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Key Points:
- Accurate and reliable solution procedures for Phase Retrieval problems are of interest in very many cases
- All the available solution procedures are subject to the possible trapping into "false solutions"
- A new point of view and a new procedure allowing some increased robustness with respect to trapping are proposed and discussed

Correspondence to:
A. R. Laganà,
antonia.lagana@unirc.it

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Phase retrieval by constrained power inflation and signum flipping

A. R. Laganà[1,2], A. F. Morabito[1,2], and T. Isernia[1,2]

[1]DIIES Department, Mediterranea University of Reggio Calabria, Reggio Calabria, Italy, [2]Consorzio Nazionale Interuniversitario per le Telecomunicazioni, Parma, Italy

Abstract In this paper we consider the problem of retrieving a signal from the modulus of its Fourier transform (or other suitable transformations) and some additional information, which is also known as "Phase Retrieval" problem. The problem arises in many areas of applied Sciences such as optics, electron microscopy, antennas, and crystallography. In particular, we introduce a new approach, based on power inflation and tunneling, allowing an increased robustness with respect to the possible occurrence of false solutions. Preliminary results are presented for the simple yet relevant case of one-dimensional arrays and noisy data.

1. Introduction

The phase retrieval problem is of significant interest in many areas of theoretical and applied science such as electron microscopy [*Misell*, 1973], astronomy [*Fienup and Dainty*, 1987], crystallography and optics [*Millane*, 1990], antenna characterization and diagnostics [*Yaccarino and Rahamat-Samii*, 1999; *Álvarez et al.*, 2010], inverse scattering [*Bucci et al.*, 2006], and in many other fields where the full knowledge of a complex function is needed but phase measurements are not available or not convenient.

Very many different contributions have appeared on the subject focusing on conditions such as to guarantee the uniqueness of the solution [*Barakat and Newsam*, 1984; *Isernia et al.*, 1995; *Fannjiang*, 2012, and references therein], on procedures such as to guarantee the recovery of such a solution [*Isernia et al.*, 1996a; *Marchesini*, 2007], as well as on both issues [*Seldin and Fienup*, 1990; *Isernia et al.*, 1999]. Assuming some condition guaranteeing the theoretical uniqueness of a solution holds true, we focus herein on the problem of developing solution procedures able to retrieve, in a robust and accurate fashion, the unknown signal.

Over the years, very many different approaches have been proposed for solving phase retrieval problems.

A number of popular and widespread procedures can be classified into the class of alternating projections [*Brègman*, 1965]. In such a class of procedures, one alternatively enforces the expected properties of the solution in the data domain (enforcing the measured amplitudes) and in the unknown domain (enforcing properties such as support and/or positivity and/or the expected amplitude distribution). With a number of different variants, the approach has been used in electron microscopy [*Misell*, 1973], optics [*Gerchberg and Saxton*, 1972; *Elser*, 2003], antenna diagnostics [*Bucci et al.*, 1990], and in very many other cases. Unfortunately, the sets defining the properties enforced in the two domains are nonconvex, so that the procedures are subject to trapping into false solutions. A smart way to overcome, at least from a practical point of view, such a problem, is the so-called "hybrid input-ouput" procedure by *Fienup* [1982], which is still very popular and adopted widespread [*Marchesini*, 2007].

A second large class of procedures is based on the minimization of some cost functional by means of some local or global optimization. Notably, some of the above approaches can be reinterpreted in terms of the (local) minimization of a suitable cost function [see *Barakat and Newsam*, 1984]. Unfortunately, because of the very large number of unknowns dealt with, and of the so called "No Free Lunch theorem," [*Wolpert and Macready*, 2010] solution procedures based on global optimization cannot guarantee a correct recovery in the finite time one has at his disposal. On the other side, the nonlinearity of the overall cost functional also results in nonquadratic (and usually nonconvex) cost functional, so that solution procedures based on local optimization can also get trapped into local minima corresponding to "false solutions" of the problem. In this respect, some progress has been provided in a number of papers by Pierri and colleagues [*Isernia et al.*, 1996a, 1999] which show how the formulation of the problem in terms of inversion of a quadratic operator allows some understanding of the false solutions problems, as well as effective counteractions.

A more recent approach relies onto the relaxation of the original Nonconvex Problem into a Convex one at the price of the introduction of a large number of auxiliary variables [*Candes et al*., 2012]. The approach, originally introduced in *Isernia et al*. [1988, 1989], has been independently rediscovered and considerably improved in *Candes et al*. [2012], where a crucial additional constraint on the auxiliary variables is also taken into account. Unfortunately, the nonlinear nature of the additional constraints and the very large number of auxiliary variables are such that the proposed procedure is not the end of the story, so that alternative points of view and solution procedures are indeed of interest.

In the following, we introduce a further approach based on the (constrained) local maximization of a cost functional coadjuvated by a “tunneling” technique allowing to escape from the attraction region of a local optimum to the attraction region of a different (and hopefully better) candidate solution. The approach takes advantage of the fact that the introduced cost functional is quadratic with respect to the unknowns, which allows a simple skip from an attraction region to another by a simple change of sign of a low number of properly defined variables.

It is worth noting that the philosophy underlying the present work is completely different from the one suggested by Isernia et al. in previous works [*Isernia et al*., 1999, 1996b, 1996a]. In those latter, the idea was that of Having a number of independent measures and a priori information such that the functional defining the solution had a single optimum. In this paper, we are able to minimize the amount of measures and a priori information by admitting that the cost functional still has different optima, but one is able to find the global optimum (corresponding to the ground truth) by means of some smart computationally efficient procedure.

2. Statement of the Problem

A very general formulation of phase retrieval problems can be given as follows.

If $f(x)$ is a one-dimensional, two-dimensional, or even a higher-dimensional unknown complex signal, and T is the Fourier transform operator, the phase retrieval problem consists of determining $f(x)$, from $|F(u)|$, where

$$F(u) = |F(u)| \exp[j\varphi(u)] = T[f(x)] \tag{1}$$

In order to fix ideas, we focus in the following on the case where $f(x)$ is a discrete source (such as a uniformly spaced array), so that $F(u)$ can be eventually interpreted as the corresponding array factor. However, all the presented ideas and procedures can be extended in a conceptually easy fashion as in many other cases (see section 6).

The remainder of the paper is as follows. In section 3 we briefly review our inspiring concept. Then, the proposed approach is introduced and developed in section 4. Finally, some numerical results confirming the interest of the approach are presented in section 5. Conclusions and suggestions for further development follow.

3. Lessons Learnd From a Canonical Case

In order to understand the rationale of the procedure proposed in section 4, it is useful to recall some results which hold true in the canonical problem of recovering a discrete one-dimensional source from the (square) amplitude of its Fourier transform.

As well known, the array factor $F(u)$ of the field radiated by a linear array with uniform spacing can be written as

$$\sum_{0}^{N-1} I_n e^{jnu} \tag{2}$$

with $u = \beta d \cos\theta$, d being the uniform spacing and θ the observation angle as measured from the line defining the array. I_n is the excitation of the nth array elements and N is the total element array number. As well known, expression (2) can be considered to be the restriction to the unitary circle of the so-called Schelkunoff polynomial $\sum_0^{N-1} I_n z^n$. The power pattern associated to such an array can then be written as

$$|F(u)|^2 = \sum_{1-N}^{N-1} D_p e^{jpu} \qquad D_p = D^*_{-p} \tag{3}$$

As for (2), expression (3) can be considered to be the restriction to the unitary circle of the function

$$P(z) = \sum_{1-N}^{N-1} D_p z^p \qquad D_p = D^*_{-p} \tag{4}$$

In order to recover the unknown sequence $I_1 \dots I_N$, (and assuming for the time being that no additional information besides the square amplitude distribution is available), one can solve the auxiliary problem:

$$\sum_{1-N}^{N-1} D_p e^{jpu_i} = M^2(u_i) \qquad D_p = D^*_{-p} \tag{5}$$

where $M^2(u_i)$ is the measured sample of $|F(u)|^2$ for $u = u_i$, to find the auxiliary variables D_p. Then, by means of the factorization of $P(z)$, one can split $P(z)$ (and hence $|F(u)|^2$) in two factors, each being the complex conjugate of the other on the unitary circle, and hence turn back to the unknowns of actual interest (i.e., the I_n coefficients) (see *Isernia et al*. [1998] for more details).

Notably, for the specific problem at hand, many different $\{I_n\}$ sets can correspond to the unique $\{D_p\}$ set of (power) coefficients, so that, even leaving aside the so-called trivial ambiguities [*Seldin and Fienup*, 1990] the solution is not unique. In fact, one will have a number of solutions equal to $2^{(M/2)}$, where M is the number of roots of the polynomial $P(z)$ outside of the unitary circle. However, a small additional information is usually sufficient to discriminate among the candidate $\{I_n\}$ sets. For example, the solution is univocally determined with a prior knowledge of one of the coefficients (but for very special cases). As a consequence, the overall problem can be effectively solved by finding (eventually) a large set of candidate solutions and picking then the most appropriate one.

Unfortunately, the above strategy cannot be used when fields cannot be represented in terms of polynomials (such as the case of near fields) and/or in two-dimensional problems, where polynomials are not factorable but for a zero measure set [*Barakat and Newsam*, 1984]. Then, our efforts have been devoted to developing a solution strategy not using polynomials and possibly able to manage, in effective fashion, the false solution problem.

In order to develop such a new approach, it is fruitful to have a look at the above problem from the point of view of the optimization of a cost functional written in terms of the original unknowns $I_1 \dots I_N$. For example, following [*Isernia et al*., 1996b] one could look for a best fitting among $\left|\sum_0^{N-1} I_n e^{jnu}\right|^2$ and the data $M^2(u)$, thus minimizing

$$\Psi = \left\| \left| \sum_{0}^{N-1} I_n e^{jnu} \right|^2 - M^2(u) \right\|^2 \tag{6}$$

Coherently with the result above, and leaving aside trivial ambiguities, such a functional will have $2^{(M/2)}$ global minima. Interestingly, because of the nature of the relationship among the actual and auxiliary unknowns, no other minima can occur. However, the consideration of a further (penalty) term enforcing the additional information which is needed for theoretical uniqueness will allow to come to a cost functional having a single global minimum. For example, if the additional information is given by the a priori knowledge that the excitation of the mth element is equal to $\overline{I_m}$, this will be the case with the functional:

$$\widetilde{\Psi} = \Psi + \alpha \left| I_m - \overline{I_m} \right|^2 \tag{7}$$

where α is a positive constant. Notably, even a very small value of α allows to get a functional having just a single point where the functional is exactly zero. However, the functional is still fourth order in terms of the unknowns, so that local minima may still occur. As a consequence, a solution procedure based on the minimization of $\widetilde{\Psi}$ may still get stuck in local minima, which will be located close to the global minima of Ψ in case of sufficiently small values of α.

However, the above theory offers a strategy to escape from such local minima. In fact, one can compute the roots of the polynomial associated to the current solution; substitute one of the its zeroes not lying on the unitary circle (say z_i) with the value $(1/z_i)^*$, where the asterisk indicates conjugation; and compute the corresponding polynomial and array factor (i.e., a new set of excitations).

Interestingly, the new sets of excitations will have exactly the same power distribution of the previous one, so that the value of Ψ keeps unaltered. However, the new tentative solution will lie in a different attraction region of the functional Ψ (or $\widetilde{\Psi}$) so that a tunneling can be achieved between the starting and the new attraction region without losing anything in terms of data fitting. By iterating local optimization and tunneling, one will have an ever decreasing behavior of the cost functional, thus hopefully reaching the ground truth solution after a number of tunneling operations. In so doing, some smart choice of the zeroes to be considered is needed. In this respect, the consideration of the so called "dominating zeroes," i.e., the ones departing to a larger extent from the unitary circle, seem to be the more appropriate choice.

4. A "Constrained Power Inflation and Tunneling" Procedure

The key points of the above optimization procedure are the consideration of a cost functional to be optimized and the possibility to have a tunneling in such a way as to escape from the possible occurrence of false solutions. Unfortunately, as near fields or fields radiated by two-dimensional sources cannot be expressed in terms of one-dimensional polynomials, which is the only case where one can safely factorize square amplitude distributions, some alternative tunneling technique has to be considered for some reasonable cost function to be optimized. Obviously, such a cost function should enforce data fitting and the additional a priori information which is eventually available.

As far as the cost function is concerned, a quadratic function seems to be a convenient choice, as one can eventually exploit theory and tools available for quadratic forms. Then, our alternative approach to the Phase Retrieval problems at hand is as follows:

Find f(x) such that:

$$\mathrm{OF} = \Sigma_0^S |Tf(x)|_s^2 \quad \text{is maximized} \tag{8}$$

Subject to

$$|Tf(x)|^2 \leq M^2(u) \tag{9}$$

where the index s in (8) denotes a sample of the corresponding quantity, and constraint (9), which will be discretized on a dense grid in practical instances allows to avoid divergence of the procedure. Notably, apart from an unessential constraint, a sufficiently large value of S implies that OF is the L_2 energy of the function Tf. According to common rules, both in (8) as well as in the discretization of (9), we use a number of samples which is roughly 10 times the number of unknown excitations.

Interestingly, the proposed procedure can be interpreted as an inflation of the function $|Tf(x)|^2$ until it reaches the right values corresponding to a perfect fitting (equivalence) of the two functions in (6). Moreover, constraint (6) defines a convex set. [*Isernia et al.*, 1998].

Obviously, the difficulties related to the nonlinearity of the problem are still present. In fact, the maximization of a quadratic function in a convex set is known to be an Non-deterministic Polynomial hard problem [*Garey and Johnson*, 1979], which witnesses the difficulty of the problem.

On the other side, both the physical interpretation as on "inflation" technique, as well as the mathematical structure of the problem, offer some interesting possibility. In this respect, it proves convenient to observe that in the simple case at hand of uniformly spaced arrays (with a half a wavelength spacing), the cost function to be maximized can be written, by virtue of the Parseval theorem, as

$$\mathrm{OF} = \sum_{n=1}^{N} (Re\{I_n\})^2 + (Im\{I_n\})^2 \tag{10}$$

However, in view of the arguments in section 3, the simple optimization of (10) (subject to constraints (9), does not guarantee the existence of a single global optimum, as the solution of the underlying problem is not unique. A possible way to come to a theoretically unique solution (as in section 2) is to fix the value of one of the excitations, say I_m. Notably, in such a way, one avoids both the nonuniqueness due to a constant phase term as well as the one related to zero flipping (see section 3).

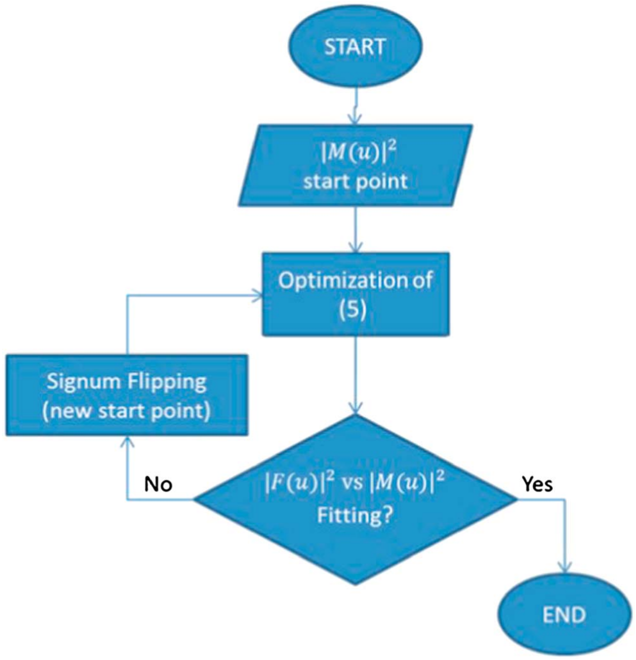

Figure 1. Flowchart of the proposed phase retrieval procedure.

Hence, the problem can be formulated as

Maximize:

$$\mathrm{OF}' = \sum_{\substack{n=1 \\ n \neq m}}^{N} (Re\{I_n\})^2 + (Im\{I_n\})^2 \quad (11)$$

subject to constraints (9) as well as to

$$I_m = \overline{I_m} \quad (12)$$

$\overline{I_m}$ being the actual value of I_m which is supposed to be known.

Then, by paralleling the procedure at the end of section 4, a possible strategy to optimize the objective function is as follows (see also flow chart in Figure 1):

1. In a first step, starting from a random point, optimize by means of a local optimization technique function (11) subject to constraints deriving from the discretization of (9);
2. Then, if the $M^2(u)$ data are not satisfactorily fitted, change the sign of one (or more) of the $Re(I_n)$, $Im(I_n)$ variables. Such a step can be interpreted as a tunneling of the cost function meant to jump into a different attraction basin. In fact, the "signum-flipping" operation will leave the value of the cost function unaltered. However, because of the need to fulfill constraints deriving from (9), the new set of unknowns needs to be projected onto the convex set defined from (9), so that some temporary worsening of the cost functional generally occurs;
3. Iterate the procedure if needed.

Of course, in all cases some clever strategy is needed in order to explore the different possible combinations of signs for the $Re(I_n)$, $Im(I_n)$ variables.

5. Numerical Results

In order to illustrate the effectiveness of the proposed strategy, we tested it in a number of cases dealing with one-dimensional arrays and corresponding far-field measurements. While the above theory is developed for the case of "ideal" (i.e., noiseless) data, we consider herein the case where each amplitude measurement is affected by errors due to measurement noise. In particular, in each sampling point, measurement errors have been simulated by adding a white Gaussian noise on the real and imaginary part of the signal and then considering the resulting modulus.

The presence of noise will have an impact on the above procedure as well as on its effectiveness. First, a relaxing of the "mask" function at the right-hand side of (9) has been used to take into account that the measured squared amplitude may be smaller than the ground truth. Moreover, one has to stop the procedure as soon as the current trial solution is consistent with measurement errors rather than looking for some "perfect" fitting. Such a last circumstance also has a theoretical drawback. In fact, in case of noise-free data, a "perfect fitting" is possible, and correctness of what is found is then granted by theoretical uniqueness. Opposite to that, noise on data implies that a perfect fitting is not any more possible (or it makes no sense anyway), so that points corresponding to local optima in case of noise-free data cannot be anymore discriminated as such, and the procedure may be trapped in these points.

On the other side, as long as local optima have a value of the functional considerably different from the expected value, the proposed approach still makes sense, and two different actions may be taken to

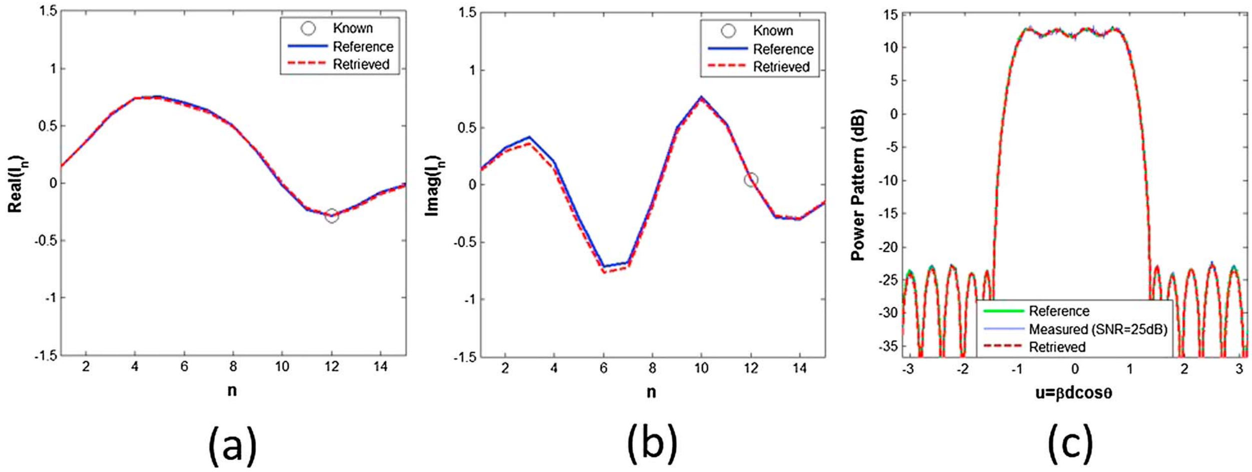

Figure 2. Results achieved in the first test case by exploiting a random starting point. (a) Real part of the reference (blue line) and retrieved (red line) excitation distributions. (b) Imaginary part of the reference (blue line) and retrieved (red line) excitation distributions. (c) Reference (green line), noise-corrupted (blue line), and retrieved (red line) power patterns.

guarantee robustness and accuracy of the technique. First, one has to increase as much as possible the accuracy of the measurements, so that an accurate (almost perfect) fitting is possible. In this respect, by exploiting the bandlimitedness of square amplitude distributions, oversampling may help. Second, following *Isernia et al*. [1999], one can use a number of a priori information (or a number of independent measurements) such that the local optima of the functional are incompatible with noise and a priori, so that one can identify them as “false” and look (through flipping) for the actual solution.

In summary, in the proposed strategy the lower the level of noise, the lower the number of independent information which is needed (up to the minimum required for theoretical uniqueness). For increasing levels of noise, a larger number of independent information is instead needed (up to the maximum discussed in *Isernia et al*. [1999]) to ensure robustness against false solutions.

Examples correspond to fields generated by linear equispaced arrays (with a spacing equal to half a wavelength) and, according to formulations (11) and (12) above, one of the excitations is supposed to be known. Each simulated retrieval required in the average 1.4 s to be performed by a calculator having an Intel Core i7-

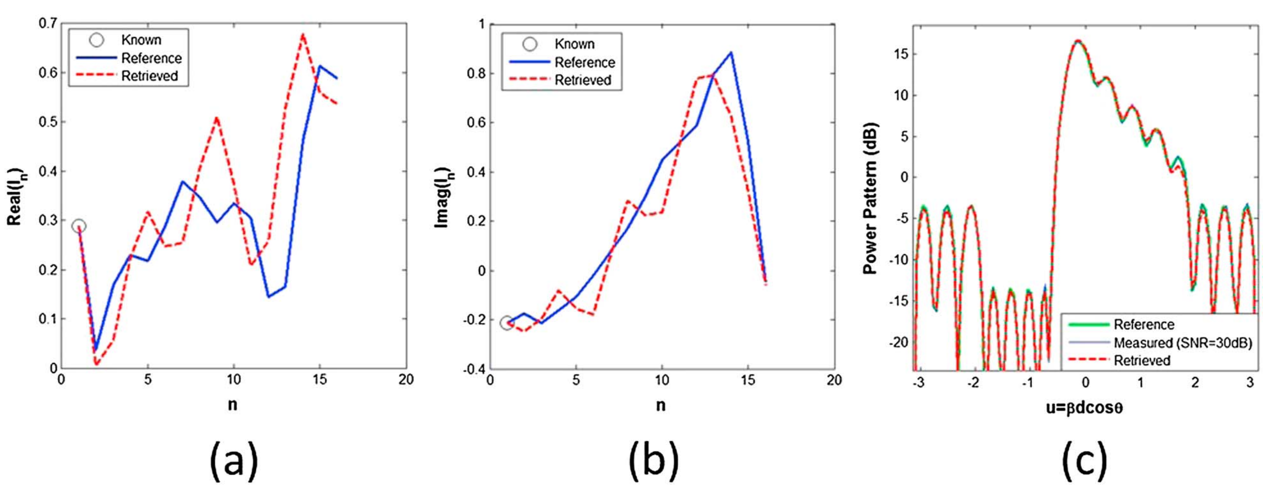

Figure 3. Results achieved in the second test case by exploiting a random starting point. (a) Real part of the reference (blue line) and retrieved (red line) excitation distributions. Imaginary part of the reference (blue line) and retrieved (red line) excitation distributions. (c) Reference (green line), noise-corrupted (blue line), and retrieved (red line) power patterns.

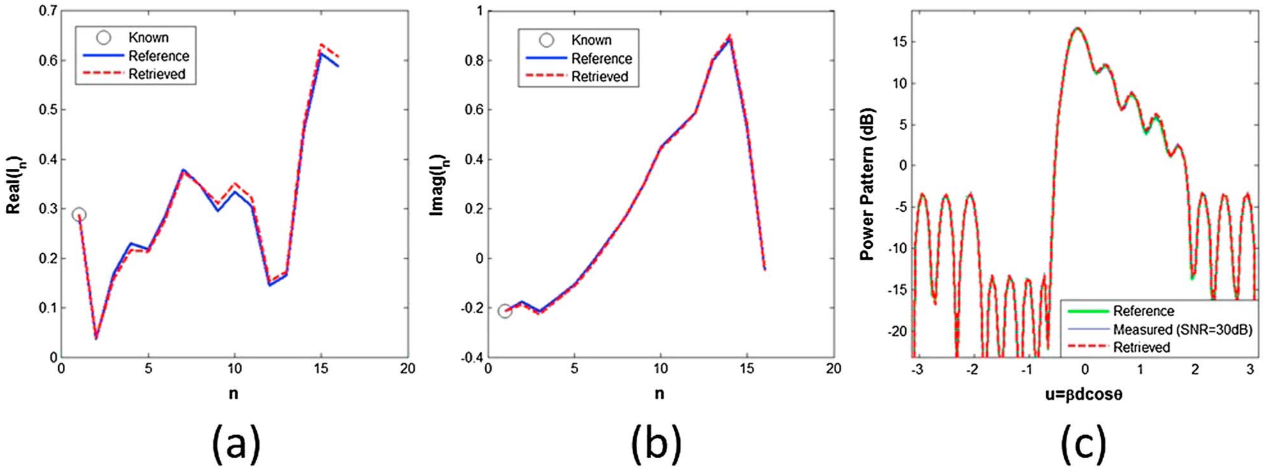

Figure 4. Results achieved in the second test case by exploiting as starting point the excitations achieved by copying the ones shown in Figure 3 and performing a signum flipping of the coefficient having the smallest amplitude. (a) Real part of the reference (blue line) and retrieved (red line) excitation distributions. (b) Imaginary part of the reference (blue line) and retrieved (red line) excitation distributions. (c) Reference (green line), noise-corrupted (blue line), and retrieved (red line) power patterns.

3537U 2.50 GHz CPU and a 10 GB RAM. In all the examples where the signum flipping is needed, we found it convenient, after a large number of different trials, to change the sign of the real and/or imaginary part of the excitations having a lower amplitude, and a single signum flipping led to successful reconstructions. The effectiveness of such a strategy can be explained by noting that a change of sign in the unknowns has no effect on the cost function (11), but it may imply a violation of constraints (9), so that the subsequent steps of the optimization procedure (performed by interior point methods [*Press et al.*, 2007]) will also modify all other variables in order to be consistent with constraints. Then, changing the sign of the smallest unknown will have the least possible effects in terms of violations of constraints and will keep the objective function as large as possible during the overall constrained optimization. As a matter of fact, all the alternative strategies we tested had worse performances.

The first example is concerned with the “flat-top” pattern shown in Figure 2c (green line), which has been generated in *Isernia et al.* [1998] through a linear array composed by 15 isotropic elements. The real and imaginary parts of the reference excitation distribution are respectively shown in Figures 2a and 2b. By adding a measurement error corresponding to a signal-to-noise ratio (SNR) equal to 25 dB (see blue curve in Figure 2c), assuming known the twelfth excitation, and using a random set of excitations as starting point of the phase retrieval procedure, at the end of the first local optimization we achieved the pattern and excitations depicted in Figures 2a–2c (red curves). As it can be seen, despite the presence of noise on the data, a correct retrieval of the array excitations has been achieved without reiterating the optimization procedure.

The second example is concerned with the “square-cosecant” pattern shown in Figure 3c (green line), which has been generated in *Elliott and Stern* [1984] through a linear array composed by 16 isotropic elements. The real and imaginary parts of the reference excitation distribution are respectively shown in Figure 3a and 3b. By adding a measurement error corresponding to a signal-to-noise ratio (SNR) equal to 30 dB (see blue curve in Figure 3c), assuming known the first excitation, and using a random set of excitations as starting point of the phase retrieval procedure, at the end of the first local optimization we achieved the pattern and excitations depicted in Figures 3a–3c (red curves). As it can been seen, the reconstructed power pattern results slightly different from the reference one, while a more significant inconsistency is present between reference and retrieved excitations. However, by applying a single signum flipping (operated on both the real and imaginary parts of the coefficient having the smallest estimated amplitude) to the retrieved excitations, and running for a second time the proposed procedure by using these new excitations as starting point, we achieved the results shown in Figure 4. In particular, a perfect power pattern matching (Figure 4c) and a correct retrieval of the complex array excitations (Figure 4a and 4b) have been reached. This result confirms not only the effectiveness of the proposed optimization approach but also the efficiency of the signum-flipping operation in “escaping” from the attraction region of a local optimum to the attraction region of a better solution.

6. Conclusions

A completely new approach to the solution of phase retrieval problems has been introduced, discussed, and tested in the simple yet significant case of discrete one-dimensional sources. The approach is based on the global optimization of a cost function which, differently from more common approaches, is based on the combination of local optimization plus tunneling operations. These latter allow to jump through the different attraction regions of the cost function to be maximized. Preliminary results dealing with noisy data confirm the interest of the approach.

Notably, opposite to *Isernia et al*. [1999] and related, we do not need a functional having a single optimum, as we can escape from a number of false solutions by means of tunneling. This represents a considerable progress with respect to previous approaches, as a reduced number of a priori/measurement will be required. Of course, one still needs that these false solutions can be recognized as such, so that false solution having a value of the functional close to the expected one cannot be avoided anyway.

It is also worth to note that the proposed approach can be extended in a conceptually easy fashion to operators other than the Fourier transforms and to unknown signals other than the excitations of a linear array (with half a wavelength spacing). In fact, in all cases we can consider the same cost function as in (8) and the same constraints as in (9). Then, the cost function to be maximized can be written as a Positive Definite Quadratic Form as

$$\mathrm{OF} = \sum_{1}^{2P} \sigma_i^2 (x_i)^2 \tag{13}$$

where x_i, $i = 1,\ldots,2P$, are suitable real auxiliary variables derived from transforming the original quadratic form into its canonical form, P is the number of (complex) unknowns, and σ_i^2 are nonnegative real quantities. Then, flipping can be performed, in case of need, on the sign of the auxiliary variables.

Such an extension includes both cases of continuous or discrete unknown signals. By a proper choice of the operator T, the above formulation includes many cases of actual interest in electromagnetics, such as antenna diagnostics or characterization [*Álvarez et al*., 2010], laser beam characterization [*Cutolo et al*., 1992], Synthetic Aperture Radar processing [*Isernia et al*., 1996c], inverse scattering [*Bucci et al*., 2006], and many others.

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Data access may be requested through the following e-mail: andrea.morabito@unirc.it.

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