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Structural symmetry and boundary conditions for nonlocal symmetrical problems

Aurora Angela Pisano · Paolo Fuschi

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Abstract The paper deals with the determination of the symmetric model and proper boundary conditions for solving nonlocal elastic symmetric structures. The above concepts, in the context of nonlocal integral elasticity, turn out to be different with respect to the standard ones, classically applied when dealing with local elastic symmetric structures. Indeed, when only a symmetric portion of the structure is analyzed, the nonlocal effects induced by the remaining (cut) portions are lost, this necessitates the consideration of an *enlarged symmetric model* on which appropriate *nonlocal boundary conditions* have to be imposed. It has to be pointed out how the width of such an enlarged model depends on the nonlocal material parameters, while the correct unknown nonlocal boundary conditions are here obtained and enforced by an iterative procedure. The accuracy of the proposed approach in solving nonlocal structural symmetric problems is tested with the aid of two numerical examples.

Keywords Symmetric structures · Nonlocal integral elasticity · Nonlocal boundary conditions.

1 Introduction and motivations

In the field of solid mechanics and continuum media, the relevant literature of the last few decades has shown a growing interest for nonlocal models. This interest is justified by the attempt to solve paradoxes/problems related to the application of the classical local continuum theories. Examples are,

A.A. Pisano, P. Fuschi
Department Patrimonio, Architettura, Urbanistica,
University Mediterranea of Reggio Calabria, Italy
Tel.: +39-0965-1696303
Fax: +39-0965-1696550
E-mail: aurora.pisano@unirc.it

among others: the singularity of the stress field predicted by the local elastic fracture mechanics theory ([6], [3]) at a tip of a crack; or the inability to consistently describe the mechanical behaviour of some innovative materials, like nanotubes or nanocomposites [11]; or other complex materials, such as those involved in biomechanics ([10], [14]); and, in general, all those problems where the diffusive processes arising at nanoscale level influence the mechanical behaviour of the material at macroscopic level. Nonlocal continuum theories seem to overcome the above drawbacks thanks to the presence, in the constitutive nonlocal laws, of additional internal material parameters. In this context, starting from the early papers of Edelen and Eringen ([4], [5], [3]), a variety of formulations have been proposed which, without pretending to be exhaustive, can be summarized as follows: the integral [7] and the gradient [2] approaches, the continualizations procedures [1], the peridynamic models [13]. The above formulations are theoretically and mathematically more complex with respect to the local ones and, consequently, their numerical application results computationally more challenging and burdensome. Then, there is the need, more than in the local context, to take advantage of all the computing strategies that can reduce the analysis cost. Among such strategies, one, certainly effective, consists in the exploitation of structural symmetries that, as well known, can half or more the structural model to be analyzed. For this reason, structural symmetries are extensively used also in nonlocal applications where, the classical (local) simplified schemes with the related assumptions and boundary conditions are used in a straightforward manner. Unluckily, it can be shown that these assumptions lead to an incorrect solution of the original nonlocal problem. Indeed, when only a symmetric portion of the structure is analysed, the nonlocal effects induced by the neglected portions are lost. In order to recover the correct solution, the paper proposes the use of an *enlarged symmetric model* and the definition of appropriate *nonlocal boundary conditions*. The width of such an enlarged model depends on the nonlocal material parameters, while the correct nonlocal boundary conditions are obtainable iteratively. The determination of the nonlocal boundary conditions, proposed in this paper, provides the answer to an open question posed by the authors in a previous article [9] on the subject. Even if the arguments defined above are of general validity in the nonlocal context, in the following reference it is made to a nonlocal elasticity model of integral type and, more precisely, to an enhanced version of the two-phases nonlocal integral Eringen model, already used by the authors in previous researches ([12], [8]). Two symmetrical structures, made up of nonlocal elastic material, are analysed in order to highlight both the incoherencies arising in the results when a classical symmetric structural portion with classical boundary conditions is considered and to show the effectiveness of the proposed remedies. The numerical examples are carried out by means of a nonlocal finite element method (NL-FEM), developed and implemented by the authors in [8]. For the sake of clarity, details on the constitutive assumptions as well as on the NL-FEM are briefly recalled in the following sections.

2 The strain-difference nonlocal integral model

To describe the constitutive behavior of the material the structural elements are made with, the strain-difference nonlocal integral model, proposed in [12], is employed. It is a two-phases local-nonlocal model that, for a continuum body, occupying a volume V, has the shape:

$$\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon} + \alpha \, \mathcal{A} \left[\triangle \left(\mathbf{D} : \mathcal{A}(\triangle \boldsymbol{\varepsilon}) \right) \right]. \tag{1}$$

In the above expression $\mathbf{D}$ is the elastic moduli tensor in its standard form, while $\mathcal{A}(\cdot)$ is a nonlocal operator defined as:

$$\mathcal{A}(\boldsymbol{\varepsilon})|_{\mathbf{x}} := \int_V g(\mathbf{x}, \mathbf{x}')\, \boldsymbol{\varepsilon}(\mathbf{x}') \, \mathrm{d}\, \mathbf{x}' \qquad \forall \, \mathbf{x} \in V, \tag{2}$$

and

$$\mathcal{A}(\triangle \boldsymbol{\varepsilon})|_{\mathbf{x}} := \int_V g(\mathbf{x}, \mathbf{x}') \left[\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x}) \right] \mathrm{d}\, \mathbf{x}' \qquad \forall \, \mathbf{x} \in V. \tag{3}$$

The function $g(\mathbf{x}, \mathbf{x}')$ entering equations (2) and (3) is known as attenuation function. It plays a crucial role in the nonlocal model taking into account, for each point $\mathbf{x}$ in V, the nonlocal effects coming from a neighborhood subdomain centered at $\mathbf{x}$. In many nonlocal models, as the one hereafter employed, the function $g(\mathbf{x}, \mathbf{x}')$ satisfies the normalization condition in the shape $\int_{V_\infty} g(\mathbf{x}, \mathbf{x}') \, \mathrm{d}\, V' = 1$.
The literature offers different analytical expressions for the attenuation function $g(\mathbf{x}, \mathbf{x}')$ which essentially depends on the Euclidean distance between points $\mathbf{x}$ and $\mathbf{x}'$ of the domain as well as on a nonlocal material parameter, say ℓ, measure of the domain seat of the diffusive process. Then, the proposed constitutive model contains two nonlocal parameters that are α and ℓ. The two parameters possess a different physical meaning; precisely: while α gives the proportion of the nonlocal phase in the constitutive model, ℓ defines the so called influence distance, say L_R, i.e. the “length” defining the “width of the continuum region” within which long distance nonlocal interactions act. The nonlocal effects increase with α and ℓ, while the material model identifies with the local one for α and/or ℓ approaching to zero.
By substituting (2) and (3) in (1) the strain-difference model can be rewritten as

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) - \alpha \int_V \mathbf{J}(\mathbf{x}, \mathbf{x}') : \left[\boldsymbol{\varepsilon}(\mathbf{x}') - \boldsymbol{\varepsilon}(\mathbf{x}) \right] \mathrm{d}\mathbf{x}', \tag{4}$$

where the following positions hold:

$$\gamma(\mathbf{x}) = \int_V g(\mathbf{x}, \mathbf{x}') \, \mathrm{d}V' \tag{5}$$

$$\mathbf{k}(\mathbf{x}, \mathbf{x}') = \int_V g(\mathbf{x}, \mathbf{z}) g(\mathbf{x}', \mathbf{z}) \mathbf{D}(\mathbf{z}) \, \mathrm{d}\mathbf{z} \tag{6}$$

$$\mathbf{J}(\mathbf{x}, \mathbf{x}') = \left[\gamma(\mathbf{x}) \mathbf{D}(\mathbf{x}) + \gamma(\mathbf{x}') \mathbf{D}(\mathbf{x}) \right] g(\mathbf{x}, \mathbf{x}') - \mathbf{k}(\mathbf{x}, \mathbf{x}'). \tag{7}$$

From equation (4) it appears evident the dependency of the nonlocal phase by the strain difference field. It has been proved in [12] that such kind of model is thermodynamically consistent and that it is possible to realistically describe the constitutive behavior of a nonlocal elastic material occupying a finite domain.

3 The nonlocal finite element method

This section briefly recalls some theoretical and computational aspects related to the numerical tool utilized in solving the analyzed nonlocal elastic symmetric structures. It comes in a nonlocal version of the standard finite element method, known as nonlocal finite element method (NL-FEM). The NL-FEM has been theoretically presented in [12] and implemented, in its enhanced version, in [8]. The two above quoted papers give all the details on the method and the Reader is invited to refer to them to gain a full understanding. Nevertheless, the method is here recalled both for general sake of clarity, and because its correct knowledge helps in the comprehension of the strategy proposed later on to solve nonlocal elastic symmetric structures.

To proceed further, let us consider a nonlocal elastic body occupying a volume V whose boundary surface is S. The body is subjected to body forces $\mathbf{b}(\mathbf{x})$ in V and surface tractions $\mathbf{t}(\mathbf{x})$ on S_t. Kinematic boundary conditions $\mathbf{u}\,(\mathbf{x}) = \bar{\mathbf{u}}(\mathbf{x})$ are also specified on $S_u = S - S_t$. The pertinent boundary value problem (BVP) is defined by the standard equilibrium and compatibility equations, together with the nonlocal stress strain relation given in (4). It has been shown ([12],[8]) that for such nonlocal BVP the associated nonlocal total potential energy functional can be written as:

$$
\begin{aligned}
\mathit{\Pi}\left[\mathbf{u}(\mathbf{x})\right] = \frac{1}{2}\int_V \nabla\mathbf{u}(\mathbf{x}) : \mathbf{D}(\mathbf{x}) : \nabla\mathbf{u}(\mathbf{x})\,\mathrm{d}\,V + \\
+\frac{\alpha}{2}\int_V \nabla\mathbf{u}(\mathbf{x}) : \gamma^2(\mathbf{x})\mathbf{D}(\mathbf{x}) : \nabla\mathbf{u}(\mathbf{x})\,\mathrm{d}\,V + \\
-\frac{\alpha}{2}\int_V\int_V \nabla\mathbf{u}(\mathbf{x}) : \mathbf{J}(\mathbf{x},\mathbf{x}') : \nabla\mathbf{u}(\mathbf{x}')\,\mathrm{d}\,V'\,\mathrm{d}\,V + \\
-\int_V \mathbf{b}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\,V - \int_{S_t}\mathbf{t}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\,S.
\end{aligned} \tag{8}
$$

The displacement field $\mathbf{u}(\mathbf{x})$ that minimizes (8), together with $\boldsymbol{\varepsilon} = \nabla^s\mathbf{u}$ and $\boldsymbol{\sigma}$ obtained from (4), furnish the solution of the boundary value problem. Functional (8) allows to derive a consistent nonlocal FEM formulation which, for the discretized problem and with reference to the global DOFs of the system all collected in the vector $\mathbf{U}$, leads to the following global system of algebraic equations:

$$\widetilde{\mathbf{K}}\,\mathbf{U} = \mathbf{F}, \tag{9}$$

formally appearing as the one of the standard (local) FEM. In eq.(9) $\mathbf{F}$ denotes the standard *global vector* of the *element nodal forces*, while $\widetilde{\mathbf{K}}$ denotes a peculiar (novel with respect to the standard FEM) matrix, namely the *nonlocal global stiffness* matrix. More detailed, if the displacement and strain fields are described, as usual, in terms of element nodal displacements, that is, referring to element $\#n$ in terms let's say of the element vector $\mathbf{d}_n$, the following standard relations hold:

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}_n(\mathbf{x})\mathbf{d}_n, \quad \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{B}_n(\mathbf{x})\mathbf{d}_n \qquad \forall\, \mathbf{x} \in V_n \tag{10}$$

with $\mathbf{N}_n(\mathbf{x})$ collecting the element shape functions and $\mathbf{B}_n(\mathbf{x})$ their derivatives. By substituting eqs.(10) in (8), after some algebra, it is easy to obtain a discretized form of functional $\mathit{\Pi}\,[\mathbf{u}(\mathbf{x})]$, precisely:

$$\mathit{\Pi}\,[\mathbf{d}_n] = \frac{1}{2}\sum_{n=1}^{N_e} \mathbf{d}_n{}^{\mathsf{T}}\mathbf{k}_n^{loc}\mathbf{d}_n + \\ \frac{\alpha}{2}\sum_{n=1}^{N_e}\left[\mathbf{d}_n{}^{\mathsf{T}}\mathbf{k}_n^{nonloc}\mathbf{d}_n - \sum_{m=1}^{N_e}\mathbf{d}_n{}^{\mathsf{T}}\mathbf{k}_{nm}^{nonloc}\,\mathbf{d}_m\right] - \sum_{n=1}^{N_e}\mathbf{d}_n{}^{\mathsf{T}}\mathbf{f}_n \tag{11}$$

where:

$$\mathbf{k}_n^{loc} = \int_{V_n} \mathbf{B}_n{}^{\mathsf{T}}(\mathbf{x})\mathbf{D}(\mathbf{x})\mathbf{B}_n(\mathbf{x})\mathrm{d}V_n, \tag{12}$$

$$\mathbf{f}_n = \int_{V_n} \mathbf{N}_n{}^{\mathsf{T}}(\mathbf{x})\mathbf{b}(\mathbf{x})\mathrm{d}V_n + \int_{S_{t(n)}} \mathbf{N}_n{}^{\mathsf{T}}(\mathbf{x})\mathbf{t}(\mathbf{x})\mathrm{d}V_n, \tag{13}$$

are the element stiffness matrix and element nodal force vector, respectively, in their standard format. The operators with apex "nonloc" are instead a *set of element nonlocal matrices* given by:

$$\mathbf{k}_n^{nonloc} = \int_{V_n} \mathbf{B}_n{}^{\mathsf{T}}(\mathbf{x})\boldsymbol{\gamma}^2(\mathbf{x})\mathbf{D}(\mathbf{x})\mathbf{B}_n(\mathbf{x})\mathrm{d}V_n, \tag{14}$$

$$\mathbf{k}_{nm}^{nonloc} = \int_{V_n}\int_{V_m} \mathbf{B}_n{}^{\mathsf{T}}(\mathbf{x})\mathbf{J}(\mathbf{x},\mathbf{x}')\mathbf{B}_m(\mathbf{x}')\mathrm{d}V_m\mathrm{d}V_n. \tag{15}$$

Through inspection of equations (14) and (15), it appears how the "nonlocal behavior" of the element generates a set of element nonlocal stiffness matrices affected by contributions coming from other elements in the mesh and this by the presence of operators $\boldsymbol{\gamma}(\mathbf{x}), \mathbf{J}(\mathbf{x},\mathbf{x}')$ and, even more explicitly, by the presence of $\mathbf{B}_m(\mathbf{x})$ in the cross-stiffness element matrix given by eq.(15). In the latter expression, for the current element $\#n$, $\#m$ ranges over all the elements in the mesh. The summation symbol in (11) mimics the assembling procedure, carried on with the usual identification procedure of the local DOFs (nodal element displacements $\mathbf{d}_n$) with the global ones ($\mathbf{U}$). Such assembling eventually produces the nonlocal global stiffness matrix of eq.(9) which turns out to be the sum of two contributions:

$$\widetilde{\mathbf{K}} = \mathbf{K}_{_{loc}} + \mathbf{K}_{_{nonloc}} \tag{16}$$

where $\mathbf{K}_{loc}$ is the standard local global stiffness matrix, while $\mathbf{K}_{nonloc}$ is a nonlocal part. The latter, takes into account, for each point $\mathbf{x}$, the effects of diffusive processes arising in all the other points, $\mathbf{x}'$, of the discretized domain V. The matrix $\widetilde{\mathbf{K}}$ is proved to be symmetric and positive definite; moreover, $\mathbf{K}_{loc}$ and $\mathbf{K}_{nonloc}$ are banded matrices, but $\mathbf{K}_{nonloc}$ possesses a larger bandwidth due to the contributions coming from the cross integrations. It is important to outline how these cross integrations are performed only with regards to a limited zone of the domain individuated by the influence distance L_R, and this because the nonlocal operators γ and $\mathbf{J}$, being functions of g, vanish beyond L_R. Each nonlocal element is influenced indeed only by the nonlocal effects of a certain number of neighbors elements.

4 Symmetric structures and nonlocal elasticity

4.1 Mechanical modeling and boundary conditions

Symmetry and antisymmetry are well known properties advantageously usable when dealing with structural problems. The exploitation of such properties allow to significantly reduce the part of structure to consider for the analysis and then allow to shoot down computational time and memory required for its numerical analysis. In fact, taking advantage from one or more symmetries owned by the structural problem, the solution provided by the analysis of a reduced (symmetric) portion of the structure, if properly mirrored with respect to the lines of symmetry, furnishes the solution of the entire structure. As known, the above assertion turns out to be true if, along the symmetry lines (planes), are applied the correct boundary conditions reproducing the "restraints" or the "freedoms" imposed on the analysed part by the ones removed. In local context, the handling of a structural symmetric problem is well established either in terms of portion of structure to analyze, which is simply the one bordered by the lines (planes) of symmetry, either in terms of boundary conditions to be enforced along such lines (planes). Conversely, in nonlocal context further considerations are needed, proving that the straightforward application of the standard rules of symmetry leads to incorrect results, [9].
A first consideration is related to the implications given by the hypothesized constitutive model (4). To remove the portion of the structure beyond the symmetry lines means to *neglect the nonlocal effects* exerted by the cut portions on the analyzed one. Indeed, bearing in mind the meaning of the influence distance, L_R, the nonlocal effects neglected are those of a sub-domain, of constant width L_R, beyond the symmetry lines and belonging to the removed portion of the structure. This consideration induces to conceive for the analysis an *enlarged symmetric model* obtained by adding to the standard symmetric one, just the above mentioned sub-domain, that is a *boundary symmetric zone* of constant width L_R.
A second, crucial, consideration concerns the definition of the appropriate boundary conditions to apply on the enlarged symmetric model. Such condi-

tions have to be now enforced not only along the symmetry line (cut) but also *within* the boundary symmetric zone. In this sense they can be named nonlocal boundary conditions (NL-BCs). It has been proved in [9] that the correct solution is obtainable only if the NL-BCs applied in the boundary symmetric zones coincide with the displacements evaluated in the corresponding zones of the original, entire, structure. Of course, the above result has only a theoretical validity, being such boundary conditions part of the solution. In the present work, the indeterminacy of the NL-BCs is solved by means of an *iterative procedure* applied to a NL-FE discretization of the nonlocal enlarged symmetric structure. At a first analysis (iteration) the enlarged symmetric model is considered with only the standard boundary conditions applied along the symmetry axes. In the second analysis, the displacements evaluated in the first analysis within the zone symmetric to the boundary symmetric zone, are applied as NL-BCs, and so on from one iteration to another till the results of two subsequent iterations are equal within a certain tollerance. The procedure, despite the lack of a theoretical proof of convergence, is validated from the numerical evidences, and is hereafter summarized in a flow-chart style.

Flowchart. Iterative scheme to enforce the NL-BCs

- *Initialization*
  Set k=0
  Set geometry, material data and loading of the enlarged symmetric model
  Set local BCs along the symmetry lines
  Set NL-BCs within the boundary symmetric zone, say $\bar{d}_i^{(0)} = 0$
  Perform NL-FE analysis and output nodal displacements ${d_i}^{(0)}$, stresses, strains ...
  Read $\bar{d}_i^{sym(0)}$, namely the nodal displacements within the zone symmetric to the boundary symmetric zone
- *Start Iteration*
  Set $k = k + 1$
  Apply the NL-BCs $\bar{d}_i^{(k)} = \bar{d}_i^{sym(k-1)}$
  Perform NL-FE analysis and output nodal displacements ${d_i}^{(k)}$, stresses, strains ...
  If $|{d_i}^{(k)} - {d_i}^{(k-1)}| \geq$ Toll
  Read $\bar{d}_i^{sym(k)}$ and perform a new iteration
  Else
  Exit
  Endif
  End Iteration

It is worth noting that an alternative "direct enforcement" of the NL-BCs would be to set the *unknown* displacements of the nodes falling within the boundary symmetric zone *equal* to the *unknown* displacements of the corresponding (symmetric) nodes falling within the zone symmetric to the boundary symmetric zone. The equality of such couples of unknowns concerns their moduli being the couples' terms of opposite sign by symmetry. Such an operation would imply a reordering/reduction of the global solving equation system, namely the entries in the relevant rows and columns of the global stiffness matrix pertaining to a couple of equal unknowns have to be added/subtracted to take into account the enforced equivalence. The main difference between the iterative and the direct enforcement of the NL-BCs, both sharing an initial identification of the couples of symmetric nodes, is that the proposed procedure requires an iterative updating of assigned displacements without actions on the assembled global stiffness matrix built from the beginning taking into account the presence of nodes with assigned DOFs; the direct enforcement requires instead a reordering of the global matrix after assembling. The Authors experienced the direct enforcement of the NL-BCs that, however, resulted in being computationally very cumbersome already for problems with few DOFs.

4.2 Numerical examples

The numerical examples analyze the two square plates in tension sketched in Figures 1a and 3a, respectively. The plates have side dimension $L = 5a$ and thickness t and for both of them it is assumed that the constituent material satisfies the stress-strain law given in (4), i.e. the material is nonlocal elastic. The nonlocal material parameters are set ℓ=0.1 cm and α=50 and the attenuation function has the following bi-exponential form:

$$\mathbf{g}(\mathbf{x}, \mathbf{x}') = \frac{1}{2\,\pi\,\ell^2\,t} \exp\left(\frac{-|\mathbf{x} - \mathbf{x}'|}{\ell}\right). \tag{17}$$

Moreover, it is assumed that the plates are both nonhomogeneous, possessing one and two inclusions, respectively.

The first example presents a square central inclusion of side dimension a, as shown in Fig.1a, which also reports the constraints acting on the upper and lower edges together with the prescribed displacements, $\bar{u}_x = 0.001$ cm, applied on the left and right edges. For the posed problem, two central axes of symmetry can be easily identified, so in a local context, only a quarter of the structure, as for example the one highlighted in the quarter upper right of Fig.1a, could be analyzed. Conversely, in the addressed context, in which a nonlocal material is employed, the symmetric scheme to be analyzed is the one reported in Fig.1b in which two boundary symmetric zones of constant width $L_R = 11\ell$ (hatched areas beyond the symmetry lines) have been added to the standard symmetric quarter model. The value $L_R = 11\ell$, also known as *computational influence distance* (see again [8] for further details), guarantees the normalization condition (recalled in Section 2) that has to be met by the assumed bi-exponential form of the attenuation function (17). Precisely, with this choice the loss on the unit value is equal to 0.02 %.

To solve the problem the NL-FEM is applied on the discretized model qualitatively drawn in Fig.1c. In this last figure are also sketched with white arrows

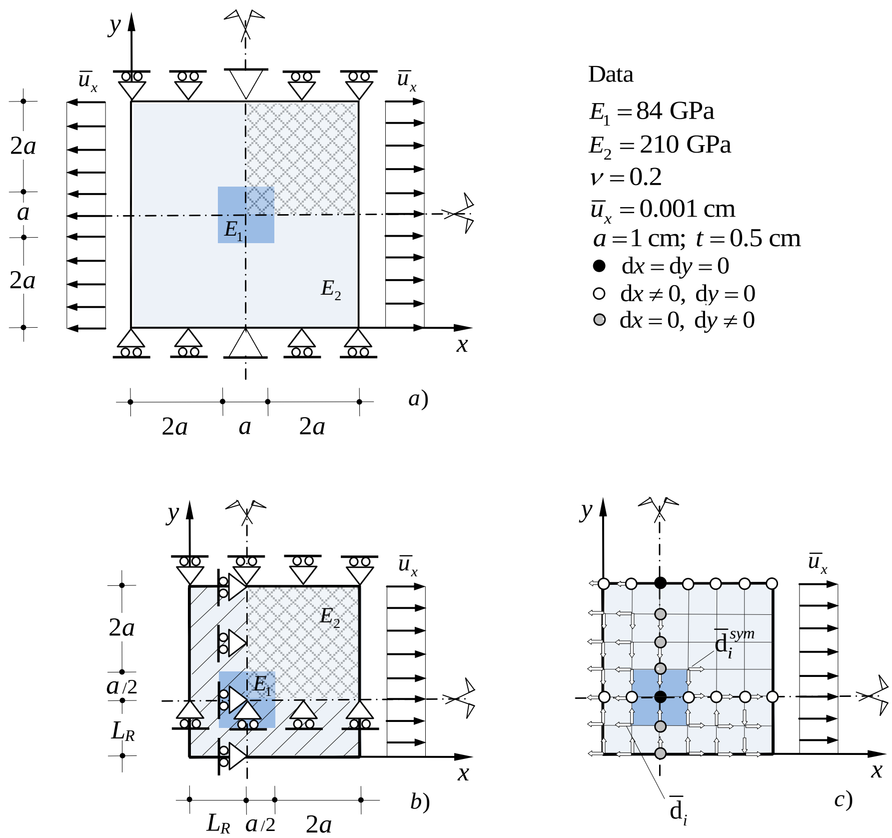

Fig. 1 Double symmetric square plate with one inclusion. Mechanical model of: a) entire structure; b) enlarged symmetric structure; c)NL-FE model of the enlarged symmetric structure with $\bar{d}_i^{sym}$ denoting the nodal displacements computed within the zone symmetric to the boundary symmetric zone and used to apply iteratively the NL-BCs, in terms of $|\bar{d}_i|=|\bar{d}_i^{sym}|$.

the NL-BCs (nodal element displacements) applied on the boundary symmetric zones and obtained following the discussed iterative procedure. In order to perform the analysis, a uniform mesh with 256 isoparametric 8-nodes NL-FEs is considered. The results, given in terms of strain components profiles ε_x and ε_y, are plotted in Fig.2 in correspondence to the two symmetry axes. The inspection of Fig.2 validates the arguments discussed throughout the paper, indeed, the solution obtained by considering the standard symmetric model (SSM) of the structure deviates from the reference one, evaluated by considering the whole model (WM), while the solution given by the enlarged symmetric model (ESM), endowed with the appropriate NL-BCs enforced iteratively, converges, after some iterations, to the correct one.

The same considerations can be repeated by observing the results provided by the second example. The latter consists in a plate having the same geometry, constraints and material properties of the first one, but characterized by the presence a double inclusion, as shown in Fig.3a. Moreover, a uniform traction, with $\bar{\sigma}_x = 85$ MPa, is applied along the left and right edges. This second example has been conceived with the purpose of validating the proposed procedure also when two inclusions, each falling in the influence zone of the other, are present. The above circumstance produces a more complex nonlocal solution that the enlarged symmetric model has to be able to pursue. Again, the mechanical scheme to be considered for the analysis is the one represented by the enlarged symmetric model of Fig.3b with applied NL-BCs in the same iterative fashion. The results, obtained with the pertinent NL-FE model of Fig.3c, are shown in Fig.4. Also in this case, the proposed procedure succeeds in solving the nonlocal symmetric structure correctly.

5 Conclusions

The paper has addressed the problem of symmetric structures in the context of nonlocal elasticity of integral type. It has been pointed out that classical symmetric models and related boundary conditions, commonly utilized also in the field of structures made of nonlocal materials, produce incorrect results. The physical reason of such incorrectness can be traced to the loss of the nonlocal effects exerted by the removed symmetric portion on the reduced (standard symmetric) one. To recover the correct solution, the paper has proposed the use of an enlarged symmetric model. The latter has been obtained by adding, beyond the symmetry lines of the standard symmetric model, boundary symmetric zones of width equal to the (nonlocal) influence distance. Such distance, related to the internal length of the nonlocal material, defines at which extent the nonlocality effects diffuse. The promoted remedy, whose effectiveness was shown in a recent contribution of the authors, has been drastically improved by the appropriate definition of the nonlocal boundary (symmetry) conditions. Such conditions, to be enforced within the boundary symmetric zones, are part of the solution and are here specified and applied through an iterative procedure. The numerical evidences seem to confirm the effectiveness of the entire procedure.

6 Compliance with Ethical Standards:

The authors declare that they have no conflict of interest.
The authors declare that this study did not receive funding.

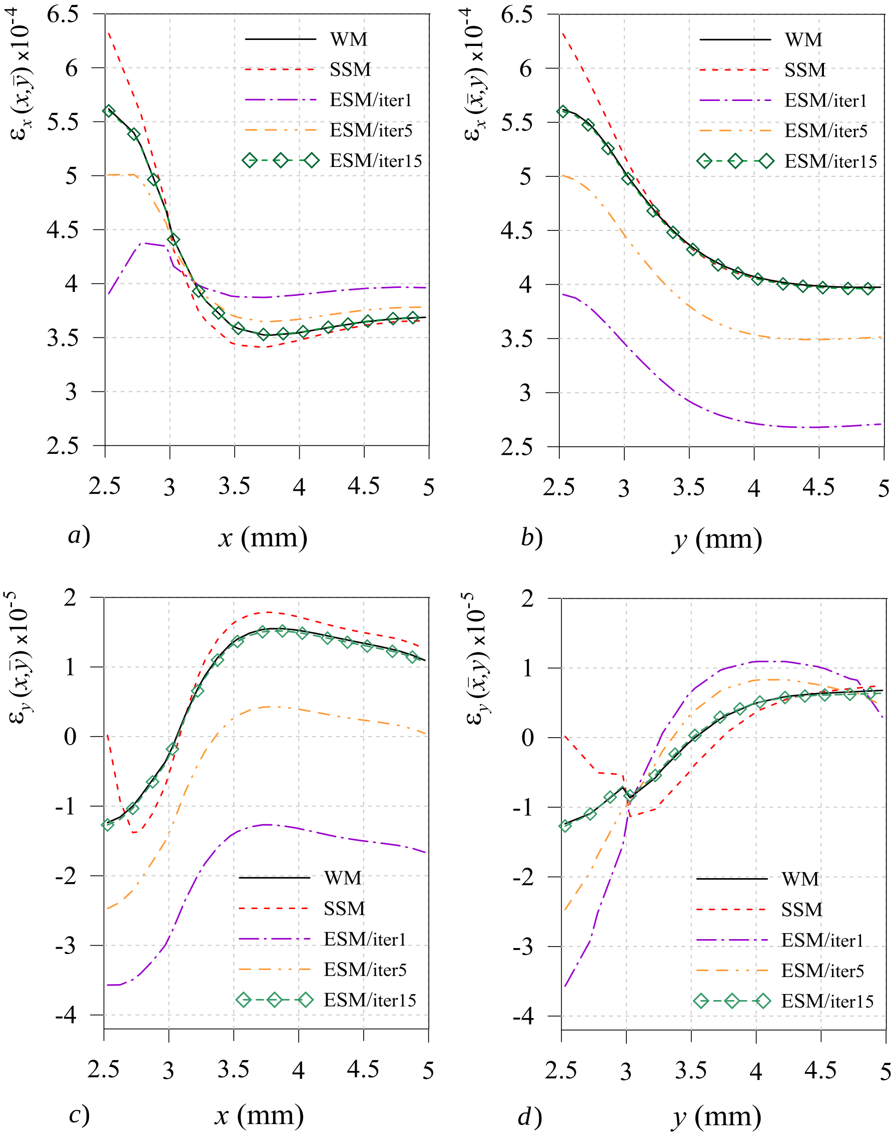

Fig. 2 Double symmetric square plate with one inclusion. Strain component profiles along the symmetry axes: ϵ_x, a) and b); ϵ_y, c) and d). Solution computed on the whole model of the plate (WM); solution computed on the standard symmetric model of the plate (SSM); solution computed on the enlarged symmetric model of the plate (ESM) at iterations 1, 5, 15.

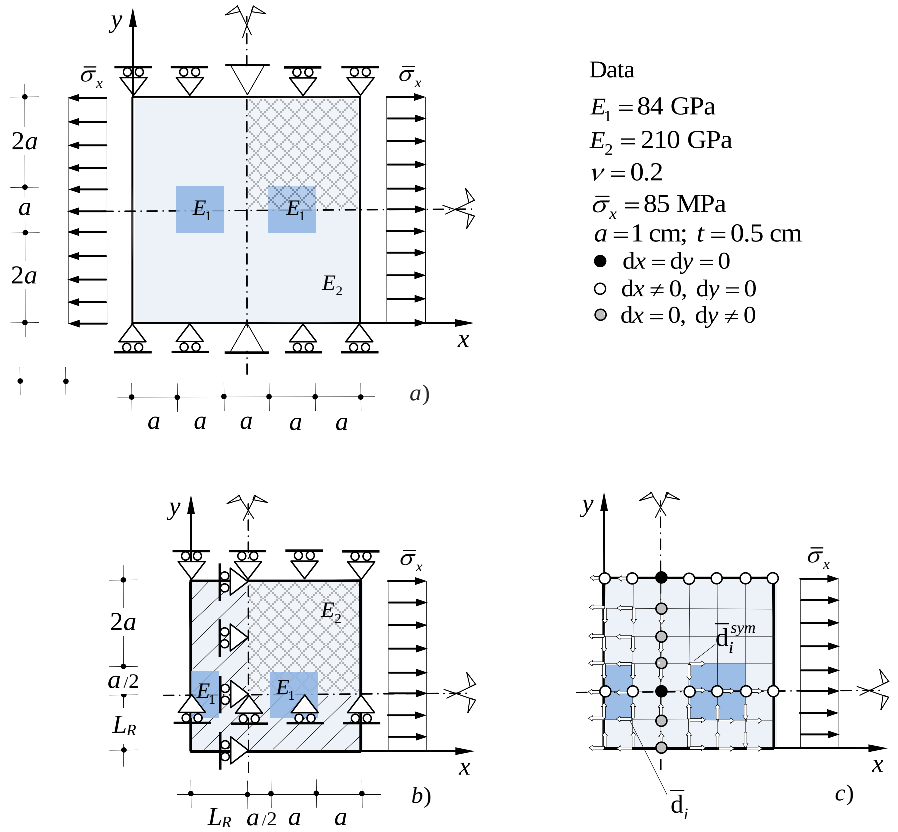

Fig. 3 Double symmetric square plate with two inclusions. Mechanical model of: a) entire structure; b) enlarged symmetric structure; c)NL-FE model of the enlarged symmetric structure with $\bar{d}_i^{sym}$ denoting the nodal displacements computed within the zone symmetric to the boundary symmetric zone and used to apply iteratively the NL-BCs, in terms of $|\bar{d}_i|$=$|\bar{d}_i^{sym}|$.

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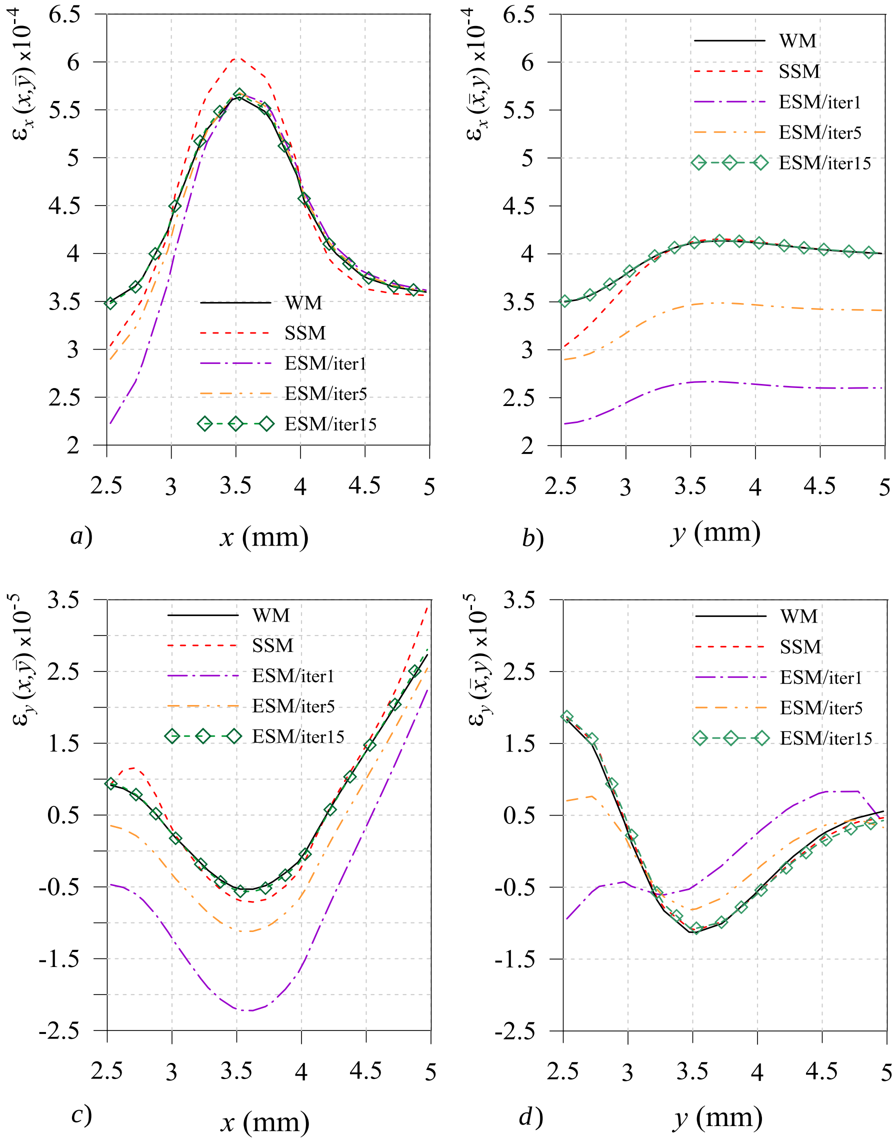

Fig. 4 Double symmetric square plate with two inclusions. Strain component profiles along the symmetry axes: ϵ_x, a) and b); ϵ_y, c) and d). Solution computed on the whole model of the plate (WM); solution computed on the standard symmetric model of the plate (SSM); solution computed on the enlarged symmetric model of the plate (ESM) at iterations 1, 5, 15.