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A STRAIN-DIFFERENCE BASED NONLOCAL ELASTICITY THEORY FOR SMALL-SCALE SHEAR-DEFORMABLE BEAMS WITH PARAMETRIC WARPING

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Abstract

The strain-difference based nonlocal elasticity theory devised by the authors (Polizzotto et al., 2006) is applied to homogeneous isotropic beams subjected to static loads. Shear deformation is taken into account and a warping parameter ω is used to fix the warping shape of the cross sections. On letting ω vary from zero to infinite, a continuous family of beam models is generated, which spans from the Euler-Bernoulli beam ($\omega = 0$) to the Timoshenko beam ($\omega \to \infty$), and identifies itself with the Reddy beam for $\omega = 2$. Taking as basic unknowns the *axial stretching* e, the *Euler-Bernoulli curvature* χ^{EB} and the *shear curvature* η, the boundary-value problem proves to be governed by three *uncoupled* integral equations whose input terms contain, beside the load data, eight arbitrary constants. These equations are solved by addressing a set of eight *uncoupled auxiliary integral equations independent of the boundary conditions*, each of which is either a Fredholm integral equation of

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the second kind, or is more complex but has strong similarities with the latter type of equation. This makes it possible to express (e, χ^{EB}, η), the axial and transverse displacements (u, w) and the shear angle γ to within the mentioned constants, which is useful to enforce the eight boundary conditions. The numerical solutions for simple beam problems are reported and graphically illustrated with particular concern to size effects and to their sensitivity to shear deformation.
Keywords: Nonlocal elasticity, Shear-deformable beams, Size effects

1. INTRODUCTION

The Eringen's nonlocal elasticity theory, either in its fully integral form (Eringen, 1972a,b, 1976, 1987; Eringen and Edelen, 1972), or in its differential form (Eringen, 1983), constitutes an appealing conceptual framework widely employed over the last fifty years for the study of size effects of small-scale structures. The importance of this research work for engineering applications is well documented by an extensive literature of which here we mention a few representative works as (Eringen, 1972a,b, 1976, 1983, 1987, 2002; Eringen and Edelen, 1972; Aydogdu, 2009; Eltaher et al., 2016; Eptaimeros et al., 2016; Faroughi et al., 2017; Gibson et al., 2007; Kumar et al., 2008; Peddieson et al., 2003; Pin Lu et al., 2007; Rafii-Tabar et al., 2016; Reddy, 2007, 2010; Sudak, 2003; Wang and Arash, 2014; Xu et al., 2016; Ma et al., 2008; Tuna and Kirca, 2016; Tuna et al., 2019).

As it emerges from the above reported literature, nonlocal continuum theories were (and still are) widely applied for the evaluation of the response of structural models as simple beams and plates used as sensors and actuators within micro- and nano-technologies. Contrary to the common expectation that "smaller is stiffer", it was found that the Eringen's nonlocal differential theory applied to beams and plates generally predicts softening size effects, or even no size effects at all as in the case of a cantilever beam under a point load (Peddieson et al., 2003). Analogous anomalies were also found for beams and plates under free vibrations (Pin Lu et al., 2006) and buckling conditions (Sudak, 2003), and for rods in tension (Benvenuti and Simone, 2013).

Over the years, the Eringen's nonlocal theories have been suitably modified in order to avoid the mentioned anomalies. Among these modified theories, we mention the two phase local/nonlocal model (Eringen, 1972b, 1987; Altan , 1989; Polizzotto, 2001; Benvenuti and Simone, 2013; Khodabakhshi and Reddy, 2015; Wang et al., 2016); solutions admitting non-continuous displacement fields (Challamel et al., 2016); hybrib models formed by the Eringen's model coupled with a strain gradient one (Challamel and Wang, 2008; Khodabakhshi and Reddy, 2015; Xu et al., 2017a,b); solutions of the Eringen's nonlocal differential problem accompanied by special boundary conditions guaranteeing the equivalence with the related purely nonlocal problem (Fernández-Sáez et al., 2016; Wang et al., 2016); the strain-difference model similar to the local/nonlocal two-phase one, but with a suitable strain difference field as source of the nonlocal phase (Polizzotto, 2002). The latter model was subsequently further elaborated and given the form on which the present paper is centered (Polizzotto et al., 2006).

A clear explanation of the limits of the Eringen's nonlocal integral theory was given by Romano et al. (2017). It was pointed out that, on the one hand, this theory leads to Fredholm integral equations of the first kind, hence it may lead to not well-posed boundary-value problems, and that, on the other hand, the solution of the nonlocal differential problem is in general different from the solution, if any, of the nonlocal integral model, except that the previously mentioned special boundary conditions may be enforced, but this is not always allowed by the equilibrium conditions of the beam.

A further discussion on the drawbacks encountered with the Eringen's nonlocal elasticity models was reported in a recent paper by the authors (Fuschi et al., 2019). Among other, it was pointed out that the requisite called "locality recovery condition" that is the property of a given nonsimple constitutive model to respond like a simple model that founds itself in a state of uniform strain Polizzotto et al. (2006), is not satisfied by the Eringen's nonlocal model. Also an explanation was given about the reason why this model predicts softening size effects in the majority of structural configurations. Indeed, on interpreting the Eringen's nonlocal integral model as a tool to redistribute the source (Hookean) local stress within a body, the redistribution is incomplete within a boundary layer of finite thickness of the body (but even within the whole body if sufficiently small), hence a fraction of the source local stress is there wasted with a consequent loss in stiffness.

The strain-difference based nonlocal elasticity theory advanced by Polizzotto et al. (2006) belongs to the same nonlocal strain-integral model family as the Eringen's nonlocal model, but it is exempt from all drawbacks of the latter model. It in fact is stress saving (that is, the stress redistribution is complete), it leads to a Fredholm integral equation of the second kind (hence to well-posed boundary-value problems), it complies with the locality recovery condition (that is, the stress response is uniform under any uniform strain state), it generally predicts stiffening size effects.

In Fuschi et al. (2019), the above strain-difference based nonlocal elasticity theory was applied to Euler-Bernoulli beams under quasi-static loads for the evaluation of the inherent size effects. Stiffening size effects were predicted in all studied beam cases (including the cantilever beam under point load at the free end), with results comparable with those obtainable by the widely accepted strain gradient elasticity theory. The solution procedure consists in a discrete numerical solution of a few mutually independent Fredholm integral equations of the second kind. The proposed method lends itself to various forms of generalizations to shear-deformable beams and to buckling conditions. Shear-deformable beams are considered in the present paper.

In order to account for shear deformation of a beam, it is required to assign a warping mode of the cross section in such a way that shear strain varies continuously from its maximum value at the neutral axis points and leads to zero value of the shear stress at the top and bottom surfaces of the beam. For this purpose, the longitudinal local displacement is enriched by extra addends in the form of polynomial functions of the transverse coordinate (see e.g. Reddy (2007)), or trigonometric functions (see e.g. Vidal and Polit (2010)).

An alternative warping model is the *parametric warping* proposed by Polizzotto (2015, 2018), whereby a continuous family of warping shapes is governed by a real scalar parameter, say $0 \leq \omega \leq \infty$. On letting ω vary, a continuous (or discrete) family of shear-deformable beams is generated, which spans from the Euler-Bernoulli beam for $\omega = 0$, to the Timoshenko one for $\omega \to \infty$, whereas it identifies itself with the Reddy beam model (Reddy, 2007) for $\omega = 2$. The latter warping model is adopted in the present paper and for the first time applied to nonlocal elastic shear deformable beams obeying the strain-difference based nonlocal elasticity theory.

The *objective* of the present paper is to extend to axial- and shear-deformable beams the study previously implemented by the authors (Fuschi et al., 2019) to evaluate size effects in small-scale Euler-Bernoulli beams subjected to quasi-static loads. For this aim, the strain-difference based nonlocal elasticity theory is applied under the hypotheses of small displacements and isotropic material. The main intent is to ascertain the influence of shear deformation on size effects.

The *method* herein adopted is based on an analytical/numerical procedure; it is centered on a particular choice of the basic unknown variables. These are the beam's axial stretching e, the Euler-Bernoulli curvature χ^{EB} (bending curvature pertaining to the Euler-Bernoulli beam) and the shear curvature η (first derivative of the shear angle γ). By this choice, the boundary-value problem will be shown to be governed by three *uncoupled* integral equations, of which two (related to e and χ^{EB}) are Fredholm integral equations of the second kind, the other (related to η) possesses strong similarity with this type of integral equations. The input terms of these governing equations are each expressed as a linear combination of some *auxiliary loading conditions* with in total as many constant coefficients as there are boundary conditions. Taking profit from the linearity of the problem and of the superposition principle, an *auxiliary integral equation* technique is envisioned, by which the beam's deformations (e, χ^{EB}, η), along with the axial and transverse displacements (u, w) and the shear angle γ are determined to within the mentioned constant coefficients, available to accommodate the inherent boundary conditions.

By a simple application to a cantilever beam it will be shown that the response of the beam is notably influenced by shear deformation and that therefore taking into account shear deformation may constitute a paramount issue for beam analysis.

In Section 2, some preliminaries of the beam model based on the strain-difference nonlocal elasticity theory are presented together with the essentials of the parametric warping technique. The constitutive equations of the beam model are also reported together with the warping coefficients. The beam problem is addressed in Section 3, where the equilibrium equations and the boundary conditions are reported together with the governing uncoupled integral equations. In Section 4, the auxiliary integral equations are derived and discussed. In Section 5, a generalization of the Navier formula for the normal stress and the Jourawski formula for the shear stress are presented. In Section 6 applications of the theory to beam problems are presented and illustrated. Conclusions are drawn in the last Section 7.

A standard notation is used throughout. The meaning of particular symbols used on occasion will be given in the text at their first appearance. The symbol := means equality by definition.

2. PRELIMINARIES TO THE BEAM AND CONSTITUTIVE EQUATIONS

A straight beam of length L is considered which is referred to Cartesian orthogonal axes (x, y, z) with x coinciding with the beam axis, z along the beam height, y in the width direction. The cross section is rectangular, of hight h and area S, see Figure 1. The kinematics of the beam is described

Figure 1: Geometrical sketch of the beam model.

by the (small) displacements

$$\left.\begin{array}{l} u_x(x,z) = u(x) - zw'(x) + \Theta(z)\gamma(x) \\ \\ u_y \equiv 0, \qquad u_z(x,z) = w(x) \end{array}\right\} \tag{1}$$

Here, $u(x)$ is the axial displacement, $w(x)$ the transverse displacement, $\gamma(x)$ the shear angle, that is, the cross section relative (anticlockwise) rotation with respect to the normal to the deflected axis measured at the centroid of the cross section; whereas $\Theta(z)$ denotes the *shear warping function* (specified later on) and $w'(x) := \mathrm{d}w(x)/\mathrm{d}x$. Therefore, there are only two meaningful strain components, that is,

$$\left.\begin{array}{l} \varepsilon_{xx}(x,z) = e(x) + z\chi(x) + \Theta(z)\eta(x) \\ \\ 2\,\varepsilon_{xz}(x,z) = \Theta'(z)\gamma(x) \end{array}\right\} \tag{2}$$

where $e(x) = u'(x)$ (axial stretching), $\chi(x) = -w''(x)$ (bending curvature), and $\eta(x) = \gamma'(x)$ (shear curvature).

2.1. **The Strain-Difference Based Nonlocal Model**

The strain-difference based nonlocal model is described in (Polizzotto et al., 2006) for 3D elastic solids and there also specialized for beam models, see also (Fuschi et al., 2019). The fundamental mathematical ingredients are the influence function $g_\ell(x,\bar{x})$ and the weight function $\Gamma_\ell(x)$, which are expressed as

$$g_\ell(x,\bar{x}) := \frac{1}{2\ell}\exp\big(-\frac{|x-\bar{x}|}{\ell}\big) \tag{3}$$

and

$$\Gamma_\ell(x) := \int_0^L g_\ell(x,\bar{x})\,\mathrm{d}\bar{x} = 1 - \frac{1}{2}\Big[\exp\Big(-\frac{x}{\ell}\Big) + \exp\Big(-\frac{L-x}{\ell}\Big)\Big] \tag{4}$$

where ℓ denotes the beams's internal length parameter ($\ell < L$). Also, there is a kernel function expressed as

$$\kappa(x,\bar{x}) := \Big[\Gamma_\ell(x) + \Gamma_\ell(\bar{x})\Big]g_\ell(x,\bar{x}) - \int_0^L g_\ell(x,p)g_\ell(\bar{x},p)\,\mathrm{d}p \tag{5}$$

along with a (local phase density) function

$$s(x) := 1 + \alpha\,\Gamma_\ell^2(x) \tag{6}$$

where α is a material constant to be identified via experimental tests on the material in use and whose influence on the nonlocal solution was addressed in Fuschi et al. (2015).

With the above definitions in mind, the (nonlocal) stresses generated by a specified strain field within the isotropic elastic beam according to the mentioned strain-difference model are

$$\left.\begin{aligned} \sigma_{xx}(x,z) &= E\mathcal{J}[\varepsilon_{xx}](x,z) \\ \\ \sigma_{xz}(x,z) &= \mu\mathcal{J}[2\varepsilon_{xz}](x,z) \end{aligned}\right\} \tag{7}$$

where E = Young modulus, μ = shear modulus, whereas $\mathcal{J}$ is an integral operator defined as

$$\mathcal{J}[\phi](x,\ldots) := s(x)\phi(x,\ldots) - \alpha\int_0^L \kappa(x,\bar{x})\phi(\bar{x},\ldots)\mathrm{d}\bar{x} \tag{8}$$

with $\phi(x,\ldots)$ being a function of x and possibly of other space co-ordinates, or even a constant.

2.2. Warping Function

Following (Polizzotto, 2015, 2018), the warping function $\Theta(z)$ is chosen in the form

$$\Theta(z) := z - \frac{|z|^{1+\omega}\,\mathrm{sign}(z)}{(1+\omega)\left(\frac{h}{2}\right)^{\omega}} \tag{9}$$

holding for any real nonnegative value of the *warping parameter* ω, as well as for any z, but $|z| \leq h/2$. It is easily verified that for $\omega = 0$ it is $\Theta(z) \equiv 0$ (no warping, Euler-Bernoulli beam), whereas for $\omega = \infty$ it is $\Theta(z) \equiv z$ (no warping but with a nonzero shear rotation, Timoshenko beam). For easy reference the warping function $\Theta(z)$ and its derivative $\Theta'(z)$ are plotted in Figure 2 for different values of ω.

Figure 2: Warping function $\Theta(z)$ for different values of ω: a) $\Theta(z)$; b) $\Theta'(z)$

Since

$$0 \leq \Theta'(z) = 1 - \left(\frac{2\,|z|}{h}\right)^{\omega} \leq 1 \tag{10}$$

and $\Theta'(\pm\frac{h}{2}) = 0$, it results from Eq.$(2)_2$ that the shear strain at the top and bottom surfaces of the beam is vanishing, that is, $\varepsilon_{xz}(x, \pm h/2) = 0$, hence $\sigma_{xz}(x, \pm h/2) = \mu\mathcal{J}[2\varepsilon_{xz}](x, \pm h/2) = 0$, $\forall x \in (0, L)$, as it is required. The quantity

$$\Delta u_x(x, z) = \Theta(z)\gamma(x) \tag{11}$$

gives the fraction of the displacement $u_x(x, z)$ due to the warping of the cross section.

2.3. Beam's Constitutive Equations

The set of stress resultants of the shear deformable beam includes, beside the standard stress resultants, that is,

$$[N, M, Q]_x = \int_S [\sigma_{xx}, z\sigma_{xx}, \sigma_{xz}]_x \,\mathrm{d}a \tag{12}$$

two further resultants specific of the shear-deformable beam, that is,

$$\left[\widehat{M}, \widehat{Q}\right]_x = \int_S [\Theta(z)\sigma_{xx},\ \Theta'(z)\sigma_{xz}]_x \,\mathrm{d}a \tag{13}$$

where da denotes infinitesimal area measure. We shall refer to $\widehat{M}$ as the *warping stress moment*, and to $\widehat{Q}$ as the *warping shear force*. For obvious reasons, no warping axial force is allowed to exist. It can be easily verified that $\widehat{M}(x) \equiv \widehat{Q}(x) \equiv 0$ for $\omega = 0$ (Euler-Bernoulli beam), and $\widehat{M}(x) \equiv M(x)$, $\widehat{Q}(x) = Q(x)$ for $\omega = \infty$ (Timoshenko beam).

Next, substituting Eq.(2) into Eq.(7) gives the stresses expressed as

$$\left.\begin{aligned} \sigma_{xx}(x,z) &= E\left\{\mathcal{J}[e](x) + z\mathcal{J}[\chi](x) + \Theta(z)\mathcal{J}[\eta](x)\right\} \\ \sigma_{xz}(x,z) &= \mu\Theta'(z)\mathcal{J}[\gamma](x) \end{aligned}\right\} \tag{14}$$

Eq.$(14)_2$ states that a non-zero shear stress is allowed to exist at a cross section x if, and only if, the shear angle $\gamma(x) \neq 0$ and thus, a non-trivial warping deformation occurs at that cross section. This fact is a characteristic feature of the beam model under study, borrowed from the classical Euler-Bernoulli model. For application purposes, a suitable extended form of the Jourawski formula will be provided (in Section 5).

Then substituting Eq.(14) into (12) and (13) gives the beam's constitutive equations as

$$\left.\begin{aligned} N(x) &= ES\mathcal{J}[e](x) \\ M(x) &= EI\mathcal{J}[\chi + a\,\eta](x) \\ \widehat{M}(x) &= EI\mathcal{J}[a\,\chi + b\,\eta](x) \\ \widehat{Q}(x) &= \mu Sd\mathcal{J}[\gamma](x) \\ Q(x) &= \mu Sc\mathcal{J}[\gamma](x) \end{aligned}\right\} \tag{15}$$

Here, S is the cross section area, I the second area moment. The (non-dimensional) quantities (a, b, c, d) denote the *warping coefficients* which, in the present case of isotropic material and rectangular cross section, turn out to be functions of the warping parameter ω defined as

$$\left.\begin{aligned} [a,b]\,(\omega) &= \tfrac{1}{I}\int_S \left[z\,\Theta(z), \Theta^2(z)\right]_\omega \mathrm{d}a \\ \\ [c,d]\,(\omega) &= \tfrac{1}{S}\int_S \left[\Theta'(z), \Theta'^2(z)\right]_\omega \mathrm{d}a \end{aligned}\right\} \tag{16}$$

The following equality holds true

$$Q(x) = \frac{c(\omega)}{d(\omega)}\widehat{Q}(x) \quad \forall x \in (0, L) \tag{17}$$

The functions $a(\omega)$, $b(\omega)$, $c(\omega)$ and $d(\omega)$ for rectangular cross section and isotropic material are reported in (Polizzotto, 2015, 2018). For easy reference, they are also reported here, namely,

$$\left.\begin{aligned} a(\omega) &= 1 - \frac{3}{(1+\omega)(3+\omega)} \\ b(\omega) &= 1 - \frac{6}{(1+\omega)(3+\omega)} + \frac{3}{(1+\omega)^2(3+2\omega)} \\ c(\omega) &= \frac{\omega}{1+\omega} \\ d(\omega) &= \frac{2\,\omega^2}{1+3\omega+2\,\omega^2} \end{aligned}\right\} \tag{18}$$

For later use, the coefficient β is also reported here, namely,

$$\beta(\omega) := b(\omega) - a^2(\omega) = \frac{3\,\omega^2}{(1+\omega)^2(3+\omega)^2(3+2\omega)} \tag{19}$$

For notational simplicity, in the following the dependence of the warping coefficients upon ω will not be explicitly indicated, except whenever necessary for more clarity. It is worth noting that different cross sections, such as T-shaped, I-shaped, circular, can be addressed by an appropriate definition of the warping function $\Theta(z)$ fixed by Eq.(9) and of the warping coefficients here defined by Eqs.(16).

3. THE BEAM PROBLEM

In this section, the beam's equilibrium equations and boundary conditions are first derived through the principle of virtual power (PVP), then the displacement governing equations are reported.

3.1. Equilibrium Equations

The beam is subjected to distributed body forces $b_x(x,z)$, $b_z(x,z)$ acting quasi-statically. Denoting by upper tildes the virtual kinematic variables, the PVP reads as

$$\begin{aligned}\int_0^L \int_S (\sigma_{xx}\,\tilde{\varepsilon}_{xx} + 2\,\sigma_{xz}\,\tilde{\varepsilon}_{xz})\,\mathrm{d}a\,\mathrm{d}x &= \int_0^L \int_S (b_x\tilde{u}_x + b_z\tilde{u}_z)\,\mathrm{d}a\,\mathrm{d}x \\ &+ \underbrace{\left[\bar{N}\,\tilde{u} + \bar{Q}\,\tilde{w} - \bar{M}\,\tilde{w}' + \bar{C}\,\tilde{\gamma}\right]_0^L}_{\text{free ends}}\end{aligned} \tag{20}$$

where $\bar{N}, \bar{Q}, \bar{M}$ and $\bar{C}$ denote assigned resultant forces and couples applied at the free ends. Equation (20) has to be satisfied identically for any choice of the virtual displacements and strains complying with Eqs.(1) and (2) along with the conditions $\tilde{u} = \tilde{w} = \tilde{w}' = \tilde{\gamma} = 0$ at the constrained ends where u, w, w' and γ are specified, that is,

$$u=\bar{u},\ w=\bar{w},\ w'=\bar{w}',\ \gamma=\bar{\gamma} \qquad \text{(constrained ends)} \tag{21}$$

Substituting (1) and (2) into (20) and operating in a straightforward manner, we can obtain the field equilibrium equations of the beam as

$$\left.\begin{aligned} &N'(x)+p_x(x)=0 \\ &M''(x)+p_z(x)+m'(x)=0 \\ &\widehat{M}'(x)-\widehat{Q}(x)+\widehat{m}(x)=0 \end{aligned}\right\} \quad \forall\, x\in(0,L) \tag{22}$$

where it is

$$\left.\begin{aligned} p_x(x) &:= \int_S b_x(x,z)\mathrm{d}a \\ p_z(x) &:= \int_S b_z(x,z)\mathrm{d}a \\ m(x) &:= \int_S z\, b_x(x,z)\mathrm{d}a \\ \widehat{m}(x) &:= \int_S \Theta(z) b_x(x,z)\mathrm{d}a \end{aligned}\right\} \tag{23}$$

The *boundary conditions* imply that at every beam end it must be:

$$\left.\begin{aligned} &\text{Either } u=\bar{u} \text{ and } N \text{ free}, &&\text{or } N=\bar{N} \text{ and } u \text{ free} \\ &\text{Either } w=\bar{w} \text{ and } M' \text{ free}, &&\text{or } M'=\bar{Q} \text{ and } w \text{ free} \\ &\text{Either } w'=\bar{w}' \text{ and } M \text{ free}, &&\text{or } M=\bar{M} \text{ and } w' \text{ free} \\ &\text{Either } \gamma=\bar{\gamma} \text{ and } \widehat{M} \text{ free}, &&\text{or } \widehat{M}=\bar{C} \text{ and } \gamma \text{ free} \end{aligned}\right\} \tag{24}$$

The boundary conditions of $(24)_4$ are explicitly affected by shear warping, the other boundary conditions are as the classical ones. It may be convenient to fix the value of the absolute rotaion ϕ at one end, then, since $\gamma = \phi + w'$, this condition is equivalent to $\gamma = \bar{\phi} + w'$ at that end while $\widehat{M}$ is free.

Next, by integration of the differential equations (22) we can obtain a closed form representation of the *class of stress resultants satisfying the field equilibrium equations* as follows:

$$\left.\begin{aligned} N(x) &= N_0(x) + ESA_1 \\ M(x) &= M_0(x) + \frac{E\,I}{L^2}\,(B_1 x + B_2 L) \\ \widehat{M}(x) &= \int_0^x \widehat{Q}(\bar{x})\mathrm{d}\bar{x} + \widehat{M}_0(x) + \frac{E\,I}{L}\,C \end{aligned}\right\} \tag{25}$$

where A_1, B_1, B_2, C are non-dimensional constants and

$$\left.\begin{aligned} N_0(x) &:= -\int_0^x p_x(\bar{x})\mathrm{d}\bar{x} \\ M_0(x) &:= -\int_0^x \left[(x-\bar{x})\,p_z(\bar{x}) + m(\bar{x})\right]\mathrm{d}\bar{x} \\ \widehat{M}_0(x) &:= -\int_0^x \widehat{m}(\bar{x})\mathrm{d}\bar{x} \end{aligned}\right\} \tag{26}$$

In the case of statically determinate beams the expressions of $N, M, \widehat{M}, Q$ prove to be uniquely determinate.

Next, substituting Eq.(15) into the left-hand side of (25), we obtain the following set of integral equations, namely,

$$\mathcal{J}[e]\,(x) = \frac{1}{ES}\,N_0(x) + A_1 \tag{27}$$

$$\mathcal{J}[\chi + a\,\eta]\,(x) = \frac{1}{EI}\,M_0(x) + \frac{1}{L^2}\,[B_1 x + B_2 L] \tag{28}$$

$$\mathcal{J}[a\,\chi + b\,\eta]\,(x) - \frac{\mu S d}{EI}\int_0^x \mathcal{J}\,[\gamma]\,(\bar{x})\mathrm{d}\bar{x} = \frac{1}{EI}\,\widehat{M}_0(x) + \frac{1}{L}C \tag{29}$$

It can be recognized that (27) and (28) exhibit a form reducible to a Fredholm integral equation of the second kind, whereas (29) exhibits a more complex form amounting to a linear combination of such integral equations. Since $\eta(x) = \gamma'(x)$, Eqs.(27-29) constitute a set of integral equations governing the beam problem useful for the evaluation of the beam deformations $e(x)$, $\chi(x)$, $\eta(x)$. Eq.(27) governs the beam axial deformation and is independent of the other two equations. Instead, the latter two equations are mutually coupled, but they can be rendered uncoupled by suitably changing the basic unknown variables of the problem.

For this purpose, let us introduce a new state variable, say χ^{EB}, defined as

$$\chi^{EB}(x) := \chi(x) + a\,\eta(x), \quad \forall\, x \in (0, L) \tag{30}$$

and let us remark that by $(15)_2$ χ^{EB} satisfies the identity

$$M(x) = EI\mathcal{J}[\chi + a\,\eta]\,(x) = EI\,\mathcal{J}\left[\chi^{EB}\right](x). \tag{31}$$

This means that $\chi^{EB}(x)$ is the *bending curvature associated to a bending moment $M(x)$ in a strain-difference nonlocal (shear-undeformable) Euler-Bernoulli beam*. For this reason, $\chi^{EB}(x)$ is here referred to as the *Euler-Bernoulli (bending) curvature.*

Hence, recalling (19), we can write the equality

$$a\,\chi(x) + b\,\eta(x) = a\,\chi^{EB}(x) + \beta\,\eta(x) \tag{32}$$

Substituting (30) and (31) into (28) and (29), respectively, and introducing the (non-dimensional) parameter

$$\lambda^2 := \frac{\mu S L^2 d}{EI} \tag{33}$$

Eqs. (28) and (29) become

$$\mathcal{J}\left[\chi^{EB}\right](x) = \frac{1}{EI}\,M_0(x) + \frac{1}{L^2}\left[B_1 x + B_2 L\right] \tag{34}$$

and

$$\begin{aligned} \beta\,\mathcal{J}[\eta]\,(x) - \tfrac{\lambda^2}{L^2}\textstyle\int_0^x \mathcal{J}[\gamma]\,(\bar{x})\mathrm{d}\bar{x} = \tfrac{1}{EI}\,\widehat{M}_0(x) + \tfrac{1}{L}C_1 \\ -a\,\left[\tfrac{1}{EI}\,M_0(x) + \tfrac{1}{L^2}B_1 x\right] \end{aligned} \tag{35}$$

where the substitution $C_1 = C - aB_2$ has been operated. Since $\eta = \gamma'$, Eqs. (34) and (35) are mutually *uncoupled integral equations* for the unknown variables χ^{EB} and η, respectively.

We also observe the following:

i) For the Euler-Bernoulli beam ($\omega = 0$), since correspondingly it is $a = b = \beta = d = 0$ and $\chi^{EB} = \chi$, then (34) identifies with the equation pertaining to the Euler-Bernoulli model, whereas (35) disappears.

ii) For the Timoshenko beam ($\omega \to \infty$), since correspondingly it is $a = b = d = 1$ and $\beta = 0$, then (34) remains as it is but with $\chi^{EB} = \chi + \eta = \phi'$ (Timoshenko curvature), whereas (35) becomes

$$\frac{\lambda^2}{L^2}\int_0^x \mathcal{J}[\gamma]\,(\bar{x})\mathrm{d}\bar{x} = \frac{1}{EI}\left[a\,M_0(x) - \widehat{M}_0(x)\right] + \frac{a}{L^2}B_1 x - \frac{1}{L}C_1 \tag{36}$$

which governs the Timoshenko beam problem.

4. SOLUTION METHOD BY AUXILIARY INTEGRAL EQUATIONS

Equations (27), (34) and (35) contain some unknown constants to be determined by the beam's boundary conditions; these equations cannot thus be solved in general in the form in which they are, except in the case of statically determinate beams. Due to the linearity of the concerned equations, a suitable solution method consists in expressing the unknown functions $e(x)$, $\chi^{EB}(x)$ and $\eta(x)$ each as a linear combination of *auxiliary unknown functions*, similar to the related source terms on the right-hand side of the governing integral equations. This solution method is in more details explained in the next sub-sections, starting with Eq.(27) associated to the beam's axial stretching.

4.1. **Axial Stretching Equation (27)**

Looking at the right-hand side of (27), let the unknown stretching $e(x)$ be expressed in the form

$$e(x) = e_0(x) + A_1\, e_1(x) \tag{37}$$

where $e_0(x)$ and $e_1(x)$ are auxiliary response functions, whereas A_1 is the *same* constant appearing on the right-hand side of (27). Substituting (37) into (27) gives

$$\mathcal{J}\left[e_0\right](x) - \frac{1}{ES}\, N_0(x) + A_1 \left\{\mathcal{J}\left[e_1\right](x) - 1\right\} = 0 \tag{38}$$

Since this equation must hold for arbitrary values of A_1, then two mutually independent *auxiliary integral equations* are generated, that is,

$$\mathcal{J}[e_n](x) = R_n(x) \quad (n = 0, 1) \tag{39}$$

where

$$R_n(x) := \begin{cases} N_0(x)/(ES) & (n = 0) \\ \\ 1 & (n = 1) \end{cases} \tag{40}$$

The above equations are Fredholm integral equations of the second kind with a symmetric, positive definite kernel, which are known to admit each a unique solution (Tricomi, 1985; Polyanin and Manzhirov, 2008). Notably, these solutions can be obtained by means of a routine numerical method, *independently of the beam's boundary conditions.*

Next, the axial displacement $u(x)$ can be readly derived by integration of the differential equation $e(x) = u'(x)$. We can write

$$u(x) = u_0(x) + A_1\, u_1(x) + A_2\, L \tag{41}$$

where A_2 is a further (non-dimensional) constant, whereas

$$u_n(x) := \int_0^x e_n(\bar{x})\mathrm{d}\bar{x} \quad (n = 0, 1) \tag{42}$$

Indeed, the axial displacement $u(x)$ is determined to within the constants A_1 and A_2, for which the boundary conditions of the beam problem must be invoked.

4.2. **Bending Equation (34)**

Equation (34) is not affected by the explicit presence of the warping parameter ω, hence it is formally like the equation pertaining to the Euler-Bernoulli beam model. For this reason the unknown variable χ^{EB} has been called "Euler-Bernoulli curvature", but one has to have in mind that, in virtue of (30), χ^{EB} does not coincide with the bending curvature χ, except that the beam is shear-undeformable, hence $\eta \equiv 0$, as it is the case for the Euler-Bernoulli beam.

In analogy to the previous sub-section, let χ^{EB} be split as

$$\chi^{EB}(x) = \chi_0^{EB}(x) + B_1\, \chi_1^{EB}(x) + B_2\, \chi_2^{EB}(x) \tag{43}$$

where B_1 and B_2 are the same constants of (34). Then, substituting (43) into (34) permits us to write a set of three auxiliary integral equations as

$$\mathcal{J}\left[\chi_n^{EB}\right](x) = U_n(x), \quad (n = 0, 1, 2) \tag{44}$$

where

$$U_n(x) := \begin{cases} M_0(x)/(EI) & (n = 0) \\ \\ x/L^2 & (n = 1) \\ \\ 1/L & (n = 2) \end{cases} \tag{45}$$

By these (mutually independent) equations, the auxiliary response functions χ_0^{EB}, χ_1^{EB} and χ_2^{EB} can be uniquely determined again *independently of the* *beam boundary conditions.*

For later use, it is useful to construct an auxiliary Euler-Bernoulli transverse displacement function, say $w^{EB}(x)$, related to χ^{EB} through the differential relation $\chi^{EB} = -\left(w^{EB}\right)''(x)$. Then, by integration of the latter equation and recalling (43) we can write

$$w^{EB}(x) = w_0^{EB}(x) + B_1\, w_1^{EB}(x) + B_2\, w_2^{EB}(x) + B_3\, x + B_4\, L \tag{46}$$

in which B_3 and B_4 are further (non-dimensional) constants, and

$$w_n^{EB}(x) := -\int_0^x (x - \bar{x})\, \chi_n^{EB}(\bar{x}) \mathrm{d}\bar{x} \quad (n = 0, 1, 2) \tag{47}$$

Therefore, the displacement $w^{EB}(x)$ is so derived to within the four constants B_1, B_2, B_3, B_4.

4.3. **Shear Deformation Equation (35)**

Applying again the procedure used before, let $g(x)$ be a function such that $g'(x) = \eta(x), \forall x \in (0, L)$ and let η and g be decoupled as

$$\left.\begin{aligned} \eta(x) &= \eta_0(x) + B_1\eta_1(x) + C_1\eta_2(x) \\ g(x) &= g_0(x) + B_1 g_1(x) + C_1 g_2(x) \end{aligned}\right\} \tag{48}$$

Here, $\eta_n(x)$ and $g_n(x)$, $(n = 0, 1, 2)$, are mutually related by the integral equations

$$g_n(x) = \int_0^x \eta_n(p)\, \mathrm{d}p, \ (n = 0, 1, 2), \quad \forall x \in (0, L) \tag{49}$$

Substituting (48) in (35) gives

$$\beta \mathcal{J}[\eta_0]\,(x) - \tfrac{\lambda^2}{L^2} \int_0^x \mathcal{J}[g_0]\,(\bar{x})\mathrm{d}\bar{x} + \tfrac{1}{EI}\left[a\, M_0(x) - \widehat{M_0}(x)\right]$$

$$+ B_1 \left\{\beta \mathcal{J}[\eta_1]\,(x) - \tfrac{\lambda^2}{L^2} \int_0^x \mathcal{J}[g_1]\,(\bar{x})\mathrm{d}\bar{x} + \tfrac{a}{L^2}\, x\right\} \tag{50}$$

$$+ C_1 \left\{\beta \mathcal{J}[\eta_2]\,(x) - \tfrac{\lambda^2}{L^2} \int_0^x \mathcal{J}[g_2]\,(\bar{x})\mathrm{d}\bar{x} - \tfrac{1}{L}\right\} = 0$$

From (50) we can obtain the auxiliary integral equations

$$\beta \mathcal{J}[\eta_n]\,(x) - \frac{\lambda^2}{L^2} \int_0^x \mathcal{J}[g_n]\,(\bar{x})\mathrm{d}\bar{x} = V_n(x) \quad (n = 0, 1, 2) \tag{51}$$

in which (49) holds along with

$$V_n(x) := \begin{cases} -\tfrac{1}{EI}\left[a\, M_0(x) - \widehat{M_0}(x)\right] & (n = 0) \\ -a\, x/L^2 & (n = 1) \\ a/L & (n = 2) \end{cases} \tag{52}$$

Equations (51) exhibit a form more complex than the other auxiliary integral equations met before. They are however linear and possess several characteristics of a Fredholm integral equation of the second kind, which encourage us to assume that the equations in question may be uniquely solved. Next, assuming that the auxiliary response functions be known, we can construct the shear angle $\gamma(x)$ by writing

$$\gamma(x) = g_0(x) + B_1 g_1(x) + C_1 g_2(x) + C_2 \tag{53}$$

where the $g_n(x)(n = 0, 1, 2)$, are given by (49) and C_2 is an arbitrary (non-dimensional) constant.

It remains to determine the beam deflection $w(x)$. For this purpose, let us recall (30) and let us introduce a *shear potential function* $G(x)$ (of dimension a length) satisfying the condition

$$G'(x) = \gamma(x) = g_0(x) + B_1 g_1(x) + C_1 g_2(x) + C_2 \tag{54}$$

Therefore, since $\eta(x) = G''(x)$, recalling that $\chi(x) = -w''(x)$ and $\chi^{EB}(x) = -\left(w^{EB}\right)''(x)$, by integration we can rewrite (30) in the form

$$w(x) = w^{EB}(x) + a\,G(x), \qquad \forall\, x \in (0, L) \tag{55}$$

Indeed, the shear potential function $G(x)$ constitutes a scaled shear deflection of the beam. Next, by (53) and by integration of (54) we can write

$$G(x) = G_0(x) + B_1 G_1(x) + C_1 G_2(x) + C_2 x \tag{56}$$

where

$$G_n(x) := \int_0^x (x - \bar{x})\, \eta_n(\bar{x})\, \mathrm{d}\bar{x}, \quad (n = 0, 1, 2) \tag{57}$$

No additive integration constant is introduced into (56) since G can be determined to within an additive constant.

Next, substituting (46) and (56) into (55) gives $w(x)$ cast in the form

$$\begin{aligned} w(x) = {} & w_0(x) + B_1 w_1(x) + B_2 w_2(x) \\ & + (C_1 - B_2)\, a\, G_2(x) + (B_3 + a\, C_2)\, x + B_4 L \end{aligned} \tag{58}$$

where the $w_n(x)$ are given by

$$w_n(x) := w_n^{EB}(x) + a\, G_n(x), \quad (n = 0, 1, 2) \tag{59}$$

The deflection response function $w(x)$ is thus represented to within the six constants B_1, B_2, B_3, B_4, C_1, C_2.

4.4. **Solution Procedure**

After the details presented previously within the present section, a solution scheme emerges quite naturally. Indeed, the analysis for every loaded beam case, proceeds with three sequential computational steps, that is,

(a) Preliminary solution of the eight (mutually independent) auxiliary integral equations (39), (44) and (51). This computation does not require the knowledge of the boundary conditions, (the load conditions affect only the zero-th auxiliary integral equation).

(b) Construction of the response functions $u(x)$, $w(x)$, $\gamma(x)$ of Eqs. (41), (55) and (54), respectively.

(c) Enforcing the boundary conditions (24) for every specific loaded beam case, such as to evaluate the eight constants A_1, A_2, B_1, B_2, B_3, B_4, C_1, C_2 and thus to complete the solution procedure.

It is worth noting that the eight auxiliary integral equations of step (a) are of two different forms. One of them belongs to Eqs.(39) and (44) and, cast in a non-dimensional form, reads as

$$s(x)\phi(x) - \alpha \int_0^1 K(x, y)\phi(y)\,\mathrm{d}y = F(x) \tag{60}$$

in which $0 \leq (x, y) \leq 1$ and the kernel $K(x, y) := L\,\kappa(x, y)$. Additionally, the unknown $\phi(y)$ represents in turn the original unknowns $e_n(x)$ and $L\,\chi_n^{EB}(x)$, whereas $F(x)$ represents $R_n(x)$ and $LU_n(x)$. Eq. (60) may be easily transformed to take on the typical form of a Fredholm integral equation of the second kind, but this is believed not necessary for the numerical computation.

The other (non-dimensional) form of integral equation, met on addressing (51), reads as

$$\begin{aligned}&\beta\left[s(x)\phi(x) - \alpha\int_0^1 K(x,y)\phi(y)\,\mathrm{d}y\right] - \lambda^2\int_0^x \{s(y)\,\Phi(y)\\ &\qquad -\alpha\int_0^1 K(y,p)\,\Phi(p)\,\mathrm{d}p\}\,\mathrm{d}y = F(x)\end{aligned} \tag{61}$$

where

$$\Phi(p) := \int_0^p \phi(q)\mathrm{d}q \tag{62}$$

Here again $0 \leq (x, y, p, q) \leq 1$, $K(x, y) := L\kappa(x, y)$, whereas $\phi(x)$ is representative of $L\eta_n(x)$, $F(x)$ of $LV_n(x)$. This latter integral equation is more complex than (60), but we presume that it may be solved through the numerical method used for (60).

4.5. **Numerical Algorithm**

The numerical algorithm used to solve the integral equations of the form (60) is the Nystrom method reported by (Press et al. (1997), pp.782–785). The solution is obtained in the form of a linear equation system, whereby the main point is the choice of the quadrature points x_i, $(i = 1, 2, \ldots, N)$, along with the weights W_i (Gauss-Legendre quadrature rule).

Equation (60) is approximated in the form

$$s(x)\phi(x) - \alpha \sum_{j=1}^{N} W_j \, K(x, x_j) \, \phi_j = F(x) \tag{63}$$

where $\phi_j = \phi(x_j)$. Then, on enforcing (63) at every $x = x_i$, gives

$$\sum_{j=1}^{N} (s_j \, \delta_{ij} - \alpha W_j \, K_{ij}) \; \phi_j = F_i, \quad (i = 1, 2, \ldots, N) \tag{64}$$

which is a set of N linear equations in the unknowns ϕ_i. Then, substituting the discrete values ϕ_i into (63) the solution function $\phi(x)$ is obtained, which generally is well-conditioned, unless α is very close to an eigenvalue of (60).

The numerical algorithm to solve the integral equations of the form (61) is equally inspired to the mentioned Nystrom method. Eq.(61) is approximated in the form

$$\begin{aligned} &\beta \left[s(x)\phi(x) - \alpha \textstyle\sum_{j=1}^{N} W_j \, K(x, x_j) \, \phi_j \right] \\ &-\lambda^2 \textstyle\sum_{j=1}^{i} W_j \, \left(s_j \, \Phi_j - \alpha \sum_{k=1}^{N} W_k \, K_{jk} \, \Phi_k \right) = F(x) \end{aligned} \tag{65}$$

where

$$\Phi_m := \Phi(x_m) = \sum_{l=1}^{m} W_l \, \phi_l, \quad (l = j, k) \tag{66}$$

On enforcing (65) at every x_i we get a linear equation system as

$$\sum_{l=1}^{N} \Big[\beta\ (s_l\, \delta_{il} - \alpha W_l\, K_{il})$$

$$- \lambda^2\, W_l\, \sum_{j=1}^{i}\, W_j\, \sum_{k=1}^{N} (s_k\, \delta_{jk} - \alpha W_k\, K_{jk})\Big]\, \phi_l = F_i, \quad (i = 1, 2, \ldots, N) \tag{67}$$

which enables one to evaluate the discrete values ϕ_l. Substituting these ϕ_l into (65) and (66) we then can compute $\phi(x)$.

5. COMPUTATION OF THE STRESSES

The normal stresses σ_{xx} can be computed by Eq. $(14)_1$. Also, by (15) written in inverted form, that is,

$$\left.\begin{aligned} \mathcal{J}[e](x) &= \frac{N(x)}{ES} \\ \mathcal{J}[\chi](x) &= \frac{1}{EI\beta}\left[bM(x) - a\widehat{M}(x)\right] \\ \mathcal{J}[\eta](x) &= \frac{1}{EI\beta}\left[-aM(x) + \widehat{M}(x)\right] \end{aligned}\right\} \tag{68}$$

we can rewrite $(14)_1$ in the equivalent form

$$\sigma_{xx}(x,z) = \frac{N(x)}{S} + \frac{bz - a\Theta(z)}{\beta}\frac{M(x)}{I} + \frac{-az + \Theta(z)}{\beta}\frac{\widehat{M}(x)}{I} \tag{69}$$

This is a generalization of the Navier formula of classical beam theory; it coincides with an analogous formula given by Polizzotto (2015).

An alternative, perhaps more expressive, form of the stress formula (69) can be obtained by $(14)_1$, but modified by (30) whereby $\chi(x) = \chi^{EB}(x) - a\eta(x)$. Then, noting that $M(x) = EI\mathcal{J}[\chi^{EB}](x)$ we obtain

$$\sigma_{xx}(x,z) = \underbrace{\frac{N(x)}{S} + \frac{zM(x)}{I}}_{\text{Navier stress addend}} + \underbrace{E\left[\Theta(z) - az\right]\mathcal{J}[\eta](x)}_{\text{warping stress addend}} \tag{70}$$

Let us note that, since $\forall\, z, |z| \leq h/2$, it is

$$\Theta(z) - az = 0 \quad \text{for } \omega = 0,\, \omega = \infty \tag{71}$$

the warping stress addend of Eq.(70) is vanishing for both the Euler-Bernoulli beam and the Timoshenko one. Also note that the mentioned warping stress addend gives zero contributions to the stress resultants $N(x)$ and $M(x)$ in Eq.(70), since in fact it is

$$\int_S [\Theta(z) - az]\,\mathrm{d}a = \int_S z\,[\Theta(z) - az]\,\mathrm{d}a = 0 \quad \forall\,\omega,\, \forall x \in (0, L) \tag{72}$$

The shear stress σ_{xz} given by $(14)_2$, recalling (10), can be rewritten as

$$\sigma_{xz}(x,z) = \mu\Theta'(z)\gamma(x) = \mu\gamma(x)\left[1 - \left(\frac{2\,|z|}{h}\right)^{\omega}\right] \tag{73}$$

Indeed, at every cross section, the shear stress predicted by the present model is proportional to the shear angle $\gamma(x)$, which is a consequence of the fact that *no shear strain is exhibited under pure bending conditions.*

Eq.(73) is not useful for application purposes. In analogy with the classical Euler-Bernoulli beam, we can replace (73) with

$$\sigma_{xz}(x,z) = \sigma_{xz}^{\mathrm{eql}}(x,z) \tag{74}$$

where $\sigma_{xz}^{\rm eql}(x,z)$ is the shear stress in local equilibrium with the predicted normal stress $\sigma_{xx}(x,z)$ of (69). Since $\sigma_{xy}=0$, assuming $b_x=p_x=m=\widehat{m}=0$, the local equilibrium equation reads

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xz}^{\rm eql}}{\partial z}=0 \tag{75}$$

Hence we have

$$\sigma_{xz}^{\rm eql}(x,z)=-\int_{-h/2}^{z}\frac{\partial}{\partial x}\sigma_{xx}(x,\bar{z}){\rm d}\bar{z} \tag{76}$$

Substituting (69) into (76), since $N'\equiv 0$, we obtain

$$\sigma_{xz}^{\rm eql}(x,z)=\frac{T(x)}{\beta BI}\left[bX(z)-aY(z)\right]+\frac{\widehat{Q}(x)}{\beta BI}\left[-aX(z)+Y(z)\right] \tag{77}$$

Here, since in general $M'(x)\neq Q(x)$, we have posited

$$T(x):=M'(x) \tag{78}$$

We have also introduced the quantities

$$X(z):=\int_{z}^{h/2}z{\rm d}a=\frac{1}{8}Bh^2\left[1-\left(\frac{2\,z}{h}\right)^2\right] \tag{79}$$

which is the classical first area moment, and

$$Y(z):=\int_{z}^{h/2}\Theta(z){\rm d}a=X(z)-\frac{Bh^2}{4(1+\omega)(2+\omega)}\left[1-\left(\frac{2\,z}{h}\right)^{2+\omega}\right] \tag{80}$$

Eqs.(77), (78) and (79) coincide with analogous formulas given by Polizzotto (2015) for local shear-deformable beams. As shown in the latter quoted paper, (77) reduces to the classical Jourawski formula for $\omega\to 0$. At a cross section x where $\gamma(x)=0$, hence $\widehat{Q}(x)=0$, Eq.(77) *loses the contribution from warping, but saves that from bending proportional to* $T(x)$.

6. APPLICATIONS

This section is devoted to the application of the proposed theory to a simple beam model as a cantilever beam under a point load P at the free end, which indeed is known from the literature due to paradoxes encountered through the application of the nonlocal-differential theory by Eringen (Eringen, 1983; Peddieson et al., 2003). It was proved by Fuschi et al. (2019) that instead no paradoxes of any sort occur with the strain-difference based nonlocal theory; it is proved here that the same occurs if shear deformation is taken into account.

The boundary conditions (24) are here specified as follows

$$\left.\begin{array}{c} w(0) = w'(0) = M(L) = 0, \quad M'(L) = P \\[2em] \gamma(0) = 0, \quad \widehat{M}(L) = 0 \end{array}\right\} \tag{81}$$

In writing the condition $\gamma(0) = 0$ it is presumed that the clamping constraint at $x = 0$ is able to impede the formation of a shear angle and thus to provide the necessary reaction couple $\widehat{M}(0)$. Instead at $x = L$ we have assumed that no constraints of any sort do exist and that therefore the shear angle is allowed to form up freely; hence an external warping couple $\widehat{M}$ may be applied at $x = L$, but here we have chosen $\widehat{M}(L) = 0$.

In the case of statically determinate beams the computational procedure simplifies somewhat but for more clarity we try to follow the proposed computational scheme. By the equilibrium conditions we can write

$$N(x) \equiv 0, \quad M(x) = -P\,(L - x)\,, \quad M'(x) = P \tag{82}$$

By the boundary conditions $(81)_3$ and $(81)_4$, the constants B_1 and B_2 are determined as

$$B_1 = -B_2 = \frac{PL^2}{EI} \tag{83}$$

while the boundary conditions $(81)_1$, $(81)_2$ and $(81)_5$ give

$$B_3 = B_4 = C_2 = 0 \tag{84}$$

Finally, from $(81)_6$ by using $(25)_3$, $(15)_4$, (53) and recalling that $C = C_1 + aB_2$, we get

$$C_1 = \frac{\frac{aPL^2}{EI} - \frac{\mu SdL^2}{EI}\left[\frac{1}{L}\int_0^L \mathcal{J}[g_0](\bar{x})\mathrm{d}\bar{x} + \frac{PL^2}{EI}\frac{1}{L}\int_0^L \mathcal{J}[g_1](\bar{x})\mathrm{d}\bar{x}\right]}{1 + \frac{\mu SdL^2}{EI}\frac{1}{L}\int_0^L \mathcal{J}[g_2](\bar{x})\mathrm{d}\bar{x}} \tag{85}$$

The numerical solution of Eqs.(44) and (51) allows to get the response functions $w(x)$ and $\gamma(x)$ given by Eqs.(53) and (55) respectively, after substituition of the above determined constants pertinent to the addressed beam case.

6.1. Description of the Obtained Results

The predicted response of the cantilever beam is graphically illustrated in Figures 3(a,b), 4(a,b), 5(a-d).

In Figure 3(a), the *maximum deflection ratio* $\left[w(L) - w_0(L)\right]/w_0(L)$, where $w_0(L) = PL^3/(3EI)$, is plotted as a function of the warping parameter ω, $0 \leq \omega \leq 40$, for fixed values $h/L = 1, E/\mu = 10, \alpha = 50$, but for different values of the internal length $\ell/L = 0; 0.05; 0.1; 0.15; 0.2$. In Figure 3(b), the same maximum deflection ratio is plotted as a function of the internal length $0 \leq \ell/L \leq 0.2$, for different values of the warping parameter

$\omega = 0; 1; 2; 3; 5; 10$. These two groups of plots give a clear idea of the sensitivity of the maximum deflection of the beam to the combined action of the internal length and the shear deformation through the warping parameter. Figure 3(a) shows that the increasing of ω causes some softening effect, which seems to be a natural consequence of the fact that $\omega > 0$ means more deformation of the Euler-Bernoulli beam, but there is a notable stiffening effect with the increasing of ℓ/L, as clearly shown by both Figures 3(a,b). The stiffening effects induced by the increasing of the internal length ℓ/L is clearly shown in Figure 3(b), where the curve corresponding to $\omega = 0$ coincides with the one reported in Fuschi et al. (2019).

In Figure 4(a), the quantity $\mu S\gamma(x)/P$, proportional to the *shear angle* $\gamma(x)$, is plotted as a function of x/L for differen values of the warping parameter $\omega = 0; 1; 2; 3; 5; 10$, and for fixed values $\ell/L = 0.1$; $h/L = 1$; $E/\mu = 10$ and $\alpha = 50$. In Figure 4(b), the *maximum shear angle* $\gamma(L)$ is plotted as a function of $\omega \geq 0$, for different values of ℓ/L but fixed values of $h/L = 1$, $E/\mu = 10$, $\alpha = 50$. Since $\gamma(L) = 0$ for $\omega = 0$, $\gamma(L)$ increases, with ω increasing, from zero to a pick value at $\omega = 1$ (starting plotting point), then $\gamma(L)$ decreases tending asymptotically to the value featuring the Timoshenko beam.

In Figures 5(a-c) the normal stress ratio $\sigma_{xx}(x, z)/\sigma_0$ given by (69), or by (70), is reported as a function of $2z/h$ at $x = 0, L/2, L$, respectively. $\sigma_0 = 6PL/(Bh^2)$ is the maximum normal stress at $x = 0$ in the Euler-Bernoulli beam.

In Figure 5(d) the shear stress ratio $\sigma_{xz}(x, z)/\mu$ given by (73) is reported as a function of $2z/h$ at $x = L$. As observed in Section 5, at every cross section x the shear stress, if computed by (77), is the sum of two contributions, of which one is proportional to $\gamma(x)$, the other is proportional to the shear force $P = M'(x)$. The latter contribution is non-zero everywhere, even at $x = 0$ where $\gamma(0) = 0$.

Figure 3: Cantilever beam under point load at the free end. Deflection ratio, $(3EIw(L)/PL^3) - 1$, for fixed values of $h/L = 1$; $E/\mu = 10$ and $\alpha = 50$, plotted: a) as function of the warping parameter ω and different values of internal length, ℓ/L; b) as a function of the internal length ℓ/L and different values of the warping parameter ω.

Figure 4: Cantilever beam under point load at the free end: a) Shear angle γ plotted as a function of x/L, for different values of the warping parameter ω, at a fixed value of the internal length $\ell/L = 0.1$; b) Maximum shear angle $\gamma(L)$ plotted as a function of ω for different values of the internal length ℓ/L. In all plots it is $h/L = 1$; $E/\mu = 10$ and $\alpha = 50$.

Figure 5: Cantilever beam under point load at the free end. Stress diagrams at cross section plotted as a function of $2z/h$, for different values of the warping parameter ω, at fixed values of the internal length $\ell/L = 0.1$, slenderness ratio, $h/L = 1$ and for $E/\mu = 10$, $\alpha = 50$: a-c) Normal stresses σ_{xx} at: $x = 0$, $x = L/2$, $x = L$, respectively; d) Shear stresses σ_{xz} (only contribution proportional to shear angle γ) at $x = L$.

7. CONCLUSION

In the present paper, the strain-difference based nonlocal elasticity theory devised by the authors has been applied to small-scale beams under static loads taking into account shear deformation. For this purpose the parametric warping method has been applied whereby a real warping parameter $\omega \geq 0$ is used to fix the warping shape of the cross section. On letting ω vary, a continuous family of beam models is obtained spanning from the Euler-Bernoulli beam ($\omega = 0$) to the Timoshenko one ($\omega = \infty$). The exploitation of the warping parameter ω enables one to have a global view on the influence of shear deformation on the beam response. A central point of the method is the choice of the basic unknowns, i.e. the axial stretching, the Euler-Bernoulli bending curvature and the shear curvature, with which three uncoupled Fredholm integral equations of the second kind (or similar to it) are obtained, for which a numerical solution method has been adopted.

The main result of the present paper indicates that within the framework of nonlocal beam structural models —but likely also for plate and shell models— shear deformation has a notable influence upon size effects and that therefore shear deformation can hardly be disregarded for a correct prediction of these effects. The proposed method lends itself to generalizations to stability and vibration problems, taking into account anisotropy of the material. These extensions are the subject of an ongoing research.

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