Symmetric Structures Made of a Nonlocal Elastic Material

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The paper focuses on the analysis of symmetric structures in the context of nonlocal integral elasticity of Eringen-type. In particular, it highlights how the standard (local-type) concept of structural symmetry cannot be applied in a straightforward manner, but it has to be redefined involving an enlarged symmetric model of the structure. Such enlarged model is indeed able to take into account the nonlocal effects exerted on the (standard) symmetric portion of the structure chosen for the analysis by the portion neglected. The appropriate boundary conditions that have to be applied to the enlarged symmetric model for guaranteeing the exact matching between the mirrored symmetric solution and the complete one, are also discussed. Two numerical examples are solved by means of a nonlocal version of the finite element method and the results obtained are critically discussed.

Keywords: Nonlocal integral elasticity; Structural symmetry; Enlarged symmetric model; Nonlocal finite element method.

1. Introduction

Nonlocal continuum approaches are nowadays widely spread in those fields of solid mechanics dealing with problems where phenomena arising at a micro– and/or a nano–scale affect the macroscopic mechanical behavior of the analyzed structure or structural element (see e.g. [Rogula, 1982], [Eringen, 1999], [Bažant and Cedolin, 2010]). The main peculiarity of such nonlocal continuum approaches is indeed the introduction of an *internal length material scale*, able to take into account the phenomena imputable to the micro– or nano–structure, while keeping the hypothesis of *continuity*. The ways to introduce the internal length are several giving rise to several different approaches all belonging to the well known microcontinuum field theories, extension of classical field theories (see e.g. [Eringen, 1999]). Attention is hereafter

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restricted to *nonlocal elasticity* for which several approaches can be envisaged. The *gradient approach*, [Askes and Aifantis, 2011], the *integral approach*, [Eringen, 2002], the *peridynamic model*, [Silling, 2000], the *continualization procedures*, [Andrianov *et al.*, 2010], are, among many others, well established formulations.

The scrutiny of such approaches is out of the scope of the present paper whose main goal is to share some remarks concerning the way to apply classical concepts of *structural symmetry*, usually employed to simplify the analysis of a mechanical problem, in the context of nonlocal elasticity. The structural symmetry here referred to is the one related to the circumstance that many practical engineering problems contain points, lines, or planes of symmetry which divide the domain into two or more identically shaped subdomains and the solution is identical in each subdomain so allowing the analysis of just one of the symmetric portions of the analyzed element. The remarks carried on in the present paper possess a general validity in a nonlocal elasticity context but they are referred to a nonlocal approach of integral type, precisely to a nonlocal elastic strain-integral model known as *strain-difference-based* model of Eringen-type [Polizzotto *et al.*, 2006]. A nonlocal version of the well known finite element method (named NL-FEM), developed by the authors (see e.g.[Fuschi *et al.*, 2015] and references therein) and related to the above strain-difference-based constitutive model is used to carry on the numerical analyses.

With the aid of two simple examples it is shown how the classical concept of symmetry axis of local elasticity does not suffice, while the usual boundary static and kinematic conditions imposed along the lines or planes of symmetry, to consistently analyze the reduced symmetric structural portion, have to be re-interpreted in a nonlocal sense. The hypotheses of infinitesimal displacements and loads acting in a quasi-static manner are assumed throughout the paper. To render the paper self-contained, an abridged description of the referred constitutive model and of the related NL-FEM, deeply expounded in [Fuschi *et al.*, 2015], are given next also to share a common terminology.

Notation. A compact notation is used throughout, with bold-face letters for vectors and tensors. The "dot" and "colon" products between vectors and tensors denote simple and double index contraction operations, respectively. For instance: $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\,\varepsilon_{ij}$, $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \{\sigma_{ij}\,n_j\}$, $\boldsymbol{D} : \boldsymbol{\varepsilon} = \{D_{ijhk}\,\varepsilon_{hk}\}$. The symbol := means equality by definition. The matrix notation is employed at Section 2.2 for adhering to a standard FEM terminology. Other symbols will be defined in the text where they appear for the first time.

2. Theoretical Background

2.1. *A nonlocal elastic constitutive model*

Let us consider an isotropic nonhomogeneous nonlocal elastic material occupying a volume V referred to a $3D$ Euclidean space. By hypothesis the nonlocal features of the material are expressed by, [Polizzotto *et al.*, 2006]:

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) - \alpha \int_V \boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}') : [\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})] \,\mathrm{d}\,\boldsymbol{x}' \quad \forall\,(\boldsymbol{x},\,\boldsymbol{x}') \in V, \tag{1}$$

which simply states that the stress response $\boldsymbol{\sigma}(\boldsymbol{x})$ to a given strain field $\boldsymbol{\varepsilon}(\boldsymbol{x})$ is the sum of two contributions. The first one, of *local nature*, is governed by the standard, symmetric and positive definite elastic moduli tensor $\boldsymbol{D}(\boldsymbol{x})$. The latter is assumed variable in space to deal, if necessary, with a nonhomogeneous material. The second contribution, of *nonlocal integral nature*, depends on *the strain difference field* $\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})$ through the symmetric nonlocal operator $\boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}')$ defined next. A material parameter α also enters the constitutive relation (1) controlling the proportion of the nonlocal addition. In a nanocomposite for example α should be related to the volumetric percentage, or to the concentration, of nanoparticles in the polymer matrix. In a completely different context, such that of the human spongy bones for example, α should be related to the density and diameter of the struts decreasing with age. Other physical interpretations of α can certainly be envisaged but this is out of the goal of the paper and α is hereafter meant just as a material parameter that has to be calibrated by laboratory experiments or identification procedures. It is worth noting that for any uniform strain field the nonlocal contribution vanishes and the stress recovers the local value. Moreover, in contrast to the original Eringen model [Eringen, 2002] of which model (1) can be considered a generalization, undesired boundary effects or numerical instabilities are eliminated by the strain-difference formulation in agreement with some experimental findings on thin wires in tension (see e.g. [Fleck *et al.*, 1994]).

The nonlocal tensor $\boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}')$ is defined as:

$$\boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}') := [\gamma(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x}) + \gamma(\boldsymbol{x}')\boldsymbol{D}(\boldsymbol{x}')]\, g(\boldsymbol{x}, \boldsymbol{x}') - \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x}') \quad \forall\,(\boldsymbol{x},\,\boldsymbol{x}') \in V, \tag{2}$$

with:

$$\gamma(\boldsymbol{x}) := \int_V g(\boldsymbol{x}, \boldsymbol{x}') \,\mathrm{d}\,V'; \tag{3a}$$

$$\boldsymbol{q}(\boldsymbol{x}, \boldsymbol{x}') := \int_V g(\boldsymbol{x}, \boldsymbol{z}) g(\boldsymbol{x}', \boldsymbol{z}) \boldsymbol{D}(\boldsymbol{z}) \,\mathrm{d}\,V^z. \tag{3b}$$

All the above operators are defined on the base of a thermodynamical consistent formulation (see e.g. [Polizzotto *et al.*, 2006] for a deeper comprehension) but, by inspection of Eqs.(1-3), it appears evident that a crucial role is indeed played by the positive scalar *attenuation function*

$$g(\boldsymbol{x}, \boldsymbol{x}') := g\left(|\boldsymbol{x} - \boldsymbol{x}'|/\ell\right). \tag{4}$$

As suggested by Eq.(4) the function $g(\boldsymbol{x}, \boldsymbol{x}')$, whose analytical shape has to be fixed, depends on the Euclidean *distance* between points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in V and on an *internal length material scale*, say ℓ. It simply assigns a "weight" to the nonlocal effects induced at the field point $\boldsymbol{x}$ by a phenomenon acting at the source point $\boldsymbol{x}'$.

A typical choice for $g(\boldsymbol{x}, \boldsymbol{x}')$ is, for example, the so-called bi-exponential function hereafter adopted, namely:

$$g(\boldsymbol{x}, \boldsymbol{x}') := \lambda \exp\left(-|\boldsymbol{x} - \boldsymbol{x}'|/\ell\right), \tag{5}$$

where $\lambda = 1/(2\pi\ell^2 t)$ (t being the structural thickness) denotes a constant evaluated by the normalization condition $\int_{V_\infty} g(\boldsymbol{x}, \boldsymbol{x}')\, \mathrm{d}\, V' = 1$. The function $g(\boldsymbol{x}, \boldsymbol{x}')$ turns into a Dirac delta for $\ell \to 0$, i.e. in the case of a local elastic material; it rapidly decreases with increasing distance, eventually vanishing beyond the so-called *influence distance*, say L_R, the latter being a multiple of the internal length ℓ. The choice of the analytical shape of $g(\boldsymbol{x}, \boldsymbol{x}')$, together with the assumed values of ℓ and L_R, is obviously a matter concerning the connection between the macroscopic nonlocal continuum model and the real material behavior at small (atomistic) scale. These choices, defining the extent of nonlocality in the continuum model, are related to the considered nonlocal material or even to the addressed mechanical problem, see e.g. [Gosh *et al*, 2014] or [Allegri and Scarpa, 2014], just to quote recent contributions to this concern.

2.2. *The nonlocal finite element method (NL-FEM)*

Let us consider a solid body/structure occupying in its undeformed state the volume V whose boundary surface is S. The structure is, by hypothesis, made of a nonlocal elastic material obeying Eq.(1). It is also subjected to given body forces $\boldsymbol{b}\,(\boldsymbol{x})$ in V and surface tractions $\boldsymbol{t}\,(\boldsymbol{x})$ on S_t. Kinematic boundary conditions $\boldsymbol{u}\,(\boldsymbol{x}) = \bar{\boldsymbol{u}}\,(\boldsymbol{x})$ are also specified on $S_u = S - S_t$. The governing equations of such nonlocal boundary value problem (NL-BVP) read:

$$\mathrm{div}\, \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{0} \quad \text{in } V; \qquad \boldsymbol{\sigma}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{t}(\boldsymbol{x}) \quad \text{on } S_t; \tag{6}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{x}) = \nabla^s \boldsymbol{u}(\boldsymbol{x}) \quad \text{in } V; \qquad \boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) \quad \text{on } S_u; \tag{7}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{D}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{x}) - \alpha \int_V \boldsymbol{\mathcal{J}}(\boldsymbol{x}, \boldsymbol{x}') : [\boldsymbol{\varepsilon}(\boldsymbol{x}') - \boldsymbol{\varepsilon}(\boldsymbol{x})]\, \mathrm{d}\, x' \qquad \text{in } V; \tag{8}$$

where Eqs.(6) and (7) are the equilibrium and compatibility equations in their standard (local) format, Eq.(8) is the discussed nonlocal constitutive relation (1) here rewritten for completeness.

It is possible to show [Polizzotto *et al*, 2006] that, assuming infinitesimal displacements and loads acting in a quasi-static manner, Equations (6-8) are the optimality conditions of the following *nonlocal total potential energy functional*:

$$
\begin{aligned}
\Pi\left[\boldsymbol{u}(\boldsymbol{x})\right] := & \frac{1}{2}\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{D}(\boldsymbol{x}) : \nabla\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\,V + \\
& + \frac{\alpha}{2}\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \gamma^2(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x}) : \nabla\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\,V + \\
& - \frac{\alpha}{2}\int_V\int_V \nabla\boldsymbol{u}(\boldsymbol{x}) : \boldsymbol{\mathcal{J}}\left(\boldsymbol{x},\boldsymbol{x}'\right) : \nabla\boldsymbol{u}(\boldsymbol{x}')\,\mathrm{d}\,V'\,\mathrm{d}\,V + \\
& - \int_V \boldsymbol{b}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\,V - \int_{S_t} \boldsymbol{t}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\,S.
\end{aligned}
\tag{9}
$$

Looking at the r.h.s. of Eq.(9), the first term represents the *local elastic energy*, the second and the third terms express the *nonlocal elastic energy*, the forth and fifth ones give the *potential energy of the external loads*. Following a standard rationale of the finite element method formulation, a discrete form of (9) can be obtained in terms of *element nodal displacements* vector, equivalent *element nodal forces* vector and *element stiffness matrices*, precisely:

$$
\begin{aligned}
\Pi\left[\boldsymbol{d_n}\right] = & \frac{1}{2}\sum_{n=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{k}_n^{loc}\,\boldsymbol{d}_n + \frac{\alpha}{2}\sum_{n=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{k}_n^{nonloc}\,\boldsymbol{d}_n + \\
& - \frac{\alpha}{2}\sum_{n=1}^{N_e}\sum_{m=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{k}_{nm}^{nonloc}\,\boldsymbol{d}_m - \sum_{n=1}^{N_e} \boldsymbol{d}_n^T\,\boldsymbol{f}_n
\end{aligned}
\tag{10}
$$

where $\boldsymbol{d}_n$ and $\boldsymbol{d}_m$ are the nodal displacements vectors of elements $\#n$ and $\#m$ respectively, being N_e the number of finite elements in which V has been subdivided and the following positions hold true:

$$\boldsymbol{k}_n^{loc} := \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x})\,\boldsymbol{D}\left(\boldsymbol{x}\right)\boldsymbol{B}_n(\boldsymbol{x})\,\mathrm{d}\,V_n; \tag{11}$$

$$\boldsymbol{k}_n^{nonloc} := \int_{V_n} \boldsymbol{B}_n^T(\boldsymbol{x})\,\gamma^2(\boldsymbol{x})\,\boldsymbol{D}\left(\boldsymbol{x}\right)\boldsymbol{B}_n(\boldsymbol{x})\,\mathrm{d}\,V_n; \tag{12}$$

$$\boldsymbol{k}_{nm}^{nonloc} := \int_{V_n}\int_{V_m} \boldsymbol{B}_n^T(\boldsymbol{x})\,\boldsymbol{\mathcal{J}}\left(\boldsymbol{x},\boldsymbol{x}'\right)\boldsymbol{B}_m(\boldsymbol{x}')\,\mathrm{d}\,V_m\,\mathrm{d}\,V_n; \tag{13}$$

$$\boldsymbol{f}_n := \int_{V_n} \boldsymbol{N}_n^T(\boldsymbol{x})\,\boldsymbol{b}(\boldsymbol{x})\,\mathrm{d}\,V_n + \int_{S_{t(n)}} \boldsymbol{N}_n^T(\boldsymbol{x})\,\boldsymbol{t}(\boldsymbol{x})\,\mathrm{d}\,S_n. \tag{14}$$

In the above expressions $\boldsymbol{N}_n(\boldsymbol{x})$ and $\boldsymbol{B}_n(\boldsymbol{x})$ denote the matrices of the $n-th$ element shape functions and their Cartesian derivatives, respectively. V_n and S_n are the volume and the surface boundary of element $\#n$.

By inspection of Eqs.(11-14), all referred to a current element $\#n$, it is easy recognizable that positions (11) and (14) possess the standard format of the element (local) stiffness matrix and equivalent nodal force vector, positions (12) and (13) define instead a *nonlocal element stiffness* matrix and a *set* of *nonlocal self*$-$ (for

$n = m$) and *cross–stiffness* (for $n \neq m$) *element matrices*, respectively. The nonlocal nature of (12) resides on the presence of the nonlocal operator $\gamma(\boldsymbol{x})$, while the nonlocal nature of (13), beside the nonlocal operator $\boldsymbol{\mathcal{J}}\left(\boldsymbol{x}, \boldsymbol{x}'\right)$, is witnessed by the simultaneous presence of matrix $\boldsymbol{B}_n$ pertaining to the current element $\#n$ and matrix $\boldsymbol{B}_m$ pertaining to the generic element $\#m$ in the FE mesh (for each $\#n$, $\#m$ ranges over the whole mesh). This allows to account for the nonlocal effects exerted by all elements in the mesh on the $n-th$ current one. It is worth noting that, taking into account the analytical shape of the nonlocal operators $\gamma(\boldsymbol{x})$ and $\boldsymbol{\mathcal{J}}\left(\boldsymbol{x}, \boldsymbol{x}'\right)$ as well as of the circumstance that the attenuation function $g\left(\boldsymbol{x}, \boldsymbol{x}'\right)$ vanishes beyond the influence distance L_R, the cross–contributions to the nonlocal element matrices are *only* those pertaining to the elements *neighbors* of the current one. All the FEs far (with respect to L_R) from the current element do not exert any nonlocal influence.

Once again a standard rationale would allow to rephrase functional (10) in terms of global DOFs and, by minimization, would furnish the solving global linear equation system whose peculiarity is a *nonlocal global stiffness* matrix collecting all the self– and cross–contributions of the FEs in the mesh. More details can be found in [Fuschi *et al*, 2015] and are here omitted to avoid repetitions.

3. The Structural Symmetry

3.1. *Advantages of symmetry*

Many problems of engineering interest are characterized by a solution that is identical within two or more identically shaped subdomains located, within the domain of definition of the pertinent boundary value problem (BVP), by points, lines or planes of symmetry. It is hereafter meant that the BVP is well posed and admits a unique solution. It is also meant that the constituent material is isotropic and elastic. For the solution to be symmetrical, three groups of data must posses the same symmetry, precisely: *i*) those concerning the shape of the domain defining the geometry; *ii*) those related to the acting loads and boundary conditions; *iii*) those pertaining to the physical properties of the constituent material which affect the coefficients of the governing equations. It is usually geometric symmetry that first catches the analyst's eye, it is then necessary to check the physical properties and loads to see if they also possess the same symmetry. If so, only a symmetric portion of the original problem, or structural element, needs to be analyzed with a great reduction of computational cost and time. It is also well known that the removal of the symmetrical portions from the original domain creates new boundaries along the lines or planes of symmetry and appropriate boundary conditions along these lines have to be specified since they were interior to the original domain. Referring, for simplicity, to a plane mechanical problem with only one line of symmetry and, consequently, to only two symmetrical plane portions, a typical rationale is based on the circumstance that the portion on one side is a mirror reflection (in terms of geometry, material properties, loading and boundary conditions) of the one on the other side and the solutions in the two portions must also be mirror reflections. For such symmetric mechanical problem, governed by a well posed BVP in the elastic context here referred, the symmetry and smoothness of the exact solution, in terms of function and all derivatives, jointly imply conditions of continuity which furnish the appropriate boundary conditions along the new boundary or line of symmetry.

If the problem under study is of *nonlocal nature*, in the sense that the pertinent governing equations are in the shape of Eqs.(6-8) at Section 2.2, the above three requirements on the data groups characterizing a symmetric solution apply unaltered. Nevertheless, at least two questions arise: Question #1) Within a nonlocal elastic context is it allowed to cut along the symmetry line and to remove one of the symmetric portions of the analyzed structure? Question #2) What are the appropriate boundary conditions to be considered in the analysis of the selected symmetrical portion which guarantee that the symmetric (half) computed solution, once mirrored, exactly matches the complete solution as in local elasticity?

The symmetrical portion removed has, with no doubts, a (nonlocal) influence on the one selected for the analysis. Precisely, taking into account the constitutive assumption expressed by Eq.(1) and the meaning of the attenuation function, defined by Eq.(4) and here assumed in the shape of Eq.(5), the boundary zone adjacent to the symmetry line, of wideness L_R and belonging to the neglected symmetrical portion indeed affects the mechanical behaviour of its mirror reflection on the portion to be analyzed. On the basis of this observation, the answers are quite obvious from a theoretical point of view. Answer #1) The symmetry line persists, but it is necessary to consider an "extended symmetric portion" obtained by adding to the standard (local) symmetric portion to be analyzed a, let's say, "symmetrical boundary zone" of wideness L_R falling within the portion to be removed. Answer #2) The boundary conditions to be applied for the analysis of the extended symmetric portion have to be "smeared" within the symmetrical boundary zone. If, from a theoretical point of view, the above answers are quite intuitive and exhaustive their implications on the solving procedure are not so trivial. In the next Section such implications are discussed through the analysis of two simple examples solved numerically via the NL-FEM described in Section 2.2.

3.2. *Symmetry within the nonlocal elastic context*

A first example concerns the nonhomogeneous square plate shown in Fig.1a. The plate's side length is equal to 5 cm while its thickness is equal to 0.5 cm. The plate is made up by a nonlocal elastic material, in particular a central square part of the plate, of sides $a = 1$ cm, has Young's modulus $E_1 = 84$ GPa while the remaining part has Young's modulus $E_2 = 260$ GPa; a Poisson's coefficient $\nu = 0.2$ is assumed for the whole structure. The nonlocal parameter α is assumed equal to 50, while two different values of ℓ (in cm) are considered, precisely $\ell = 0.1$ and $\ell = 0.05$. Moreover, the bi-exponential attenuation function given by Eq.(5) is adopted with a computational influence distance $L_R = 11\ell$. The plate is fixed at the left edge (i.e. at $x = 0$ cm) and it is subjected to a uniform prescribed displacement $\bar{u}_x = 0.001$ cm at the opposite edge ($x = 5$ cm). Free are the upper and lower edges. This example has been already analyzed by the authors in [Fuschi *et al.*, 2015] with the aim of showing the validity of the strain-difference-based model as well as of the NL-FEM approach. However, in the above quoted paper the plate has been analyzed in its entirely do not taking into account its symmetry, circumstance that it will be investigated here. The structure of Fig.1a is indeed symmetric, in the sense specified in the previous section, with respect to an horizontal central axis. If the nonlocal nature of the problem is ignored, it should be studied using the classical reduced (half) symmetric scheme given in Fig.1c with appropriate boundary conditions enforced along the symmetry axis, namely the ones sketched in the same figure. The related FE models of the entire structure and of the symmetric (half) one should be those qualitatively drawn in Figs.1b and 1d, respectively. The results obtained from the two models, complete and reduced, would be fully coincident to each other as well known for a local problem. On the other hand, if the structure is treated as it is, that is made of a nonlocal material, the results obtained from models a) and c) turn out to be more different as much more are the nonlocal effects. The truthfulness of such assertion is deducible by the results, given in terms of strain component profiles ε_x, ε_y and ε_{xy} plotted at the mid-plate horizontal section, $\bar{y} \backsimeq 2.5$ cm, and at the mid-plate vertical section, $\bar{x} \backsimeq 2.5$ cm, drawn in Figs.2a-f for models b), complete, and d), reduced. In particular: the structure of Fig.1a has been discretized into 400 FEs, obtained subdividing both sides of the plates in 20 equal parts. For the chosen values of ℓ such mesh can be proved to be sufficiently accurate in order to ensure the mesh independency of the results. The model of the symmetric (half) structure, Fig.1d, is consequently made of 200 FEs. By inspection of Figure 2 appears evident how, for all the strain profiles, the half symmetric model furnishes a solution that *deviates* from the expected one. This is due to the *missing nonlocal effects* exerted on the chosen symmetric portion of the structure by the one removed. A deficiency inherent to the classical symmetric (half) scheme when used in the nonlocal context. In order to recover the nonlocal effects of the portion to be removed and obtain the correct nonlocal elastic solution, by mirroring the one computed on a symmetric reduced model, the symmetric scheme of Fig.1c has to be *enlarged*, with respect to the symmetry axis, by adding a *symmetrical boundary zone* whose wideness is equal to the computational influence distance L_R as shown in Fig.1e; beyond L_R all the nonlocal effects vanish. This boundary zone is highlighted in Fig.1e by the filled area below the symmetry axis, being the related FE model the one of Fig.1f for which 320 FEs have been used. For what concerns the appropriate boundary conditions to be enforced on the enlarged symmetric portion of the structure, it appears sensible to assume that, beyond the standard (local-type) ones along the cut or symmetry axis, some sort of *nonlocal boundary conditions* have to be *smeared* within the symmetrical boundary zone.

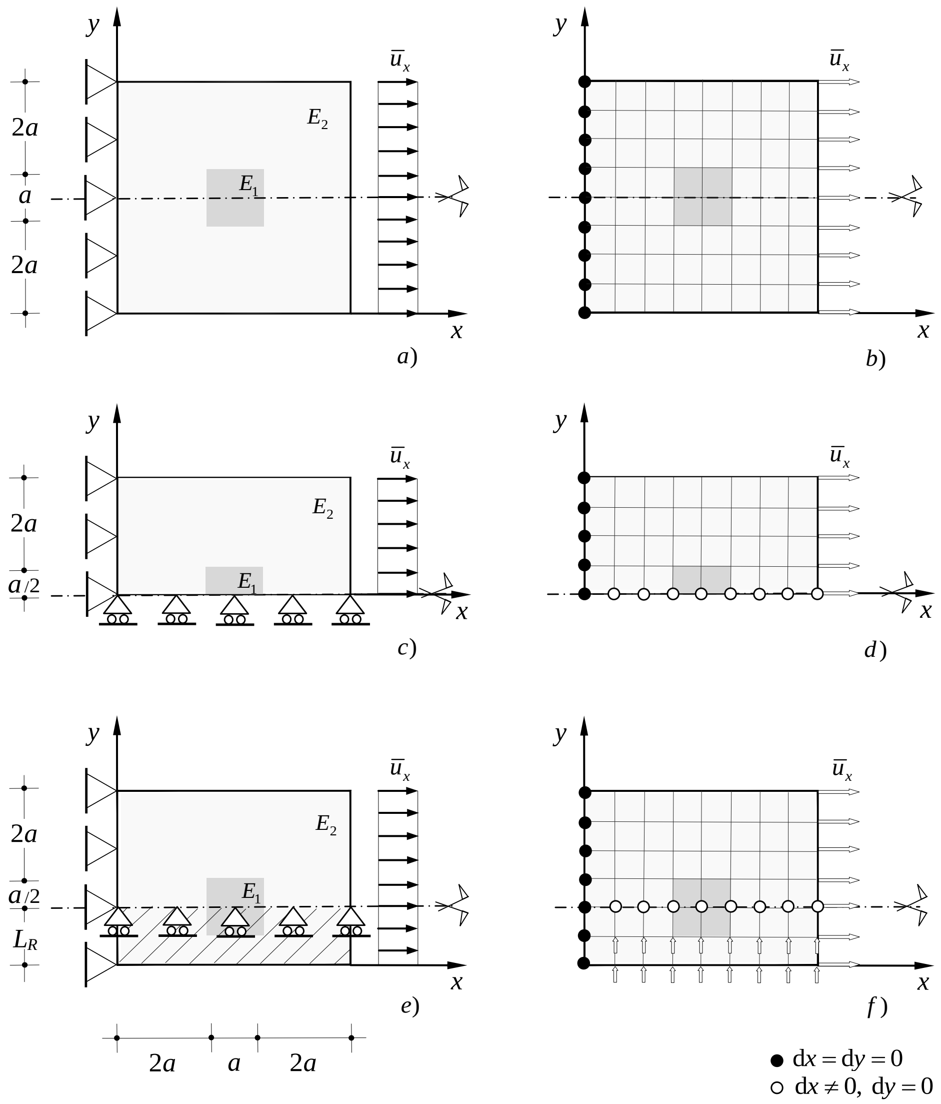

Fig. 1. Nonlocal elastic symmetric square plate under tension with piecewise constant Young modulus. Mechanical model of: *a*) whole structure; *c*) symmetric half structure; *e*) enlarged symmetric half structure. *b*), *d*) and *f*) FE models of structures *a*), *c*) and *e*), respectively.

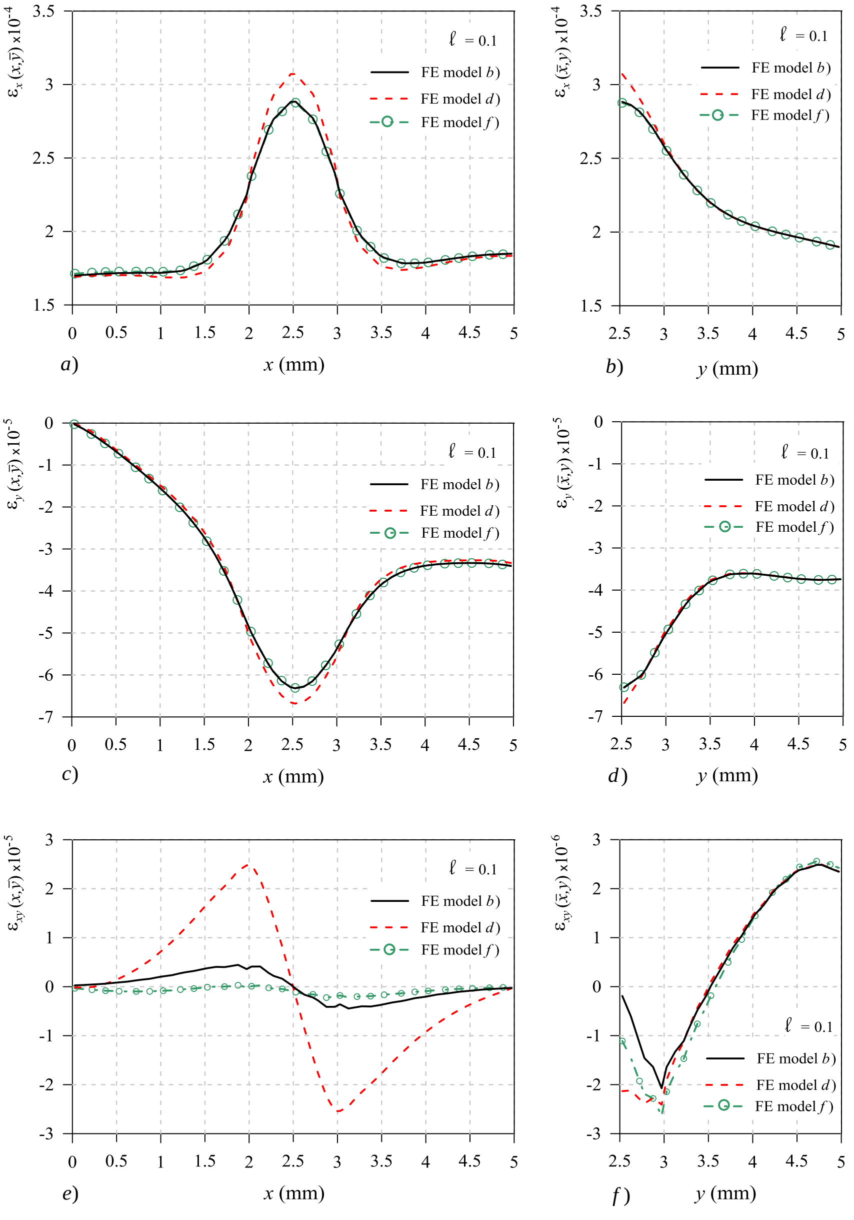

Fig. 2. Nonlocal elastic symmetric square plate ($\ell = 0.1$ cm). Strain components profiles ε_x, ε_y, ε_{xy}, versus x at $y \simeq 2.5$ on the left side and versus y at $x \simeq 2.5$ on the right side. Solution referred to the entire plate (solid lines), solution referred to the half symmetric portion (dashed lines), solution referred to the enlarged half symmetric portion (dashed lines with circles).

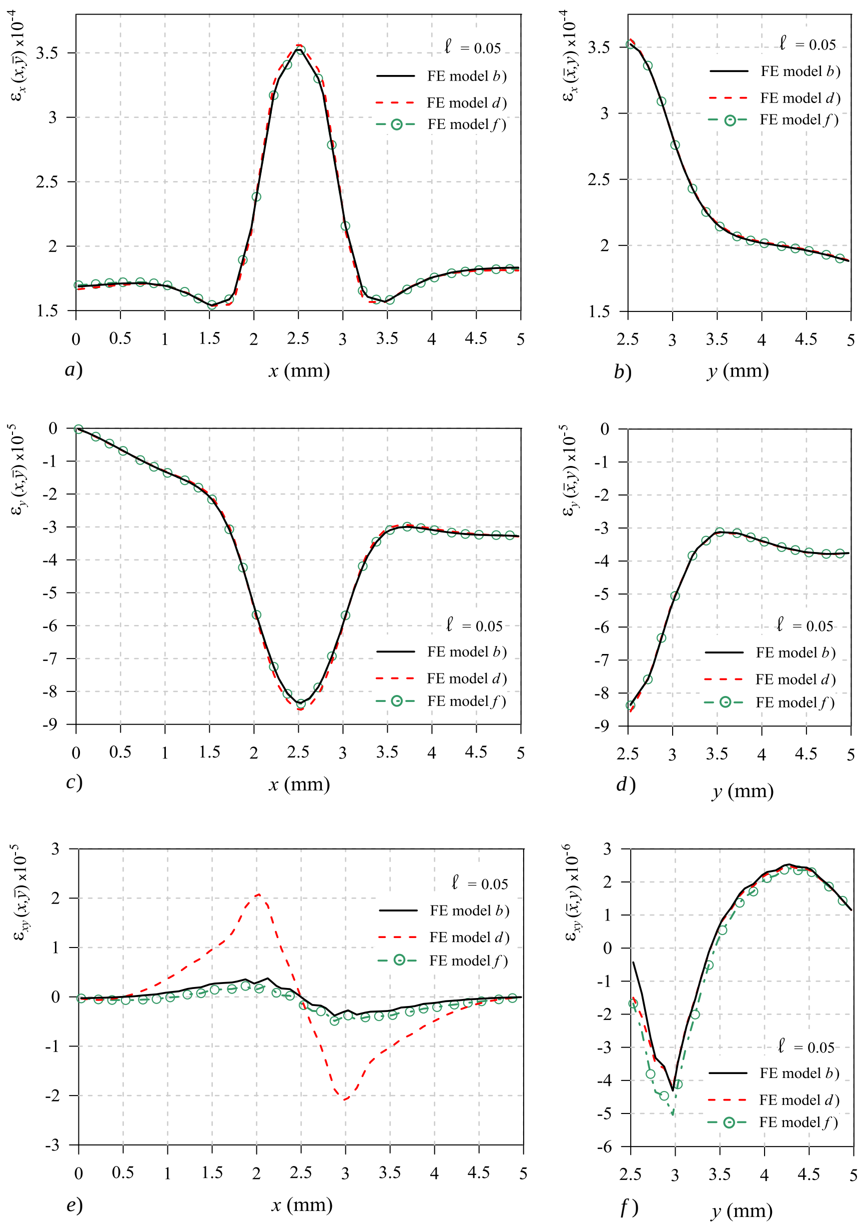

Fig. 3. Nonlocal elastic symmetric square plate ($\ell = 0.05$ cm). Strain components profiles ε_x, ε_y, ε_{xy}, versus x at $y \simeq 2.5$ on the left side and versus y at $x \simeq 2.5$ on the right side. Solution referred to the entire plate (solid lines), solution referred to the half symmetric portion (dashed lines), solution referred to the enlarged half symmetric portion (dashed lines with circles).

Once again it is essential to take into account the nonlocal effects exerted by phenomena arising within the portion usually removed on the one chosen for the analysis. Appropriate nonlocal boundary conditions for model of Fig.1e, or for its FE mesh of Fig.1f, turn out to be the vertical displacements, varying from zero at the symmetry axis to a maximum value at the lower edge of the symmetrical boundary zone, exhibited by the FEs' nodes falling within this zone. Such smeared boundary conditions are sketched in Fig.1f. It is obvious that the above displacements are unknown, they are part of the solution of the addressed nonlocal symmetric problem and the proposed rationale cannot be applied if the solution has to be searched just on the enlarged symmetric nonlocal scheme. This point remains an open question. Nevertheless the truthfulness of applying smeared displacements is proved by imposing the vertical nodal displacements given, in the zone corresponding to the enlarged symmetrical boundary zone, by the solution of the original (entire) structure solved on the scheme of Fig.1a. The numerical results are plotted in Figures 2e and f by the dashed lines with circles. The enlarged symmetric model, endowed with the discussed nonlocal (smeared) boundary conditions, gives exactly the expected solution for the strain profiles ε_x and ε_y and a very good approximation of the strain profile ε_{xy}.

It is worth noting that, even if the discussed drawback seems ignored by many numerical examples presented in the relevant literature, which utilize ℓ values of the same order of the smallest structural dimension, it has likely a minor influence when smaller values (with respect to the problem dimensions) of the internal length ℓ are assumed. Such circumstance is illustrated in Figs.3a-f which provide the results obtained for the symmetric plate of Figs.1a-f assuming $\ell = 0.05$ and a FE mesh of the enlarged symmetric scheme of 280 elements. In this case, the solution given by the standard (local-type) symmetric scheme of model in Figs.1c and 1d, can be still considered a good approximation of the correct solution for the components ε_x and ε_y while it keeps its inaccuracy for ε_{xy}. Such results highlight how for *small* values of ℓ, the concept of structural symmetry, as known in classical local problems, may be applied but with the due precautions.

A second numerical example considers a nonhomogeneous square plate having geometrical characteristics and material properties equal to the ones used in the first example, but showing a double symmetry. The plate is subjected to prescribed uniform displacements, equal to $\bar{u}_x = 0.001$ cm and of opposite sign, acting at both left and right edges and it is constrained to the upper and lower edges so that the vertical displacements are there impeded as shown in Fig.4a. Considering the geometrical, material, loading and boundary conditions, this structure is bisymmetric with respect to the two central horizontal and vertical axes. Such bisymmetry, within a local treatment, should allow to analyze only a quarter of the plate, as the one shown in Fig.4c. However, also in this case, if the problem is treated as nonlocal, the models of Figs.4a and 4c, through the corresponding FE models of Figs.4b and 4d, do not lead to the same results as it has to be.

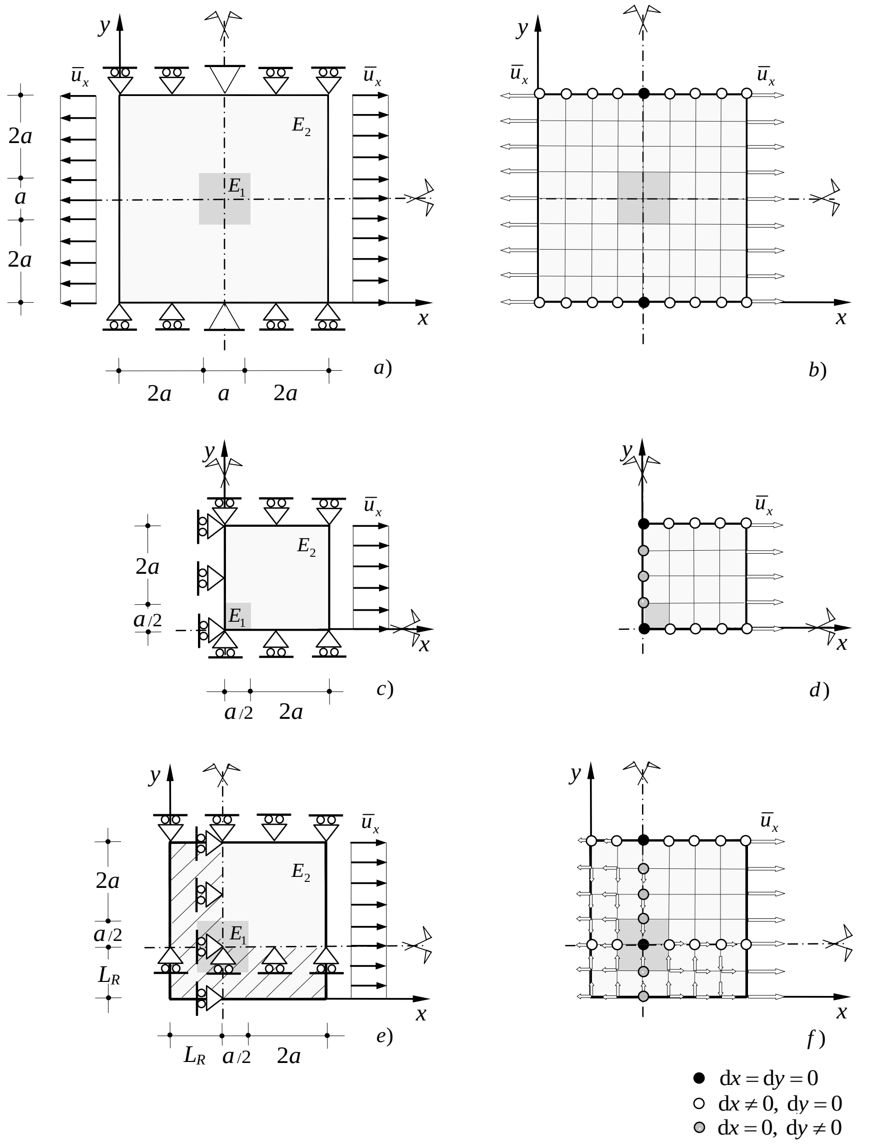

Fig. 4. Nonlocal elastic bi-symmetric square plate under tension with piecewise constant Young modulus. Mechanical model of: *a*) whole structure; *c*) symmetric quarter structure; *e*) enlarged symmetric quarter structure. *b*), *d*) and *f*) FE models of structures *a*), *c*) and *e*), respectively.

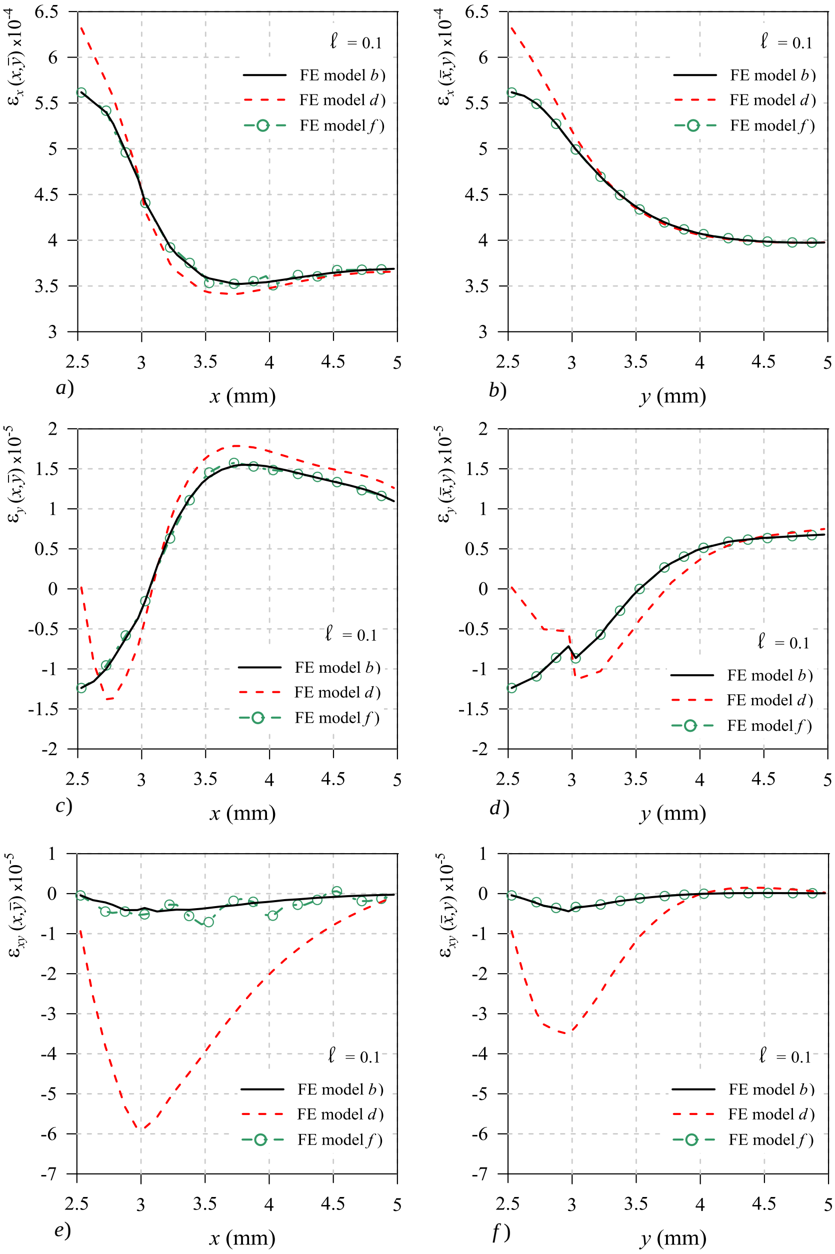

Fig. 5. Nonlocal elastic bi-symmetric square plate ($\ell = 0.1$cm). Strain components profiles ε_x, ε_y, ε_{xy}, versus x at $y \simeq 2.5$ on the left side and versus y at $x \simeq 2.5$ on the right side. Solution referred to the entire plate (solid lines), solution referred to the quarter symmetric portion (dashed lines), solution referred to the enlarged quarter symmetric portion (dashed lines with circles).

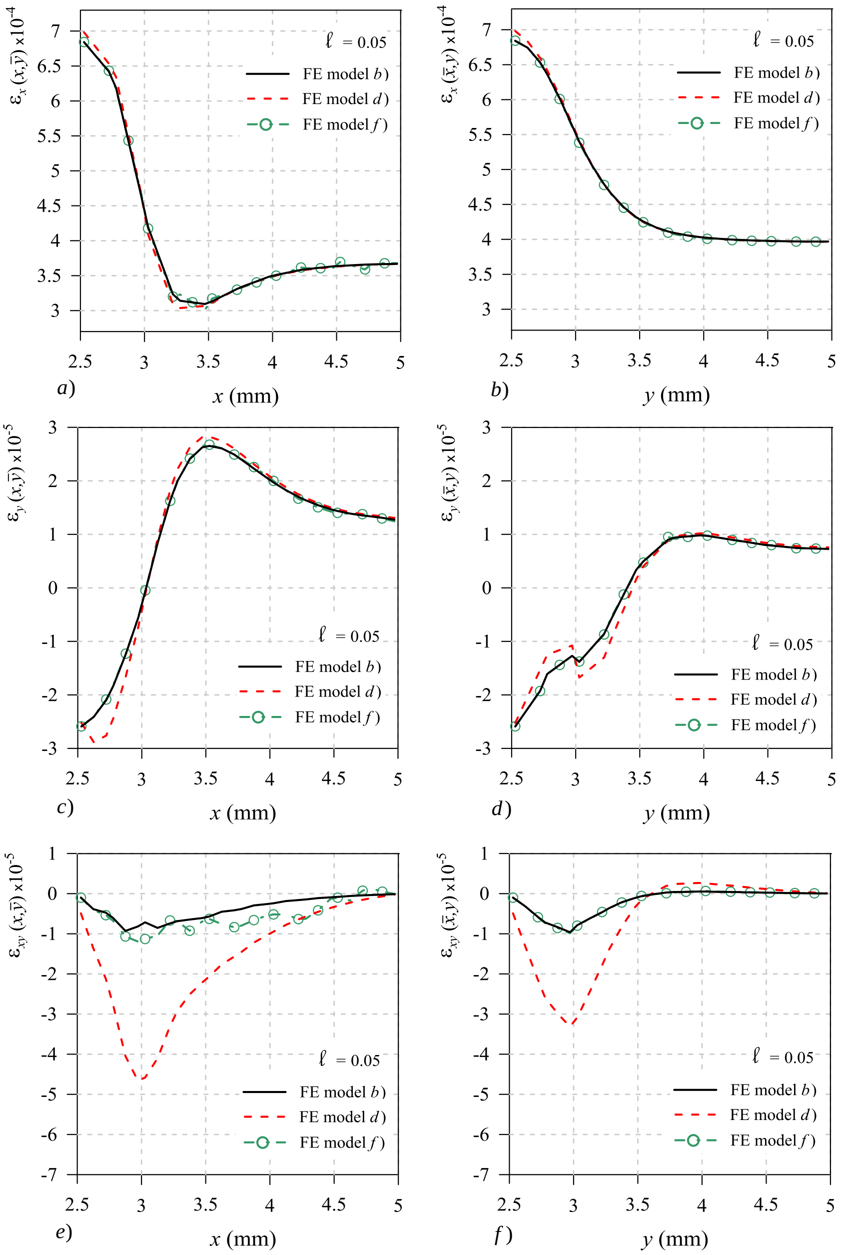

Fig. 6. Nonlocal elastic bi-symmetric square plate ($\ell = 0.05$cm). Strain components profiles ε_x, ε_y, ε_{xy}, versus x at $y \simeq 2.5$ on the left side and versus y at $x \simeq 2.5$ on the right side. Solution referred to the entire plate (solid lines), solution referred to the quarter symmetric portion (dashed lines), solution referred to the enlarged quarter symmetric portion (dashed lines with circles).

Following the rationale of the previous example, the symmetric scheme of Fig.4c has to be enlarged by including two boundary zones of wideness L_R as sketched in Figs.4e and f. Moreover, beyond the local-type constraints applied along the cuts (lines of symmetry), within these zones have to be applied some smeared displacements evaluated, once again, knowing the solution for the original (entire) structure. In order to obtain the expected results, the enlarged symmetric model of Fig.4f has eventually to be considered. The pertinent smeared boundary conditions being in this case the horizontal and vertical nodal displacements given, in the zone corresponding to the enlarged symmetrical boundary zone, by the solution of the original (entire) structure. The numerical results given in terms of strain components ε_x, ε_y and ε_{xy} are shown in Figs.5a-f and 6a-f for $\ell = 0.1$ and $\ell = 0.05$ respectively. Also in this case the entire structure is discretized using an uniform mesh of 400 FEs, the quarter symmetric portion is discretized with 100 FEs, while the enlarged symmetric model consists of 256 FEs when $\ell = 0.1$ and 196 FEs when $\ell = 0.05$. The obtained results highlight, as for the first example, that the entire model and the enlarged symmetric model give rise to the same results, slight differences are registered only for the strain component ε_{xy}. On the contrary, when the quarter (local-type) symmetric model is used all the results deviates from the ones given by the analysis of the entire structure. As before, small values of ℓ mitigate the highlighted drawbacks but do not eliminate them if the nonlocal solution is computed on a reduced local-type symmetric portion of the structure.

4. Conclusions

Structural symmetry simplifies the analysis of many problems of engineering interest allowing the analyst to study a symmetric portion of the entire pertinent mechanical model. The reduced (halved or more) geometrical dimensions of such symmetric portion allow to reduce the DOFs of the corresponding discretized model adopted for the numerical analysis or, with an equal number of DOFs, allows a more accurate description of the problem. The solution determined on the analyzed symmetric portion, once mirrored with respect to the line or plane of symmetry, gives the complete solution of the problem.

The paper has focused on some incoherencies arising when structural symmetry, and the well known related simplifications, are used for structures made of an elastic material of nonlocal integral type. The key-point is that nonlocality of an elastic material, postulated for taking into account within a macroscopic treatment phenomena arising at a micro– or nano–structural level, implies that what is measured or evaluated at a point in the domain is affected by what is happening in the neighboring points. The removal of one or more symmetric portions to isolate the reduced (simplified) model to be analyzed implies the loss on nonlocal effects exerted on the chosen symmetric portion of the structure by the ones removed. The solution so computed, once mirrored with respect to the line or plane of symmetry, does not matches the one obtainable on the complete model of the structure.

Such circumstance has been illustrated by the analysis of two simple structures and an enlarged symmetric model has been conceived to recover the missing nonlocal effects. The enlarged model is obtained by adding to the standard (local-type) symmetric portion a boundary symmetric zone originally belonging to the portion to be removed. The wideness of the added zone being strictly related to the internal length material scale as well as to the analytical shape of the kernel used within the integral constitutive model. The appropriate boundary conditions to be applied for guaranteeing the mechanical equivalence between the reduced model and the original structure have been also discussed introducing a sort of nonlocal (smeared) boundary conditions whose definition is actually a point left open to discussion. The required conditions, are themselves part of the solution to be worked out. The proposed remedies, aside the one concerning the definition of the (smeared) nonlocal boundary conditions here furnished by the known complete solution of the addressed examples but deserving further investigations for a general applicability, appear effective to eliminate the described inconveniences that likely, as deducible from the given results, are more evident as much are the nonlocality effects. The latter are strictly related to the internal length material scale value which has to be small enough with respect to structural dimensions. Such requirement is not satisfied in many existing numerical examples and, to the authors knowledge, the tackled problem of structural symmetry within nonlocal elasticity has not been highlighted in the relevant literature. Nevertheless, the remarks carried on throughout the paper do not pretend to be exhaustive, the main goal of the work being the will of sharing what has been experienced by the authors in solving numerically nonlocal elastic symmetric structures.

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