Communications in Applied and Industrial Mathematics, ISSN 2038-0909, e-376
DOI: 10.1685/journal.caim.376

Approximate multiplicative controllability for degenerate parabolic problems with Robin boundary conditions

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Communicated by Nicola Bellomo

Abstract

In this work we study the global approximate multiplicative controllability for a weakly degenerate parabolic Cauchy-Robin problem. The problem is weakly degenerate in the sense that the diffusion coefficient is positive in the interior of the domain and is allowed to vanish at the boundary, provided the reciprocal of the diffusion coefficient is summable. In this paper, we will show that the above system can be steered, in the space of square-summable functions, from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear static controls. Moreover, we extend the above result relaxing the sign constraint on the initial-state.

Keywords: Approximate controllability, weakly degenerate parabolic equations, Robin boundary condition, bilinear control, Sturm-Liouville systems.

AMS Subject Classification: 35K65, 93B05, 34B24

1. Introduction.

Motivations.

In Control theory, boundary and interior locally distributed controls are usually employed (see, e.g., [1], [2], [3], [4], [5], [6]). These controls are additive terms in the equation and have localized support. However, such models are unfit to study several interesting applied problems such as chemical reactions controlled by catalysts, and also smart materials, which are able to change their principal parameters under certain conditions. This explains the growing interest in *multiplicative controllability*. General references for *multiplicative controllability* are, e.g., [7], [8], [9], [10], [11], [12].

Received 2011 06 14. Accepted 2011 08 26. Published 2011 11 28.

This note is inspired by [13], [14] and [15]. In [13] A.Y. Khapalov studied the global nonnegative approximate controllability of the one dimensional *non-degenerate* semilinear convection-diffusion-reaction equation governed in a bounded domain via the bilinear control $\alpha \in L^\infty(Q_T)$. In [14], the same approximate controllability property is derived in suitable classes of functions that change sign.
Degenerate parabolic problems are related to several applied models. For instance, in climatology, the so-called Budyko-Sellers model studies the role played by continental and oceanic areas of ice on climate change. The one-dimensional version of such a model reduces to an equation of the form

$$u_t - \big((1-x^2)u_x\big)_x = g(t,x)\, h(x,u) + f(t,x), \qquad x \in (-1,1),$$

where g, f and h are given functions and h may even be discontinuous. Here, the coefficient $a(x) = 1 - x^2$ vanishes at the boundary and the problem is strongly degenerate, in the sense that $\frac{1}{a} \notin L^1(-1,1)$.
In [15], we have obtained approximate controllability results with multiplicative control for strongly degenerate parabolic equations. In this article we extend the results of [15] to weakly degenerate equations, that is, assuming $\frac{1}{a} \in L^1(-1,1)$.

Problem formulation.

Let us consider the following *Cauchy-Robin* weakly degenerate boundary linear problem in divergence form, governed in the bounded domain $(-1,1)$ by means of the *bilinear control* $\alpha(t,x)$

$$\text{(1)} \qquad \begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in } \; Q_T \;=\; (0,T)\times(-1,1) \\ \begin{cases} \beta_0 v(t,-1) + \beta_1 a(-1) v_x(t,-1) = 0 & t \in (0,T) \\ \gamma_0\, v(t,1) \;+\; \gamma_1\, a(1)\, v_x(t,1) = 0 & t \in (0,T) \end{cases} \\ v(0,x) = v_0(x) & x \in (-1,1) \end{cases}$$

We assume that

i. $v_0 \in L^2(-1,1)$
ii. $\alpha \in L^\infty(Q_T)$
iii. $a \in C^0([-1,1]) \cap C^1(-1,1)$ fulfills the following properties
   (a) $a(x) > 0 \;\forall\, x \in (-1,1), \quad a(-1) = a(1) = 0$
   (b) $\frac{1}{a} \in L^1(-1,1)$
iv. $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R},\; \beta_0^2 + \beta_1^2 > 0,\; \gamma_0^2 + \gamma_1^2 > 0$, satisfy the sign condition

DOI: 10.1685/journal.caim.376

(a) $\beta_0\beta_1 \leq 0$ and $\gamma_0\gamma_1 \geq 0$.

Under the assumptions *iii.)* we say that the problem (1) is weakly degenerate.

2. Main goals.

We are interested in studying the multiplicative controllability of problem (1) by the *bilinear control* $\alpha(t,x)$. In particular, for the above linear problem, we will discuss results guaranteeing global nonnegative approximate controllability in large time (for multiplicative controllability see [13], [11], [14], [15]).
Now we recall one definition from control theory.

Definition 2.1. *We say that the system (1) is nonnegatively globally approximately controllable in $L^2(-1,1)$, if for every $\varepsilon > 0$ and for every nonnegative $v_0(x)$, $v_d(x) \in L^2(-1,1)$ with $v_0 \not\equiv 0$ there are a $T = T(\varepsilon, v_0, v_d)$ and a bilinear control $\alpha(t,x) \in L^\infty(Q_T)$ such that for the corresponding solution $v(t,x)$ of (1) we obtain*

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \leq \varepsilon\,.$$

In this work at first the *nonnegative global approximate controllability* result is obtained for the linear system (1) in the following theorem.

Theorem 2.1. *The linear system (1) is nonnegatively approximately controllable in $L^2(-1,1)$ by means of static controls in $L^\infty(-1,1)$. Moreover, the corresponding solution to (1) remains nonnegative at all times.*

Then, the results present in Theorem 2.1 can be extended to a larger class of initial states.

Theorem 2.2. *For any $v_d \in L^2(-1,1), v_d \geq 0$ and any $v_0 \in L^2(-1,1)$ such that*

$$\langle v_0, v_d\rangle_{L^2(-1,1)} > 0, \tag{2}$$

for every $\varepsilon > 0$, there are $T = T(\varepsilon, v_0, v_d) \geq 0$ and a static bilinear control, $\alpha = \alpha(x)$, $\alpha \in L^\infty(-1,1)$ such that

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \leq \varepsilon\,.$$

Remark 2.1. The solution $v(t,x)$ of the problem (1) in the assumptions of Theorem 2.2 does not remain nonnegative in Q_T, like in Theorem 2.1, but it can also assume negative values.

In the following, we will sometimes use $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ instead of $\|\cdot\|_{L^2(-1,1)}$ and $\langle\cdot,\cdot\rangle_{L^2(-1,1)}$.

3. Well-posedness in weighted Sobolev spaces.

In order to deal with the well-posedness of problem (1), it is necessary to introduce the following Sobolev weighted spaces

$$H^1_a(-1,1) := \{u \in L^2(-1,1) : u \text{ absolutely continuous in } [-1,1], \\ \sqrt{a}u_x \in L^2(-1,1)\}$$

and

$$H^2_a(-1,1) := \{u \in H^1_a(-1,1) |\, au_x \in H^1(-1,1)\},$$

respectively with the following norms

$$\|u\|^2_{H^1_a} := \|u\|^2_{L^2(-1,1)} + |u|^2_{1,a} \text{ and } \|u\|^2_{H^2_a} := \|u\|^2_{H^1_a} + \|(au_x)_x\|^2_{L^2(-1,1)},$$

where $|u|_{1,a} := \|\sqrt{a}u_x\|_{L^2(-1,1)}$ is a seminorm.

In this note we consider the following space

$$\mathcal{B}(0,T) = C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_a(-1,1)),$$

where let us define the following norm

$$\|u\|^2_{\mathcal{B}(0,T)} := \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{L^2(-1,1)} + 2\int_0^T \int_{-1}^1 a(x)u_x^2 dx\,,\ \forall u \in \mathcal{B}(0,T)\,.$$

In [16] the following result is obtained.

Lemma 3.1.

$$H^1_a(-1,1) \hookrightarrow L^2(-1,1) \qquad \textit{with compact embedding.}$$

A similar result is obtained in [15] in the strongly degenerate case.

DOI: 10.1685/journal.caim.376

We now recall the existence and uniqueness result for system (1) obtained in [17] (see also [16]). Let us consider, first, the operator $(A_0, D(A_0))$ defined by

$$
(3)\qquad \begin{cases} D(A_0) = \left\{ u \in H^2_a(-1,1) \,\middle|\, \begin{cases} \beta_0 u(-1) + \beta_1 a(-1) u_x(-1) = 0 \\ \gamma_0\, u(1) \;+\; \gamma_1\, a(1)\, u_x(1) = 0 \end{cases} \right\} \\ \\ A_0 u = (a u_x)_x\,, \quad \forall\, u \in D(A_0)\,. \end{cases}
$$

Observe that A_0 is a closed, self-adjoint, dissipative operator with dense domain in $L^2(-1,1)$. Therefore, A_0 is the infinitesimal generator of a C_0- semigroup of contractions in $L^2(-1,1)$.

Next, given $\alpha \in L^\infty(-1,1)$, let us introduce the operator

$$
(4)\qquad \begin{cases} D(A) = D(A_0) \\ \\ A = A_0 + \alpha I\ . \end{cases}
$$

For such an operator we have the following proposition.

Proposition 3.1.

- *$D(A)$ is compactly embedded and dense in $L^2(-1,1)$.*
- *$A : D(A) \longrightarrow L^2(-1,1)$ is the infinitesimal generator of a strongly continuous semigroup, e^{tA}, of bounded linear operators on $L^2(-1,1)$.*

Observe that problem (1) can be recast in the Hilbert space $L^2(-1,1)$ as

$$
(5)\qquad \begin{cases} u'(t) = A\, u(t)\,, \qquad t > 0 \\ \\ u(0) = u_0\ . \end{cases}
$$

where A is the operator in (4).

We recall that a *weak solution* of (5) is a function $u \in C^0([0,T]; L^2(-1,1))$ such that for every $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is absolutely continuous on $[0,T]$ and

$$
\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^* v\rangle\,,
$$

for almost $t \in [0,T]$ (see [18]).

Theorem 3.1. *For every $\alpha \in L^\infty((0,T)\times(-1,1))$ and every $u_0 \in L^2(-1,1)$, there exists a unique weak solution $u \in \mathcal{B}(0,T)$ to (1), which coincides with $e^{tA}u_0$.*

4. Auxiliary lemmas.

Let $A = A_0 + \alpha I$, where the operator A_0 is defined in (3) and $\alpha \in L^\infty(-1,1)$. Since A is self-adjoint and $D(A) \hookrightarrow L^2(-1,1)$ is compact (see Proposition 3.1), we have the following (see also [19]).

Lemma 4.1. *There exists an increasing sequence with $\{\lambda_k\}_{k\in\mathbb{N}}$, $\lambda_k \longrightarrow +\infty$ as $k \to \infty$, such that the eigenvalues of A are given by $\{-\lambda_k\}_{k\in\mathbb{N}}$, and the corresponding eigenfunctions $\{\omega_k\}_{k\in\mathbb{N}}$ form a complete orthonormal system in $L^2(-1,1)$.*

In this note we obtain the following result

Lemma 4.2. *Let A be the operator defined in (4) with $\alpha = \alpha_*$*

$$\begin{cases} D(A) = D(A_0) \\ \\ A = A_0 + \alpha_* I \ , \end{cases} \tag{6}$$

and let $\{\lambda_k\}, \{\omega_k\}$ be the eigenvalues and eigenfunctions of A, respectively, given by Lemma 4.1. Let $v \in D(A)$ be such that $v > 0$ on $(-1,1)$, and $\alpha_*(x) = -\frac{(a(x)v_x(x))_x}{v(x)} \in L^\infty(-1,1)$. Then

$$\lambda_1 = 0 \quad \textit{and} \quad |\omega_1| = \frac{v}{\|v\|}.$$

Moreover, $\frac{v}{\|v\|}$ and $-\frac{v}{\|v\|}$ are the only eigenfunctions of A with norm 1 that do not change sign in $(-1,1)$.

Remark 4.1. Problem (6) is equivalent to the following Sturm-Liouville system

$$\begin{cases} (a(x)\omega_x)_x + \alpha_*(x)\omega + \lambda\,\omega = 0 \qquad \text{in} \qquad (-1,1) \\ \\ \begin{cases} \beta_0\omega(-1) + \beta_1 a(-1)\omega_x(-1) = 0 \\ \\ \gamma_0\,\omega(1) \ + \ \gamma_1\,a(1)\,\omega_x(1) = 0 \end{cases} \end{cases} .$$

DOI: 10.1685/journal.caim.376

Proof. (of Lemma 4.2)
STEP.1 We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad \text{and} \qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions of the operator (6) (see Lemma 4.1). Therefore,

$$\langle \omega_k, \omega_h \rangle = 0, \qquad \text{if } h \neq k\,. \tag{7}$$

One can check, by easy calculations, that $\frac{v(x)}{\|v\|}$ is an eigenfunction of A associated with the eigenvalue $\lambda = 0$. Since $\frac{v}{\|v\|}$ has norm 1 and $v(x) > 0$ on $(-1, 1)$, we have that

$$\exists\, k_* \in \mathbb{N} : \omega_{k_*}(x) = \frac{v(x)}{\|v\|} > 0 \text{ or } \omega_{k_*}(x) = -\frac{v(x)}{\|v\|} < 0,\ \forall x \in (-1,1)\,. \tag{8}$$

Writing (7) with $k = k_*$ we obtain

$$\langle \omega_{k_*}, \omega_h \rangle = \int_{-1}^{1} \omega_{k_*}(x)\omega_h(x)dx = 0, \qquad \forall h \neq k_*\,. \tag{9}$$

Therefore, considering (8) and (9), we observe that ω_{k_*} is the only eigenfunction of the operator defined in (6) that doesn't change sign in $(-1, 1)$.

STEP.2 Let us now prove that

$$k_* = 1\,,$$

that is, $\lambda_1 = 0$. Recall that

$$\lambda_1 = \min_{u\in D(A)\setminus\{0\}} \frac{-\langle Au, u\rangle}{\|u\|^2},$$

where

$$\langle Au, u\rangle = \int_{-1}^{1} \left((a\, u_x)_x\, u + \alpha_*\, u^2\right)\, dx = [au_x\, u]_{-1}^{1} - \int_{-1}^{1} a\, u_x^2\, dx + \int_{-1}^{1} \alpha_*\, u^2\, dx\,.$$

By Lemma 4.1, since $\lambda_{k_*} = 0$, it is sufficient to prove that $\lambda_1 \geq 0$, or

$$\int_{-1}^{1} \alpha_*\, u^2\, dx + [au_x\, u]_{-1}^{1} \leq \int_{-1}^{1} a\, u_x^2\, dx, \qquad \forall\, u \in H_a^1(-1, 1)\,. \tag{10}$$

If $\beta_1\gamma_1 \neq 0$, using the Robin boundary conditions, we have

$$[au_x\, u]_{-1}^{1} = a(1)u_x(t,1)\,u(t,1) - a(-1)u_x(t,-1)\,u(t,-1) = \frac{-\gamma_0}{\gamma_1}u^2(t,1) + \frac{\beta_0}{\beta_1}\,u^2(t,-1).$$

Integrating by parts, keeping in mind that $\beta_1\gamma_1 \neq 0$, we have

$$\begin{aligned}
\int_{-1}^{1} \alpha_*\, u^2\, dx &= -\int_{-1}^{1} \frac{(a\,v_x)_x}{v}\, u^2\, dx = -\left[a\,v_x \frac{u^2}{v}\right]_{-1}^{1} + \int_{-1}^{1} a\,v_x \left(\frac{u^2}{v}\right)_x\, dx \\
&= -a(1)\,v_x(1)\frac{u^2(t,1)}{v(1)} + a(-1)\,v_x(-1)\frac{u^2(t,-1)}{v(-1)} \\
&\quad + \int_{-1}^{1} a\,v_x \frac{2uu_x}{v}\, dx - \int_{-1}^{1} a\,v_x^2 \left(\frac{u^2}{v^2}\right)\, dx \\
&= \frac{\gamma_0}{\gamma_1}v(1)\frac{u^2(t,1)}{v(1)} - \frac{\beta_0}{\beta_1}v(-1)\frac{u^2(t,-1)}{v(-1)} \\
&\quad + 2\int_{-1}^{1} \sqrt{a}\,\frac{v_x}{v}u\sqrt{a}u_x\, dx - \int_{-1}^{1} a\,v_x^2 \left(\frac{u^2}{v^2}\right)\, dx \\
&\leq \frac{\gamma_0}{\gamma_1}u^2(t,1) - \frac{\beta_0}{\beta_1}u^2(t,-1) \\
&\quad + \int_{-1}^{1} a\,\left(\frac{v_x u}{v}\right)^2 dx + \int_{-1}^{1} au_x^2\, dx - \int_{-1}^{1} a\,v_x^2 \left(\frac{u^2}{v^2}\right)\, dx \\
&= -\,[au_x\, u]_{-1}^{1} + \int_{-1}^{1} au_x^2\, dx\,,
\end{aligned}$$

from which (10) follows. In fact, (10) holds true even for $\beta_1\gamma_1 = 0$, as one can show by similarly argument. □

For the proof of Theorem 2.1 the following Lemma is necessary.

Lemma 4.3. *Let $T > 0$, $\alpha \in L^\infty(Q_T)$, let $v_0 \in L^2(-1,1)$, $v_0(x) \geq 0$ a.e. $x \in (-1,1)$ and let $v \in \mathcal{B}(0,T)$ be the solution to the linear system*

$$\begin{cases}
v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in } Q_T = (0,T)\times(-1,1) \\
\begin{cases} \beta_0 v(t,-1) + \beta_1 a(-1)v_x(t,-1) = 0 & t \in (0,T) \\ \gamma_0\, v(t,1) + \gamma_1\, a(1)\, v_x(t,1) = 0 & t \in (0,T) \end{cases} \\
v(0,x) = v_0(x) & x \in (-1,1)\ ,
\end{cases}$$

DOI: 10.1685/journal.caim.376

Then

$$v(t,x) \geq 0, \quad \forall (t,x) \in Q_T\,.$$

Proof. Let $v \in \mathcal{B}(0,T)$ be the solution to the system (1), and we consider the positive-part and the negative-part (see Appendix). It is sufficient to prove that

$$v^-(t,x) \equiv 0 \qquad\quad \text{in } Q_T\,.$$

Multiplying both members equation of the problem (1) by v^- and integrating it on $(-1,1)$ we obtain

$$\int_{-1}^{1} \left[v_t v^- - (a(x)v_x)_x v^- - \alpha v v^-\right] dx = 0. \tag{11}$$

Recalling the definition v^+ and v^-, we obtain

$$\int_{-1}^{1} v_t v^- dx = \int_{-1}^{1} (v^+ - v^-)_t v^- dx = -\int_{-1}^{1} (v^-)_t v^- dx = -\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} (v^-)^2 dx\,.$$

Integrating by parts and applying Theorem 6.1 (see Appendix), we obtain $v^- \in H^1_a(-1,1)$ and the following equality

$$\int_{-1}^{1} (a(x)v_x)_x v^-\, dx = [a(x)v_x v^-]_{-1}^{1} - \int_{-1}^{1} a(x) v_x (-v)_x\, dx$$

If $\beta_1\gamma_1 \neq 0$, using the Robin boundary conditions and the sign assumptions, we have

$$[a(x)v_x v^-]_{-1}^{1} = a(1)v_x(t,1)v^-(t,1) - a(-1)v_x(t,-1)v^-(t,-1) =$$

$$= -\frac{\gamma_0}{\gamma_1} v(t,1)v^-(t,1) + \frac{\beta_0}{\beta_1} v(t,-1)v^-(t,-1) \geq 0\,.$$

If $\beta_1\gamma_1 = 0$([a]), proceeding similarly, we obtain

$$[a(x)v_x v^-]_{-1}^{1} \geq 0.$$

We also have

$$\int_{-1}^{1} \alpha v v^- dx = -\int_{-1}^{1} \alpha (v^-)^2 dx$$

[a] In the particular case $\beta_1 = \gamma_1 = 0$ we have $[a(x)v_x v^-]_{-1}^{1} = 0$. Indeed, in this case the problem (1) is reduced to a Cauchy-Dirichlet problem.

and therefore (11) becomes

$$-\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(v^-)^2dx+\int_{-1}^{1}\alpha(v^-)^2dx=[a(x)v_xv^-]_{-1}^{1}+\int_{-1}^{1}a(x)v_x^2\geq 0,$$

from which

$$\frac{d}{dt}\int_{-1}^{1}(v^-)^2dx\leq 2\int_{-1}^{1}\alpha(v^-)^2dx\leq 2\|\alpha\|_\infty\int_{-1}^{1}(v^-)^2dx.$$

From the above inequality, applying Gronwall's lemma we obtain

$$\int_{-1}^{1}(v^-(t,x))^2dx\leq e^{2t\|\alpha\|_\infty}\int_{-1}^{1}(v^-(0,x))^2dx\,.$$

Since

$$v(0,x)=v_0(x)\geq 0\,,$$

we have

$$v^-(0,x)=0.$$

Therefore,

$$v^-(t,x)=0,\qquad\qquad \forall(t,x)\in Q_T.$$

From this, as we mentioned initially, it follows that

$$v(t,x)=v^+(t,x)\geq 0\qquad \forall(t,x)\in Q_T.$$

□

5. Proofs of main goals.

We are now ready to prove our main result.

Proof. (of Theorem 2.1)
STEP.1 Let A_0 be the operator defined in (3), to prove Theorem 2.1 it is sufficient to consider the set of target states

$$v_d\in D(A_0),\ v_d>0 \text{ on } (-1,1) \text{ such that } \frac{(a\,v_{dx})_x}{v_d}\in L^\infty(-1,1)\,. \tag{12}$$

Indeed, regularizing by convolution, every function $v_d\in L^2(-1,1), v_d\geq 0$ can be approximated by a sequence of strictly positive $C^\infty([-1,1])-$functions.
Then, fixing $\varepsilon>0$, we can find a function $v_d^\varepsilon\in C^\infty([-1,1]), v_d^\varepsilon>0$ in

DOI: 10.1685/journal.caim.376

$[-1,1]$ such that $\|v_d - v_d^\varepsilon\| \le \frac{\varepsilon}{2}$.
Now, let us consider $\bar\omega_1$, the first positive eigenfunction of A_0 with norm 1. Note that $\bar\omega_1$ is a solution of the following Sturm-Liouville problem

$$(13)\qquad \begin{cases} (a(x)\omega_x)_x + \lambda\,\omega = 0 & \text{in} \quad (-1,1) \\ \begin{cases} \beta_0\omega(-1) + \beta_1 a(-1)\omega_x(-1) = 0 \\ \gamma_0\,\omega(1) \;+\; \gamma_1\,a(1)\,\omega_x(1) = 0 \end{cases} \end{cases}$$

Define

$$\bar v_d^\varepsilon(x) = \xi_\sigma(x)\,\bar\omega_1(x) + (1-\xi_\sigma(x))\;v_d^\varepsilon(x), \qquad x\in[-1,1],$$

where $\xi_\sigma \in C^\infty([-1,1])$ (σ is a positive real number) is a symmetrical cut-off function

- $\xi_\sigma(-x) = \xi_\sigma(x), \qquad \forall x\in[-1,1]$
- $0 \le \xi_\sigma(x) \le 1, \qquad \forall x\in[0,1]$
- $\xi_\sigma(x) = 0, \qquad \forall x\in[0,1-\sigma]$
- $\xi_\sigma(x) = 1, \qquad \forall x\in[1-\frac{\sigma}{2},1]$.

Then,

$$\bar v_d^\varepsilon \in H_a^2(-1,1), \bar v_d^\varepsilon > 0 \text{ in } (-1,1) \text{ and } \begin{cases} \beta_0\bar v_d^\varepsilon(-1) + \beta_1 a(-1)\bar v_{dx}^\varepsilon(-1) = 0 \\ \gamma_0\,\bar v_d^\varepsilon(1) \;+\; \gamma_1\,a(1)\,\bar v_{dx}^\varepsilon(1) = 0 \end{cases}$$

Moreover, taking into account that there is $\sigma>0$ such that

$$\|v_d^\varepsilon - \bar v_d^\varepsilon\|^2 \le \int_{-1}^{-1+\sigma} \left(\bar\omega_1(x) - v_d^\varepsilon(x)\right)^2\,dx + \int_{1-\sigma}^{1} \left(\bar\omega_1(x) - v_d^\varepsilon(x)\right)^2\,dx \le \frac{\varepsilon^2}{4},$$

we have

$$\|v_d - \bar v_d^\varepsilon\| \le \|v_d - v_d^\varepsilon\| + \|v_d^\varepsilon - \bar v_d^\varepsilon\| \le \varepsilon\,.$$

Finally, since $\frac{(a(x)\bar\omega_{1x}(x))_x}{\bar\omega_1(x)} = -\bar\lambda_1\ \forall x\in(-1,1)$ ([b]), we have

$$\frac{(a\,\bar v_{dx}^\varepsilon)_x}{\bar v_d^\varepsilon} \in L^\infty(-1,1)\,.$$

STEP.2 Taking any nonzero, nonnegative initial state $v_0\in L^2(-1,1)$ and any target state v_d as described in (12) in STEP.1, let us set

$$(14)\qquad \alpha_*(x) = -\frac{(a(x)v_{dx}(x))_x}{v_d(x)}, \qquad x\in(-1,1).$$

[b] $-\bar\lambda_1$ is the first eigenvalue of the Sturm-Liouville problem (13).

Then, by (12),

$$\alpha_* \in L^\infty(-1,1)\,.$$

We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad \text{and} \qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions of the spectral problem $A\omega + \lambda\omega = 0$, with $A = A_0 + \alpha_* I$ (see Lemma 4.1), where as first eigenfunction we take the one which is positive in $(-1,1)$.

We can see, by Lemma 4.2, that

$$\lambda_1 = 0 \qquad \text{and} \qquad \omega_1(x) = \frac{v_d(x)}{\|v_d\|} > 0,\ \forall x \in (-1,1)\,. \tag{15}$$

<u>STEP.3</u> Let us now choose the following static bilinear control

$$\alpha(x) = \alpha_*(x) + \delta,\ \forall x \in (-1,1), \quad \text{with } \delta \in \mathbb{R} \ (\delta \text{ to be determined below}).$$

Adding $\delta \in \mathbb{R}$ in the coefficient α_* there is a shift of the eigenvalues corresponding to α_* from $\{-\lambda_k\}_{k\in\mathbb{N}}$ to $\{-\lambda_k + \delta\}_{k\in\mathbb{N}}$, but the eigenfunctions remain the same for α_* and $\alpha_* + \delta$.
The corresponding solution of (1), for this particular bilinear coefficient α, has the following Fourier series representation

$$\begin{aligned} v(t,x) = \sum_{k=1}^{\infty} e^{(-\lambda_k+\delta)t}\langle v_0, \omega_k\rangle \omega_k(x) \\ = e^{\delta t}\langle v_0, \omega_1\rangle \omega_1(x) + \sum_{k>1} e^{(-\lambda_k+\delta)t}\langle v_0, \omega_k\rangle \omega_k(x)\,. \end{aligned}$$

Let

$$r(t,x) = \sum_{k>1} e^{(-\lambda_k+\delta)t}\langle v_0, \omega_k\rangle \omega_k(x)$$

where, recalling that $\lambda_k < \lambda_{k+1}$, we obtain

$$-\lambda_k < -\lambda_1 = 0 \quad \text{ for ever } k \in \mathbb{N},\ k > 1\,.$$

Owing to (15),

$$\begin{aligned} \|v(t,\cdot) - v_d\| &\le \left\| e^{\delta t}\langle v_0, \omega_1\rangle \omega_1 - \|v_d\|\omega_1 \right\| + \|r(t,x)\| \\ &= \left| e^{\delta t}\langle v_0, \omega_1\rangle - \|v_d\| \right| + \|r(t,x)\|\,. \end{aligned} \tag{16}$$

Since $-\lambda_k < -\lambda_2$, $\forall k > 2$, applying Bessel's inequality we have

$$\begin{aligned}(17)\quad \|r(t,x)\|^2 &\leq e^{2(-\lambda_2+\delta)t}\sum_{k>1}|\langle v_0,\omega_k\rangle|^2\|\omega_k(x)\|^2\\ &= e^{2(-\lambda_2+\delta)t}\sum_{k>1}\langle v_0,\omega_k\rangle^2 \leq e^{2(-\lambda_2+\delta)t}\|v_0\|^2.\end{aligned}$$

Fixed $\varepsilon > 0$, we choose $T_\varepsilon > 0$ such that

$$e^{-\lambda_2 T_\varepsilon} = \varepsilon\frac{\langle v_0, v_d\rangle}{\|v_0\|\|v_d\|^2}. \tag{18}$$

Since $v_0 \in L^2(-1,1)$, $v_0 \geq 0$ and $v_0 \not\equiv 0$ in $(-1,1)$ and by (15), we obtain

$$\langle v_0,\omega_1\rangle = \int_{-1}^{1} v_0(x)\omega_1(x)dx > 0. \tag{19}$$

Then, it is possible choose δ_ε so that

$$e^{\delta_\varepsilon T_\varepsilon}\langle v_0,\omega_1\rangle = \|v_d\|,$$

that is, since $\omega_1 = \frac{v_d}{\|v_d\|}$,

$$\delta_\varepsilon = \frac{1}{T_\varepsilon}\ln\left(\frac{\|v_d\|^2}{\langle v_0, v_d\rangle}\right). \tag{20}$$

So, by (16) – (18) and (20) we conclude that

$$\|v(T_\varepsilon,\cdot) - v_d(\cdot)\| \leq e^{(-\lambda_2+\delta_\varepsilon)T_\varepsilon}\|v_0\| = e^{-\lambda_2 T_\varepsilon}\frac{\|v_d\|^2}{\langle v_0, v_d\rangle}\|v_0\| = \varepsilon\,.$$

From which we have the conclusion. □

Proof. (of Theorem 2.2) The proof of Theorem 2.1 can be adapted to Theorem 2.2, keeping in mind that, in STEP.3, inequality (19) continues to hold in this new setting. In fact we have

$$\langle v_0,\omega_1\rangle = \frac{1}{\|v_d\|}\langle v_0, v_d\rangle > 0, \text{ by assumptions (2).}$$

From this point on, one can proceed as in the proof of Theorem 2.1. □

6. Appendix.

Positive and negative part.

Given $\Omega \subseteq \mathbb{R}^n$, $v : \Omega \longrightarrow \mathbb{R}$ we consider the positive-part function

$$v^+(x) = \max(v(x), 0), \qquad \forall x \in \Omega,$$

and the negative-part function

$$v^-(x) = \max(0, -v(x)), \qquad \forall x \in \Omega.$$

Then we have the following equality

$$v = v^+ - v^- \qquad \text{in } \Omega$$

For the functions v^+ and v^- the following result of regularity in Sobolev's spaces will be useful (see [20], Appendix A).

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$, $u : \Omega \longrightarrow \mathbb{R}$, $u \in H^{1,s}(\Omega)$, $1 \leq s \leq \infty$. Then*

$$u^+, \, u^- \in H^{1,s}(\Omega)$$

and for $1 \leq i \leq n$

$$(u^+)_{x_i} = \begin{cases} u_{x_i} & \text{in } \{x \in \Omega : u(x) > 0\} \\ 0 & \text{in } \{x \in \Omega : u(x) \leq 0\} \end{cases},$$

and

$$(u^-)_{x_i} = \begin{cases} -u_{x_i} & \text{in } \{x \in \Omega : u(x) < 0\} \\ 0 & \text{in } \{x \in \Omega : u(x) \geq 0\} \end{cases}.$$

Gronwall's Lemma.

Lemma 6.1. *Gronwall's inequality (differential form). Let $\eta(t)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. $t \in [0, T]$ the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \tag{21}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds \right]$$

for all $0 \leq t \leq T$.

In particular, if $\psi(t) \equiv 0$ in (21), i.e. $\eta' \leq \phi\eta$ for a.e. $t \in [0, T]$, and $\eta(0) = 0$, then

$$\eta \equiv 0 \qquad \text{in } [0, T].$$

DOI: 10.1685/journal.caim.376

Acknowledgements.

This research has been performed in the framework of the GDRE CONEDP. The authors wish to thank Institut Henri Poincaré (Paris, France) for providing a very stimulating environment during the "Control of Partial and Differential Equations and Applications" program in the Fall 2010.

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