On a class of critical (p, q)-Laplacian problems*

Pasquale Candito
Università degli Studi di Reggio Calabria
89100 Reggio Calabria, Italy
pasquale.candito@unirc.it

Salvatore A. Marano
Università degli Studi di Catania
95125 Catania, Italy
marano@dmi.unict.it

Kanishka Perera†
Florida Institute of Technology
Melbourne, FL 32901, USA
kperera@fit.edu

Abstract

We obtain nontrivial solutions of a critical (p, q)-Laplacian problem in a bounded domain. In addition to the usual difficulty of the loss of compactness associated with problems involving critical Sobolev exponents, this problem lacks a direct sum decomposition suitable for applying the classical linking theorem. We show that every Palais-Smale sequence at a level below a certain energy threshold admits a subsequence that converges weakly to a nontrivial critical point of the variational functional. Then we prove an abstract critical point theorem based on a cohomological index and use it to construct a minimax level below this threshold.

1 Introduction and main results

The (p, q)-Laplacian operator

$$\Delta_p\, u + \Delta_q\, u = \operatorname{div}\left[\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \nabla u\right]$$

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* *MSC2010:* Primary 35J92, 35B33, Secondary 58E05

Key Words and Phrases: (p, q)-Laplacian problems, critical Sobolev exponent, nontrivial solutions, critical point theory, cohomological index

† This work was completed while the third-named author was visiting Università di Reggio Calabria, and he is grateful for the kind hospitality of the host university.

appears in a wide range of applications that include biophysics [12], plasma physics [25], reaction-diffusion equations [1, 5], and models of elementary particles [9, 4, 2]. Consequently, quasilinear elliptic boundary value problems involving this operator have been widely studied in the literature (see, e.g., [3, 17, 24, 16] and the references therein). In particular, the critical (p, q)-Laplacian problem

$$\begin{cases} -\Delta_p\, u - \Delta_q\, u = \mu\, |u|^{r-2}\, u + |u|^{p^\ast - 2}\, u & \text{in } \Omega\\[10pt] \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N > p > q > 1$, $\mu > 0$, and $p^\ast = Np/(N-p)$ is the critical Sobolev exponent, has been studied by Li and Zhang [14] in the case $1 < r < q$ and by Yin and Yang [26] in the case $p < r < p^\ast$. In the present paper we consider the question of existence of nontrivial solutions in the borderline case

$$\begin{cases} -\Delta_p\, u - \Delta_q\, u = \mu\, |u|^{q-2}\, u + \lambda\, |u|^{p-2}\, u + |u|^{p^\ast - 2}\, u & \text{in } \Omega\\[10pt] \qquad\qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

with $\mu \in \mathbb{R}$ and $\lambda > 0$. In addition to the usual difficulty of the lack of compactness associated with problems involving critical exponents, this problem is further complicated by the absence of a direct sum decomposition suitable for applying the linking theorem when μ is above the second eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_q\, u = \mu\, |u|^{q-2}\, u & \text{in } \Omega\\[10pt] \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

To overcome this difficulty, we will first prove an abstract critical point theorem based on a cohomological index that generalizes the classical linking theorem of Rabinowitz [23].

Weak solutions of problem (1.1) coincide with critical points of the C^1-functional

$$\Phi(u) = \int_\Omega \left(\frac{1}{p}\, |\nabla u|^p + \frac{1}{q}\, |\nabla u|^q - \frac{\mu}{q}\, |u|^q - \frac{\lambda}{p}\, |u|^p - \frac{1}{p^\ast}\, |u|^{p^\ast}\right) dx, \quad u \in W^{1,p}_0(\Omega), \tag{1.3}$$

where $W^{1,p}_0(\Omega)$ is the usual Sobolev space with the norm $\|u\| = \|\nabla u\|_p$ and $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. Recall that Φ satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$, or $(\mathrm{PS})_c$ for short, if every sequence $(u_j) \subset W^{1,p}_0(\Omega)$ such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$, called a $(\mathrm{PS})_c$ sequence, has a convergent subsequence. Let

$$S = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_{p^\ast}^p} > 0 \tag{1.4}$$

be the best constant for the Sobolev imbedding $W^{1,p}_0(\Omega) \hookrightarrow L^{p^\ast}(\Omega)$. Our existence results will be based on the following proposition.

Proposition 1.1. *If* $c < S^{N/p}/N$ *and* $c \neq 0$*, then every* $(\mathrm{PS})_c$ *sequence has a subsequence that converges weakly to a nontrivial critical point of* Φ*.*

Let

$$\mu_1 = \inf_{u \in W_0^{1,q}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_q^q}{\|u\|_q^q} > 0 \tag{1.5}$$

be the first eigenvalue of the eigenvalue problem (1.2). First we seek a nonnegative nontrivial solution of problem (1.1) when $\mu \leq \mu_1$. Let

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p} > 0 \tag{1.6}$$

be the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_p\, u = \lambda\, |u|^{p-2}\, u & \text{in } \Omega \\ \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our first main result is the following theorem.

Theorem 1.2. *Assume that* $1 < q < p$ *and* $p^2 < N$*. If* $0 < \lambda < \lambda_1$ *and* $\mu \leq \mu_1$*, then problem* (1.1) *has a nonnegative nontrivial solution in each of the following cases:*

(i) $N(p-1)/(N-p) \leq q < (N-p)\, p/N$,

(ii) $N(p-1)/(N-1) < q < \min\{N(p-1)/(N-p), (N-p)\, p/N\}$,

(iii) $(1 - 1/N)\, p^2 + p < N$ *and* $q = N(p-1)/(N-1)$,

(iv) $(p-1)\, p^2/(N-p) < q < N(p-1)/(N-1)$.

Now we assume that $p < q^*$, where $q^* = Nq/(N-q)$ is the critical exponent for the imbedding $W_0^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$. Then we have the following theorem.

Theorem 1.3. *Assume that* $1 < q < p < \min\{N, q^*\}$*. If* $\mu < \mu_1$*, then there exists* $\lambda^*(\mu) > 0$ *such that problem* (1.1) *has a nonnegative nontrivial solution for all* $\lambda \geq \lambda^*(\mu)$*.*

Let $u^\pm(x) = \max\{\pm u(x), 0\}$ be the positive and negative parts of u, respectively, and set

$$\Phi^+(u) = \int_\Omega \left(\frac{1}{p}\, |\nabla u|^p + \frac{1}{q}\, |\nabla u|^q - \frac{\mu}{q}\, (u^+)^q - \frac{\lambda}{p}\, (u^+)^p - \frac{1}{p^*}\, (u^+)^{p^*}\right) dx, \quad u \in W_0^{1,p}(\Omega).$$

If u is a critical point of Φ^+, then

$$\Phi^{+\prime}(u)\, u^- = \int_\Omega \left(|\nabla u^-|^p + |\nabla u^-|^q\right) dx = 0$$

and hence $u^- = 0$, so $u = u^+$ is a critical point of Φ and therefore a nonnegative solution of problem (1.1). Moreover, if $\mu \geq 0$ then $u > 0$ in Ω. Indeed, due to the critical growth of the nonlinearity, we can guarantee that u is bounded by Cianchi [6, Theorem 2], hence we apply Lieberman [15, Theorem 1.7], and Pucci and Serrin [22, Theorem 1.1.1] to get $u > 0$. Proofs of Theorems 1.2 and 1.3 will be based on constructing minimax levels of mountain pass type for Φ^+ below the threshold level given in Proposition 1.1.

Next we seek a (possibly nodal) nontrivial solution of problem (1.1) when $\mu \geq \mu_1$. We have the following theorem.

Theorem 1.4. *Assume that* $1 < q < p < \min\{N, q^*\}$. *If* $\mu \geq \mu_1$, *then there exists* $\lambda^*(\mu) > 0$ *such that problem* (1.1) *has a nontrivial solution for all* $\lambda \geq \lambda^*(\mu)$.

This extension of Theorem 1.3 is nontrivial. Indeed, the functional Φ does not have the mountain pass geometry when $\mu \geq \mu_1$ since the origin is no longer a local minimizer, and a linking type argument is needed. However, the classical linking theorem cannot be used since the nonlinear operator $-\Delta_q$ does not have linear eigenspaces. We will use a more general construction based on sublevel sets as in Perera and Szulkin [21] (see also Perera et al. [20, Proposition 3.23]). Moreover, the standard sequence of eigenvalues of $-\Delta_q$ based on the genus does not give enough information about the structure of the sublevel sets to carry out this linking construction. Therefore we will use a different sequence of eigenvalues introduced in Perera [19] that is based on a cohomological index.

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [11] is defined as follows. Let W be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of A with each u and $-u$ identified, let $f : \overline{A} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of A is defined by

$$i(A) = \begin{cases} \sup\left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\}, & A \neq \emptyset \\ 0, & A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere S^{m-1} in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{R}\mathrm{P}^{m-1} \subset \mathbb{R}\mathrm{P}^\infty$, which induces isomorphisms on H^q for $q \leq m-1$, so $i(S^{m-1}) = m$. The following proposition summarizes the basic properties of this index.

Proposition 1.5 (Fadell-Rabinowitz [11]). *The index* $i : \mathcal{A} \to \mathbb{N} \cup \{0, \infty\}$ *has the following properties:*

(i_1) *Definiteness:* $i(A) = 0$ *if and only if* $A = \emptyset$;

(i_2) *Monotonicity: If there is an odd continuous map from* A *to* B *(in particular, if* $A \subset B$*), then* $i(A) \leq i(B)$. *Thus, equality holds when the map is an odd homeomorphism;*

(i_3) *Dimension:* $i(A) \le \dim W$*;*

(i_4) *Continuity: If* A *is closed, then there is a closed neighborhood* $N \in \mathcal{A}$ *of* A *such that* $i(N) = i(A)$*. When* A *is compact,* N *may be chosen to be a* δ*-neighborhood* $N_\delta(A) = \{u \in W : dist(u, A) \le \delta\}$*;*

(i_5) *Subadditivity: If* A *and* B *are closed, then* $i(A \cup B) \le i(A) + i(B)$*;*

(i_6) *Stability: If* SA *is the suspension of* $A \ne \emptyset$*, obtained as the quotient space of* $A \times [-1, 1]$ *with* $A \times \{1\}$ *and* $A \times \{-1\}$ *collapsed to different points, then* $i(SA) = i(A) + 1$*;*

(i_7) *Piercing property: If* A*,* A_0 *and* A_1 *are closed, and* $\varphi : A \times [0, 1] \to A_0 \cup A_1$ *is a continuous map such that* $\varphi(-u, t) = -\varphi(u, t)$ *for all* $(u, t) \in A \times [0, 1]$*,* $\varphi(A \times [0, 1])$ *is closed,* $\varphi(A \times \{0\}) \subset A_0$ *and* $\varphi(A \times \{1\}) \subset A_1$*, then* $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \ge i(A)$*;*

(i_8) *Neighborhood of zero: If* U *is a bounded closed symmetric neighborhood of* 0*, then* $i(\partial U) = \dim W$*.*

The Dirichlet spectrum of $-\Delta_q$ in Ω consists of those $\mu \in \mathbb{R}$ for which problem (1.2) has a nontrivial solution. Although a complete description of the spectrum is not yet known when $N \ge 2$, we can define an increasing and unbounded sequence of eigenvalues via a suitable minimax scheme. The standard scheme based on the genus does not give the index information necessary to prove Theorem 1.4, so we will use the following scheme based on the cohomological index as in Perera [19]. Let

$$\Psi(u) = \frac{1}{\displaystyle\int_\Omega |u|^q\, dx}, \quad u \in S_q = \left\{u \in W_0^{1,q}(\Omega) : \int_\Omega |\nabla u|^q\, dx = 1\right\}.$$

Then eigenvalues of problem (1.2) on S_q coincide with critical values of Ψ. We use the standard notation

$$\Psi^a = \{u \in S_q : \Psi(u) \le a\}, \quad \Psi_a = \{u \in S_q : \Psi(u) \ge a\}, \quad a \in \mathbb{R}$$

for the sublevel sets and superlevel sets, respectively. Let $\mathcal{F}$ denote the class of symmetric subsets of S_q and set

$$\mu_k := \inf_{M \in \mathcal{F},\, i(M) \ge k}\ \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$

Then $0 < \mu_1 < \mu_2 \le \mu_3 \le \cdots \to +\infty$ is a sequence of eigenvalues of problem (1.2) and

$$\mu_k < \mu_{k+1} \implies i(\Psi^{\mu_k}) = i(S_q \setminus \Psi_{\mu_{k+1}}) = k \tag{1.7}$$

(see Perera et al. [20, Propositions 3.52 and 3.53]).

Proof of Theorem 1.4 will make essential use of (1.7) and will be based on the following abstract critical point theorem, which is of independent interest. Let W be a Banach space, let

$$S = \{u \in W : \|u\| = 1\}$$

be the unit sphere in W, and let

$$\pi : W \setminus \{0\} \to S, \quad u \mapsto \frac{u}{\|u\|}$$

be the radial projection onto S.

Theorem 1.6. *Let Φ be a C^1-functional on W and let A_0, B_0 be disjoint nonempty closed symmetric subsets of S such that*

$$i(A_0) = i(S \setminus B_0) < \infty. \tag{1.8}$$

Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that

$$\sup \Phi(A) \le \inf \Phi(B), \qquad \sup \Phi(X) < \infty,$$

where

$$\begin{aligned} &A = \{tu : u \in A_0,\, 0 \le t \le R\} \cup \{R\,\pi((1-t)\,u + tv) : u \in A_0,\, 0 \le t \le 1\}\,, \\ &B = \{ru : u \in B_0\}\,, \\ &X = \{tu : u \in A,\, \|u\| = R,\, 0 \le t \le 1\}\,. \end{aligned}$$

Let $\Gamma = \{\gamma \in C(X, W) : \gamma(X)$ is closed and $\gamma|_A = id_A\}$ and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} \Phi(u).$$

Then

$$\inf \Phi(B) \le c \le \sup \Phi(X)$$

and Φ has a $(\mathrm{PS})_c$ sequence.

Remark 1.7. Theorem 1.6, which does not require a direct sum decomposition, generalizes the linking theorem of Rabinowitz [23].

2 Preliminaries

In this preliminary section we prove Proposition 1.1 and Theorem 1.6.

Proof of Proposition 1.1. Let (u_j) be a $(\mathrm{PS})_c$ sequence. Then

$$\Phi(u_j) = \int_\Omega \left(\frac{1}{p}\, |\nabla u_j|^p + \frac{1}{q}\, |\nabla u_j|^q - \frac{\mu}{q}\, |u_j|^q - \frac{\lambda}{p}\, |u_j|^p - \frac{1}{p^*}\, |u_j|^{p^*}\right) dx = c + \mathrm{o}(1) \tag{2.1}$$

and

$$\Phi'(u_j)\, u_j = \int_\Omega \left(|\nabla u_j|^p + |\nabla u_j|^q - \mu\, |u_j|^q - \lambda\, |u_j|^p - |u_j|^{p^*}\right) dx = \mathrm{o}(1)\, \|u_j\|\,. \tag{2.2}$$

So

$$\int_\Omega \left[\left(\frac{1}{q} - \frac{1}{p}\right)(|\nabla u_j|^q - \mu\, |u_j|^q) + \left(\frac{1}{p} - \frac{1}{p^*}\right) |u_j|^{p^*}\right] dx = \mathrm{o}(1)\, \|u_j\| + \mathrm{O}(1),$$

and since $q < p < p^*$, this and the Hölder and Young inequalities yield

$$\int_\Omega |u_j|^{p^*}\, dx \le \mathrm{o}(1)\, \|u_j\| + \mathrm{O}(1).$$

Since $p > 1$, it follows from this and (2.1) that (u_j) is bounded in $W^{1,p}_0(\Omega)$. So a renamed subsequence converges to some u weakly in $W^{1,p}_0(\Omega)$, strongly in $L^s(\Omega)$ for all $1 \le s < p^*$, and a.e. in Ω. Then u is a critical point of Φ by the weak continuity of Φ'.

Suppose $u = 0$. Since (u_j) is bounded in $W^{1,p}_0(\Omega)$ and converges to 0 in $L^p(\Omega)$, (2.2) gives

$$\mathrm{o}(1) = \int_\Omega \left(|\nabla u_j|^p + |\nabla u_j|^q - |u_j|^{p^*}\right) dx \ge \|u_j\|^p \left(1 - \frac{\|u_j\|^{p^*-p}}{S^{p^*/p}}\right)$$

by (1.4). If $\|u_j\| \to 0$, then $\Phi(u_j) \to 0$, contradicting $c \ne 0$, so this implies

$$\|u_j\|^p \ge S^{N/p} + \mathrm{o}(1)$$

for a renamed subsequence. Then (2.1) and (2.2) yield

$$c = \int_\Omega \left[\left(\frac{1}{p} - \frac{1}{p^*}\right) |\nabla u_j|^p + \left(\frac{1}{q} - \frac{1}{p^*}\right) |\nabla u_j|^q\right] dx + \mathrm{o}(1) \ge \frac{S^{N/p}}{N} + \mathrm{o}(1),$$

contradicting $c < S^{N/p}/N$. □

Proof of Theorem 1.6. First we show that A (homotopically) links B with respect to X in the sense that

$$\gamma(X) \cap B \neq \emptyset \quad \forall \gamma \in \Gamma. \tag{2.3}$$

If (2.3) does not hold, then there is a map $\gamma \in C(X, W \setminus B)$ such that $\gamma(X)$ is closed and $\gamma|_A = id_A$. Let

$$\widetilde{A} = \{R\, \pi((1 - |t|)\, u + tv) : u \in A_0,\ -1 \leq t \leq 1\}$$

and note that $\widetilde{A}$ is closed since A_0 is closed (here $(1 - |t|)\, u + tv \neq 0$ since v is not in the symmetric set A_0). Since

$$SA_0 \to \widetilde{A}, \quad (u, t) \mapsto R\, \pi((1 - |t|)\, u + tv)$$

is an odd continuous map,

$$i(\widetilde{A}) \geq i(SA_0) = i(A_0) + 1 \tag{2.4}$$

by (i_2) and (i_6) of Proposition 1.5. Consider the map

$$\varphi : \widetilde{A} \times [0, 1] \to W \setminus B, \quad \varphi(u, t) = \begin{cases} \gamma(tu), & u \in \widetilde{A} \cap A \\ -\gamma(-tu), & u \in \widetilde{A} \setminus A, \end{cases}$$

which is continuous since γ is the identity on the symmetric set $\{tu : u \in A_0,\ 0 \leq t \leq R\}$. We have $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in \widetilde{A} \times [0, 1]$, $\varphi(\widetilde{A} \times [0, 1]) = \gamma(X) \cup (-\gamma(X))$ is closed, and $\varphi(\widetilde{A} \times \{0\}) = \{0\}$ and $\varphi(\widetilde{A} \times \{1\}) = \widetilde{A}$ since $\gamma|_A = id_A$. Applying (i_7) with $\widetilde{A}_0 = \{u \in W : \|u\| \leq r\}$ and $\widetilde{A}_1 = \{u \in W : \|u\| \geq r\}$ gives

$$i(\widetilde{A}) \leq i(\varphi(\widetilde{A} \times [0, 1]) \cap \widetilde{A}_0 \cap \widetilde{A}_1) \leq i((W \setminus B) \cap S_r) = i(S_r \setminus B) = i(S \setminus B_0), \tag{2.5}$$

where $S_r = \{u \in W : \|u\| = r\}$. By (2.4) and (2.5), $i(A_0) < i(S \setminus B_0)$, contradicting (1.8). Hence (2.3) holds.

It follows from (2.3) that $c \geq \inf \Phi(B)$, and $c \leq \sup \Phi(X)$ since $id_X \in \Gamma$. By a standard argument, Φ has a $(\mathrm{PS})_c$ sequence (see, e.g., Ghoussoub [13]). □

Remark 2.1. The linking construction in the above proof was used in Perera and Szulkin [21] to obtain nontrivial solutions of p-Laplacian problems with nonlinearities that interact with the spectrum. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [7]. See also Perera et al. [20, Proposition 3.23].

3 Proofs of Theorems 1.2 and 1.3

Fix $u_0 > 0$ in $W_0^{1,p}(\Omega)$ such that $\|u_0\|_{p^*} = 1$. Since $q < p < p^*$,

$$\Phi^+(tu_0) = \int_\Omega \left(\frac{t^p}{p}\, |\nabla u_0|^p + \frac{t^q}{q}\, |\nabla u_0|^q - \frac{\mu\, t^q}{q}\, u_0^q - \frac{\lambda\, t^p}{p}\, u_0^p\right) dx - \frac{t^{p^*}}{p^*} \to -\infty$$

as $t \to +\infty$. Take $t_0 > 0$ so large that $\Phi^+(t_0 u_0) \le 0$, let

$$\Gamma = \left\{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = t_0 u_0\right\}$$

be the class of paths joining 0 and $t_0 u_0$, and set

$$c := \inf_{\gamma \in \Gamma}\ \max_{u \in \gamma([0,1])}\ \Phi^+(u).$$

Lemma 3.1. *If* $0 < c < S^{N/p}/N$, *then problem* (1.1) *has a nonnegative nontrivial solution.*

Proof. By the mountain pass theorem, Φ^+ has a $(\mathrm{PS})_c$ sequence (u_j). An argument similar to that in the proof of Proposition 1.1 shows that a subsequence of (u_j) converges weakly to a nontrivial critical point u of Φ^+. □

We have the following upper bounds for c.

Lemma 3.2. *Let* $\widetilde{\lambda} = \lambda/2$.

(i) If $\int_\Omega |\nabla u_0|^p\, dx > \widetilde{\lambda} \int_\Omega u_0^p\, dx$, *then*

$$c \le \frac{1}{N}\left[\int_\Omega \left(|\nabla u_0|^p - \widetilde{\lambda}\, u_0^p\right) dx\right]^{N/p} + \left(\frac{1}{q} - \frac{1}{p}\right) \frac{\left[\displaystyle\int_\Omega \left(|\nabla u_0|^q - \mu\, u_0^q\right) dx\right]^{p/(p-q)}}{\left(\widetilde{\lambda} \displaystyle\int_\Omega u_0^p\, dx\right)^{q/(p-q)}}.$$

(ii) If $\int_\Omega |\nabla u_0|^p\, dx \le \widetilde{\lambda} \int_\Omega u_0^p\, dx$, *then*

$$c \le \left(\frac{1}{q} - \frac{1}{p}\right) \frac{\left[\displaystyle\int_\Omega \left(|\nabla u_0|^q - \mu\, u_0^q\right) dx\right]^{p/(p-q)}}{\left(\widetilde{\lambda} \displaystyle\int_\Omega u_0^p\, dx\right)^{q/(p-q)}}.$$

Proof. Since $\gamma(s) = s t_0 u_0$ is a path in Γ,

$$c \le \max_{s \in [0,1]}\ \Phi^+(s t_0 u_0) \le \max_{t \ge 0}\ \Phi^+(t u_0) \le \max_{t \ge 0}\ \left[\frac{t^p}{p} \int_\Omega \left(|\nabla u_0|^p - \widetilde{\lambda}\, u_0^p\right) dx - \frac{t^{p^*}}{p^*}\right]$$

$$+ \max_{t \ge 0}\ \left[\frac{t^q}{q} \int_\Omega \left(|\nabla u_0|^q - \mu\, u_0^q\right) dx - \frac{\widetilde{\lambda}\, t^p}{p} \int_\Omega u_0^p\, dx\right]. \quad \square$$

Proof of Theorem 1.2. Without loss of generality we may assume that $0 \in \Omega$. Let $r > 0$ be so small that $B_{2r}(0) \subset \Omega$, take a function $\psi \in C_0^\infty(B_{2r}(0), [0,1])$ such that $\psi = 1$ on $B_r(0)$, and set

$$u_\varepsilon(x) = \frac{\psi(x)}{\big(\varepsilon + |x|^{p/(p-1)}\big)^{(N-p)/p}}, \qquad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{p^*}}$$

for $\varepsilon > 0$. Then $\|v_\varepsilon\|_{p^*} = 1$ and

$$\int_\Omega |\nabla v_\varepsilon|^p \, dx = S + \mathrm{O}(\varepsilon^{(N-p)/p}), \tag{3.1}$$

$$\int_\Omega v_\varepsilon^p \, dx = \begin{cases} K\varepsilon^{p-1} + \mathrm{O}(\varepsilon^{(N-p)/p}), & p^2 < N \\ K\varepsilon^{p-1} \, |\log \varepsilon| + \mathrm{O}(\varepsilon^{p-1}), & p^2 = N \\ \mathrm{O}(\varepsilon^{(N-p)/p}), & p^2 > N \end{cases} \tag{3.2}$$

for some constant $K > 0$,

$$\int_\Omega |\nabla v_\varepsilon|^q \, dx = \begin{cases} \mathrm{O}(\varepsilon^{N(p-1)(p-q)/p^2}), & q > \frac{N(p-1)}{N-1} \\ \mathrm{O}(\varepsilon^{N(N-p)(p-1)/(N-1)\, p^2} \, |\log \varepsilon|), & q = \frac{N(p-1)}{N-1} \\ \mathrm{O}(\varepsilon^{(N-p)\, q/p^2}), & q < \frac{N(p-1)}{N-1}, \end{cases} \tag{3.3}$$

and

$$\int_\Omega v_\varepsilon^q \, dx = \begin{cases} \mathrm{O}(\varepsilon^{(p-1)[Np-(N-p)\, q]/p^2}), & q > \frac{N(p-1)}{N-p} \\ \mathrm{O}(\varepsilon^{N(p-1)/p^2} \, |\log \varepsilon|), & q = \frac{N(p-1)}{N-p} \\ \mathrm{O}(\varepsilon^{(N-p)\, q/p^2}), & q < \frac{N(p-1)}{N-p} \end{cases} \tag{3.4}$$

as $\varepsilon \to 0$ (see, e.g., Drábek and Huang [10]). Fixed $u_0 = v_\varepsilon$, we consider the correspondent critical level c, as described at the bottom of page 8. Our aim is to apply Lemma 3.1. Since $\mu \le \mu_1$ and by (1.5), (1.6), and (1.4),

$$\Phi^+(u) \ge \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p - \frac{1}{p^*}\, S^{-p^*/p}\, \|u\|^{p^*} \quad \forall u \in W_0^{1,p}(\Omega).$$

Since $\lambda < \lambda_1$ and $p^* > p$, it follows from this that 0 is a strict local minimizer of Φ^+, so $c > 0$. We will verify that in each case $c < S^{N/p}/N$ for $\varepsilon > 0$ sufficiently small by using Lemma 3.2 (i) with $u_0 = v_\varepsilon$ and (3.1)–(3.4).

(i) Since $p^2 < N$ and $q \ge N(p-1)/(N-p) > N(p-1)/(N-1)$, we have

$$c \le \frac{1}{N} \left[S - K\widetilde{\lambda}\varepsilon^{p-1} + \mathrm{O}(\varepsilon^{(N-p)/p})\right]^{N/p} + \mathrm{O}(\varepsilon^{(p-1)[N/p-q/(p-q)]}). \tag{3.5}$$

$(N-p)/p > p-1$ since $p^2 < N$, and $(p-1)[N/p - q/(p-q)] > p-1$ since $q < (N-p)\,p/N$, so the desired conclusion follows.

(ii) Since $N(p-1)/(N-1) < q < N(p-1)/(N-p)$, (3.5) still holds, and $(p-1)[N/p - q/(p-q)] > p-1$ since $q < (N-p)\,p/N$.

(iii) Since $q = N(p-1)/(N-1) < N(p-1)/(N-p)$, we have

$$c \le \frac{1}{N} \left[S - K\widetilde{\lambda}\varepsilon^{p-1} + \mathrm{O}(\varepsilon^{(N-p)/p})\right]^{N/p} + \mathrm{O}(\varepsilon^{N(N-p^2)(p-1)/(N-p)\,p}\,|\log\varepsilon|^{(N-1)\,p/(N-p)}),$$

and $N(N-p^2)(p-1)/(N-p)\,p > p-1$ since $(1-1/N)\,p^2 + p < N$.

(iv) Since $q < N(p-1)/(N-1) < N(p-1)/(N-p)$, we have

$$c \le \frac{1}{N} \left[S - K\widetilde{\lambda}\varepsilon^{p-1} + \mathrm{O}(\varepsilon^{(N-p)/p})\right]^{N/p} + \mathrm{O}(\varepsilon^{(N-p^2)\,q/p\,(p-q)}),$$

and $(N-p^2)\,q/p\,(p-q) > p-1$ since $q > (p-1)\,p^2/(N-p)$. $\square$

Proof of Theorem 1.3. We apply Lemma 3.1. Since $q < p < q^*$, $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ by the Hölder inequality and the Sobolev imbedding, so

$$T = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_q^q}{\|u\|_p^q} \ge \inf_{u \in W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_q^q}{\|u\|_p^q} > 0. \tag{3.6}$$

By (1.4), (1.5), and (3.6),

$$\Phi^+(u) \ge \frac{1}{p}\,\|u\|^p - \frac{1}{p^*}\,S^{-p^*/p}\,\|u\|^{p^*} + \frac{1}{q}\left(1 - \frac{\mu^+}{\mu_1}\right)\|\nabla u\|_q^q - \frac{\lambda}{p}\,T^{-p/q}\,\|\nabla u\|_q^p \quad \forall u \in W_0^{1,p}(\Omega),$$

where $\mu^+ = \max\{\mu, 0\}$. Since $\mu^+ < \mu_1$ and $p^* > p > q$, it follows from this that 0 is a strict local minimizer of Φ^+, so $c > 0$. It is clear from Lemma 3.2 *(ii)* that $c < S^{N/p}/N$ for $\lambda > 0$ sufficiently large. $\square$

4 Proof of Theorem 1.4

Proof of Theorem 1.4. Since $q < p$, $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ by the Hölder inequality. Let S_p denote the unit sphere of $W_0^{1,p}(\Omega)$ and let

$$\pi_p(u) = \frac{u}{\|\nabla u\|_p}, \quad u \in W_0^{1,p}(\Omega) \setminus \{0\}, \qquad \pi_q(u) = \frac{u}{\|\nabla u\|_q}, \quad u \in W_0^{1,q}(\Omega) \setminus \{0\}$$

be the radial projections onto S_p and S_q, respectively. Since $\mu \geq \mu_1$, $\mu_k \leq \mu < \mu_{k+1}$ for some $k \geq 1$. Then

$$i(\pi_q^{-1}(\Psi^{\mu_k})) = i(\pi_q^{-1}(S_q \setminus \Psi_{\mu_{k+1}})) = k \tag{4.1}$$

by (1.7). Set $M = \big\{u \in W_0^{1,q}(\Omega) : \|u\|_q = 1\big\}$. By Degiovanni and Lancelotti [8, Theorem 2.3], the set $\pi_q^{-1}(\Psi^{\mu_k}) \cup \{0\}$ contains a symmetric cone C such that $C \cap M$ is compact in $C^1(\Omega)$ and

$$i(C \setminus \{0\}) = k. \tag{4.2}$$

Since $W_0^{1,p}(\Omega)$ is a dense linear subspace of $W_0^{1,q}(\Omega)$, the inclusion $\pi_q^{-1}(S_q \setminus \Psi_{\mu_{k+1}}) \cap W_0^{1,p}(\Omega) \subset \pi_q^{-1}(S_q \setminus \Psi_{\mu_{k+1}})$ is a homotopy equivalence by Palais [18, Theorem 17], so

$$i(\pi_q^{-1}(S_q \setminus \Psi_{\mu_{k+1}}) \cap W_0^{1,p}(\Omega)) = k \tag{4.3}$$

by (4.1). We apply Theorem 1.6 to our functional Φ defined in (1.3) with

$$A_0 = \pi_p(C \setminus \{0\}) = \pi_p(C \cap M), \qquad B_0 = S_p \setminus (\pi_q^{-1}(S_q \setminus \Psi_{\mu_{k+1}}) \cap W_0^{1,p}(\Omega)),$$

noting that A_0 is compact since $C \cap M$ is compact and π_p is continuous. We have

$$i(A_0) = i(C \setminus \{0\}) = k$$

by (4.2), and

$$i(S_p \setminus B_0) = i(\pi_q^{-1}(S_q \setminus \Psi_{\mu_{k+1}}) \cap W_0^{1,p}(\Omega)) = k$$

by (4.3), so (1.8) holds.

For $u \in S_p$ and $t \geq 0$,

$$\Phi(tu) \leq \frac{t^q}{q} \int_\Omega \big(|\nabla u|^q - \mu\, |u|^q\big)\, dx - \frac{\widetilde{\lambda}\, t^p}{p} \int_\Omega |u|^p\, dx - \frac{t^p}{p} \left(\widetilde{\lambda} \int_\Omega |u|^p\, dx - 1\right), \tag{4.4}$$

where $\widetilde{\lambda} = \lambda/2$. Pick any $v \in S_p \setminus A_0$. Since A_0 is compact, so is the set

$$X_0 = \{\pi_p((1 - t)\, u + tv) : u \in A_0,\ 0 \leq t \leq 1\}$$

and hence

$$\alpha = \inf_{u \in X_0} \int_\Omega |u|^p\, dx > 0, \qquad \beta = \sup_{u \in X_0} \int_\Omega \big(|\nabla u|^q - \mu\, |u|^q\big)\, dx < \infty.$$

Let $\lambda \geq 2/\alpha$, so that $\widetilde{\lambda}\alpha \geq 1$. Then for $u \in A_0 \subset X_0$ and $t \geq 0$, (4.4) gives

$$\Phi(tu) \leq -(\mu - \mu_k)\, \frac{t^q}{q} \int_\Omega |u|^q\, dx \leq 0 \tag{4.5}$$

since $\mu \geq \mu_k$. For $u \in X_0$ and $t \geq 0$, (4.4) gives

$$\Phi(tu) \leq \frac{\beta\, t^q}{q} - \frac{\widetilde{\lambda}\alpha\, t^p}{p} \leq \left(\frac{1}{q} - \frac{1}{p}\right) \frac{(\beta^+)^{p/(p-q)}}{(\widetilde{\lambda}\alpha)^{q/(p-q)}}, \tag{4.6}$$

where $\beta^+ = \max\{\beta, 0\}$. Fix λ so large that the last expression is $< S^{N/p}/N$, take positive $R \geq (p\, \beta^+/q\, \widetilde{\lambda}\alpha)^{1/(p-q)}$, and let A and X be as in Theorem 1.6. Then it follows from (4.5) and (4.6) that

$$\sup \Phi(A) \leq 0, \qquad \sup \Phi(X) < \frac{S^{N/p}}{N}.$$

Since $p < q^*$, $W^{1,q}_0(\Omega) \hookrightarrow L^p(\Omega)$ by the Sobolev imbedding, so

$$T = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_q^q}{\|u\|_p^q} \geq \inf_{u \in W^{1,q}_0(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_q^q}{\|u\|_p^q} > 0. \tag{4.7}$$

By (1.4) and (4.7),

$$\Phi(u) \geq \frac{1}{p}\, \|u\|^p - \frac{1}{p^*}\, S^{-p^*/p}\, \|u\|^{p^*} + \frac{1}{q} \left(1 - \frac{\mu}{\mu_{k+1}}\right) \|\nabla u\|_q^q - \frac{\lambda}{p}\, T^{-p/q}\, \|\nabla u\|_q^p \quad \forall u \in \pi_p^{-1}(B_0).$$

Since $\mu < \mu_{k+1}$ and $p^* > p > q$, it follows from this that if $0 < r < R$ is sufficiently small and B is as in Theorem 1.6, then

$$\inf \Phi(B) > 0.$$

Then $0 < c < S^{N/p}/N$ and Φ has a $(\mathrm{PS})_c$ sequence by Theorem 1.6, a subsequence of which converges weakly to a nontrivial critical point of Φ by Proposition 1.1. □

Acknowledgement

The authors have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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