y)^{1/n}))\,\mathrm{d}x}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} \leqslant \varepsilon. \end{aligned} \quad (4.43)$$

Coupling (4.41) with (4.43) tells us that

$$\frac{|\int_\Omega F(u)\,\mathrm{d}x|}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} < 2\varepsilon,$$

if $\|u\|_{W_0^{1,\Phi}(\Omega)} < \delta$. Thus, (4.39) follows.

Assume next that (2.10) holds. In this case, by (3.12), there exists $\delta > 0$ such that $\|u\|_{L^\infty(\Omega)} < t_\varepsilon$ if $\|u\|_{W_0^{1,\Phi}(\Omega)} < \delta$. Equation (4.41) then yields

$$\frac{|\int_\Omega F(u)\,\mathrm{d}x|}{\int_\Omega \Phi(\nabla u)\,\mathrm{d}x} < \varepsilon,$$

and (4.39) follows also in this case. □

Our last preparatory result in view of the proof of theorem 2.1 is contained in the following lemma.

Lemma 4.8. *Let $\Phi \in C^1(\mathbb{R}^n)$ be an n-dimensional Young function such that $s_\Phi < \infty$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function fulfilling (2.4) and (2.5). If $u \in C_0^1(\Omega)$ and u does not vanish identically, then*

$$\lim_{t\to\infty} \int_\Omega (\Phi(t\nabla u) - F(tu))\,\mathrm{d}x = -\infty. \quad (4.44)$$

Proof. Owing to (A 21), for every $\varepsilon > 0$, there exists $M \geqslant 0$ such that

$$\Phi(t\xi) \leqslant \Phi(\xi) t^{s_\Phi + \varepsilon} \quad \text{if } t \geqslant 1 \text{ and } |\xi| \geqslant M. \quad (4.45)$$

By (2.5), if ε and $\alpha > 0$ are chosen in such a way that $\liminf_{t\to\pm\infty} tf(t)/F(t) > \alpha > s_\Phi + \varepsilon$, then

$$\frac{tf(t)}{F(t)} \geqslant \alpha \quad \text{if } |t| \text{ is sufficiently large.}$$

Owing to assumption (2.4) and to the last inequality, there exist $a, b > 0$ such that

$$F(t) \geqslant a|t|^\alpha - b \quad \text{for } t \in \mathbb{R}. \tag{4.46}$$

Now, let u be as in the statement and let $t \geqslant 1$. Owing to (4.45) and (4.46),

$$\begin{aligned} J_\Phi(tu) &\leqslant \int_{\{|\nabla u|\leqslant M\}} \Phi(t\nabla u)\,\mathrm{d}x + \int_{\{|\nabla u|>M\}} \Phi(t\nabla u)\,\mathrm{d}x - \int_\Omega a|tu|^\alpha\,\mathrm{d}x + b|\Omega| \\ &\leqslant \int_{\{|\nabla u|\leqslant M\}} \Phi\left(tM\frac{\nabla u}{|\nabla u|}\right)\mathrm{d}x + t^{s_\Phi+\varepsilon}\int_{\{|\nabla u|>M\}} \Phi(\nabla u)\,\mathrm{d}x \\ &\quad - at^\alpha \|u\|^\alpha_{L^\alpha(\Omega)} + b|\Omega| \\ &\leqslant t^{s_\Phi+\varepsilon}\left[\int_{\{|\nabla u|\leqslant M\}} \Phi\left(M\frac{\nabla u}{|\nabla u|}\right)\mathrm{d}x + \int_{\{|\nabla u|>M\}} \Phi(\nabla u)\,\mathrm{d}x\right] \\ &\quad - at^\alpha \|u\|^\alpha_{L^\alpha(\Omega)} + b|\Omega|, \end{aligned}$$

where $\nabla u/|\nabla u|$ is taken to be 0 if $\nabla u = 0$. Equation (4.44) follows, inasmuch as $s_\Phi + \varepsilon < \alpha$. □

We are now in a position to accomplish the proof of theorem 2.1.

Proof of theorem 2.1. Corollary 4.1 and proposition 4.5 ensure that the functional J_Φ, defined by (1.2), is of class C^1 and satisfies the Palais–Smale condition (4.2). Lemmas 4.7 and 4.8 tell us that conditions (4.4) and (4.5), respectively, are fulfilled. Thus, J_Φ satisfies the assumptions of the mountain pass theorem stated above, and hence J_Φ has a non-trivial critical point $u \in W_0^{1,\Phi}(\Omega)$, which is a solution to (1.1).

The boundedness of u follows from an application of [1, theorem 4.1] (see also [2]). □

5. Special instances

In this section we apply theorem 2.1 to some classes of functions Φ, which govern the differential operator in the equation in (1.1), with a distinctive structure, including those corresponding to (1.4), (1.7) and (1.9). In particular, the novelties of our conclusions in comparison with the existing literature are pointed out.

5.1. Isotropic growth

Consider first the isotropic case in which Φ is radial, namely

$$\Phi(\xi) = A(|\xi|) \quad \text{for } \xi \in \mathbb{R}^n, \tag{5.1}$$

where A is as in (1.7). Problem (1.1) then reads

$$\left.\begin{aligned} -\operatorname{div}\left(\frac{A'(|\nabla u|)}{|\nabla u|}\nabla u\right) &= f(u) \quad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.2}$$

Owing to (5.1), we have that $\Phi_*(t) = A(t)$, and $\Phi_n(t) = A_n(t)$, where

$$A_n(t) = A(H_A^{-1}(t)) \quad \text{for } t \geqslant 0,$$

and

$$H_A(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau\right)^{(n-1)/n} \qquad \text{for } t \geqslant 0.$$

Moreover, $i_\Phi = i_A$ and $s_\Phi = s_A$, where we have set

$$i_A = \liminf_{t\to\infty} \frac{tA'(t)}{A(t)}, \qquad s_A = \limsup_{t\to\infty} \frac{tA'(t)}{A(t)}.$$

By theorem 2.1, problem (5.2) has a non-trivial solution, provided that

$$1 < i_A, \qquad s_A < \infty,$$

$$\liminf_{t\to\pm\infty} \frac{tf(t)}{F(t)} > s_A,$$

$$\lim_{t\to 0} \frac{tf(t)}{A(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

and either

$$\int^\infty \left(\frac{\tau}{A(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau < \infty,$$

or

$$\int^\infty \left(\frac{\tau}{A(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau = \infty, \qquad \int_0 \left(\frac{\tau}{A(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau < \infty,$$

and

$$\lim_{t\to\pm\infty} \frac{tf(t)}{A_n(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0.$$

This result strengthens [8, theorem 1.1], where an analogous conclusion is derived with the function A_n replaced with another function that, in general, can grow more slowly near ∞ (see, for instance, the next example). Furthermore, in [8] the assumption that $s_A < \infty$ is replaced with the more stringent assumption that $\sup_{t\geqslant 0} tA'(t)/A(t) < \infty$.

5.2. Isotropic power-type growth

Let us further adapt problem (5.2) to functions A having an explicit asymptotic behaviour near ∞. Assume first that

$$A(t) = \frac{1}{p}t^p \quad \text{for } t \geqslant 0,$$

for some $p \in (1, n)$. With this choice of A, problem (5.2) agrees with the classical Dirichlet problem for the p-Laplace equation

$$\left.\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\ \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.3}$$

We then obtain that a non-trivial solution exists, provided that f fulfils the following conditions:

$$\begin{gathered} \liminf_{t\to\pm\infty} \frac{tf(t)}{F(t)} > p, \\ \lim_{t\to 0} \frac{f(t)}{|t|^{p-1}} = 0, \\ \lim_{t\to\pm\infty} \frac{f(t)}{|t|^{p^*-1}} = 0, \end{gathered} \tag{5.4}$$

where $p^* = np/(n-p)$, the Sobolev conjugate associated with p. Note that this conclusion somewhat augments standard results for the p-Laplace equation, which require that $\lim_{t\to\pm\infty} f(t)/|t|^{q-1} = 0$ for some $q < p^*$.

If $p > n$, the same result holds, without assumption (5.4).

In the borderline case when $A(t) = t^n$ for every $t \geqslant 0$, theorem 2.1 does not apply, since

$$\int^{\infty} \left(\frac{\tau}{A(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau = \infty \quad \text{and} \quad \int_0 \left(\frac{\tau}{A(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau = \infty.$$

However, if $A(t) = t^n$ for large t, but the latter integral converges, then theorem 2.1 entails that a non-trivial solution to problem (5.2) exists, provided that

$$\begin{gathered} \liminf_{t\to\pm\infty} \frac{tf(t)}{F(t)} > n, \\ \lim_{t\to 0} \frac{tf(t)}{A(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0, \end{gathered}$$

and

$$\lim_{t\to\pm\infty} \frac{tf(t)}{\mathrm{e}^{\lambda|t|^{n'}}} = 0 \quad \text{for every } \lambda > 0.$$

In [8] the stronger assumption

$$\lim_{t\to\pm\infty} \frac{tf(t)}{\mathrm{e}^{\lambda|t|}} = 0 \quad \text{for every } \lambda > 0$$

was instead required, as well as a more stringent condition at 0 than

$$\int_0 \left(\frac{\tau}{A(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau < \infty.$$

5.3. Anisotropic growth in split form

Here, we deal with anisotropic functions Φ with a split structure, namely functions given by

$$\Phi(\xi) = \sum_{i=1}^{n} A_i(|\xi_i|) \quad \text{for } \xi \in \mathbb{R}^n,$$

where $A_i\colon [0,\infty) \to [0,\infty)$, $i = 1,\ldots,n$, are strictly convex, continuously differentiable functions vanishing at 0. In this case, problem (1.1) takes the form

$$\left.\begin{aligned} -\sum_{i=1}^{n}(A_i'(u_{x_i}))_{x_i} = f(u) \quad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.5}$$

The function Φ_* is equivalent to the convex function $\hat{A}\colon [0,\infty) \to [0,\infty)$, whose inverse is given by

$$\hat{A}^{-1}(r) = \left(\prod_{i=1}^{n} A_i^{-1}(r)\right)^{1/n} \quad \text{for } r \geqslant 0,$$

in the sense that there exist positive constants c_1 and c_2 such that

$$\hat{A}(c_1 t) \leqslant \Phi_*(t) \leqslant \hat{A}(c_2 t) \quad \text{for } t \geqslant 0; \tag{5.6}$$

see [6, (1.9)]. Hence, the Sobolev conjugate Φ_n is in turn equivalent to the function $\hat{A}_n$, given by

$$\hat{A}_n(t) = \hat{A}(H_{\hat{A}}^{-1}(t)) \quad \text{for } t \geqslant 0,$$

where now

$$H_{\hat{A}}(t) = \left(\int_0^t \left(\frac{\tau}{\hat{A}(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau\right)^{(n-1)/n} \quad \text{for } t \geqslant 0.$$

Furthermore, one can show that

$$i_\Phi = \min_{1\leqslant i\leqslant n} i_{A_i}, \qquad s_\Phi = \max_{1\leqslant i\leqslant n} s_{A_i}. \tag{5.7}$$

Hence, an application of theorem 2.1 tells us that problem (5.5) has a non-trivial solution, provided that

$$1 < \min_{1\leqslant i\leqslant n} i_{A_i}, \qquad \max_{1\leqslant i\leqslant n} s_{A_i} < \infty,$$

$$\liminf_{t\to\pm\infty} \frac{tf(t)}{F(t)} > \max_{1\leqslant i\leqslant n} s_{A_i},$$

$$\lim_{t\to 0} \frac{tf(t)}{\hat{A}(\lambda|t|)} = 0 \quad \text{for every } \lambda > 0,$$

and either

$$\int^{\infty} \left(\frac{\tau}{\hat{A}(\tau)}\right)^{1/(n-1)} \mathrm{d}\tau < \infty,$$

or

$$\int^{\infty}\left(\frac{\tau}{\hat{A}(\tau)}\right)^{1/(n-1)}\mathrm{d}\tau=\infty,\qquad \int_0\left(\frac{\tau}{\hat{A}(\tau)}\right)^{1/(n-1)}\mathrm{d}\tau<\infty$$

and

$$\lim_{t\to\infty}\frac{tf(t)}{\hat{A}_n(\lambda t)}=0\quad\text{for every }\lambda>0.$$

5.4. Anisotropic power-type growth

Consider the standard instance of (5.5), when

$$A_i(t)=\frac{1}{p_i}t^{p_i}\quad\text{for }t\geqslant 0,$$

for some powers $p_i>1$, $i=1,\dots,n$. Namely,

$$\Phi(\xi)=\sum_{i=1}^{n}\frac{1}{p_i}|\xi_i|^{p_i}\quad\text{for }\xi\in\mathbb{R}^n.$$

Thus, problem (1.1) agrees with

$$\left.\begin{aligned}-\sum_{i=1}^{n}(|u_{x_i}|^{p_i-2}u_{x_i})_{x_i}&=f(u)\quad\text{in }\Omega,\\ u&=0\qquad\text{on }\partial\Omega.\end{aligned}\right\}\tag{5.8}$$

Note that here

$$i_{A_i}=s_{A_i}=p_i\quad\text{for }i=1,\dots,n.$$

Owing to (5.6),

$$\Phi_*(t)\approx t^{\overline{p}}\quad\text{for }t\geqslant 0,$$

where the relation '$\approx$' means that the two sides are bounded by each other up to multiplicative constants independent of t, and $\overline{p}$ is the harmonic average of the powers p_i, defined via (1.10). In particular, when $\overline{p}<n$, one has that

$$\Phi_n(t)\approx t^{\overline{p}^*}\quad\text{for }t\geqslant 0,$$

where $\overline{p}^*=n\overline{p}/(n-\overline{p})$, the Sobolev conjugate of $\overline{p}$.

By the result of § 5.3, we can thus conclude that problem (5.8) has a non-trivial solution, provided that

$$\liminf_{t\to\pm\infty}\frac{tf(t)}{F(t)}>\max_{1\leqslant i\leqslant n}p_i,$$

$$\lim_{t\to 0}\frac{f(t)}{|t|^{\overline{p}-1}}=0,$$

and either $\overline{p}>n$, or $\overline{p}<n$ and

$$\lim_{t\to\pm\infty}\frac{f(t)}{|t|^{\overline{p}^*-1}}=0.\tag{5.9}$$

This recovers [11, theorem 4] and extends it in that, unlike [11], here we are not assuming that f is just a power. Let us point out that in [11] the sharpness of assumption (5.9) is also shown. This is accomplished by proving, via suitable anisotropic Pohozaev-type identities, the non-existence of non-trivial solutions to (5.8), in suitable classes of domains, in the case of nonlinearities f of the form $f(t) = t^{q-1}$ with $q > \overline{p}^*$.

5.5. Anisotropic power-logarithmic-type growth

We deal here with a somewhat more general case than that of § 5.4, corresponding to (5.5) with the choice

$$A_i(t) = \frac{1}{p_i} t^{p_i} \log^{\alpha_i}(c+t) \quad \text{for } t \geqslant 0, \tag{5.10}$$

where $p_i > 1$, $\alpha_i \in \mathbb{R}$, $i = 1, \dots, n$, and c is a positive constant, sufficiently large (depending on the powers p_i and α_i) for all functions A_i to be convex. Thus,

$$\Phi(\xi) = \sum_{i=1}^{n} \frac{1}{p_i} |\xi_i|^{p_i} \log^{\alpha_i}(c+|\xi_i|) \quad \text{for } \xi \in \mathbb{R}^n.$$

Note that

$$i_{A_i} = s_{A_i} = p_i \quad \text{for } i = 1, \dots, n$$

if the functions A_i are given by (5.10). Let $\overline{p}$ be defined as in (1.10), and let $\overline{\alpha}$ be defined as

$$\overline{\alpha} = \frac{\overline{p}}{n} \sum_{i=1}^{n} \frac{\alpha_i}{p_i}.$$

One can verify, via (5.6), that

$$\Phi_*(t) \approx \begin{cases} t^{\overline{p}} & \text{near } 0, \\ t^{\overline{p}} \log^{\overline{\alpha}}(c+t) & \text{near } \infty, \end{cases}$$

up to multiplicative constants independent of t. Moreover, if $\overline{p} < n$, then conditions (2.7) and (2.8) are fulfilled, and

$$\Phi_n(t) \approx t^{\overline{p}^*} \log^{\overline{\alpha} n/(n-\overline{p})}(c+t) \quad \text{near } \infty,$$

where $\overline{p}^*$ denotes the Sobolev conjugate of $\overline{p}$.

The result of § 5.3 ensures that problem (5.5) has a non-trivial solution, provided that

$$\liminf_{t \to \pm\infty} \frac{tf(t)}{F(t)} > \max_{1 \leqslant i \leqslant n} p_i,$$
$$\lim_{t \to 0} \frac{f(t)}{|t|^{\overline{p}-1}} = 0,$$

and either $\overline{p} > n$, or $\overline{p} < n$ and

$$\lim_{t \to \pm\infty} \frac{f(t)}{|t|^{\overline{p}^*-1} \log^{\overline{\alpha} n/(n-\overline{p})}(|t|)} = 0.$$

Appendix A. Young functions and n-dimensional Young functions

Standard Young functions have been extensively treated in the literature. Notation and properties involving Young functions, which are exploited in this paper, are recalled in the first part of this appendix. For a comprehensive treatment of this matter we refer the reader to the monographs [13, 17, 18].

The Young conjugate of a Young function A is the Young function $\widetilde{A}$ defined as

$$\widetilde{A}(s) = \sup\{st - A(t) \colon t \geqslant 0\} \quad \text{for } s \geqslant 0.$$

One has that $\widetilde{\widetilde{A}} = A$ for any Young function A.

On denoting by A^{-1} the (generalized) left-continuous inverse of A, one has that

$$t \leqslant \widetilde{A}^{-1}(t)A^{-1}(t) \leqslant 2t \quad \text{for } t \geqslant 0. \tag{A 1}$$

Hence,

$$\frac{A(t)}{t} \leqslant \widetilde{A}^{-1}(A(t)) \leqslant 2\frac{A(t)}{t} \quad \text{for } t > 0. \tag{A 2}$$

If A is a Young function, then

$$\lambda A(t) \leqslant A(\lambda t) \quad \text{for } \lambda \geqslant 1 \text{ and } t \geqslant 0.$$

A Young function A is said to satisfy the Δ_2-condition near ∞ if it is finite valued and there exist constants $C \geqslant 2$ and $M \geqslant 0$ such that

$$A(2t) \leqslant CA(t) \quad \text{if } t \geqslant M. \tag{A 3}$$

A Young function A is said to dominate another Young function B near ∞ if there exist constants $c > 0$ and $M \geqslant 0$ such that

$$B(t) \leqslant A(ct) \quad \text{if } t \geqslant M. \tag{A 4}$$

If (A 4) holds with $M = 0$, then we say that A dominates B globally. Two Young functions A and B are called equivalent near ∞ (globally) if they dominate each other near ∞ (globally).

The function B is said to increase essentially more slowly than A near ∞ if it is finite valued and

$$\lim_{t\to\infty} \frac{B(\lambda t)}{A(t)} = 0 \quad \text{for every } \lambda > 0. \tag{A 5}$$

Condition (A 5) is equivalent to

$$\lim_{s\to\infty} \frac{A^{-1}(s)}{B^{-1}(s)} = 0. \tag{A 6}$$

The theory of n-dimensional Young functions seems to be much less developed than that of standard Young functions. Contributions to this topic can be found in [19–22, 25]. The remaining part of this appendix is devoted to definitions and proofs of some results on this subject, which are not straightforward consequences of parallel results for usual Young functions.

For technical reasons, we distinguish between Young functions and 1-dimensional Young functions. However, extending a Young function to an even function on the whole of $\mathbb{R}$ results in a 1-dimensional Young function; conversely, the restriction of a 1-dimensional Young function to $[0,\infty)$ is a Young function. Thus, any definition or result concerning Young functions translates into a corresponding definition or result for 1-dimensional Young functions, and vice versa.

Given a Young function A, the function $\mathbb{R}^n \ni \xi \mapsto A(|\xi|)$ is an (isotropic) n-dimensional Young function. Moreover, given an n-dimensional Young function Φ, and a point $\xi \in \mathbb{R}^n$, the function $[0,\infty) \ni t \mapsto \Phi(t\xi)$ is a Young function. If Φ is an n-dimensional Young function, then

$$\lambda\Phi(\xi) \leqslant \Phi(\lambda\xi) \quad \text{for } \lambda \geqslant 1 \text{ and } \xi \in \mathbb{R}^n. \tag{A 7}$$

An n-dimensional Young function Φ is said to satisfy the Δ_2-condition near ∞ if it is finite valued and there exist constants $C \geqslant 2$ and $M \geqslant 0$ such that

$$\Phi(2\xi) \leqslant C\Phi(\xi) \quad \text{if } |\xi| \geqslant M. \tag{A 8}$$

The function Φ is said to satisfy the ∇_2-condition near ∞ if there exist constants $C > 2$ and $M \geqslant 0$ such that

$$\Phi(2\xi) \geqslant C\Phi(\xi) \quad \text{if } |\xi| \geqslant M. \tag{A 9}$$

The global Δ_2-condition and the global ∇_2-condition are defined accordingly, with $M = 0$.

Our applications mainly require properties of functions satisfying these conditions near ∞, and we thus focus on this case in what follows. The relevant properties have, however, global counterparts, which are usually simpler to prove.

Proposition A.1. *Let Φ be an n-dimensional Young function.*

(i) *$\Phi \in \Delta_2$ near ∞ if and only if it is finite valued and there exist constants $M \geqslant 0$ and $k > 1$ such that*

$$\Phi(k\xi) \leqslant 2\Phi(\xi) \quad \text{if } |\xi| \geqslant M. \tag{A 10}$$

(ii) *$\Phi \in \nabla_2$ near ∞ if and only if there exist constants $M \geqslant 0$ and $k > 1$ such that*

$$\Phi(k\xi) \geqslant 2k\Phi(\xi) \quad \text{if } |\xi| \geqslant M. \tag{A 11}$$

Proof. (i) Assume that (A 8) holds. Set $\delta = 1/(C-1) \leqslant 1$, and $k = 1+\delta$. By the convexity of Φ and (A 8), we have that

$$\begin{aligned}\Phi(k\xi) = \Phi((k-2\delta+2\delta)\xi) &= \Phi((1-\delta)\xi + \delta(2\xi))\\ &\leqslant (1-\delta)\Phi(\xi) + \delta\Phi(2\xi)\\ &\leqslant (1-\delta)\Phi(\xi) + \delta C\Phi(\xi) = 2\Phi(\xi) \quad \text{if } |\xi| \geqslant M,\end{aligned}$$

namely, (A 10) holds.

Conversely, assume that (A 10) is in force. Fix $m \in \mathbb{N}$ such that $k^m \geqslant 2$, and choose $C = 2^m$. Iterating (A 10) tells us that

$$\Phi(2\xi) \leqslant \Phi(k^m\xi) \leqslant 2^m\Phi(\xi) = C\Phi(\xi) \quad \text{if } |\xi| \geqslant M.$$

Hence, (A 8) follows.

(ii) Assume that (A 9) holds. Fix $m \geqslant 1$ such that $(\frac{1}{2}C)^m > 2$, namely $C^m > 2^{m+1}$, and choose $k = 2^m$. Thus, by iteration of (A 9), we deduce that

$$\Phi(\xi) \leqslant \frac{1}{C^m}\Phi(2^m\xi) < \frac{1}{2^{m+1}}\Phi(2^m\xi) = \frac{1}{2k}\Phi(k\xi) \quad \text{if } |\xi| \geqslant M.$$

namely, (A 11) holds.

Finally, suppose that (A 11) holds. Thus,

$$\Phi\left(\frac{2\xi}{k}\right) \leqslant \frac{1}{2k}\Phi(2\xi) \quad \text{if } |\xi| \geqslant \frac{Mk}{2}. \tag{A 12}$$

Set $C = 4k/(2k-1) > 2$, and $M_1 = Mk/2$. Owing to the convexity of Φ, to inequality (A 12), and to (A 7),

$$\Phi(\xi) \leqslant \frac{1}{4k}\Phi(2\xi) + \frac{k-1}{2k}\Phi(2\xi) = \frac{1}{C}\Phi(2\xi) \quad \text{if } |\xi| \geqslant M_1.$$

This establishes property (A 9). □

An n-dimensional Young function Ψ is said to dominate another n-dimensional Young function Φ near ∞ if there exist constants $c > 0$ and $M \geqslant 0$ such that

$$\Phi(\xi) \leqslant \Psi(c\xi) \quad \text{if } |\xi| \geqslant M. \tag{A 13}$$

If (A 13) holds with $M = 0$, then we say that Ψ dominates Φ globally. Two n-dimensional Young functions Ψ and Φ are called equivalent near ∞ (globally) if they dominate each other near ∞ (globally).

The Young conjugate of Φ is the n-dimensional Young function $\widetilde{\Phi}$ given by

$$\widetilde{\Phi}(\eta) = \sup\{\eta\cdot\xi - \Phi(\xi) \colon \xi \in \mathbb{R}^n\} \quad \text{for } \eta \in \mathbb{R}^n. \tag{A 14}$$

One has that

$$\widetilde{\widetilde{\Phi}} = \Phi. \tag{A 15}$$

Proposition A.2. *Let Φ be an n-dimensional Young function. Then Φ is finite valued if and only if*

$$\lim_{|\eta|\to\infty} \frac{\widetilde{\Phi}(\eta)}{|\eta|} = \infty. \tag{A 16}$$

Proof. Let Φ be finite valued. Assume, by contradiction, that (A 16) fails. Then there exist a constant $c > 0$ and a sequence $\{\eta_k\} \subset \mathbb{R}^n$ such that $\lim_{k\to\infty}|\eta_k| = \infty$, and $\widetilde{\Phi}(\eta_k) \leqslant c|\eta_k|$ for $k \in \mathbb{N}$. Since $\eta_k/|\eta_k| = 1$, there exists a subsequence (still denoted by $\{\eta_k\}$), and $\theta \in \mathbb{R}^n$, with $|\theta| = 1$, such that $\lim_{k\to\infty}\eta_k/|\eta_k| = \theta$. In particular, this limit implies that $\lim_{k\to\infty}\theta\cdot\eta_k/|\eta_k| = |\theta|^2 = 1$. Now, fix any $t > c$. Therefore,

$$\begin{aligned}\Phi(t\theta) = \widetilde{\widetilde{\Phi}}(t\theta) = \sup_{\eta\in\mathbb{R}^n}[t\theta\cdot\eta - \widetilde{\Phi}(\eta)] &\geqslant \sup_{k\in N}[t\theta\cdot\eta_k - \widetilde{\Phi}(\eta_k)]\\ &\geqslant \sup_{k\in N}\left[\left(t\frac{\theta\cdot\eta_k}{|\eta_k|} - c\right)|\eta_k|\right] = \infty.\end{aligned}$$

This contradicts the fact that Φ is finite valued.

Conversely, assume that (A 16) holds. Then,

$$\Phi(\xi) = \widetilde{\widetilde{\Phi}}(\xi) = \sup_{\eta\in\mathbb{R}^n} [\xi\cdot\eta - \widetilde{\Phi}(\eta)] \leqslant \sup_{\eta\in\mathbb{R}^n} [|\eta||\xi| - \widetilde{\Phi}(\eta)] = \sup_{\eta\in\mathbb{R}^n} \left[|\eta|\left(|\xi| - \frac{\widetilde{\Phi}(\eta)}{|\eta|}\right)\right]. \tag{A 17}$$

Fix any $\xi \in \mathbb{R}^n$. By (A 16), there exists $K > 0$ such that $\widetilde{\Phi}(\eta)/|\eta| \geqslant 2|\xi|$ if $|\eta| > K$, whence

$$|\eta|\left(|\xi| - \frac{\widetilde{\Phi}(\eta)}{|\eta|}\right) \leqslant 0 \quad \text{if } |\eta| > K. \tag{A 18}$$

From (A 17) and (A 18) we deduce that

$$\Phi(\xi) \leqslant \sup_{|\eta|\leqslant K} \left[|\eta|\left(|\xi| - \frac{\widetilde{\Phi}(\eta)}{|\eta|}\right)\right] \leqslant K|\xi| < \infty,$$

and hence Φ is finite valued. □

The following corollary is a straightforward consequence of proposition A.2 and of (A 15).

Corollary A.3. *Let Φ be an n-dimensional Young function. Then*

$$\Phi \textit{ is finite valued and } \lim_{|\xi|\to\infty} \frac{\Phi(\xi)}{|\xi|} = \infty$$

if and only if

$$\widetilde{\Phi} \textit{ is finite valued and } \lim_{|\eta|\to\infty} \frac{\widetilde{\Phi}(\eta)}{|\eta|} = \infty.$$

Proposition A.4. *Let Φ and Ψ be n-dimensional Young functions. Assume that Ψ is finite valued and there exists $M_0 \geqslant 0$ such that*

$$\Phi(\xi) \geqslant \Psi(\xi) \quad \textit{if } |\xi| \geqslant M_0. \tag{A 19}$$

Then there exists $M_1 \geqslant 0$ such that

$$\widetilde{\Phi}(\eta) \leqslant \widetilde{\Psi}(\eta) \quad \textit{if } |\eta| \geqslant M_1. \tag{A 20}$$

Conversely, assume that $\lim_{|\xi|\to\infty} \Phi(\xi)/|\xi| = \infty$ and (A 20) holds for some $M_1 \geqslant 0$. Then (A 19) holds for some $M_0 \geqslant 0$.

Proof. Assume that Ψ is finite valued, and (A 19) is in force. By proposition A.2, there exists $M_1 \geqslant 0$ such that $\widetilde{\Psi}(\eta) > M_0|\eta|$ if $|\eta| \geqslant M_1$. Thus, for any such η,

$$\begin{aligned}
\widetilde{\Phi}(\eta) = \sup_{\xi\in\mathbb{R}^n} [\xi\cdot\eta - \Phi(\xi)] &= \max\left\{ \sup_{|\xi|\leqslant M_0} [\xi\cdot\eta - \Phi(\xi)], \sup_{|\xi|>M_0} [\xi\cdot\eta - \Phi(\xi)]\right\} \\
&\leqslant \max\left\{M_0|\eta|, \sup_{|\xi|>M_0} [\xi\cdot\eta - \Psi(\xi)]\right\} \\
&\leqslant \max\{M_0|\eta|, \widetilde{\Psi}(\eta)\} \\
&= \widetilde{\Psi}(\eta).
\end{aligned}$$

Inequality (A 20) is thus established.

Conversely, assume that $\lim_{|\xi|\to\infty}\Phi(\xi)/|\xi| = \infty$, and (A 20) holds. Then there exists $M_0 \geqslant 0$ such that $\Phi(\xi) > M_1|\xi|$ whenever $|\xi| \geqslant M_0$. By (A 15),

$$\begin{aligned}\Psi(\xi) = \widetilde{\widetilde{\Psi}}(\xi) &= \max\Big\{\sup_{|\eta|\leqslant M_1}[\xi\cdot\eta - \widetilde{\Psi}(\eta)],\ \sup_{|\eta|>M_1}[\xi\cdot\eta - \widetilde{\Psi}(\eta)]\Big\}\\ &\leqslant \max\Big\{M_1|\xi|,\ \sup_{|\eta|>M_1}[\xi\cdot\eta - \widetilde{\Phi}(\eta)]\Big\}\\ &\leqslant \max\{M_1|\xi|, \Phi(\xi)\}\\ &= \Phi(\xi) \quad \text{if } |\xi| \geqslant M_0,\end{aligned}$$

namely, (A 19) holds. □

Proposition A.5. *Let $\Phi \in C^1(\mathbb{R}^n)$ be an n-dimensional Young function.*

(i) *$\Phi \in \Delta_2$ near ∞ if and only if $s_\Phi < \infty$.*

(ii) *$\Phi \in \nabla_2$ near ∞ if and only if $i_\Phi > 1$.*

Proof. Given any $\xi \in \mathbb{R}^n$, define the continuously differentiable Young function A by

$$A(t) = \Phi(t\xi) \quad \text{for } t \geqslant 0.$$

Note that $A'(t) = \Phi_\xi(t\xi)\cdot\xi$ for all $t \geqslant 0$.

(i) Assume that $s_\Phi < \infty$. Then for every $\varepsilon > 0$ there exists $M \geqslant 0$ such that

$$\frac{\Phi_\xi(\xi)\cdot\xi}{\Phi(\xi)} < s_\Phi + \varepsilon \quad \text{if } |\xi| \geqslant M.$$

Given any $\xi \in \mathbb{R}^n$ such that $|\xi| \geqslant M$, we have that

$$\frac{A'(t)}{A(t)} \leqslant \frac{s_\Phi + \varepsilon}{t} \quad \text{if } t \neq 0,$$

whence

$$A(t) \leqslant t^{s_\Phi+\varepsilon}A(1) \quad \text{if } t \geqslant 1. \tag{A 21}$$

The choice $t = 2$ in the last inequality yields

$$\Phi(2\xi) \leqslant 2^{s_\Phi+\varepsilon}\Phi(\xi) \quad \text{if } |\xi| \geqslant M.$$

This tells us that $\Phi \in \Delta_2$ near ∞.

Conversely, assume that $\Phi \in \Delta_2$ near ∞. Since $A'(t)$ is non-negative and non-decreasing,

$$\Phi(2\xi) = A(2) = \int_0^2 A'(t)\,\mathrm{d}t \geqslant \int_1^2 A'(t)\,\mathrm{d}t \geqslant A'(1) = \Phi_\xi(\xi)\cdot\xi \quad \text{for } \xi \in \mathbb{R}^n.$$

Thus, inasmuch as $\Phi \in \Delta_2$ near ∞, there exist $C \geqslant 2$ and $M \geqslant 0$ such that

$$C\Phi(\xi) \geqslant \Phi(2\xi) \geqslant \Phi_\xi(\xi)\cdot\xi \quad \text{if } |\xi| \geqslant M.$$

This shows that $s_\Phi \leqslant C$.

(ii) Assume that $i_\Phi > 1$. Then for every $\varepsilon \in (0, i_\Phi - 1)$ there exists $M > 0$ such that

$$\frac{\Phi_\xi(\xi)\cdot\xi}{\Phi(\xi)} > i_\Phi - \varepsilon > 1 \quad \text{if } |\xi| \geqslant M.$$

Hence, given any $\xi \in \mathbb{R}^n$ such that $|\xi| \geqslant M$,

$$\frac{A'(t)}{A(t)} \geqslant \frac{i_\Phi - \varepsilon}{t} \quad \text{if } t \geqslant 1.$$

Therefore, $A(t) \geqslant t^{i_\Phi-\varepsilon}A(1)$ if $t \geqslant 1$. Since $i_\Phi - \varepsilon > 1$, we may choose $t = 2^{1/(i_\Phi-\varepsilon-1)}$ in the last inequality. So doing, we obtain that

$$\Phi(2^{1/(i_\Phi-\varepsilon-1)}\xi) \geqslant 2^{(i_\Phi-\varepsilon)/(i_\Phi-\varepsilon-1)}\Phi(\xi) \quad \text{if } |\xi| \geqslant M.$$

Thus, $\Phi \in \nabla_2$ near ∞, since (A 11) holds with $k = 2^{1/(i_\Phi-\varepsilon-1)}$.

Conversely, assume that $\Phi \in \nabla_2$ near ∞. By (A 9), there exist $C > 2$ and $M \geqslant 0$ such that

$$C\int_0^1 A'(t)\,\mathrm{d}t = C\Phi(\xi) \leqslant \Phi(2\xi) = \int_0^2 A'(t)\,\mathrm{d}t \quad \text{if } |\xi| \geqslant M.$$

Consequently,

$$C\int_0^2 A'(t)\,\mathrm{d}t \leqslant \int_0^2 A'(t)\,\mathrm{d}t + C\int_1^2 A'(t)\,\mathrm{d}t \quad \text{if } |\xi| \geqslant M,$$

and hence

$$(C-1)\Phi(2\xi) \leqslant C\int_1^2 A'(t)\,\mathrm{d}t \leqslant CA'(2) = C\Phi_\xi(2\xi)\cdot\xi \quad \text{if } |\xi| \geqslant M.$$

Altogether,

$$1 < \frac{2(C-1)}{C} \leqslant \frac{\Phi_\xi(2\xi)\cdot 2\xi}{\Phi(2\xi)} \quad \text{if } |\xi| \geqslant M,$$

whence $i_\Phi > 1$. □

PROPOSITION A.6. *Let Φ be a finite-valued n-dimensional Young function such that* $\lim_{|\xi|\to\infty}\Phi(\xi)/|\xi| = \infty$.

(i) *$\Phi \in \Delta_2$ near ∞ if and only if $\widetilde{\Phi} \in \nabla_2$ near ∞.*

(ii) *$\Phi \in \nabla_2$ near ∞ if and only if $\widetilde{\Phi} \in \Delta_2$ near ∞.*

Proof. By corollary A.3 and (A 15), it suffices to prove part (ii). Assume that $\Phi \in \nabla_2$ near ∞. Let k and M be as in (A 11). Define the n-dimensional Young function Φ_1 as $\Phi_1(\xi) = \Phi(k\xi)/(2k)$ for $\xi \in \mathbb{R}^n$. Then

$$\Phi_1(\xi) \geqslant \Phi(\xi) \quad \text{if } |\xi| \geqslant M, \tag{A 22}$$

and

$$\widetilde{\Phi}_1(\eta) = \frac{\widetilde{\Phi}(2\eta)}{2k} \quad \text{for } \eta \in \mathbb{R}^n.$$

Owing to (A 22) and proposition A.4, there exists $M_1 > 0$ such that $\widetilde{\Phi}_1(\eta) \leqslant \widetilde{\Phi}(\eta)$ if $|\eta| \geqslant M_1$. This implies that $\widetilde{\Phi} \in \Delta_2$ near ∞.

Conversely, assume that $\widetilde{\Phi} \in \Delta_2$ near ∞, and let $C > 2$ and $M \geqslant 0$ be as in (A 8). Define the n-dimensional Young function Ψ by $\Psi(\eta) = \widetilde{\Phi}(2\eta)/C$ for $\eta \in \mathbb{R}^n$. By corollary A.3, $\lim_{|\xi|\to\infty} \Psi(\xi)/|\xi| = \infty$. We can thus argue as above, and find $M_1 > 0$ such that

$$\frac{1}{C}\Phi\left(\frac{C\xi}{2}\right) = \widetilde{\Psi}(\xi) \geqslant \widetilde{\widetilde{\Phi}}(\xi) = \Phi(\xi) \quad \text{if } |\xi| \geqslant M_1.$$

Hence, $\Phi \in \nabla_2$ near ∞. □

PROPOSITION A.7. *Let $\Phi \in C^1(\mathbb{R}^n)$ be an n-dimensional Young function. If*

$$\lim_{|\xi|\to\infty} \frac{\Phi(\xi)}{|\xi|} = \infty,$$

then

$$\widetilde{\Phi}(\Phi_\xi(\xi)) \leqslant \Phi(2\xi) \quad \textit{for } \xi \in \mathbb{R}^n. \tag{A 23}$$

Proof. If $\xi = 0$, then (A 23) holds trivially. Assume now that $\xi \neq 0$. Set

$$\eta = \Phi_\xi(\xi). \tag{A 24}$$

The function $g\colon \mathbb{R}^n \to \mathbb{R}$, defined as

$$g(\zeta) = \eta \cdot \zeta - \Phi(\zeta) \quad \text{for } \zeta \in \mathbb{R}^n,$$

is concave. Moreover, owing to our assumptions, $\lim_{|\zeta|\to\infty} g(\zeta) = -\infty$. Thus, it attains its maximum at every point ζ where its gradient vanishes, namely at those points such that $\eta = \Phi_\xi(\zeta)$. In particular, by (A 24), g attains its maximum at ξ. Therefore,

$$\widetilde{\Phi}(\Phi_\xi(\xi)) = \widetilde{\Phi}(\eta) = \max g = g(\xi) = \Phi_\xi(\xi) \cdot \xi - \Phi(\xi) \leqslant \Phi_\xi(\xi) \cdot \xi \leqslant \Phi(2\xi).$$

Hence, inequality (A 23) follows. □

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