Three solutions for parametric problems with nonhomogeneous $(a, 2)$-type differential operators and reaction terms sublinear at zero

Pasquale Candito[*], Leszek Gasiński[†], and Roberto Livrea[*]

June 5, 2019

Keywords: $(a, 2)$-operator, constant sign solutions, nodal solutions, multiplicity of solutions, nonhomogeneous operator.
Abstract: We consider parametric Dirichlet problems driven by the sum of a Laplacian and a nonhomogeneous differential operator ($(a, 2)$-type equation) and with a reaction term which exhibits arbitrary polynomial growth and a nonlinear dependence on the parameter. We prove the existence of three distinct nontrivial smooth solutions for small values of the parameter, providing sign information for them: one is positive, one is negative and the third one is nodal.
2010 AMS Subject Classification: 35J20, 35J60, 35J92

1 Introduction

In this paper we study the following Dirichlet problem

$$\begin{cases} -\operatorname{div} a(\nabla u) - \Delta u = f_\lambda(x, u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$P_{f,\lambda}$}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a $C^{2,\alpha}$ boundary $\partial\Omega$, $0 < \alpha \leq 1$, $-\operatorname{div}(a(\nabla u))$ is a nonhomogeneous operator with $a : \mathbb{R}^N \to \mathbb{R}^N$ continuous, strictly monotone satisfying certain regularity conditions which are listed in hypotheses $H(a)$ below and $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $\lambda > 0$, $x \to f_\lambda(x, s)$ is measurable and for almost all $x \in \Omega$, $\lambda > 0$, $s \to f_\lambda(x, s)$ is continuous) involving a positive parameter λ.

The operator $-\operatorname{div}(a(\nabla u))$ generalizes the p-Laplacian operator to a possibly nonhomogeneous setting. The sum $-\operatorname{div} a(\nabla u) - \Delta u$ forms the so called $(a, 2)$-type operator and generalizes in a natural way the $(p, 2)$-operator, which arises in problems of mathematical physics: see [3] (quantum physics), [35] (double phase problems in elasticity theory), [6], [33] (plasma physics). Some recent results on

[*]The research was supported by INdAM – GNAMPA.
[†]The research was supported by the National Science Center of Poland under Project No. 2015/19/B/ST1/01169.

existence and multiplicity of solutions for $(p,2)$-equations are obtained in [1], [7], [16], [23], [24], [25], [26], [30], [31], [34]. In our setting, the $(p,2)$-Laplacian is a particular case of the $(a,2)$-operator provided that $2 < p < +\infty$, see Examples 2.4 **(a)**. However, the main difference between these two operators is located in the "nonlinear part", that is, the operator $a(\cdot)$ may be nonhomogeneous (the novelty here is given by $H(a)(i)$), on the contrary of the p-Laplacian, may be nonomogeneous. A meaningful example of nonhomogeneous operator is the (p,q)-Laplacian, see Example 2.4 **(b)**. Moreover, Examples 2.4 **(c)** and **(d)** involve nonlinear nonhomogeneous differential operators that cannot be reduced to a $p-$Laplacian type operator.

The aim of this paper is to establish the existence of at least three nontrivial solutions for problem $(P_{f,\lambda})$ under a suitable sublinear conditions at zero on the reaction term f_λ and without assuming any asymptotic condition at infinity (Theorem 3.3). Hence, a global supercritical growth on f_λ is also allowed (Theorem 3.1) and, as it is well known, this is not a standard situation. Indeed, a critical and/or a supercritical growth condition at infinity produce, for instance, a lack of compactness which makes more difficult the application of the classical tools of nonlinear analysis. Here, by the way of a suitable combination of sub-super solutions and truncation techniques, we adopt the direct methods in calculus of variations, in conjunction with Lieberman's regularity results [21], the strong maximum principle and the boundary point Lemma of Pucci-Serrin (Theorems 2.8, 2.9), to obtain the existence of at least one strictly positive and one strictly negative solution, see Theorem 3.1 and the preparatory Lemmas 2.6 and 2.10. In particular, adapting a reasoning of [16], we exploit the strong regularity property of the solutions of the Laplace equation to construct suitable sub-super solutions for problem $(P_{f,\lambda})$. However, here the conditions at zero on the reaction term are slightly more general (Lemma 2.11).

At the best of our knowledge, there are not other papers dealing with $(a,2)$-operators and the result concerning the existence of a third nodal solution (Theorem 4.1) seems to be new also for a $(p,2)$-equation.

Finally, in comparison with the above mentioned papers and the references therein, see also [17], [19] and [27], the main differences that one could point out consist in:

(I) a more specific assumption on $\partial\Omega$;

(II) the particular and new structure of the $(a,2)$-operator;

(III) suitable conditions on the reaction term, so that $f_\lambda(x,\cdot)$ can assume both linear (see $H(f)(ii)$) or sublinear (see $H(f)(ii)'$) behaviour near at zero;

(IV) $f_\lambda(x,\cdot)$ does not satisfies any particular asymptotic condition at infinity.

2 Mathematical background and preliminary lemmas

In the study of problem $(P_{f,\lambda})$ in addition to the Sobolev space $W_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\| = \|\nabla u\|_p, \quad \forall u \in W_0^{1,p}(\Omega),$$

we will also use the ordered Banach space

$$C_0^1(\overline{\Omega}) \ = \ \left\{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\right\},$$

whose positive cone is given by

$$(C_0^1(\Omega))_+ \ = \ \left\{u \in C_0^1(\overline{\Omega}) : \ u(x) \geqslant 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior, given by

$$D_+ \ = \ \left\{u \in (C_0^1(\Omega))_+ : \ u(x) > 0 \text{ for all } x \in \Omega, \ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\right\}.$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

Next, we introduce the conditions on the function $a(\cdot)$ involved in the definition of the differential operator. So, let $\eta \in C^1(0, +\infty)$ be a function satisfying

$$0 \ < \ \widehat{c} \ \leqslant \ \frac{t\eta'(t)}{\eta(t)} \ \leqslant \ c_0 \quad \forall t > 0 \tag{2.1}$$

$$c_1 t^{p-1} \ \leqslant \ \eta(t) \ \leqslant \ c_2(1 + t^{p-1}) \quad \forall t > 0, \tag{2.2}$$

with $\widehat{c}, c_0, c_1, c_2 > 0$ and $p > 2$, see Remark 2.1. Denote with $|y|$ the euclidian norm of $y \in \mathbb{R}^N$. The hypotheses on the function $a(\cdot)$ are the following:

$H(a)$: *$a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and*

(i) $a_0 \in C^1(0, +\infty)$, the function $t \longmapsto ta_0(t)$ is strictly increasing,

$$\lim_{t \searrow 0} \frac{ta_0'(t)}{a_0(t)} = A_0 \in \mathbb{R},$$

and there exist two constants $\varrho_1, \varrho_2 \in (0, 1)$ such that

$$\lim_{t \searrow 0} t^{\varrho_1} a_0'(t) = 0 \quad \text{and} \quad \lim_{t \searrow 0} \frac{a_0(t)}{t^{\varrho_2}} = 0; \tag{2.3}$$

(ii) there exists $c_3 > 0$, such that

$$|\nabla a(y)| \ \leqslant \ c_3 \frac{\eta(|y|)}{|y|} \quad \forall y \in \mathbb{R}^N \setminus \{0\};$$

(iii) we have

$$\left(\nabla a(y)\xi, \ \xi\right)_{\mathbb{R}^N} \ \geqslant \ \frac{\eta(|y|)}{|y|}\|\xi\|^2 \quad \forall y \in \mathbb{R}^N \setminus \{0\}, \ \xi \in \mathbb{R}^N;$$

(iv) if $G_0(t) = \int_0^t s a_0(s)\,ds$ for all $t > 0$, then there exist $\tau \in (1,p]$ and $\sigma \in (0,+\infty)$ such that

$$\lim_{t \searrow 0} \frac{\tau G_0(t)}{t^\tau} \;=\; \sigma.$$

Remark 2.1. Assumptions $H(a)$ force to have $2 < \tau \le p$. Indeed, from the second limit in (2.3) it follows that

$$a_0(t) \longrightarrow 0 \text{ as } t \longrightarrow 0, \tag{2.4}$$

and, if it was $1 < \tau \le 2$, by the L'Hôpital's rule one would have

$$\sigma = \lim_{t \searrow 0} \frac{\tau G_0(t)}{t^\tau} = \lim_{t \searrow 0} \frac{a_0(t)}{t^{\tau-2}} = 0,$$

in contradiction with $H(a)(iv)$.

Remark 2.2. It is clear from the above hypotheses that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing. If we set

$$G(y) \;=\; G_0(|y|) \quad \forall y \in \mathbb{R}^N,$$

then $G(\cdot)$ is convex and

$$\nabla G(y) \;=\; G_0'(|y|)\frac{y}{|y|} \;=\; a_0(|y|)y \;=\; a(y) \quad \forall y \in \mathbb{R}^N \setminus \{0\}.$$

Therefore $G(\cdot)$ is the primitive of $a(\cdot)$.

The above hypotheses on $a(\cdot)$ lead to the following lemma summarizing the main properties of the function $a(\cdot)$ (see [18, Lemma 3.2 and Corollary 3.3]).

Lemma 2.3. *If hypotheses $H(a)(i)-(iii)$ hold, then*
(a) *the function $y \longmapsto a(y)$ is maximal monotone and strictly monotone;*
(b) *there exists $c_4 > 0$, such that*

$$|a(y)| \;\leqslant\; c_4\big(1 + |y|^{p-1}\big) \quad \forall y \in \mathbb{R}^N;$$

(c) *we have*

$$\big(a(y), y\big)_{\mathbb{R}^N} \;\geqslant\; \frac{c_1}{p-1}|y|^p \quad \forall y \in \mathbb{R}^N;$$

(d) *there exists $c_5 > 0$, such that*

$$\frac{c_1}{p(p-1)}|y|^p \;\leqslant\; G(y) \;\leqslant\; c_5\big(1 + |y|^p\big) \quad \forall y \in \mathbb{R}^N.$$

Next we present some examples of maps $a(\cdot)$ which satisfy hypotheses $H(a)$ above. These examples illustrate the generality of our conditions on $a(\cdot)$.

Example 2.4. The following maps $y \longmapsto a(y)$ satisfy hypotheses $H(a)$.
(a) $a(y) = |y|^{p-2}y$ with $2 < p < +\infty$. This map corresponds to the p-Laplacian differential operator defined by

$$\Delta_p u \;=\; \operatorname{div}\,(|\nabla u|^{p-2}\nabla u) \quad \forall u \in W^{1,p}(\Omega).$$

Note that hypothesis $H(a)(i)$ holds with $\varrho_1 \in (\max\{0, 3-p\}, 1)$ and $\varrho_2 \in (0, \min\{p-2, 1\})$.

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $2 < q < p < +\infty$. This map corresponds to the (p,q)-Laplace differential operator defined by

$$\Delta_p u + \Delta_q u \quad \forall u \in W^{1,p}(\Omega).$$

Note that hypothesis $H(a)(i)$ holds with $\varrho_1 \in (\max\{0, 3-q\}, 1)$ and $\varrho_2 \in (0, \min\{q-2, 1\})$.

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y - y$ with $4 \leq p < +\infty$. This map corresponds to the generalized p-mean curvature differential operator plus the Laplace operator defined by

$$\operatorname{div}\,((1 + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u) - \Delta u \quad \forall u \in W_0^{1,p}(\Omega).$$

Hypothesis $H(a)(i)$ holds with any $\varrho_1, \varrho_2 \in (0,1)$ and with

$$\lim_{t \searrow 0} \frac{t a_0'(t)}{a_0(t)} \;=\; 2.$$

(d) $a(y) = |y|^{p-2}y + \frac{|y|^{p-2}y}{1+|y|^p}$ with $2 < p < +\infty$. This map corresponds to the following differential operator

$$\Delta_p u + \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{1 + |\nabla u|^p}\right) \quad \forall u \in W_0^{1,p}(\Omega),$$

which arises in problem of plasticity. Also here hypothesis $H(a)(i)$ holds with $\varrho_1 \in (\max\{0, 3-p\}, 1)$ and $\varrho_2 \in (0, \min\{p-2, 1\})$.

Let $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear function defined by

$$A(u) = -\operatorname{div} a(\nabla u), \quad \forall u \in W_0^{1,p}(\Omega),$$

that is

$$\langle A(u), y\rangle \;=\; \int_\Omega \big(a(\nabla u),\ \nabla y\big)_{\mathbb{R}^N}\, dx, \quad \forall u, y \in W_0^{1,p}(\Omega).$$

We have the following properties of A (see [11, p. 746]).

Proposition 2.5. *If hypotheses $H(a)(i)-(iii)$ hold, then $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is bounded (maps bounded sets to bounded ones), continuous, strictly monotone (hence maximal monotone) and of type $(S)_+$, i.e., if $u_n \longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \;\leqslant\; 0,$$

then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$.

Now, we recall some basic definitions and results concerning the following Dirichlet problem

$$\begin{cases} -\operatorname{div} a(\nabla u) - \Delta u = \widehat{f}(x,u), & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$P_{\widehat{f}}$}$$

where $a \in C^1(0,+\infty)$ is a function satisfying hypotheses $H(a)$ and $\widehat{f}:\Omega\times\mathbb{R}\to\mathbb{R}$ is a function with subcritical growth, namely it satisfies the following hypotheses:

$\underline{H(\widehat{f})}$: $\widehat{f}:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

(i) there exist $\alpha \in L^\infty(\Omega)_+$, $c>0$ and $1 \leqslant r < p^*$, s.t.

$$|\widehat{f}(x,s)| \leqslant \alpha(x) + c|s|^{r-1}, \quad \text{for a.a. } x\in\Omega \text{ and all } s\in\mathbb{R},$$

where $p^* = \frac{pN}{N-p}$, if $p<N$ and $p^* = +\infty$, if $p \geqslant N$.

(ii) $\widehat{f}(x,0) = 0$ for almost all $x\in\Omega$,

We recall that the Nemytskij map corresponding to a measurable function $\widehat{f}:\Omega\times\mathbb{R}\to\mathbb{R}$ is indicated as

$$N_{\widehat{f}}(u)(\cdot) = \widehat{f}(\cdot, u(\cdot)), \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Set

$$\widehat{F}(x,s) \;=\; \int_0^s \widehat{f}(x,t)\,dt, \quad \text{for all } (x,s)\in\Omega\times\mathbb{R}.$$

It is well-known that the critical points of the C^1-functional

$$I(u) \;=\; \int_\Omega G(\nabla u(x))\,dx + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega \widehat{F}(x,u(x))\,dx \quad \forall u \in W_0^{1,p}(\Omega),$$

are the weak solutions of problem $(P_{\widehat{f}})$, i.e., $u\in W_0^{1,p}(\Omega)$ is a weak solution of problem $(P_{\widehat{f}})$ if

$$A(u) - \Delta u = N_{\widehat{f}}(u), \quad \text{in } W^{-1,p'}(\Omega).$$

We say that $u \in W^{1,p}(\Omega)$ is a super (sub) solution of problem $(P_{\widehat{f}})$ if $u_{|\partial\Omega} \geqslant 0$ $(u_{|\partial\Omega} \leqslant 0)$ and

$$A(u) - \Delta u \geqslant (\leqslant) N_{\widehat{f}}(u), \quad \text{in } W^{-1,p'}(\Omega).$$

We impose $u \geqslant 0$ (resp. $u \leqslant 0$) on $\partial\Omega$ in the sense of trace operator.

Now, our aim is to localize some critical points of the functional I, see [5].

Let $\underline{u}$ and $\overline{u}$ be two functions in $W_0^{1,p}(\Omega)$, with $\underline{u} \leqslant \overline{u}$. We consider the following three Carathéodory functions $\widehat{f^{\overline{u}}}, \widehat{f_{\underline{u}}}, \widehat{f_{\underline{u}}^{\overline{u}}} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined, for every $(x,s) \in \Omega \times \mathbb{R}$ by

$$\widehat{f^{\overline{u}}}(x,s) = \begin{cases} \widehat{f}(x,s), & s \leqslant \overline{u}(x); \\ \widehat{f}(x,\overline{u}(x)), & s > \overline{u}(x), \end{cases} \qquad \widehat{f_{\underline{u}}}(x,s) = \begin{cases} \widehat{f}(x,\underline{u}(x)), & s < \underline{u}(x); \\ \widehat{f}(x,s), & s \geqslant \underline{u}(x), \end{cases}$$

$$\widehat{f_{\underline{u}}^{\overline{u}}}(x,s) = \begin{cases} \widehat{f}(x,\underline{u}(x)), & s < \underline{u}(x); \\ \widehat{f}(x,s), & \underline{u}(x) \leqslant s \leqslant \overline{u}(x); \\ \widehat{f}(x,\overline{u}(x)), & s > \overline{u}(x). \end{cases}$$

Moreover, denote by $\widehat{F^{\overline{u}}}$, $\widehat{F_{\underline{u}}}$ and $\widehat{F_{\underline{u}}^{\overline{u}}}$ the primitives of $\widehat{f^{\overline{u}}}$, $\widehat{f_{\underline{u}}}$ and $\widehat{f_{\underline{u}}^{\overline{u}}}$ respectively, (for instance, $\widehat{F^{\overline{u}}}(x,\xi) = \int_0^\xi \widehat{f^{\overline{u}}}(x,s)ds$ for every $(x,\xi) \in \Omega \times \mathbb{R}$). We consider the following functionals defined on $W_0^{1,p}(\Omega)$,

$$\begin{aligned} I^{\overline{u}}(w) &= \int_\Omega G(\nabla w)\,dx + \frac{1}{2}\|\nabla w\|_2^2 - \int_\Omega \widehat{F^{\overline{u}}}(x,w)dx, \\ I_{\underline{u}}(w) &= \int_\Omega G(\nabla w)\,dx + \frac{1}{2}\|\nabla w\|_2^2 - \int_\Omega \widehat{F_{\underline{u}}}(x,w)\,dx, \\ I_{\underline{u}}^{\overline{u}}(w) &= \int_\Omega G(\nabla w)\,dx + \frac{1}{2}\|\nabla w\|_2^2 - \int_\Omega \widehat{F_{\underline{u}}^{\overline{u}}}(x,w)dx \end{aligned}$$

for all $w \in W_0^{1,p}(\Omega)$. Such functionals are weakly lower semicontinuous and continuously Gateâux differentiable on $W_0^{1,p}(\Omega)$.

Let $x \in \mathbb{R}$. We set $x^\pm := \max\{\pm x, 0\}$ and for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in W_0^{1,p}(\Omega)$, $|u| = u^+ + u^-$ and $u = u^+ - u^-$.

Lemma 2.6. *Let $\underline{u}$ and $\overline{u}$ be respectively a sub-solution and a super-solution of problem $(P_{\widehat{f}})$. Then we have:*

1) *If u is a critical point of $I^{\overline{u}}$ in $W_0^{1,p}(\Omega)$, then $u \leqslant \overline{u}$.*

2) *If u is a critical point of $I_{\underline{u}}$ in $W_0^{1,p}(\Omega)$, then $\underline{u} \leqslant u$.*

3) *Provided that $\underline{u} \leqslant \overline{u}$, if w is a critical point of $I_{\underline{u}}^{\overline{u}}$ in $W_0^{1,p}(\Omega)$, then one has that $\underline{u} \leqslant w \leqslant \overline{u}$.*

Proof. We only show that 1) holds, the proof of 2) is similar, while 3) follows at once combining 1) and 2). Let u be a critical point of $I^{\overline{u}}$. Since $I^{\overline{u}}$ is a C^1-functional, this means that

$$A(u) - \Delta u \;=\; N_{\widehat{f^{\overline{u}}}}(u).$$

Testing such equation with $(u-\overline{u})^+ \in W_0^{1,p}(\Omega)$ and using the fact that $\overline{u}$ is a super-solution for problem $(P_{\widehat{f}})$, we have

$$\begin{aligned} \left\langle A(u)-\Delta u,(u-\overline{u})^+\right\rangle &= \int_\Omega \widehat{f^{\overline{u}}}(x,u(x))(u-\overline{u})^+\,dx \\ = \int_\Omega \widehat{f^{\overline{u}}}(x,\overline{u}(x))(u-\overline{u})^+\,dx &\leqslant \left\langle A(\overline{u})-\Delta\overline{u},(u-\overline{u})^+\right\rangle, \end{aligned}$$

which forces

$$\begin{aligned} &\int_{\{\overline{u}<u\}} \langle a(u)-a(\overline{u}),\nabla u-\nabla\overline{u}\rangle\,dx + \|\nabla(u-\overline{u})^+\|_2^2 \\ = &\left\langle A(u)-A(\overline{u})-\Delta u+\Delta\overline{u},(u-\overline{u})^+\right\rangle \leqslant 0. \end{aligned}$$

On the other hand, due to Proposition 2.5, we have that the operator A is strictly monotone that implies

$$\int_{\{\overline{u}<u\}} \langle a(u)-a(\overline{u}),\nabla u-\nabla\overline{u}\rangle\,dx \geqslant 0.$$

Putting together the last two inequalities, we have that $|\{\overline{u}<u\}|_{\mathbb{R}^N}=0$. Hence, we conclude that $u\leqslant\overline{u}$ in $W_0^{1,p}(\Omega)$. $\square$

The following proposition is a modification of the result due to Gasiński-Papageorgiou [13, Proposition 2.6] and its proof can be obtained using the regularity results due to Lieberman [21].

Proposition 2.7. *If $u_0\in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of I, i.e., there exists $r_1>0$ s.t.*

$$I(u_0)\leqslant I(u_0+\varphi), \qquad \text{for all } \varphi\in C_0^1(\overline{\Omega}) \text{ with } \|\varphi\|_{C_0^1(\overline{\Omega})}\leqslant r_1,$$

then $u_0\in C_0^{1,\eta}(\overline{\Omega})$ with $\eta\in(0,1)$ and it is a local $W_0^{1,p}(\Omega)$-minimizer of I, i.e., there exists $r_2>0$ s.t.

$$I(u_0)\leqslant I(u_0+\varphi), \qquad \text{for all } \varphi\in W_0^{1,p}(\Omega) \text{ with } \|\varphi\|_{W_0^{1,p}(\Omega)}\leqslant r_2.$$

A further analysis, based on the previous proposition, on the maximum principle, and on the boundary point lemma of Pucci-Serrin ([29]), leads to some qualitative properties of suitable critical points of I. For the reader convenience, before to detail these properties, we recall suitable versions of the regularity results, due to Pucci-Serrin, when the following differential inequality is considered

$$\operatorname{div}(\tilde{a}(|\nabla u|)\nabla u)+\tilde{b}(x,u)\leq 0 \tag{2.5}$$

in Ω, where

$(\tilde{a})_1$ $\tilde{a}\in C^1(\mathbb{R}^+)$;

$(\tilde{a})_2$ *$t \longmapsto t\tilde{a}_0(t)$ is strictly increasing in $\mathbb{R}^+$ and $t\tilde{a}_0(t) \longrightarrow 0$ as $t \searrow 0$;*

while $\tilde{b} \in L^\infty_{\rm loc}(\Omega \times \mathbb{R}^+)$ is such that

$(\tilde{b})_1$ $\tilde{b}(x,s) \geq -b(s)$ for a.a. $x \in \Omega$ and for all $s \geq 0$,

with b being a function such that

$(b)_1$ $b(0) = 0$ and b is continuous and non-decreasing on some interval $(0, \delta_1)$, $\delta_1 > 0$.

Theorem 2.8 (**Strong maximum principle** [29], page 111)**.** *Suppose that*

$$\lim_{t \searrow 0} \frac{t\tilde{a}'(t)}{\tilde{a}(t)} = 0. \tag{2.6}$$

Let $(\tilde{a})_1$, $(\tilde{a})_2$, $(\tilde{b})_1$ and $(b)_1$ be satisfied. For the strong maximum principle to be valid for (2.5) it is sufficient that

$$\int_0^{\delta_1} \frac{1}{H^{-1}(B(s))}\, ds = \infty, \tag{2.7}$$

where $H(t) = t^2\tilde{a}(t) - \int_0^t \xi\tilde{a}(\xi)\, d\xi$, $t \geq 0$, and $B(s) = \int_0^s b(t)\, dt$.

Theorem 2.9 (**Boundary point lemma** [29], page 120)**.** *Assume (2.6). Suppose that $(\tilde{a})_1$, $(\tilde{a})_2$, $(\tilde{b})_1$ and $(b)_1$ hold and that (2.7) is satisfied.*

Let u be a C^1 solution of (2.5) in $\overline{\Omega}$, with $u > 0$ in Ω and $u(x) = 0$, where $x \in \partial\Omega$. If Ω satisfies an interior sphere condition at x, then $\frac{\partial u}{\partial n} < 0$ at x.

Let us now point out a variational property of certain solutions of $(P_{\widehat{f}})$.

Lemma 2.10. *Let $\underline{u}$ and $\overline{u}$ be as in Lemma 2.6. Assume that there exists $\delta > 0$ such that*

$$s\widehat{f}(x,s) \geq 0 \quad \text{for a.a. } x \in \Omega, \text{ for all } s \in [-\delta, \delta]. \tag{2.8}$$

Then we have:

(1) If $0 = \underline{u} < \overline{u}$ and $u_0 \in W^{1,p}_0(\Omega)$ is a nontrivial global minimizer for $I^{\overline{u}}_0$, then u_0 is a local minimizer of $I^{\overline{u}}$ and $u_0 \in D_+$.

(2) If $\underline{u} < \overline{u} = 0$ and $u_0 \in W^{1,p}_0(\Omega)$ is a nontrivial global minimizer for $I^0_{\underline{u}}$, then u_0 is a local minimizer of $I_{\underline{u}}$ and $u_0 \in -D_+$.

Proof. Let us prove only (1), the proof of (2) being similar. Let $u_0 \in W^{1,p}_0(\Omega)$ be a nontrivial global minimizer for $I^{\overline{u}}_0$. From Lemma 2.6 it follows that $u_0 \in [0, \overline{u}]$, hence it is a weak solution of $(P_{\widehat{f}})$. Applying the results of [10], see also [20, p. 286], we have that $u_0 \in L^\infty(\Omega)$. Hence, from the regularity theory of Lieberman [21] one has that

$$u_0 \in C^1_0(\overline{\Omega}). \tag{2.9}$$

We claim that
$$u_0 \in D_+. \tag{2.10}$$
To verify (2.10) put
$$\tilde{a}(t) = a_0(t) + 1 \text{ for all } t > 0$$
and observe that, in view of $H(a)$, both $(\tilde{a})_1$ and $(\tilde{a})_2$ hold (Remark 2.1). Moreover, in view of $H(a)(i)$
$$\lim_{t\searrow 0} \frac{t\tilde{a}'(t)}{\tilde{a}(t)} = \lim_{t\searrow 0} \frac{ta_0'(t)}{a_0(t)} \cdot \frac{a_0(t)}{a_0(t)+1} = 0,$$
namely (2.6) holds.

Reasoning as in [4, Lemma 3.1], from $H(\widehat{f})(i)$ and (2.8) it follows that for any $M > 0$ there exists $c_M > 0$ such that
$$\widehat{f}(x,s) + c_M s^{p-1} \geq 0 \quad \text{for a.a. } x \in \Omega, \text{ all } s \in [0, M]. \tag{2.11}$$
Indeed, fixed $M > 0$, if $M \leq \delta$ then (2.11) trivially holds with arbitrary $c_M > 0$, since
$$\widehat{f}(x,s) + s^{p-1} \geq \widehat{f}(x,s) \geq 0 \quad \text{for a.a. } x \in \Omega, \text{ all } s \in [0, M] \subseteq [0, \delta].$$
If $M > \delta$, put $K = \|\alpha\|_\infty + c|M|^{r-1}$, $c_M = \max\left\{1, \frac{K}{\delta^{p-1}}\right\}$ and observe that
$$\widehat{f}(x,s) + c_M s^{p-1} \geq \widehat{f}(x,s) + s^{p-1} \geq \widehat{f}(x,s) \geq 0 \quad \text{for a.a. } x \in \Omega, \text{ all } s \subseteq [0, \delta]. \tag{2.12}$$
Moreover,
$$-\widehat{f}(x,s) \leq |\widehat{f}(x,s)| \leq K \leq \frac{K}{\delta^{p-1}} s^{p-1} \leq c_M s^{p-1} \quad \text{for a.a. } x \in \Omega, \forall s \in [\delta, M]. \tag{2.13}$$
Hence, (2.12) and (2.13) imply (2.11).

Consider $M = \|u_0\|_\infty$, then one has
$$\operatorname{div} \tilde{a}(|\nabla u_0|\nabla u_0) = \operatorname{div} a(\nabla u_0) + \Delta u_0 = -\widehat{f}(x, u_0) \leq c_M |u_0|^{p-1},$$
for a.a. $x \in \Omega$. Namely, u_0 solves (2.5), where $\tilde{b}(x,s) = -c_M s^{p-1}$, so that $(\tilde{b})_1$ is verified with $b = -\tilde{b}$.

For every $t > 0$, exploiting $H(a)(iii)$ and (2.2), with $y = (t, 0, \ldots, 0)$ and $\xi = (1, 0, \ldots, 0)$, one has
$$\begin{aligned} c_1 t^{p-2} &\leq \frac{\eta(t)}{t} = \frac{\eta(\|y\|)}{\|y\|}\|\xi\|^2 \leq \big(\nabla a(y)\xi,\ \xi\big)_{\mathbb{R}^N} \\ &= a_0'(\|y\|)\frac{y_i\, y_j}{\|y\|}\xi_i\, \xi_j + a_0(\|y\|)\delta_{ij}\xi_i\, \xi_j \\ &= \frac{a_0'(t)}{t}t^2 + a_0(t), \end{aligned}$$

that is

$$t^2\, a_0'(t) + t\, a_0(t) \geq c_1 t^{p-1} \quad \text{for all } t > 0,$$

and integrating one has

$$t^2 a_0(t) - \int_0^t \xi\, a_0(\xi)\; d\xi \geq \frac{c_1}{p} t^p \quad \text{for all } t > 0,$$

that leads to

$$\begin{aligned} H(t) &= t^2\, \tilde{a}(t) - \int_0^t \xi\, \tilde{a}(\xi)\; d\xi \\ &= t^2\, a_0(t) + t^2 - \int_0^t \xi\, a_0(\xi)\; d\xi - \frac{t^2}{2} \\ &= t^2 a_0(t) - \int_0^t \xi\, a_0(\xi)\; d\xi + \frac{t^2}{2} \\ &\geq \frac{c_1}{p} t^p \quad \text{for all } t > 0. \end{aligned} \tag{2.14}$$

A direct computation shows that

$$H'(t) = t\, a_0(t) \left(1 + \frac{t\, a_0'(t)}{a_0(t)}\right) + t \quad \text{for all } t > 0,$$

and, in view of $H(a)(i)$, there exits $\delta_2 > 0$ such that H is continuous and strictly increasing in $(0, \delta_2]$. Put $H_0(t) = \frac{c_1}{p} t^p$ for all $t \in (0, \delta_2]$. Then,

$$H^{-1}(s) \leq H_0^{-1}(s) \quad \text{for all } s \in (0, H_0(\delta_2)]. \tag{2.15}$$

If not, let $\overline{s} \in (0, H_0(\delta_2)]$ be such that

$$H^{-1}(\overline{s}) > H_0^{-1}(\overline{s}).$$

Hence, thanks to the monotonicity of H and in view of (2.14), we achieve

$$\overline{s} > H(H_0^{-1}(\overline{s})) \geq H_0(H_0^{-1}(\overline{s})) = \overline{s},$$

a contradiction, and so (2.15) holds. At this point one has

$$\frac{1}{H^{-1}(B(s))} \geq \frac{1}{H_0^{-1}(B(s))} = \left(\frac{c_1}{c_M}\right)^{1/p} \frac{1}{s} \quad \text{for all } s \in (0, (c_1/c_M)^{1/p} \delta_2).$$

Finally, if $\delta_1 \in (0, (c_1/c_M)^{1/p} \delta_2)$, it is clear that (2.7) holds. Hence, we can apply Theorem 2.8 and get $u_0 > 0$ in Ω. Taking in mind (2.9), because of Theorem 2.9 one can conclude that claim (2.10) is verified.

Let U be a C^1-neighborhood of u_0 such that $u_0 \in U \subset D_+$. Then

$$I^{\overline{u}}(u_0) = I_0^{\overline{u}}(u_0) \leqslant I_0^{\overline{u}}(u) = I^{\overline{u}}(u),$$

for every $u \in U$, that is u_0 is a $C_0^1(\Omega)$ local minimizer of $I^{\overline{u}}$. Therefore, Proposition 2.7 ensures that u_0 is also a $W_0^{1,p}(\Omega)$ local minimizer of $I^{\overline{u}}$.

□

The next lemma will be useful for producing nontrivial solutions.

Lemma 2.11. *Let $\widehat{f}:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying conditions $H(\widehat{f})$. Let $a:\mathbb{R}^N\to\mathbb{R}^N$ be an operator fulfilling hypotheses $H(a)$. Assume that*

(i) there exist $\delta_0>0$ and $c>0$ with $c>\lambda_1/2$, where λ_1 is the first eigenvalue of $(-\Delta, W_0^{1,2}(\Omega))$ such that

$$c|s|\leqslant \widehat{F}(x,s),\quad \text{for all } |s|\leqslant\delta_0 \text{ and for a.a. } x\in\Omega.$$

Then, zero is not a local $W_0^{1,p}(\Omega)$-minimizer for the functional I.

Proof. By using hypothesis $H(a)(iv)$, we have that there exists $\bar{\delta}_0\in(0,\delta_0)$ such that

$$\frac{G_0(t)}{t^\tau}<\frac{2\sigma}{\tau},\quad \text{for all}\quad 0<t<\bar{\delta}_0. \tag{2.16}$$

Let ϕ_1 be the positive eigenfunction related to λ_1 and normalized in $L^2(\Omega)$. Recall that $\phi_1\in D_+$. Hence, for every

$$0<\rho<\bar{\rho}:=\min\left\{\frac{\bar{\delta}_0}{\max\limits_{x\in\Omega}|\phi_1(x)|},\frac{\bar{\delta}_0}{\max\limits_{x\in\Omega}|\nabla\phi_1(x)|},\left[\frac{\tau}{2\sigma\|\nabla\phi_1\|_\tau^\tau}\left(c-\frac{\lambda_1}{2}\right)\right]^{1/(\tau-2)}\right\},$$

owing to (2.16) and (i), one has

$$\begin{aligned}
I(\rho\phi_1) &= \int_\Omega G_0(|\nabla\rho\phi_1(x)|)\,dx+\frac{1}{2}\|\nabla\rho\phi_1(x)\|_2^2-\int_\Omega\widehat{F}(x,\rho\phi_1(x))\,dx\\
&\leqslant \frac{2\sigma\rho^\tau}{\tau}\int_\Omega|\nabla\phi_1(x)|^\tau\,dx+\frac{\rho^2}{2}\int_\Omega|\nabla\phi_1(x)|^2\,dx-c\rho^2\int_\Omega|\phi_1(x)|^2\,dx\\
&= \rho^2\left(\frac{2\sigma\rho^{\tau-2}}{\tau}\int_\Omega|\nabla\phi_1(x)|^\tau\,dx+\frac{\lambda_1}{2}-c\right).
\end{aligned}$$

From this, recall also that, as observed in Remark 2.1, $\tau>2$, we see that

$$I(\rho\phi_1)<0=I(0),$$

for every $\rho\in(0,\bar{\rho})$, that is, the zero function is not a local $C_0^1(\overline{\Omega})$-minimizer for I. By the embedding of $C_0^1(\overline{\Omega})$ in $W_0^{1,p}(\Omega)$, it is clear that $\rho\phi_1\to0$ in $W_0^{1,p}(\Omega)$, as $\rho\to0^+$. Hence, the conclusion is achieved. □

Remark 2.12. From the proof of Lemma 2.11 it follows that condition $H(a)(iv)$ could be replaced by the more general

$$\lim_{t\searrow0}\frac{G(t)}{t^\tau}=0$$

for some $\tau\in(1,p)$, provided $\widehat{F}(x,\cdot)$ satisfies a more restrictive condition, namely it is (γ)-linear at zero, with $\gamma\in(1,\min\{\tau,2\})$. This kind of conditions will be assumed in Section 4.

3 Multiplicity Results

In this section we proof the existence of at least three nontrivial smooth solutions for problem $(P_{f,\lambda})$, staring with the two of constant sign. The assumptions for the nonlinearity f are the following:

$H(f)$: *For every $\lambda > 0$, $f_\lambda\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f_\lambda(x,0) = 0$ for almost all $x \in \Omega$ and*

(i) there exists $\tilde{c} > 0$ such that for every $\lambda > 0$

$$|f_\lambda(x,s)| \leqslant a_\lambda(x) + \tilde{c}|s|^{r_\lambda - 1}$$

for almost all $x \in \Omega$, all $s \in \mathbb{R}$, with $a_\lambda \in L^\infty(\Omega)_+$ and $\|a_\lambda\|_\infty \longrightarrow 0$ as $\lambda \searrow 0$, as well as $2 < r_\lambda < +\infty$ and $r_\lambda \longrightarrow r > 2$ as $\lambda \searrow 0$;

(ii) for every $\lambda > 0$, there exists $\theta_\lambda > \frac{\lambda_1}{2}$ such that

$$\liminf_{s \to 0} \frac{f_\lambda(x,s)}{s} = \theta_\lambda$$

uniformly for a.a. $x \in \Omega$.

We start with the existence of two nontrivial constant sign solutions.

Theorem 3.1. *If hypotheses $H(a)$ and $H(f)$ hold, then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ problem $(P_{f,\lambda})$ admits at least two nontrivial constant sign smooth solutions*

$$u_\lambda \;\in\; D_+ \quad \textit{and} \quad v_\lambda \;\in\; -D_+.$$

Proof. First we consider the following auxiliary Dirichlet problem

$$\left\{ \begin{array}{rcll} -\Delta e(z) & = & 1 & \text{in } \Omega, \\ e|_{\partial\Omega} & = & 0. & \end{array} \right. \tag{3.1}$$

This problem has a unique solution $e \in D_+$. In fact since we assumed that $\partial\Omega$ is a C^3-manifold, standard regularity theory (see Troianiello [32, Theorem 3.23, page 189]) implies that $e \in C^2(\overline{\Omega})$.

Claim 1. There exists $\varrho > 1$ such that

$$M_A \;=\; \sup_{t \in [0,1]} \frac{\|A(te)\|_\infty}{t^\varrho} \;<\; +\infty \tag{3.2}$$

We have

$$\begin{aligned} A(te) &= -\operatorname{div}\left(a_0(|\nabla te|)\nabla te\right) \\ &= -t \cdot \sum_{i=1}^N \left[a_0'(t|\nabla e|) \frac{t(\nabla e, \nabla \frac{\partial e}{\partial x_i})_{\mathbb{R}^N}}{|\nabla e|} \frac{\partial e}{\partial x_i} + a_0(|t\nabla e|) \frac{\partial^2 e}{\partial x_i^2} \right] \end{aligned}$$

$$= \quad -\sum_{i=1}^{N}\left[t^{2-\varrho_1}t^{\varrho_1}a_0'(t|\nabla e|)\frac{(\nabla e,\nabla\frac{\partial e}{\partial x_i})_{\mathbb{R}^N}}{|\nabla e|}\frac{\partial e}{\partial x_i}+t^{1+\varrho_2}\frac{a_0(|t\nabla e|)}{t^{\varrho_2}}\frac{\partial^2 e}{\partial x_i^2}\right].$$

As $e\in C^2(\overline{\Omega})$, we have that all first and second partial derivatives of e are continuous and thus bounded on $\overline{\Omega}$. Using also hypothesis $H(a)(i)$, we get that $A(te)\in C(\overline{\Omega})$ and (3.2) holds with $\varrho=\min\{2-\varrho_1,1+\varrho_2\}>1$. This proves Claim 1.

Claim 2. There exists $\lambda^*>0$ such that for every $\lambda\in(0,\lambda^*)$, we can find $\xi_0^\lambda\in(0,1)$ for which we have

$$\|a_\lambda\|_\infty+c(\xi_0^\lambda\|e\|_\infty)^{r_\lambda-1} \;<\; \xi_0^\lambda-(\xi_0^\lambda)^\varrho M_A,$$

where M_A and $\varrho>1$ are given by Claim 1.

Arguing by contradiction, suppose that we can find a sequence $\{\lambda_n\}_{n\geqslant1}\subseteq(0,1)$ such that $\lambda_n\searrow 0$ and

$$\|a_{\lambda_n}\|_\infty+c(\xi\|e\|_\infty)^{r_{\lambda_n}-1} \;\geqslant\; \xi-\xi^\varrho M_A \quad \forall n\geqslant1,\ \xi>0.$$

Letting $n\to+\infty$ and using hypothesis $H(f)(i)$ we obtain

$$c(\xi\|e\|_\infty)^{r-1} \;\geqslant\; \xi(1-\xi^{\varrho-1}M_A),$$

so

$$c\xi^{r-2}\|e\|_\infty^{r-1} \;\geqslant\; 1-\xi^{\varrho-1}M_A.$$

But recall that $r>2$, $\varrho>1$ and $\xi>0$ is arbitrary. So, we let $\xi\searrow 0$ and we reach a contradiction. This proves Claim 2.

Fix $\lambda\in(0,\lambda^*)$ and let $\overline{u}_\lambda=\xi_0^\lambda e\in D_+\cap C^2(\overline{\Omega})$. From Claims 1 and 2 and hypothesis $H(f)(i)$ we have

$$\begin{aligned}A\overline{u}_\lambda(x)-\Delta\overline{u}_\lambda(x) \;&\geqslant\; \xi_0^\lambda-\|A(\xi_0^\lambda e)\|_\infty \;=\; \xi_0^\lambda-(\xi_0^\lambda)^\varrho\frac{\|A(\xi_0^\lambda e)\|_\infty}{(\xi_0^\lambda)^\varrho}\\ &\geqslant\; \xi_0^\lambda-(\xi_0^\lambda)^\varrho M_A \;\geqslant\; \|a_\lambda\|_\infty+c(\xi_0^\lambda\|e\|_\infty)^{r-1}\\ &\geqslant\; f_\lambda(x,\overline{u}_\lambda(x)) \quad \text{for a.a. } x\in\Omega. \end{aligned} \tag{3.3}$$

Hence, we have that $\overline{u}_\lambda$ is a super-solution of problem $(P_{f,\lambda})$ and $\underline{u}_\lambda=0$ is obviously a sub-solution.

For $\lambda\in(0,\lambda^*)$ we consider truncation $(f_\lambda)_0^{\overline{u}_\lambda}$ of the reaction $f_\lambda(x,\cdot)$: and the C^1-functional $(I_\lambda)_0^{\overline{u}_\lambda}\colon W_0^{1,p}(\Omega)\longrightarrow\mathbb{R}$ defined by

$$(I_\lambda)_0^{\overline{u}_\lambda}(u) \;=\; \int_\Omega G(\nabla u)\,dz+\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega(\widehat{F}_\lambda)_0^{\overline{u}_\lambda}(z,u)\,dz \quad \forall u\in W_0^{1,p}(\Omega).$$

Evidently $(I_\lambda)_0^{\overline{u}_\lambda}$ is coercive (Lemma 2.3(d)) and by the Sobolev embedding theorem, we see that it is also sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\lambda\in W_0^{1,p}(\Omega)$ such that

$$(I_\lambda)_0^{\overline{u}_\lambda}\,(u_\lambda)=\inf_{u\in W_0^{1,p}(\Omega)}(I_\lambda)_0^{\overline{u}_\lambda}\,(u), \tag{3.4}$$

and Lemma 2.6 ensures that

$$u_\lambda \in [0, \overline{u}_\lambda], \tag{3.5}$$

where $[0, \overline{u}_\lambda] = \{u \in W_0^{1,p}(\Omega) : 0 \leqslant u(x) \leqslant \overline{u}_\lambda(x)$ for almost all $x \in \Omega\}$. Assumption $H(f)(ii)$ implies that condition (i) of Lemma 2.11 holds with $\widehat{F} = (\widehat{F}_\lambda)_0^{\overline{u}_\lambda}$, so that, u_λ is nontrivial. Moreover, Lemma 2.10, implies that

$$u_\lambda \in D_+$$

and it is a local minimizer of $(I_\lambda)^{\overline{u}_\lambda}$ that concludes the first part of the proof.

In a similar fashion, using this time $\underline{u}_\lambda = -\overline{u}_\lambda \in (-D_+) \cap C^2(\overline{\Omega})$, we produce a negative solution $v_\lambda \in -D_+$. □

Remark 3.2. We wish to explicitly point out that assumption $(H)(f)(ii)$ is verified when, in particular, $f_\lambda(x, \cdot)$ is sublinear at zero, namely, for example, if

$(ii)'$ *for every $\lambda > 0$, there exist $\gamma_\lambda \in (1, 2)$, $\theta_\lambda > 0$ such that*

$$\liminf_{s \to 0} \frac{f_\lambda(x, s)}{|s|^{\gamma_\lambda - 2} s} = \theta_\lambda$$

uniformly for a.a. $x \in \Omega$.

In fact, in this case, the more restrictive condition holds

$$\lim_{s \to 0} \frac{f_\lambda(x, s)}{s} = +\infty \tag{3.6}$$

uniformly for a.a. $x \in \Omega$, for every $\lambda > 0$.

We conclude pointing out a further multiplicity result, provided f is sublinear at zero.

Theorem 3.3. *If hypotheses $H(a)$ and $H(f)(i)$ hold in addition to (3.6). Then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ problem $(P_{f,\lambda})$ admits at least three distinct nontrivial smooth solutions*

$$u_\lambda \in D_+, \; v_\lambda \in -D_+ \quad \textit{and} \quad \widehat{w}_\lambda \in [\underline{u}_\lambda, \overline{u}_\lambda] \cap C_0^1(\overline{\Omega}).$$

Proof. From Theorem 3.1 we obtain $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ there exists the solutions $u_\lambda \in D_+$ and $v_\lambda \in -D_+$. Fix $\lambda \in (0, \lambda^*)$ and let us prove the existence of a third nontrivial solution $\widehat{w}_\lambda \in [\underline{u}_\lambda, \overline{u}_\lambda] \cap C_0^1(\overline{\Omega})$. Clearly, because of Lemma 2.6, we can obtain our conclusion verifying that

$$I_{\underline{u}_\lambda}^{\overline{u}_\lambda} \text{ admits a nontrivial critical point } \widehat{w}_\lambda \text{ such that } \widehat{w}_\lambda \neq v_\lambda \text{ and } \widehat{w}_\lambda \neq u_\lambda, \tag{3.7}$$

where $\underline{u}_\lambda$ and $\overline{u}_\lambda$ are as in the proof of Theorem 3.1. Preliminary, we observe that from the proof of Theorem 3.1 we can emphasize the following further properties: u_λ and v_λ are nonzero global minimizers of $I_0^{\overline{u}_\lambda}$ and $I^0_{\underline{u}_\lambda}$ respectively, such that

$$I^0_{\underline{u}_\lambda}(v_\lambda) < 0 \quad I_0^{\overline{u}_\lambda}(u_\lambda) < 0. \tag{3.8}$$

Hence, since $u_\lambda \in D_+ \cap [0, \overline{u}_\lambda]$ and $v_\lambda \in -D_+ \cap [\underline{u}_\lambda, 0]$, it is clear that they are local minimizers of $I^{\overline{u}_\lambda}_{\underline{u}_\lambda}$ with respect to the $C^1_0(\overline{\Omega})$-topology. Thus, applying Proposition 2.7, we can conclude that

$$v_\lambda \text{ and } u_\lambda \text{ are } W^{1,p}_0(\Omega) - \text{local minimizers of } I^{\overline{u}_\lambda}_{\underline{u}_\lambda}. \tag{3.9}$$

Furthermore, observe that

$$I^{\overline{u}_\lambda}_{\underline{u}_\lambda} \text{ satisfies the Palais-Smale condition.} \tag{3.10}$$

In fact, let $\{w_n\} \subset W^{1,p}_0(\Omega)$ be a sequence such that $\{I^{\overline{u}_\lambda}_{\underline{u}_\lambda}(w_n)\}$ is bounded and $(I^{\overline{u}_\lambda}_{\underline{u}_\lambda})'(w_n) \longrightarrow 0$. Hence, since $I^{\overline{u}_\lambda}_{\underline{u}_\lambda}$ is coercive (exploit the definition of the truncation and condition d) of Lemma 2.3), there exists $w \in W^{1,p}_0(\Omega)$ such that $w_n \longrightarrow w$ weakly in $W^{1,p}_0(\Omega)$ and $w_n \longrightarrow w$ in $L^p(\Omega)$ (where a subsequence is considered if necessary). Observe that

$$\begin{aligned}
\limsup_{n\to+\infty} \langle A(w_n), w_n - w\rangle &= \limsup_{n\to+\infty} [\langle A(w_n), w_n - w\rangle + \langle -\Delta w, w_n - w\rangle] \\
&\le \limsup_{n\to+\infty} [\langle A(w_n), w_n - w\rangle + \langle -\Delta w_n, w_n - w\rangle] \\
&= \lim_{n\to+\infty} \int_\Omega f^{\overline{u}_\lambda}_{\underline{u}_\lambda}(x, w_n(x))(w_n(x) - w(x))\, dx = 0,
\end{aligned}$$

where we exploited the monotonicity of $-\Delta$, the convergence of $(I^{\overline{u}_\lambda}_{\underline{u}_\lambda})'(w_n)$, the definition of the truncation $f^{\overline{u}_\lambda}_{\underline{u}_\lambda}(x,\cdot)$, assumption $H(f)(i)$ and the convergence properties of $\{w_n\}$. At this point, (3.10) follows directly from Proposition 2.5.

Summarizing, we can apply [28, Corollary 1] to the C^1-functional $I^{\overline{u}_\lambda}_{\underline{u}_\lambda}$, so that it possesses a third critical point $\widehat{w}_\lambda$, being

$$I^{\overline{u}_\lambda}_{\underline{u}_\lambda}(\widehat{w}_\lambda) = \mu = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I^{\overline{u}_\lambda}_{\underline{u}_\lambda}(\eta(t)),$$

where $\Gamma = \{\eta \in C^0([0,1], W^{1,p}_0(\Omega)) : \ \eta(0) = v_\lambda,\ \eta(1) = u_\lambda\}$. Let us now show that assuming (3.7) false we achieve a contradiction. The negation of (3.7), in combination with (3.9), implies that

$$K = \left\{ w \in W^{1,p}_0(\Omega) : \ (I^{\overline{u}_\lambda}_{\underline{u}_\lambda})'(w) = 0 \right\} = \{v_\lambda,\ 0,\ u_\lambda\}, \tag{3.11}$$

namely, $\widehat{w}_\lambda = 0$ and, in particular,

$$\mu = 0. \tag{3.12}$$

We will conclude producing a path $\widehat{\eta} \in \Gamma$ such that

$$\max_{t\in[0,1]} I^{\overline{u}_\lambda}_{\underline{u}_\lambda}(\widehat{\eta}(t)) < 0, \tag{3.13}$$

thus the following contradiction occurs

$$0 = \mu \leq \max_{t \in [0,1]} I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(\widehat{\eta}(t)) < 0.$$

Let $z_1,\ z_2 \in C_0^1(\Omega)$ be two linearly independent functions, normalized in $W_0^{1,p}(\Omega)$ and such that

$$\|z_1\| = \|z_2\| = 1,\ z_1 \in -(C_0^1(\overline{\Omega}))_+,\ z_2 \in (C_0^1(\overline{\Omega}))_+.$$

Put $Z = \text{span}\,\{z_1,\ z_2\}$ and consider suitable positive constants $d_i,\ i = 1, \ldots, 4,$ such that for every $z \in Z$

$$d_1\|z\| \leq \|z\|_{C_0^1(\overline{\Omega})} \leq d_2\|z\|,$$

$$d_3\|z\| \leq \|z\|_2 \leq d_4\|z\|.$$

Fix $M > k_2^2/d_3^2$, where k_2 is the constant of the embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$ (remember that $p > 2$), and exploit (3.6) to find $\bar{\delta} = \bar{\delta}(\lambda, M) > 0$ such that

$$\frac{f_\lambda(x, s)}{s} \geq M$$

for a.a. $x \in \Omega$ and every $s \in [-\bar{\delta}, \bar{\delta}] \setminus \{0\}$. Hence,

$$F_\lambda(x, s) \geq \frac{M}{2} s^2$$

for a.a. $x \in \Omega$ and every $s \in [-\bar{\delta}, \bar{\delta}]$.

Assumption $H(a)(iv)$ assures the existence of $\bar{\bar{\delta}} > 0$ such that

$$G_0(t) \leq \frac{2\sigma}{\tau} t^\tau \quad \forall t \in [0, \bar{\bar{\delta}}].$$

Recalling that $u_\lambda \in D_+$ and $v_\lambda \in -D_+$, fix $\varepsilon \in (0, \min\{\bar{\delta}, \bar{\bar{\delta}}\})$ such that

$$u_\lambda + \varepsilon B_{C_0^1}(0, 1) \subset (C_0^1(\overline{\Omega}))_+, \quad -v_\lambda + \varepsilon B_{C_0^1}(0, 1) \subset (C_0^1(\overline{\Omega}))_+, \tag{3.14}$$

where $B_{C_0^1}(0, 1) = \Big\{u \in C_0^1(\overline{\Omega}) :\ \|u\|_{C_0^1(\overline{\Omega})} \leq 1\Big\}$.

Fix $\rho \in \left(0, \min\left\{\frac{\varepsilon}{d_2}, \left[\frac{\tau}{4\sigma k_\tau}(M d_3^2 - k_2^2)\right]^{1/(\tau-2)}\right\}\right)$, where k_τ is constant of the embedding $L^p(\Omega) \hookrightarrow L^\tau(\Omega)$. Put

$$S_\rho(Z) = \{z \in Z :\ \|z\| = \rho\}$$

and observe that for every $z \in S_\rho(Z)$

$$\|z\|_{C_0(\overline{\Omega})} \leq \bar{\delta}, \quad \|z\|_{C_0^1(\overline{\Omega})} \leq \varepsilon, \tag{3.15}$$

in addition to

$$v_\lambda(x) \leq z(x) \leq u_\lambda(x), \quad \forall x \in \overline{\Omega}, \tag{3.16}$$

as well as

$$I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(z) < 0. \tag{3.17}$$

In fact, if $z \in S_\rho(Z)$ one has

$$\|z\|_{C_0(\overline{\Omega})} \le \|z\|_{C_0^1(\overline{\Omega})} \le d_2\|z\| = d_2\rho \le \varepsilon < \bar{\delta}$$

and (3.15) holds. Moreover, taking in mind (3.14),

$$u_\lambda - z \ge 0, \quad -v_\lambda + z \ge 0,$$

namely (3.16) holds. Finally,

$$\begin{aligned}
I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(z) &= \int_\Omega G_0(|\nabla z(x)|)\, dx + \frac{1}{2}\|\nabla z\|_2^2 - \int_\Omega F_\lambda(x, z(x))\, dx \\
&\le \frac{2\sigma}{\tau}\|\nabla z\|_\tau^\tau + \frac{k_2^2}{2}\|z\|^2 - \frac{M}{2}\|z\|_2^2 \\
&\le \frac{2\sigma}{\tau}k_\tau^\tau\|z\|^\tau + \frac{1}{2}\left(k_2^2 - Md_3^2\right)\|z\|^2 \\
&= \rho^2\left(\frac{2\sigma}{\tau}k_\tau^\tau\rho^{\tau-2} + \frac{k_2^2 - Md_3^2}{2}\right) < 0
\end{aligned}$$

and (3.17) holds too.

Put $\widehat{z}_1 = \rho z_1$ and $\widehat{z}_2 = \rho z_2$. It is obvious that $\widehat{z}_i \in S_\rho(Z)$ $(i = 1, 2)$. Moreover,

$$m_{1,\lambda} = I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(v_\lambda) = I_{\underline{u}_\lambda}^0(v_\lambda) \le I_{\underline{u}_\lambda}^0(\widehat{z}_1) = I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(\widehat{z}_1) = \widehat{\mu}_1 < 0.$$

From (3.11) it follows that

$$m_{1,\lambda} < \widehat{\mu}_1,$$

and every $\nu \in (m_{1,\lambda}, \widehat{\mu}_1)$ is not a critical value of $I_{\underline{u}_\lambda}^0$ (see also Lemma 2.6). If

$$S(I_{\underline{u}_\lambda}^0, \widehat{\mu}_1) = \{w \in W_0^{1,p}(\Omega) :\ I_{\underline{u}_\lambda}^0(w) \le \widehat{\mu}_1\},$$

applying the second deformation lemma to $I_{\underline{u}_\lambda}^0$, there exists a suitable $\eta \in C^0([0,1] \times S(I_{\underline{u}_\lambda}^0, \widehat{\mu}_1), S(I_{\underline{u}_\lambda}^0, \widehat{\mu}_1))$ such that $\eta(0, w) = w$, $\eta(1, w) = v_\lambda$ for every $w \in S(I_{\underline{u}_\lambda}^0, \widehat{\mu}_1)$ and $I_{\underline{u}_\lambda}^0(\eta(t, w)) \le I_{\underline{u}_\lambda}^0(w)$ for every $t \in [0, 1]$ and $w \in S(I_{\underline{u}_\lambda}^0, \widehat{\mu}_1)$. Let us define the path $\eta_- : [0, 1] \to W_0^{1,p}(\Omega)$ by putting

$$\eta_-(t)(x) = \min\{\eta(t, \widehat{z}_1)(x), 0\}$$

for every $t \in [0, 1]$, $x \in \Omega$. Obviously $\eta_- \in C^0([0, 1], W_0^{1,p}(\Omega))$ such that $\eta_-(0) =$

$\widehat{z}_1$ and $\eta_-(1)=v_\lambda$. Moreover, for every $t\in[0,1]$ one has

$$\begin{aligned}
I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(\eta_-(t)) &= \int_{\{\eta(t,\widehat{z}_1)<0\}} G_0(|\eta(t,\widehat{z}_1)(x)|)\,dx + \frac{1}{2}\int_{\{\eta(t,\widehat{z}_1)<0\}} |\nabla\eta(t,\widehat{z}_1)(x)|^2\,dx \\
&\qquad - \int_{\{\eta(t,\widehat{z}_1)<0\}} F_\lambda(x,\eta(t,\widehat{z}_1)(x))\,dx \\
&= \int_{\{\eta(t,\widehat{z}_1)<0\}} G_0(|\eta(t,\widehat{z}_1)(x)|)\,dx + \frac{1}{2}\int_{\{\eta(t,\widehat{z}_1)<0\}} |\nabla\eta(t,\widehat{z}_1)(x)|^2\,dx \\
&\qquad - \int_{\Omega} F^0_{\lambda_{\underline{u}_\lambda}}(x,\eta(t,\widehat{z}_1)(x))\,dx \\
&\le I^0_{\underline{u}_\lambda}(\eta(t,\widehat{z}_1)) \le I^0_{\underline{u}_\lambda}(\widehat{z}_1) = I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(\widehat{z}_1) = \widehat{\mu}_1 < 0
\end{aligned}$$

In similar way one can prove the existence of a path $\eta_+\in C^0([0,1],W_0^{1,p}(\Omega))$ such that $\eta_+(0)=\widehat{z}_2$, $\eta_+(1)=u_\lambda$ and $I_{\underline{u}_\lambda}^{\overline{u}_\lambda}(\eta_+(t))\le\widehat{\mu}_2<0$ for every $t\in[0,1]$.

Take a path $\eta_Z\in C^0([0,1],W_0^{1,p}(\Omega))$ having range in the (arc-wise connected) set $S_\rho(Z)$ and jointing $\widehat{z}_1$ and $\widehat{z}_2$. Finally, the juxtaposition of η_-, η_Z and η_+ produces the path $\widehat{\eta}$ stated in (3.13) and the proof is complete. □

4 Nodal solutions

We devote this section to a deeper analysis with the aim of pointing out a sign information on the third solution established in Theorem 3.3.

We will assume a slightly more restrictive condition on the nonlinear term f as well as on the function $a(\cdot)$ related to the differential operator. In particular, we will replace $H(a)(iv)$ with

$(iv)'$ *There exists $\tau\in(1,p)$ such that the function $t\mapsto G_0(t^{1/\tau})$ is convex and*

$$\lim_{t\searrow 0}\frac{G_0(t)}{t^\tau}=0.$$

Moreover, we will require

$H'(f)$: *For every $\lambda>0$, $f_\lambda\colon\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function, such that $f_\lambda(x,0)=0$ for almost all $x\in\Omega$ and*

$(i)'$ *there exists $\tilde{c}>0$ such that for every $\lambda>0$*

$$|f_\lambda(x,s)|\leqslant a_\lambda(x)+\tilde{c}|s|^{r_\lambda-1}$$

for almost all $x\in\Omega$, all $s\in\mathbb{R}$, with $a_\lambda\in L^\infty(\Omega)_+$ and $\|a_\lambda\|_\infty\longrightarrow 0$ as $\lambda\searrow 0$, as well as $p\le r_\lambda<+\infty$ and $r_\lambda\longrightarrow r\ge p$ as $\lambda\searrow 0$;

$(ii)'$ *for every $\lambda>0$, there exist $\gamma_\lambda\in(1,\min\{\tau,2\})$, $\theta_\lambda>0$ such that*

$$\liminf_{s\to 0}\frac{f_\lambda(x,s)}{|s|^{\gamma_\lambda-2}s}=\theta_\lambda$$

uniformly for a.a. $x\in\Omega$.

Theorem 4.1. *Assume that hypotheses $H(a)(i)-(iii)$, $H(a)(iv)'$ and $H'(f)$ hold. Then there exists $\lambda^*>0$ such that for every $\lambda\in(0,\lambda^*)$ problem $(P_{f,\lambda})$ admits at least three distinct nontrivial smooth solutions*

$$u^*_\lambda\in\ D_+,\ v^*_\lambda\in\ -D_+\quad \textit{and}\quad w_\lambda\in]v^*_\lambda,u^*_\lambda[\cap C^1_0(\overline{\Omega})\setminus\{0\}\ \textit{is nodal.}$$

Proof. Since $H'(f)$ implies $H(f)$, bearing in mind Remark 2.12, one can follows the same arguments of Theorem 3.3, and conclude that there exists $\lambda^*>0$ such that for every $\lambda\in(0,\lambda^*)$ problem $(P_{f,\lambda})$ admits at least two nontrivial constant sign smooth solutions

$$u_\lambda\ \in\ D_+\quad\text{and}\quad v_\lambda\ \in\ -D_+.$$

In particular, recall that

$$\underline{u}_\lambda\le v_\lambda\le 0\le u_\lambda\le\overline{u}_\lambda,$$

with $\overline{u}_\lambda=\xi^\lambda_0 e$ and $\underline{u}_\lambda=-\xi^\lambda_0 e$ for some $\xi^\lambda_0\in(0,1)$ and $e\in C^2(\overline{\Omega})$ is the unique solution of (3.1). Fix $\lambda\in(0,\lambda^*)$ and consider the nonempty sets

$$S_+(\lambda)=\{u\in W^{1,p}_0(\Omega):\ 0\le u\le\overline{u}_\lambda\text{ and }u\text{ is a solution of problem }(P_{f,\lambda})\},$$

$$S_-(\lambda)=\{u\in W^{1,p}_0(\Omega):\ \underline{u}_\lambda\le u\le 0\text{ and }u\text{ is a solution of problem }(P_{f,\lambda})\}.$$

The rest of proof is split in several steps.

Step 1. There exist $\hat{u}_\lambda\in D_+\cap[0,\overline{u}_\lambda]$ and $\hat{v}_\lambda\in -D\cap[-\overline{u}_\lambda,0]$ such that

$$v\le\hat{v}_\lambda,\quad \hat{u}_\lambda\le u$$

for every $v\in S_-(\lambda)$, $u\in S_+(\lambda)$.

Step 2. $S_+(\lambda)$ and $S_-(\lambda)$ are downward and upward directed respectively.

Step 3. $S_+(\lambda)$ admits a minimal element u^*_λ and $S_-(\lambda)$ admits a maximal element v^*_λ. In particular, $u^*_\lambda\in D_+$ is the smallest positive solution of $(P_{f,\lambda})$ and $v^*_\lambda\in -D_+$ is biggest negative solution of $(P_{f,\lambda})$.

Step 4. Problem $(P_{f,\lambda})$ admits a nontrivial solution w_λ in the ordered interval $[v^*_\lambda,u^*_\lambda]$.

Step 5. The function w_λ is a nodal solution of $(P_{f,\lambda})$.

Proof of Step 1. Let us prove the existence of $\hat{u}_\lambda$. Assumption $H'(f)$ assures that for every $M>0$ there exists $C_M>0$ such that for every $s\in[0,M]$ and uniformly for a.a. $x\in\Omega$

$$sf_\lambda(x,s)\ge\frac{\theta}{2}s^{\gamma_\lambda}-c_M s^{r_\lambda}. \tag{4.1}$$

Indeed, let $\delta \in (0, M)$ be such that

$$sf_\lambda(x,s) \geq \frac{\theta_\lambda}{2} s^{\gamma_\lambda}, \tag{4.2}$$

for every $s \in [0,\delta]$ and uniformly a.a. $x \in \Omega$. Let $c_M > 0$ be such that

$$(c_M - \tilde{c})\delta^r \geq \frac{\theta_\lambda}{2} M^{\gamma_\lambda} + \|a_\lambda\|_\infty M.$$

A direct computation shows that for all $s \in [\delta, M]$ and uniformly for a.a. $x \in \Omega$

$$(c_M - \tilde{c})s^r \geq (c_M - \tilde{c})\delta^r \geq \frac{\theta_\lambda}{2} M^{\gamma_\lambda} + \|a_\lambda\|_\infty M \geq \frac{\theta_\lambda}{2} s^{\gamma_\lambda} + a_\lambda(x)s$$

namely

$$sf(x,s) - sa_\lambda(x) - \tilde{c}s^{r_\lambda} \geq \frac{\theta_\lambda}{2} s^{\gamma_\lambda} - c_M s^{r_\lambda}. \tag{4.3}$$

Conditions (4.2) and (4.3) lead to (4.1).

Fix $M = \|\overline{u}_\lambda\|_\infty$ and claim that the function $\hat{u}_\lambda$ is the unique positive solution in $[0, \overline{u}_\lambda]$ of the auxiliary problem

$$\begin{cases} -\operatorname{div} a(\nabla u) - \Delta u = \frac{\theta_\lambda}{2}|u|^{\gamma_\lambda - 2}u - c_M|u|^{r_\lambda - 2}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{AP}$$

Put

$$J_\lambda(w) = \int_\Omega G(\nabla w)\,dx + \frac{1}{2}\|\nabla w\|_2^2 - \int_\Omega P_\lambda(x, w(x))\,dx$$

for every $w \in W_0^{1,p}$, where $P_\lambda(x,s) = \int_0^s p_\lambda(x,s)\,ds$ and

$$p_\lambda(x,s) = \begin{cases} 0 & \text{if } s \leq 0 \\ \frac{\theta_\lambda}{2} s^{\gamma_\lambda - 1} - c_M s^{r_\lambda - 1} & \text{if } 0 < s < \overline{u}_\lambda(x) \\ \frac{\theta_\lambda}{2} \overline{u}_\lambda^{\gamma_\lambda - 1}(x) - c_M \overline{u}_\lambda^{r_\lambda - 1}(x) & \text{if } s \geq \overline{u}_\lambda(x). \end{cases}$$

Arguing as in (3.3) and exploiting (4.1) one has that for a.a. $x \in \Omega$

$$A(\overline{u}_\lambda) - \Delta\overline{u}_\lambda \geq f_\lambda(x, \overline{u}_\lambda) \geq p_\lambda(x, \overline{u}_\lambda).$$

Hence $\overline{u}_\lambda$ is a super-solution of problem (AP). Moreover, it is clear that J_λ is a C^1, coercive and sequentially weakly lower semicontinuous functional. Thus, there exists $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$J_\lambda(\hat{u}_\lambda) = \inf_{W_0^{1,p}(\Omega)} J_\lambda(w).$$

A simple rearrangement of the proof of Lemma 2.11 assures that, since $\gamma_\lambda < \min\{\tau, 2\} < p < r_\lambda$,

$$J_\lambda(\hat{u}_\lambda) < 0,$$

so $\hat{u}_\lambda \neq 0$. Moreover, from Lemma 2.6 it follows that

$$0 \leq \hat{u}_\lambda \leq \overline{u}_\lambda,$$

that is $\hat{u}_\lambda$ is a nontrivial, positive solution of (AP) and the regularity theory assures that $\hat{u}_\lambda \in (C_0^1(\overline{\Omega}))_+$. Moreover, observe that (again recall that $p \leq r_\lambda$)

$$\operatorname{div} a(\nabla \hat{u}_\lambda) + \Delta \hat{u}_\lambda \leq c_M \hat{u}_\lambda^{r_\lambda - 1} \leq c_M \|\hat{u}_\lambda\|_\infty^{r_\lambda - p} \hat{u}_\lambda^{p-1}.$$

We can apply Theorem 2.9 and conclude that $\hat{u}_\lambda \in D_+$.

Let us verify the uniqueness. Observe that it is not restrictive assume that the number τ in assumption $H(a)(iv)'$ is such that $1 < \tau < 2$. Following the idea developed in [8], consider the functional $g_\tau : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$g_\tau(u) = \begin{cases} \int_\Omega G(\nabla u^{1/\tau}) dx + \frac{1}{2}\|\nabla u^{1/\tau}\|_2^2 & \text{if } u \geq 0,\ u^{1/\tau} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} g_\tau = \{u \in L^1(\Omega) :\ g_\tau(u) < +\infty\}$ and let $h \in [0,1]$. We set

$$z = \left((1-h)u_1 + hu_2\right)^{1/\tau},\ v_1 = u_1^{1/\tau},\ v_2 = u_2^{1/\tau}.$$

Thanks to [8, Lemma 1], we have

$$|\nabla z(x)| \leq \left[(1-h)|\nabla v_1(x)|^\tau + h|\nabla v_2(x)|^\tau\right]^{1/\tau} \text{ a.e. in } \Omega.$$

Thus, by the monotonicity of G_0 and condition $H(a)(iv)'$, as well as convexity of $\delta_\tau(t) = \frac{1}{2}t^{2/\tau}$ (remember that we supposed $\tau \in (1,2)$) one has

$$\begin{aligned} G(\nabla z(x)) &= G_0(|\nabla z(x)|) \leq G_0\left(\left((1-h)|\nabla v_1(x)|^\tau + h|\nabla v_2(x)|^\tau\right)^{1/\tau}\right) \\ &\leq (1-h)G_0(|\nabla v_1(x)|) + hG_0(|\nabla v_2(x)|) \end{aligned}$$

as well as

$$\begin{aligned} \frac{1}{2}|\nabla z(x)|^2 &\leq \frac{1}{2}\left((1-h)|\nabla v_1(x)|^\tau + h|\nabla v_2(x)|^\tau\right)^{2/\tau} \\ &\leq \frac{1-h}{2}|\nabla v_1(x)|^2 + \frac{h}{2}|\nabla v_2(x)|^2 \end{aligned}$$

for a.a. $x \in \Omega$, namely g_τ is convex.

Moreover, applying the Fatou's lemma one has that g_τ is lower semicontinuous.

Suppose that $u \in W^{1,p}(\Omega)$ is another positive solution in $[0, \overline{u}_\lambda]$ of problem (AP). Following the previous reasoning we have $u \in D_+$. Then, for every $\varphi \in C^1(\bar{\Omega})$ and $s \in (-1,1)$ with $|s|$ small, we have

$$u^\tau + s\varphi \in D_+ \ \cap \operatorname{dom} g_\tau.$$

Therefore, the Gateâux derivative of g_τ at u^τ in the direction φ can be computed using the chain rule

$$(g_\tau)'(u^\tau)(\varphi) = \frac{1}{\tau}\int_\Omega \frac{-\mathrm{div}a(\nabla u) - \Delta u}{u^{\tau-1}}\varphi\, dx$$

for all $\varphi \in W^{1,p}(\Omega)$ (we have used the density of $C^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$). Clearly, the preceding condition holds also for the solution $\hat{u}_\lambda$. Hence, the convexity of g_τ implies that $(g_\tau)'(\cdot)$ is monotone. Thus,

$$0 \le \int_\Omega \left[\frac{-\mathrm{div}a(\nabla u) - \Delta u}{u^{\tau-1}} + \frac{\mathrm{div}a(\nabla \hat{u}_\lambda) + \Delta \hat{u}_\lambda}{\hat{u}_\lambda^{\tau-1}}\right](u^\tau - \hat{u}_\lambda^\tau)dx =$$

$$= \int_\Omega \left[\frac{\theta_\lambda}{2}\left(\frac{1}{u^{\tau-\gamma_\lambda}} - \frac{1}{\hat{u}_\lambda^{\tau-r_\lambda}}\right) + c_M\left(\hat{u}_\lambda^{r_\lambda-\tau} - u^{r_\lambda-\tau}\right)\right](u^\tau - \hat{u}_\lambda^\tau)\, dx. \tag{4.4}$$

Taking in mind that, $\gamma_\lambda < \tau < r_\lambda$, from (4.4) it follows that

$$u = \hat{u}_\lambda,$$

and the uniqueness is proved.

Let us conclude by verifying that for every $u \in S_+(\lambda)$ one has

$$\hat{u}_\lambda \le u. \tag{4.5}$$

To this end, as above use a truncation argument putting

$$\tilde{J}_\lambda(w) = \int_\Omega G(\nabla w)\, dx + \frac{1}{2}\|\nabla w\|_2^2 - \int_\Omega \tilde{P}_\lambda(x, w(x))\, dx$$

for every $w \in W_0^{1,p}(\Omega)$, where $\tilde{P}_\lambda(x,s) = \int_0^s \tilde{p}_\lambda(x,t)\, dt$ and

$$\tilde{p}_\lambda(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ \frac{\theta_\lambda}{2}s^{\gamma_\lambda-1} - c_M s^{r_\lambda-1} & \text{if } 0 < s < u(x) \\ \frac{\theta_\lambda}{2}u^{\gamma_\lambda-1}(x) - c_M u^{r_\lambda-1}(x) & \text{if } s \ge u(x). \end{cases}$$

Observe that u is a super-solution of problem (AP) and that, by the Weierstrass theorem, $\tilde{J}_\lambda$ admits a nontrivial global minimizer $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$. In particular,

$$0 \le \tilde{u}_\lambda \le u \le \overline{u}_\lambda, \tag{4.6}$$

namely $\tilde{u}_\lambda$ is a positive solution of (AP), that is, in view of the uniqueness property, $\hat{u}_\lambda = \tilde{u}_\lambda$ and (4.5) follows from (4.6).

The existence of $\hat{v}_\lambda$ is proved similarly thanks to the symmetry of problem (AP). In particular, $\hat{v}_\lambda = -\hat{u}_\lambda$.

Proof of Step 2. Let us verify that $S_+(\lambda)$ is downward directed, namely that

for all u, $v \in S_+(\lambda)$ there exists $w \in S_+(\lambda)$ with $w \le u$, $w \le v$.

Fix u, $v \in S_+(\lambda)$. We first note that arguing as in [22, Lemma 3] and exploiting the monotonicity of A and $-\Delta$ one has that $\overline{w} = \min\{u, v\}$ is a super-solution of $(P_{f,\lambda})$ and clearly $\overline{w} \le \overline{u}_\lambda$. In particular, $\overline{w}$ is also a super-solution of (AP) and $\hat{u}_\lambda \le \overline{w} \le \overline{u}_\lambda$. Truncating with $\overline{w}$, the functional

$$I_0^{\overline{w}}(u) = \int_\Omega G(\nabla w)\,dx + \frac{1}{2}\|\nabla w\|_2^2 - \int_\Omega (F_\lambda)_0^{\overline{w}}(x, w(x))\,dx,$$

by the Weierstrass theorem, attains its negative minimum at some $w \in W_0^{1,p}(\Omega)$. The usual comparison arguments permit to conclude that $0 \le w \le \overline{w}$ and $w \in S_+(\lambda)$.

Similarly one can prove the analogous property of $S_-(\lambda)$.

Proof of Step 3. Let us verify the existence of u_λ^*. Consider a chain $\mathcal{C}$ (that is a totally ordered subset) of $S_+(\lambda)$. Thus, there exists a decreasing sequence $\{u_n\}$ in $\mathcal{C}$ (view [9, pag. 336]) such that $\tilde{u} = \inf \mathcal{C} = \inf_{n\in\mathbb{N}} u_n$. Hence, $u_n \to \tilde{u}$ for a.a. in Ω and it is clear that (see Step 1) $\hat{u}_\lambda \le \tilde{u} \le \overline{u}_\lambda$. Moreover, since

$$A(u_n) - \Delta u_n = f_\lambda(x, u_n) \tag{4.7}$$

in view of Lemma 2.3 and assumption $H'(f)$ one has there exists $C > 0$ such that

$$\frac{c_1}{p-1}\|\nabla u_n\|_p^p + \|\nabla u_n\|_2^2 = \int_\Omega f(x, u_n)u_n\,dx \le C$$

for every $n \in \mathbb{N}$, namely $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, we can suppose that

$$u_n \to \tilde{u} \text{ weakly in } W_0^{1,p}(\Omega) \text{ and } u_n \to \tilde{u} \text{ in } L^p(\Omega). \tag{4.8}$$

At this point, being $0 \le u_n \le \overline{u}_\lambda$, assumption $H'(f)$, the monotonicity of $-\Delta$ and the convergence properties of $\{u_n\}$ implies that

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \le 0.$$

Thus, from Proposition 2.5 we achieve the strong convergence of $\{u_n\}$ to $\tilde{u}$ and, as a direct consequence, passing to the limit in (4.7), $\tilde{u} \in S_+(\lambda)$ and $\mathcal{C}$ admits minimum. The Kuratowski-Zorn's lemma assures that $S_+(\lambda)$ has a minimal element u_λ^* which is nontrivial ($\hat{u}_\lambda \le u_\lambda^*$ as seen in Step 1). We conclude verifying that u_λ^* is the smallest positive solution of $(P_{f,\lambda})$ in the ordered interval $[0, \overline{u}_\lambda]$. Let $u \in S_+(\lambda)$. Since $S_\lambda(\lambda)$ is downward directed there exists $\breve{u} \in S_+(\lambda)$ such that $\breve{u} \le u$ and $\breve{u} \le u_\lambda^*$, but the minimality of u_λ^* implies that $u_\lambda^* = \breve{u} \le u$ and we are done.

The proof of the existence of the maximal element v_λ^* that is the biggest negative solution in the ordered interval $[\underline{u}_\lambda, 0]$ is similar.

Proof of Step 4. First observe that the minimality of u_λ^* and the maximality of v_λ^* imply that they are global minimizers of the functionals $I_0^{u_\lambda^*}$ and $I_{v_\lambda^*}^0$

respectively. At this point, the existence of a third nontrivial solution $w_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$ of problem $(P_{f,\lambda})$ can be obtained arguing exactly as in the proof of Theorem 3.3 with u_λ^* and v_λ^* instead of u_λ and v_λ respectively, as well as u_λ^* and v_λ^* in place of $\overline{u}_\lambda$ and $\underline{u}_\lambda$, so that the functionals $I_0^{\overline{u}_\lambda}$ and $I_{\underline{v}_\lambda}^0$ are here replaced by $I_0^{u_\lambda^*}$ and $I_{v_\lambda^*}^0$.

Proof of Step 5. Since $w_\lambda \in (v_\lambda^*, u_\lambda^*) \setminus \{0\}$ it cannot be of constant sign by virtue of the extremality properties of u_λ^* and v_λ^*.
The proof is complete. □

References

[1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, *Nodal solutions for $(p, 2)$-equations*, Trans. Amer. Math. Soc., 367 (2015), 7343–7372.

[2] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14 (1973), 349–381.

[3] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani, *Solitons in several space dimensions: Derrick's problem and infinitely many solutions*, Arch. Ration. Mech. Anal., 154 (2000), 297–324.

[4] P. Candito, S. Carl, R. Livrea, *Critical points in open sublevels and multiple solutions for parameter-depending quasilinear elliptic equations*, Adv. Differential Equations, 19 (2014), 1021–1042.

[5] P. Candito, G. D'Aguì, N. S. Papageorgiou, *Nonlinear noncoercive Neumann problems with a reaction concave near the origin*, Topol. Methods Nonlinear Anal., 47 (2016), 289–317.

[6] L. Cherfils, Y. Il′yasov, *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Commun. Pure Appl. Anal., 4 (2005), 9–22.

[7] S. Cingolani, M. Degiovanni, *Nontrivial solutions for p-Laplace equations with right hand side having p-linear growth at infinity*, Comm. Partial Differential Equations, 30 (2005), 1191–1203.

[8] J.I. Diaz, J.E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math., 305 (1987), 521–524.

[9] N. Dunford, J.T. Schwartz, *Linear Operators. I. General Theory*, Interscience Publishers, Inc., New York, (1958).

[10] X. Fan, D. Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal. 36 (1999), 295-318.

[11] L. Gasiński, N.S. Papageorgiou, Nonlinear Analysis. Series in Mathematical Analysis and Applications, 9. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[12] L. Gasiński, N.S. Papageorgiou, *Nodal and multiple constant sign solutions for resonant p-Laplacian equations with a nonsmooth potential*, Nonlinear Anal., 71:11 (2009), 5747–5772.

[13] L. Gasiński, N.S. Papageorgiou, *Multiple solutions for nonlinear coercive problems with a nonhomogeneous differential operator and a nonsmooth potential*, Set-Valued Var. Anal., 20 (2012), 417–443.

[14] L. Gasiński, N.S. Papageorgiou, *A pair of positive solutions for (p, q)-equations with combined nonlinearities*, Commun. Pure Appl. Anal., 13 (2014), 203–215.

[15] L. Gasiński, N.S. Papageorgiou, *Dirichlet (p, q)-equations at resonance*, Discrete Contin. Dyn. Syst., Series A, 34:5 (2014), 2037–2060.

[16] L. Gasiński, N.S. Papageorgiou, *Multiple solutions for $(p, 2)$-equations with resonance and concave terms*, Results Math., 74:79 (2019), doi: 10.1007/s00025-019-0996-9.

[17] T. He, P. Guo, Y. Huang, Y. Lei, *Multiple nodal solutions for nonlinear nonhomogeneous elliptic problems with a superlinear reaction*, Nonlinear Anal. Real World Appl. 42 (2018), 207-219.

[18] L. Klimczak, *Two constant sign solutions for a nonhomogeneous Neumann boundary value problem*, Ann. Univ. Paedagog. Crac. Stud. Math., 14 (2015), 47–62.

[19] S. Th. Kyritsi, D. O'Regan, N. S. Papageorgiou, *Existence of multiple solutions for nonlinear Dirichlet problems with a nonhomogeneous differential operator*, Adv. Nonlinear Stud. 10 (2010), no. 3, 631–657.

[20] O.A. Ladyzhenskaya, N. Uraltseva, Linear and Quasilinear Elliptic Equations, Vol. 46 of Mathematics in Science and Engineering, Academic Press, New York, 1968.

[21] G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations, 16 (1991), 311–361.

[22] D. Motreanu, V.V. Motreanu, N.S. Papageorgiou, *A unified approach for multiple constant sign and nodal solutions*, Advances Diff. Eqts. 12 (2007), 1363-1392.

[23] D. Mugnai, N.S. Papageorgiou, *Wang's multiplicity result for superlinear (p, q)-equations without the Ambrosetti-Rabinowitz condition*, Trans. Amer. Math. Soc., 366 (2014), 4919–4937.

[24] N.S. Papageorgiou, V.D. Rădulescu, *Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance*, Appl. Math. Optim., 69 (2014), 393–430.

[25] N.S. Papageorgiou, V.D. Rădulescu, *Resonant $(p, 2)$-equations with asymmetric reaction*, Anal. Appl., 13:5 (2015), 481–506.

[26] N.S. Papageorgiou, P. Winkert, *Resonant $(p, 2)$-equations with concave terms*, Appl. Anal., 94:2 (2015), 342–360.

[27] N.S. Papageorgiou, P. Winkert, *Nonlinear nonhomogeneous Dirichlet equations involving a superlinear nonlinearity*, Results Math. 70 (2016), no. 1-2, 3179.

[28] P. Pucci, J. Serrin, *A mountain pass theorem*, J. Differential Equations, 60 (1985), 142–149.

[29] P. Pucci, J. Serrin, The Maximum Principle, in: Progress in Nonlinear Differential Equations and their Applications, 73, Birkhäuser Verlag, Basel, 2007.

[30] M. Sun, *Multiplicity of solutions for a class of the quasilinear elliptic equations at resonance*, J. Math. Anal. Appl., 386 (2012), 661–668.

[31] M. Sun, M. Zhang, J. Su, *Critical groups at zero and multiple solutions for a quasilinear elliptic equation*, J. Math. Anal. Appl., 428 (2015), 696–712.

[32] G.M. Troianiello, Elliptic Differential Equations and Obstacle Problems, Plenum Press, New York, 1987.

[33] H. Wilhelmsson, Explosive instabilities of reaction-diffusion equations, Phys. Rev. A (3), 36:2 (1987), 965–966.

[34] D. Yang, C. Bai, *Nonlinear elliptic problem of 2-q-Laplacian type with asymmetric nonlinearities*, Electron. J. Differential Equations, 170 (2014) 1–13.

[35] V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR-Izv., 29 (1987), 33–66.

(P. Candito) Department DICEAM University of Reggio Calabria, Via Graziella (Feo Di Vito), Reggio Calabria (Italy)
E-mail address: `pasquale.candito@unirc.it`

(L. Gasinski) Department of Mathematics, Pedagogical University of Cracow, Podchorazych 2, 30-084 Cracow (Poland)
E-mail address: `leszek.gasinski@up.krakow.pl`

(R. Livrea) Department of Mathematics and Computer Science, University of Palermo, Via Archirafi, 90123 Palermo (Italy)
E-mail address: `roberto.livrea@unipa.it`