Reducing the Number of Elements in Phase-Only Reconfigurable Arrays Generating Sum and Difference Patterns

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Abstract—**A new strategy for the synthesis of one-dimensional reconfigurable sparse arrays generating sum and difference power patterns is presented. In order to reduce the weight and complexity of the beam forming network, the design procedure provides solutions in which only one set of amplitude coefficients is required and just a π phase shift is needed for half array to switch between the two radiation modes. The synthesis is cast as the solution of a Linear Programming problem plus a local optimization procedure where the sparseness of the array layout is achieved by properly exploiting the principles of Compressive Sensing. Examples concerning the design of monopulse arrays show the effectiveness of the procedure, and comparisons with respect to equi-spaced array solutions are included to quantify the savings with respect to conventional architectures.**

***Index Terms*— Compressive sensing, maximally-sparse arrays, monopulse radars, power synthesis, reconfigurable patterns.**

I. Introduction and Motivations

ARRAY antennas able to provide both sum and difference patterns, the former having one main lobe along the target direction and the latter exhibiting a null in the same direction, are strongly required in monopulse radar systems [1]. As a matter of fact, a very large number of algorithms have been proposed in the last sixty years for the synthesis of such radiating systems (see [2] and the references cited therein).

Notably, in almost all the available procedures the synthesis is solved by exploiting *uniformly-spaced* arrays. In fact, the adoption of aperiodic arrays for the generation of *multiple patterns* appears to be limited to those cases wherein pencil and/or shaped beams are concerned [3],[4], and the chance of designing reconfigurable sparse arrays able to radiate *difference* patterns has not been addressed yet. This circumstance represents an important limitation, as the 'lack of periodicity' in the array layout would offer several advantages with respect to the 'classical' equi-spaced architectures, including the decrease of the sidelobes level and the enhancement of the antenna bandwidth [3]-[6].

In the attempt of filling such a gap, this contribution presents a new approach for the synthesis of one-dimensional arrays being at the same time *sparse and reconfigurable* between *sum and difference* radiation modes. Notably, the provided solutions exhibit two key features which allow to minimize the weight and complexity of the Beam Forming Network (BFN), as requested in advanced monopulse antenna systems to be used in challenging radar applications [1]:

- reconfiguration simplicity, obtained by splitting the antenna into two symmetric parts where just a π phase shift on one of them is required to change radiation modality;
- minimization of the number of array elements, by exploiting at best the Compressive Sensing (CS) theory [7].

Moreover, by taking maximum advantage from the best results available in the independent synthesis of sum [8] and difference [9] patterns, the design procedure allows enforcing arbitrary upper-bound constraints on both the generated power patterns while being extremely fast and effective. In fact, the overall synthesis problem is stated as a *Linear Programming* (LP) one plus a fast local optimization procedure.

In the following, the synthesis approach is mathematically formulated in Section II, while interesting examples supporting the given theory are shown in Section III. Conclusions follow.

II. Mathematical Formulation

The devised synthesis procedure consists of three consecutive steps. The first one is presented in Subsection II.A and is devoted to provide a preliminary array layout, while the second and third ones are respectively discussed in Subsections II.B and II.C and are aimed at refining the solution coming out from the first step while guaranteeing the feasibility and practical implementation of the array layout.

A. First Step: CS-driven Synthesis

The preliminary array locations and excitations are synthesized by applying a CS-driven algorithm to a 'reference', dense and periodic, starting array layout. To this end, the strategies proposed in [5]-[7] are properly exploited.

Towards this aim, let us assume an array composed by $2N$ elements disposed along the x-axis with an inter-element constant distance equal to d. Moreover, let us assume that the array locations are symmetrical with respect to the origin of the x-axis. Finally, let us enumerate by subscripts $\{-N, \dots, -1\}$ and $\{1, \dots, N\}$ the excitations of the elements located on the left and on the right with respect to the array center, respectively. Under these assumptions, by denoting with $\beta=2\pi/\lambda$ the wavenumber (λ being the wavelength) and neglecting any

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element factor, the sum and difference patterns will be respectively given by:

$$F^{\Sigma}(u)=\sum_{n=-N}^{N} a_n^{\Sigma} e^{j\left[n-\frac{\mathrm{sgn}(n)}{2}\right]d\beta u} \tag{1}$$

$$F^{\Delta}(u)=\sum_{n=-N}^{N} a_n^{\Delta} e^{j\left[n-\frac{\mathrm{sgn}(n)}{2}\right]d\beta u} \tag{2}$$

with u=sinθ (θ being the observation angle with respect to the boresight direction), $a_{-N}^{\Sigma},\ldots,a_{N}^{\Sigma}$, $a_{-N}^{\Delta},\ldots,a_{N}^{\Delta}$ denoting the excitations for the two modes, and *sgn* being the sign function. Considering without any loss of generality u=0 as target direction, we identify the two sets of excitations by solving the following optimization problem in the unknowns $a_1^{\Sigma},\ldots,a_N^{\Sigma}$:

$$\min_{a_1^{\Sigma},\ldots,a_N^{\Sigma}}\left[\sum_{n=1}^{N}\left|a_n^{\Sigma}\right|\right] \tag{3}$$

subject to:

$$a_n^{\Delta}=a_n^{\Sigma},\quad a_{-n}^{\Sigma}=a_n^{\Sigma},\quad a_{-n}^{\Delta}=-a_n^{\Delta}\qquad n=1,\ldots,N \tag{4}$$

$$-\sqrt{UB^{\Sigma}(u)}\leq F^{\Sigma}(u)\leq\sqrt{UB^{\Sigma}(u)}\qquad u\in\Omega^{\Sigma} \tag{5}$$

$$-\sqrt{UB^{\Delta}(u)}\leq jF^{\Delta}(u)\leq\sqrt{UB^{\Delta}(u)}\qquad u\in\Omega^{\Delta} \tag{6}$$

$$\left.F^{\Sigma}(u)\right|_{u=0}\geq\tau \tag{7}$$

$$j\left.\frac{\partial F^{\Delta}(u)}{\partial u}\right|_{u=0}\geq\eta \tag{8}$$

being τ and η real non-negative constants set by the designer.

In the above formulation, constraints (4) ensure that the excitation distributions of the sum and difference beams are characterized by an *even* and *odd* symmetry, respectively. This property, in conjunction with the Fourier-Transform nature of (1),(2) and with the circumstance that the optimal excitations coming out from the synthesis of *single* sum or difference patterns are always *purely real* [8],[9], entails that:

1. a phase-only shift of π around the origin of the x-axis on the array excitations allows switching between the two radiation modalities;
2. the objective function (3) results equivalent to the ℓ_1-norm of *both* the sum and difference excitation sets. By virtue of the relations in [5],[6] and the theory in [7] concerning the ℓ_1-norm relaxation of ℓ_0-norm-based optimization problems, the minimization of such functional leads to the minimization of the number of active elements;
3. (1),(2) result *purely real* and a *purely imaginary* functions of u, respectively.

By virtue of point 3 above, constraints (5),(6) allow enforcing arbitrary upper-bound constraints on the two *power* patterns in the Ω^{Σ} and Ω^{Δ} spatial regions. Point 3 also entails that constraints (7),(8) guarantee in the target direction a sum-pattern amplitude at least equal to τ and a difference-pattern derivative amplitude at least equal to η.

Notably, both the objective functional (3) and the set of constraints (4)-(8) are *linear* functions of the unknowns, and hence the synthesis is reduced to a simple *LP problem*. Moreover, it should be noted that the number of actual unknowns is equal to just a quarter of the overall number of excitations, i.e., 4N, with relevant advantages in terms of computational time and optimality of the solutions.

As far as the choices of N and d are concerned, the following strategy is adopted. **First,** we choose the array size T on the basis of the power pattern constraints. In fact, by virtue of the theory about the degrees of freedom of scattered fields [10], provided $d\leq\lambda/2$, different values of N and d corresponding to the same value of T are expected to lead to very similar radiation performances in the visible part of the spectrum. **Second,** we select N according to the *Theorem 1.3* in [7], wherein (1.6) represent the mathematical condition under which CS guarantees a satisfactory reconstruction. In particular, among all the N values fulfilling such Theorem, we pick up the lowest one in order to minimize the computational burden of problem (3)-(8) above. **Third,** once T and N are selected, we identify the inter-element spacing as $d=T/(2N-1)$.

B. *Second Step: Thinning of the Array Layout*

Starting from the array layout and excitations coming out from the previous step, we perform the following operations:

i. discard the antennas having a normalized excitation amplitude lower than a threshold ε;
ii. recursively substitute each couple of remaining elements whose distance is lower than a threshold ψ with a single element placed in the middle point between them and excited by the sum of the respective excitations.

A sketch representing these operations, which lead to an actual *sparse* array layout, is given in Fig. 1. Obviously, a worsening of the radiation performance is expected after their execution. Therefore, the following step of the procedure is aimed at refining the achieved solution.

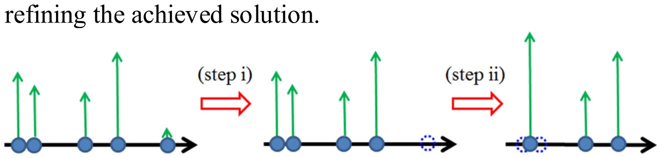

Fig. 1. Layout modifications through operations i and ii of the second step.

C. *Third Step: Final Refinement of the solution*

Due to constraints (4) and to the fact that a constant inter-element spacing is adopted in (1),(2), the array locations coming out from steps i and ii above still result symmetric with respect to the origin of x-axis and can be written as x_{-S}, …, x_{-1}, x_1, …, x_S by adopting the same numbering as above.

To recover from possible radiation losses induced by the second step of the procedure, we determine slight shifts on array locations, say $\Delta x_{-S},\ldots,\Delta x_{-1},\Delta x_1,\ldots,\Delta x_S$, and new excitations $a_{-S}^{\Sigma 1},\ldots,a_S^{\Sigma 1}$, $a_{-S}^{\Delta 1},\ldots,a_S^{\Delta 1}$, in such a way that:

$$-\sqrt{UB^{\Sigma}(u)}\leq F^{\Sigma 1}(u)\leq\sqrt{UB^{\Sigma}(u)}\qquad u\in\Omega^{\Sigma} \tag{9}$$

$$-\sqrt{UB^{\Delta}(u)}\leq jF^{\Delta 1}(u)\leq\sqrt{UB^{\Delta}(u)}\qquad u\in\Omega^{\Delta} \tag{10}$$

$$\left.F^{\Sigma 1}(u)\right|_{u=0}\geq\tau \tag{11}$$

$$j\left.\frac{\partial F^{\Delta 1}(u)}{\partial u}\right|_{u=0}\geq\eta \tag{12}$$

wherein

$$F^{\Sigma 1}(u) = \sum_{s=-S}^{S} a_s^{\Sigma 1} e^{j(x_s + \Delta x_s)\beta u} \quad (13)$$

$$F^{\Delta 1}(u) = \sum_{s=-S}^{S} a_s^{\Delta 1} e^{j(x_s + \Delta x_s)\beta u} \quad (14)$$

respectively represent the 'refined' array factors associated to the sum and difference modalities. In fact, while saving $2(N\text{-}S)$ elements with respect to the first step of the procedure, *any* intersection of constraints (9)-(12) allows restoring the fulfillment of the original requirements (5)-(8).

In order to simplify and accelerate as much as possible the research of a solution to problem (9)-(12), a further action is undertaken. In fact, due to the way the unknown location shifts appears in (13),(14), constraints (9)-(12) represent non-convex sets in the space of the unknowns. This circumstance may prevent the achievement of an optimal solution in case a local-optimization procedure is exploited. On the other side, the adoption of global-optimization algorithms may also result problematic from the computational point of view.

To resolve this issue, we have exploited the fact that, due to the previous fulfillment of constraints (5)-(8), it is expected that just 'small' location shifts are necessary to satisfy (9)-(12), so that (13),(14) can be reasonably approximated by

$$F^{\Sigma 2}(u) = \sum_{s=-S}^{S} a_s^{\Sigma 1} e^{jx_s\beta u}\left(1 + j\beta u \Delta x_s\right) \quad (15)$$

$$F^{\Delta 2}(u) = \sum_{s=-S}^{S} a_s^{\Delta 1} e^{jx_s\beta u}\left(1 + j\beta u \Delta x_s\right) \quad (16)$$

respectively. As a consequence, the last step can be formulated as in (9)-(12) by respectively using $F^{\Sigma 2}$ and $F^{\Delta 2}$ instead of $F^{\Sigma 1}$ and $F^{\Delta 1}$ and adding the convex constraint

$$-\rho \le \beta \Delta x_s \le \rho \;\; \forall s \quad (17)$$

being ρ a user-defined constant such that $0<\rho<<1$.

By so doing, the problem (9)-(12) nearly becomes of the LP class, so that the occurrence of sub-optimal solutions can be avoided while fast local optimization procedures can still be used. Finally, the adopted formulation allows the addition of the convex constraint

$$\left[(x_{S+1} + \Delta x_{S+1}) - (x_S + \Delta x_S)\right] \ge \xi \;\; s = 1,2,\ldots,S-1 \quad (18)$$

in order to avoid possible undesired mutual-coupling effects.

The opportunity of refining in the last part of the procedure the array locations allows adopting in the first step a d value much larger than the ones usually exploited by CS-based approaches and, at the same time, to avoid sequential ℓ_1-norm optimizations of the kind proposed in [5],[6]. Therefore, with respect to these powerful techniques, the procedure exhibits a considerably reduced computational burden without renouncing to the same high radiation performances.

III. Numerical Assessment

Since results are not available for linear arrays being at the same time sparse, phase-only reconfigurable between a sum and a difference radiation mode, and composed by the minimum possible number of elements, the following test cases have been aimed at proving the actual saving in array elements with respect to 'standard' solutions based on equi-spaced arrays.

In order to ensure high scanning capabilities, the synthesis has been performed by extending the width of the spectral domain to regions much larger than the visible range (i.e., $|u|>1$). Each simulation required less than 11 seconds by exploiting an Intel i7 2.5 GHz CPU and a 10 GB RAM.

The first test case concerns a sparse array composed by 28 isotropic elements located over an aperture of 17.1λ. The sum field exhibits an amplitude equal to 20 dB in the target direction and fulfills an upper bound of 0 dB for $|u|\geq 0.1$.

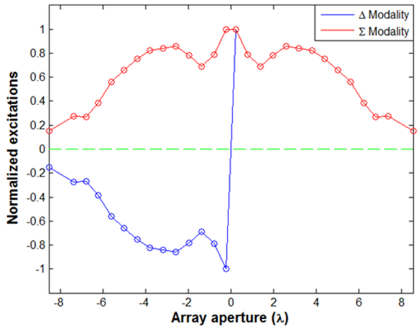

Fig. 2: First test case: synthesized excitations for the sum (red colour) and the difference (blue color) radiation modalities. Design parameters: d=0.3λ; N=29; τ=20 dB; η=46.2 dB; ρ=0.75; ξ=0.44λ; S=14; ε=0.1; ψ=0.3λ.

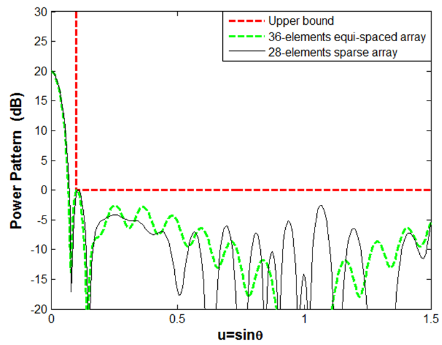

Fig. 3: Sum beam mode: power patterns respectively corresponding through (13) to the red source distribution of Fig. 2 (black color) and to the λ/2 equi-spaced array with the minimum number of elements (green color).

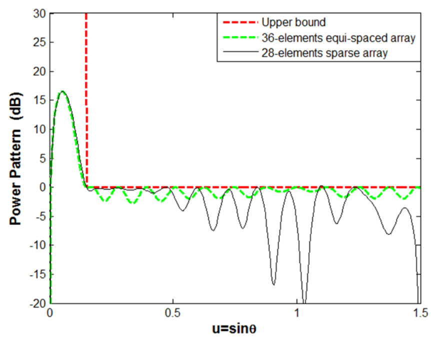

Fig. 4: Difference beam mode: power patterns respectively corresponding through (14) to the blue source distribution of Fig. 2 (black color) and to the λ/2 equi-spaced array with the minimum number of elements (green color).

The square amplitude of the derivative of the difference field is equal to 46.1 dB in the target direction. This pattern fulfills an upper bound of 0 dB for $|u|\geq 0.15$. Fig. 2 shows the achieved excitations and reports in its caption the adopted design parameters, while the corresponding phase-only reconfigurable power patterns and the exploited masks are depicted in figures 3 and 4. The overall synthesis has been performed for u ϵ [-1.5, 1.5]. The minimum, average, and maximum spacing between adjacent elements turned out being equal to 0.44λ, 0.63λ, and 1.15λ, respectively. The procedure allowed saving 22.2% of elements with respect to the case wherein a 0.5λ equi-spaced array with the minimum possible number of elements, i.e., 36, is adopted over the same aperture. The fields radiated by this equi-spaced array are superimposed to our solutions in figures 3 and 4. As it can be seen, despite the relevant elements number reduction, no price is paid in terms of radiation performance.

As a second test case, we designed an array composed by 56 isotropic elements located over an aperture of 32λ. The sum power pattern exhibits an amplitude equal to 20 dB in the target direction and fulfills an upper bound of 0 dB for $|u|\geq 0.035$. The square amplitude of the derivative of the difference field is equal to 52.2 dB in the target direction. This field fulfills an upper bound of 0 dB for $|u|\geq 0.07$. Fig. 5 shows the achieved excitations and reports in its caption the adopted design parameters, while the corresponding phase-only reconfigurable power patterns and the exploited masks are depicted in Figures 6 and 7. The synthesis has been performed for $u \in [-1.8, 1.8]$. The minimum, average, and maximum spacing between adjacent elements turned out being equal to 0.43λ, 0.58λ, and 1.28λ, respectively. The procedure allowed saving 15.2% of elements with respect to the case wherein a 0.5λ equi-spaced array with the minimum possible number of elements, i.e., 66, is adopted over the same aperture. The patterns of this equi-spaced array are superimposed to our solutions in figures 6 and 7. As in the first test case, despite the relevant elements number reduction, no worsening on radiation performance is experienced.

Notably, the constraints adopted in both examples resulted unfeasible by exploiting equi-spaced arrays having an inter-element spacing equal to 0.55λ or 0.6λ (i.e., *values equal to the average spacings of our solutions*) and the same overall length.

IV. Conclusions

An innovative approach has been proposed to the synthesis of one-dimensional, minimally-redundant sparse arrays radiating phase-only reconfigurable sum and difference patterns. By exploiting at best the Compressive Sensing principles and the fundamental results available in the synthesis of single sum and difference beams through equi-spaced arrays, the procedure resulted very fast and effective. Numerical experiments, also including investigations on large-scanning capabilities, have shown relevant array-elements savings with respect to ‘standard’ uniform layouts.

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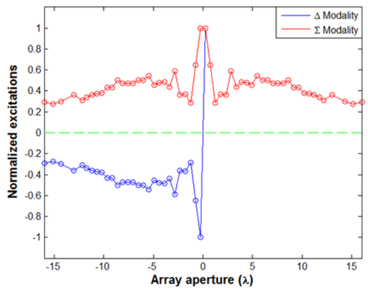

Fig. 5: Second test case: synthesized excitations for the sum (red colour) and the difference (blue color) radiation modalities. Design parameters: d=0.26λ; N=62; τ=20 dB; η=52.2 dB; ρ=0.31; ξ=0.43λ; S=28; ε=0.12; ψ=0.26λ.

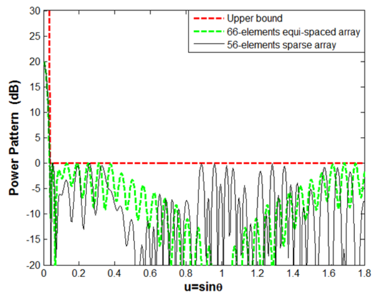

Fig. 6: Sum beam mode: power patterns respectively corresponding through (13) to the red source distribution of Fig. 5 (black color) and to the λ/2 equi-spaced array with the minimum number of elements (green color).

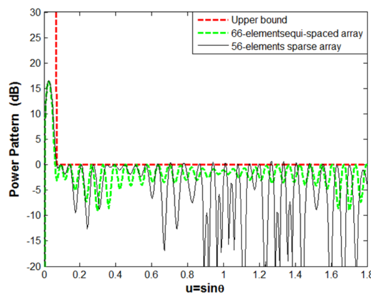

Fig. 7: Difference beam mode: power patterns respectively corresponding through (14) to the blue source distribution of Fig. 5 (black color) and to the λ/2 equi-spaced array with the minimum number of elements (green color).